

$$\frac{\partial^2 f(X, t, T)}{\partial X^2} = e^{-r(T-t)} P(X). \quad (\text{A4})$$

To understand why equation (A4) gives the RND consider the portfolio known as a 'butterfly spread'. This is given by

$$C(X + \varepsilon, t, T) - 2C(X, t, T) + C(X - \varepsilon, t, T) \quad (\text{A5})$$

Where ε in (A5) is a small increment. The portfolio is created by selling two call options at exercise price X , and by purchasing a single call option at exercise price $(X + \varepsilon)$ and another at $(X - \varepsilon)$. The portfolio makes no payout except in the interval $[X - \varepsilon, X + \varepsilon]$. Consider $1/\varepsilon^2$ shares of this portfolio; in the limit as ε tends to zero, the payoff function tends to a Dirac delta function with mass at X , thus the portfolio will pay £1/\$1 if $S_T = X$ and nothing otherwise. The price of the portfolio (A5) must be

$$\frac{1}{\varepsilon^2} [C(X + \varepsilon, t, T) - 2C(X, t, T) + C(X - \varepsilon, t, T)] \quad (\text{A6})$$

and

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon^2} [C(X + \varepsilon, t, T) - 2C(X, t, T) + C(X - \varepsilon, t, T)] \right) = \frac{\partial^2 C(X, t, T)}{\partial X^2} \quad (\text{A7})$$

Thus the risk-neutral probability that $S_T = X$ is the price of a butterfly spread centered at X , in the limit as $\varepsilon \rightarrow 0$, and this is equal to $e^{-r(T-t)} p(X)$. In reality, X is not continuous and options are only available for a limited number of exercise prices at discrete intervals. However, Breeden and Litzenberger (1978) have shown that for discrete data, finite difference methods can be used to obtain a numerical solution to equation (A4). In addition, Neuhaus (1995)¹³ has shown how the RND can be obtained via equation (A4) using finite differences.

¹³ Neuhaus, H., 1995. "The Information Content of Derivatives for Monetary Policy", Discussion Paper 3/95, Economic Research Group of the Deutsche Bundesbank.