EXPLAINING YOUNG MORTALITY

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ABSTRACT. Stochastic modeling of mortality rates focuses on fitting linear models to logarithmically adjusted mortality data from the middle or late ages. Whilst this modeling enables insurers to project mortality rates and hence price mortality products it does not provide good fit for younger aged mortality. Mortality rates below the early 20's are important to model as they give an insight into estimates of the cohort effect for more recent years of birth. It is also important given the cumulative nature of life expectancy to be able to forecast mortality improvements at all ages. When we attempt to fit existing models to a wider age range, 5-89, rather than 20-89 or 50-89, their weaknesses are revealed as the results are not satisfactory. The linear innovations in existing models are not flexible enough to capture the non-linear profile of mortality rates that we see at the lower ages. In this paper we modify an existing 4 factor model of mortality to enable better fitting to a wider age range, and using data from seven developed countries our empirical results show that the proposed model has a better fit to the actual data, is robust, and has good forecasting ability.

JEL Classification: C51, C52, C53, G22, G23, J11

Keywords and Phrases: Mortality, stochastic models, forecasting, non-linearity.

1. INTRODUCTION

In recent years there has been an increasing amount of attention put on the modeling of mortality risk as a significant risk that pension providers and insurance firms are exposed to. These development have been driven in part by the introduction of more stringent regulation and historically low rates of interest and inflation. The later has exposed longevity risk as being a significant risk in its own right and the development of innovative hedging products has allowed risk holders to unbundle longevity risk from the interest and inflation risks.

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There is a significant amount of literature on stochastic modeling of mortality rates. The impetus for the rapid development in stochastic mortality modeling started with the often used model of Lee and Carter (1992) who modeled US male data using a one factor time series approach. Many innovations of the Lee-Carter model have been developed since including, Booth *et al.* (2002), Brouhns *et al.* (2002), Girosi and King (2005), Renshaw and Haberman (2006), Cairns *et al.* (2006), Currie *et al.* (2004), Currie (2006), Hári *et al.* (2008), Tulijapurkar (2008), and Plat (2009).

Many papers propose that mortality in advanced ages is influenced by the mortality experiences at the younger age range and it is clear that the average life expectancy of a population will be affected by experience at all ages. This cumulative effect means that experience at the younger ages is important to consider when modeling the mortality experience of a population. From a demographic viewpoint it is also clear that being able to model and forecast mortality at all ages is important. Hauser and Weir (2010) and Weir (2011) state that greater attention must be given to study designs that allow early-life exposures, experiences, and characteristics to be included in the analysis of outcomes in later life. Cohort effects[1] have been identified as an important component in a mortality model and yet existing models are missing significant information on the most recent cohorts by excluding the younger ages from their models. When we fit existing models to a wider age range starting from age 5 rather than age 20 or 50, the results are not satisfactory[2] since the linear innovations are not flexible enough to capture the non-linear dynamics at the lower ages, the so called "lifestyle" mortality (accidents, drug abuse) profile. In this paper we propose a mortality model that aims to improve upon the fit quality of existing models on a wider age range whilst at the same time not losing sight of the positive aspects of existing models. In particular, using a wider age range introduces a non-linear profile of mortality and we aim to capture this in a better way.

Using the data of a range of developed countries' from 1950 - 2006 we find that the proposed model fits the data very well, is applicable to a fuller age range and captures the cohort effect. It also has a non-trivial correlation structure, captures the non-linear effects at lower ages, has no

[1]The cohort effect was identified in reports by the Government Actuary's Department (1995, 2001, 2002). These reports highlighted that the generations born between 1925 and 1945 (centered on the generation born in 1931) experienced more rapid improvement than earlier and later generations. This feature had been noted for both males and females in the UK.

[2]We show later in the paper that fitting errors more than double in some cases when a wider age range is fitted. See tables 6 and 7 model M9 for example.

robustness problems and can take into account parameter risk, while the structure of the model remains relatively simple.

The remainder of the paper is organized as follows. First, in Section 2 the background to stochastic mortality modeling is reviewed. In Section 3 an empirical comparison of existing models is conducted which further motivates the paper. In Section 4 a modification of the Plat (2009) model is proposed and its fitting and forecasting performance is assessed using the mortality data of 7 different countries. Conclusions are drawn in Section 5.

2. BACKGROUND

Due to the increasing focus on risk management and measurement for insurers and pension funds, the literature on stochastic mortality models has developed rapidly during the last twenty years. A need to measure the performance quality of these models led to the development of a range of criteria against which models could be assessed. In this section we discuss the background to stochastic mortality modeling starting with the criteria. We follow this with an overview of existing stochastic mortality models up to and including the Plat (2009) model.

In order to assess the quality of a model (from both a fitting and a forecasting perspective) we need to have a range of metrics on which we can quantify the performance of the model. A good set of criteria should allow us to quantify the performance of a mortality model against a range of aspects considered to be "good qualities" for a model of mortality rates. Cairns *et al.* (2011) proposed criteria against which a model can be assessed. For example, the model must fit the existing data well, the model must produce biologically reasonable forecasts etc. Using these criteria we can determine how good a particular model is at fitting and forecasting mortality.

Stochastic mortality models either model the central mortality rate or the initial mortality rate (see Coughlan *et al.*, 2007). Let $D_{x,t}$ be the number of people with age x that died in year t, and $E_{x,t}$, the exposure being the average population with age x in the year t, the central mortality rate[3] $m_{x,t}$ is defined as:

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}}, \tag{1}$$

[3]The initial mortality rate q_x is the probability that a person aged x dies within the next year. The different mortality measures are linked by the approximation: $q_x \approx 1 - e^{-m_x}$.

The first and most well known stochastic mortality model is that of Lee and Carter (1992):

$$\ln(m_{x,t}) = a_x + b_x\kappa_t + \epsilon_{x,t}, \tag{2}$$

where a_x and b_x are age effects and κ_t is a random period effect.[4] Applying the necessary constraints the a_x are given by

$$a_x = \frac{1}{N}\sum_{t=1}^{N} \ln m_{x,t}. \tag{3}$$

The bilinear part $b_x\kappa_t$ was then determined as the first singular component of a singular value decomposition (SVD), with the remaining information from the SVD considered to be part of the error structure. The κ_t are estimated and refitted to ensure the model maps onto historic data and the subsequent time series κ_t is used to forecast mortality rates using normal time series forecasting techniques.

Among many discussions of the Lee-Carter model, Cairns *et al.* (2006, 2009, and 2011) summarized the main disadvantages of the model. The model has one factor, resulting in mortality improvements at all ages being perfectly correlated (trivial correlation structure). For countries where a cohort effect is observed in the past, the model gives a poor fit to historical data. The uncertainty in future death rates is proportional to the average improvement rate b_x which for high ages can lead to this uncertainty being too low, since historical improvement rates have often been lower at high ages. Also, the model can result in a lack of smoothness in the estimated age effect b_x.

Despite the weaknesses of the Lee-Carter model it's simplicity has led to it being taken as a benchmark against which other stochastic mortality models can be assessed. There has been a significant amount of literature developing additions to, or modifications of, the Lee-Carter model. For example Booth *et al.* (2002), Brouhns *et al.* (2002), Lee and Miller (2001), Girosi and King (2005), De Jong and Tickle (2006), Delwarde *et al.* (2007) and Renshaw and Haberman (2003, 2006).

Mortality data is 2 dimensional with deaths and exposures being recorded by year and by age. We can therefore consider the data from three different viewpoints, the age profile (or how mortality changes from age to age), the time profile (how mortality rates for a specific

[4]This model was fitted to US mortality data for ages 0-110 between the years of 1933 and 1987.

age change over time), and more recently identified, the cohort profile (how mortality for a specific cohort of the population - those born in a particular year - changes in relation to other cohorts). The Lee Carter model identified the interaction between age and time through the one bilinear factor $b_x\kappa_t$. Many of the modifications since the Lee Carter model have sought to capture additional age-period effects or cohort effects and they can be grouped as such.

2.1. **Cohort effect additions.** Renshaw and Haberman (2006) modified the Lee-Carter model by simply adding a factor γ_{t-x} to capture effects that could be attributed to the year of birth $(t-x)$,

$$\ln(m_{x,t}) = a_x + b_x^1\kappa_t + b_x^2\gamma_{t-x} + \epsilon_{x,t}, \tag{4}$$

where κ_t is defined as before and γ_{t-x} is a random cohort effect.

The model does have a much better fit for countries such as the UK where a cohort effect has been identified, however it suffers from a lack of robustness perhaps due to the presence of more than one local maximum in the likelihood function. Among others, for instance Currie (2006) noted that if the model was fitted using data from 1961-2000 then the parameters showed qualitatively different characteristics to those obtained when fitting to data from 1981-2000. Furthermore, as noted by Currie (2006), although the model incorporates the cohort effect, for most of the simulated mortality rates the correlation structure is still trivial with the simulated cohort parameters only being relevant for the higher ages at the far end of the projection.

Following this analysis Currie (2006) applied a simplified age-period-cohort model of Clayton and Schlifflers (1987) to mortality which removed the robustness problem but at the expense of the fitting quality:

$$\ln(m_{x,t}) = a_x + \kappa_t + \gamma_{t-x} + \epsilon_{x,t}. \tag{5}$$

2.2. **Age-period effect additions.** Cairns *et al.* (2006) observed that for England & Wales and United States data, the fitted cohort effect appeared to have a trend in the year of birth. This suggested that the cohort effect was compensating for the lack of a second age-period effect, as well as trying to capture the cohort effect in the data. This led them to introduce a two factor model of mortality,

$$\text{logit}(q_{x,t}) = \kappa_t^1 + \kappa_t^2(x-\bar{x}) + \epsilon_{x,t}, \tag{6}$$

where $\bar{x}$ is the mean age in the sample range and (κ_t^1, κ_t^2) are assumed to be a bivariate random walk with drift. The two factors in this model were both period factors with no cohort effect allowed for. This was rectified in Cairns *et al.* (2009), namely capturing the cohort effect as an additional effect on top of the two age-period effects. All these models have multiple factors resulting in a non-trivial correlation structure which mirrors the reality that improvements in mortality rates are different for different age ranges. A further adaptation was also created allowing for the cohort effect to diminish over time. The main problem with these models arises from the fact that they were designed for higher ages and so ignored the modeling of mortality at the lower ages (for example the accident hump). Cairns *et al.* (2009) argue that the significant cost associated with mortality is at the older ages and thus their modeling focused on those ages. When using these models for full age ranges, the fit quality is relatively poor and the projections are biologically unreasonable.

2.3. **Age-period and cohort additions combined.** Plat (2009) wanted to develop a model which maintained the good aspects of the existing models whilst leaving out the weaker features. The result was a four factor model which took its beginnings from the Lee-Carter model and which added factors to capture the second age-period effect, as per the Cairns *et al.* (2006) model and the cohort effect, as per the Renshaw and Haberman (2006) model. The innovation in the Plat model was to then add a further period factor affecting only the lower ages and designed to allow the model to fit to the whole age range. The model specification is given by:

$$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3(\bar{x} - x)^+ + \gamma_{t-x} + \epsilon_{x,t}, \tag{7}$$

where the a_x is similar to that of the Lee-Carter model and makes sure that the overall shape of the mortality curve by age is reasonable, the κ_t^1 and κ_t^2 model the mortality rates as in the Cairns *et al.* (2006) model and the κ_t^3 models the effects specific to the lower ages only where $(\bar{x} - x)^+$ takes the value $(\bar{x} - x)$ when this is positive and zero otherwise. Finally the γ_{t-x} models the cohort effect.

The range of existing models described above meet most of the criteria set out by Cairns *et al.* (2011) and the Plat model meets all of the criteria by it's very design. However, when the age range is widened to allow for the non-linear characteristics of young mortality experience then

as far as we are aware, none of the existing models meet the above criteria adequately (although some are close). This is the starting point of this paper.

3. EMPIRICAL COMPARISON OF EXISTING MODELS

In this section we empirically compare the existing models to see their performance when the age range is widened to allow for the non-linear mortality experience at lower ages. For ease of notation we will use the naming convention established by Cairns *et al.* (2009). Table 5 in the appendix sets out the names we will use for each of the models.

We fit the models to different countries and to different age ranges for each country. The data sets[5] used are: Male mortality data during 1950-2006 for age ranges 5-89, 20-89 and 50-89 of Great Britain (GB), England & Wales (E&W), Scotland (SCO), United States (US), Australia (AUS), New Zealand (NZ), and The Netherlands (NL). Although a longer history is available for some of the countries, we have used the period 1950 - 2006 for all the countries as this data is more reliable and will allow a valid comparison with the results of Cairns *et al.* (2009 and 2011), and with Plat (2009) who used the period 1960 - 2006. The model fit is compared using the Mean Average Percentage Error (MAPE) measure and the Bayesian Information Criterion(BIC) measure.

The MAPE measures the average difference in absolute value between $\hat{m}_{x,t}$, the estimate of $m_{x,t}$, and $m_{x,t}$ itself, it is defined by:

$$\text{MAPE} = \frac{1}{NM} \sum_{x,t} \frac{\|\hat{m}_{x,t} - m_{x,t}\|}{m_{x,t}}. \tag{8}$$

where we have N time dimensions (in this case N=57) and M age dimensions (in this case M=70).

The BIC measure provides a trade-off between fit quality and parsimony of the model and it is defined as:

$$\text{BIC} = L(\hat{\phi}) - \frac{1}{2} K \ln(P), \tag{9}$$

where $L(\hat{\phi})$ is the log-likelihood of the estimated parameter $\hat{\phi}$, P is the number of observations and K is the number of parameters being estimated.

[5]The data consists of numbers of deaths $D_{x,t}$ and the corresponding exposures $E_{x,t}$ and is extracted from: www.mortality.org, see HMD 2004.

Table 1 gives a comparison of the fitting results (in terms of MAPE) to the age range 5-89. Tables 2 and 3 show the fitting results to ages 20-89 and 50-89. We see from tables 1 and 2 that when a wide age range is used (5-89 or 20-89), the Plat model M9 is not the best fitting model, however, if we exclude model M2, which suffers from robustness issues, the Plat model is confirmed to be the best fitting model over the age range 20-89. When fitting to the age ranges 5-89 and 20-89 it is important to note that the models of Cairns *et al.* (2006, 2009) do not perform very well for these age ranges, since they were designed for higher ages only. For comparison we also fit the existing models to data between 1950 and 2006 for ages 50-89 only. Table 3 shows that the Plat model still outperforms other models.

TABLE 1. The MAPE for the model fit to ages 5-89 (%)

Model	M1	M2	M3	M5	M6	M7	M9
GB	6.14	**3.91**	6.96	21.83	16.71	12.88	7.56
E&W	6.38	**4.16**	7.08	21.83	16.87	13.03	7.64
SCO	10.97	**9.28**	12.76	19.99	18.74	15.76	14.72
US	4.58	**2.96**	5.43	16.08	15.59	15.20	5.65
NL	8.99	**7.01**	7.91	23.57	17.82	12.95	7.22
AUS	7.45	**6.44**	8.80	23.86	20.46	18.52	9.61
NZ	12.32	**11.86**	13.66	27.42	25.46	23.84	13.74

TABLE 2. The MAPE for the model fit to ages 20-89 (%)

Model	M1	M2	M3	M5	M6	M7	M9
GB	14.45	**3.19**	14.53	16.53	9.93	7.60	3.27
E&W	14.34	**3.39**	14.42	16.82	10.09	7.73	3.50
SCO	15.67	**6.31**	15.70	16.45	10.32	8.81	**6.31**
US	12.47	**2.46**	12.53	14.07	7.92	6.30	2.76
NL	12.54	**4.16**	12.62	16.14	11.20	8.03	4.22
AUS	5.67	**4.56**	5.84	17.10	10.99	8.40	5.25
NZ	9.57	**8.57**	9.26	19.20	15.32	12.06	9.19

TABLE 3. The MAPE for the model fit to ages 50-89 (%)

Model	M1	M2	M3	M5	M6	M7	M9
GB	2.86	1.75	2.00	3.87	1.93	1.53	**1.36**
E&W	2.94	1.87	2.13	4.03	2.02	1.62	**1.48**
SCO	4.05	3.33	3.17	4.57	3.29	3.14	**2.82**
US	2.21	1.47	1.61	2.52	2.09	1.87	**1.41**
NL	4.05	2.39	2.59	4.01	2.19	2.16	**2.07**
AUS	3.18	3.22	3.78	3.62	2.94	2.65	**2.59**
NZ	5.64	5.46	5.94	6.35	5.81	5.78	**5.37**

We also look at the fitting results based on the BIC. Tables 6, 7, and 8 in the appendix show the BIC measures for the seven countries, based on fitting to the full age 5-89, the 20-89 age range, and the 50-89 age range, respectively. We see from the tables that it is unclear which model is the best performing using a BIC measure with the Renshaw-Haberman model, M2, showing some good fitting performances, but with models M3, M5, M6, and M9, all performing well on some countries data sets. A particular point to note at this stage (and to motivate the discussion further), is that by widening the age range from 20-89 to 5-89 we can see that for the Plat model for example, the fit quality moves from 3.27% on the 20-89 age range to 7.56% on the 5-89 age range.

To understand why the Plat model does not perform very well for the wider age range and to motivate our further analysis, we look at male data from GB and US. At first, it might be informative to split the data into the period effect and the age effect. Figures 1 and 2 plot the time effect for GB and US males at ages 15, 35, 55 and 75 with each graph showing the natural logarithm of mortality between the years 1950 and 2006. We see from figures 1 and 2 that the logarithm of mortality for both GB and US shows a markedly downward trend over time for each of the age ranges, and the mortality looks more volatile at the younger ages, in this case the 15 and 35 year old samples. This might be attributed to the small numbers of deaths at those ages and the fact that deaths at the lower ages are due to a wider range of causes influenced by "lifestyle" choices and so are not linked to general deterioration due to ill health and old age.

Focusing on specific years and looking at the mortality effect for the whole age range, in figures 3 and 4, we can see that a linear pattern does emerges beyond age 25 or so, however, looking at the mortality below that age we see a very clear non-linear pattern arising. Again this is due to "lifestyle" factors and in order to model these effects we require more flexibility in the factors than the existing model allow.

Looking at the 4 factor model of Eq. (7), the design innovation was to include the additional factor $\kappa_t^3(\bar{x} - x)^+$. This factor adds, in a linear way, an additional flexibility for ages less than the mean of the data set. In the case of Plat this would be for ages less than 55. Figures 3 and 4 show clearly that the logarithm of mortality for ages below the mean of 55 are far from linear.

As we have seen from tables 1 - 3 whilst the Plat model performs relatively well when fit to the data set from age 20, its performance dips somewhat when fitted to the larger data set. In terms of the MAPE when looking at tables 1 and 2 we find that when the wider age range

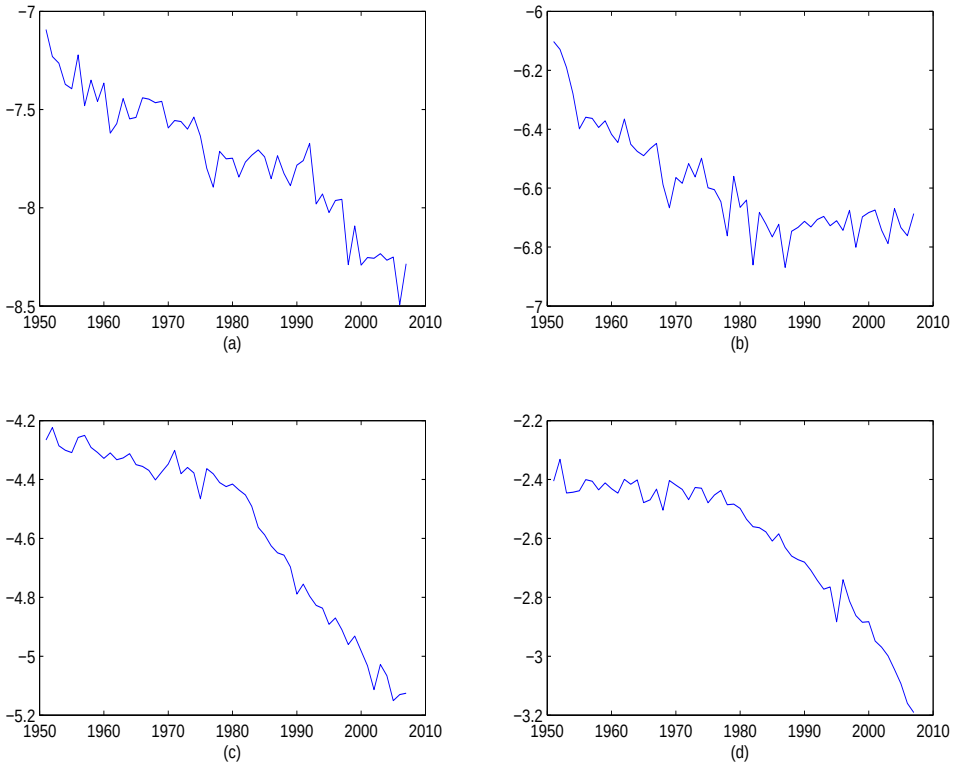

FIGURE 1. Logarithm of mortality by year for GB males aged (a) 15, (b) 35, (c) 55, and (d) 75.

is fitted the percentage error more than doubles across all countries for which we have fit the model. This implies that the addition of a fourth linear factor is inadequate when modeling mortality at lower ages. In the following section we propose a modification to the Plat model which introduces some additional flexibility into the model allowing it to be more adequately fitted to a wider age range.

4. A MODIFICATION TO THE PLAT MODEL

In this section we incorporate the non-linear features of mortality at younger ages into an adaptation of the Plat model proposing an alternative better fitting model. We show the quality of the fit of the proposed model with that of the existing models by fitting to data from a range of countries for the age ranges 5-89, 20-89 and 50-89 and for years 1950-2006.

4.1. **The model.**

We model the central mortality rate $m_{x,t}$ as:

$$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3\big((\bar{x} - x)^+ + [(\bar{x} - x)^+]^2\big) + \gamma_{t-x} + \epsilon_{x,t}, \tag{10}$$

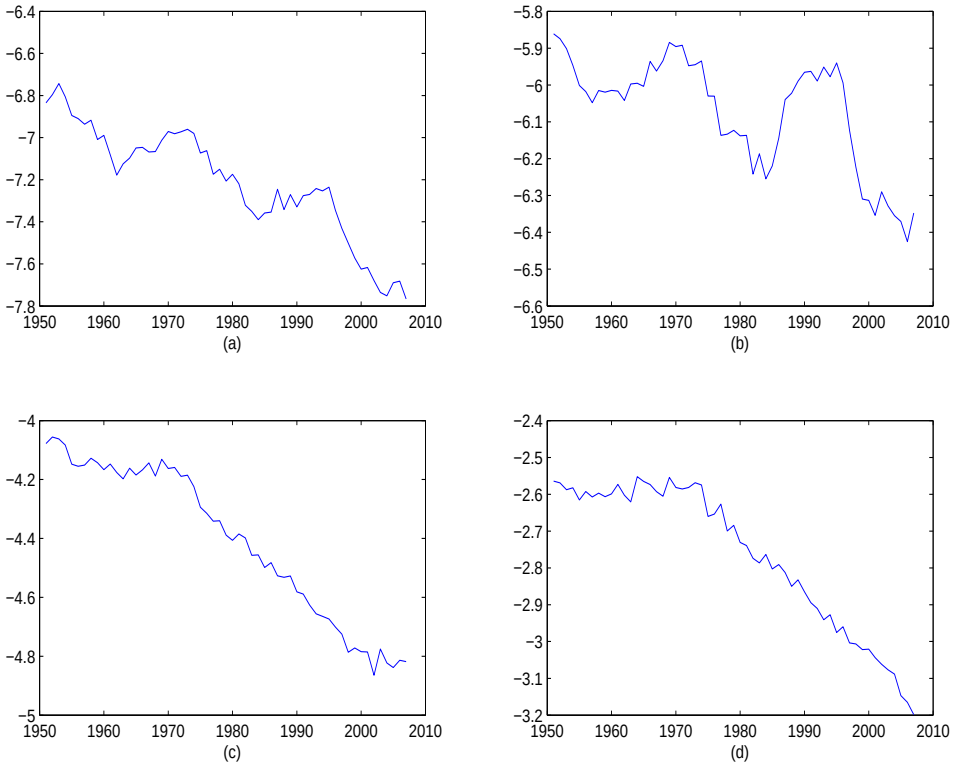

FIGURE 2. Logarithm of mortality by year for US males aged (a) 15, (b) 35, (c) 55, and (d) 75.

where a_x makes sure that the basic shape of the mortality curve over ages is in line with historical observations as in the Lee-Carter model (2) and the κ_t^1 factor represents changes in the level of mortality for all ages. Following the reasoning in Cairns *et al.* (2006), the (long-term) stochastic process for this factor should not be mean reverting. The κ_t^2 factor allows changes in mortality to vary between ages reflecting the historical observation that improvement rates can differ for different age classes and κ_t^3 models the effects specific to the lower age only as in the Plat model (7). The adjusted coefficient of κ_t^3 is designed to capture some of the non-linear effects observed at the lower ages, the "quadratic lower age effect"[6]. Finally the γ_{t-x} models the cohort effect in the same way as the models of Currie (2006) and Cairns *et al.* (2009) and Plat (2009). The proposed model (10) has 4 stochastic factors, and so has a relatively simple structure similar to the Plat (2009), Currie (2006) and the Cairns *et al.* (2006) models.

[6]We also look at the more general case $\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x}-x) + \kappa_t^3\big((\bar{x}-x)^+ + a[(\bar{x}-x)^+]^2\big) + \gamma_{t-x} + \epsilon_{x,t}$ where the parameter "a" was included to test a range of different quadratic coefficients. However, we found that the fit quality did not vary much, on both BIC and MAPE for non-zero values of "a", and we therefore focus on a model with a parameter $a = 1$. Results of general "a" are available on request.

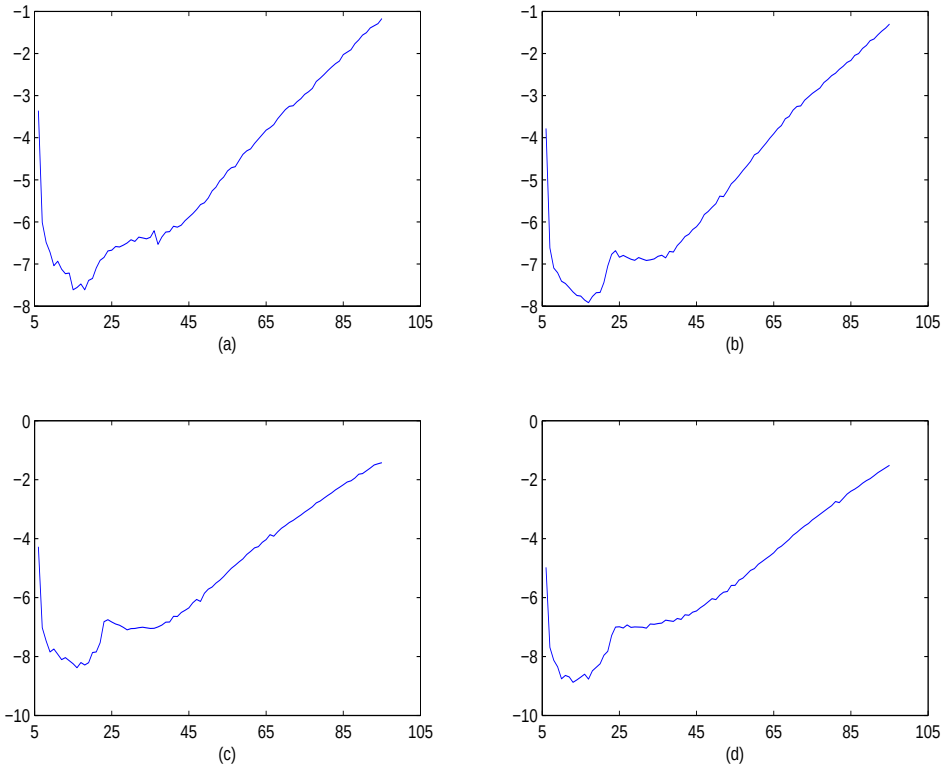

FIGURE 3. Logarithm of mortality for GB males during the years (a) 1950, (b) 1965, (c) 1980, and (d) 2005.

Historical data indicates that the dynamics of mortality rates at lower ages (up to age 40 / 50) whilst still showing a downward trend over time does show much more variation around the trend. This can be attributed in part to the small number of deaths and in part to the nature of deaths at these ages, the so called "lifestyle" mortality factors (smoking, drug abuse, alcohol abuse, car accidents and violence) for example. In the Plat model the factor κ_t^3 was added. In model (10) we modify the coefficient of κ_t^3 to capture the non-linear dynamics observed in the historical data.

The factor κ_t^1 shows a trend and is fitted with a non-stationary ARIMA process. The factors κ_t^2 and κ_t^3 allow the model to have a non-trivial correlation structure between ages. Fitting a non-stationary ARIMA-process for these factors could result (in some scenarios) in projected scenarios where the shape of the mortality curve over ages is not biologically reasonable. Therefore, a stationary (mean reverting) process will be assumed for these factors. The process for the cohort effect factor γ_{t-x} should not have a trend since we should not expect cohort effects to improve year on year. Therefore, a trendless mean reverting process will be assumed for γ_{t-x}.

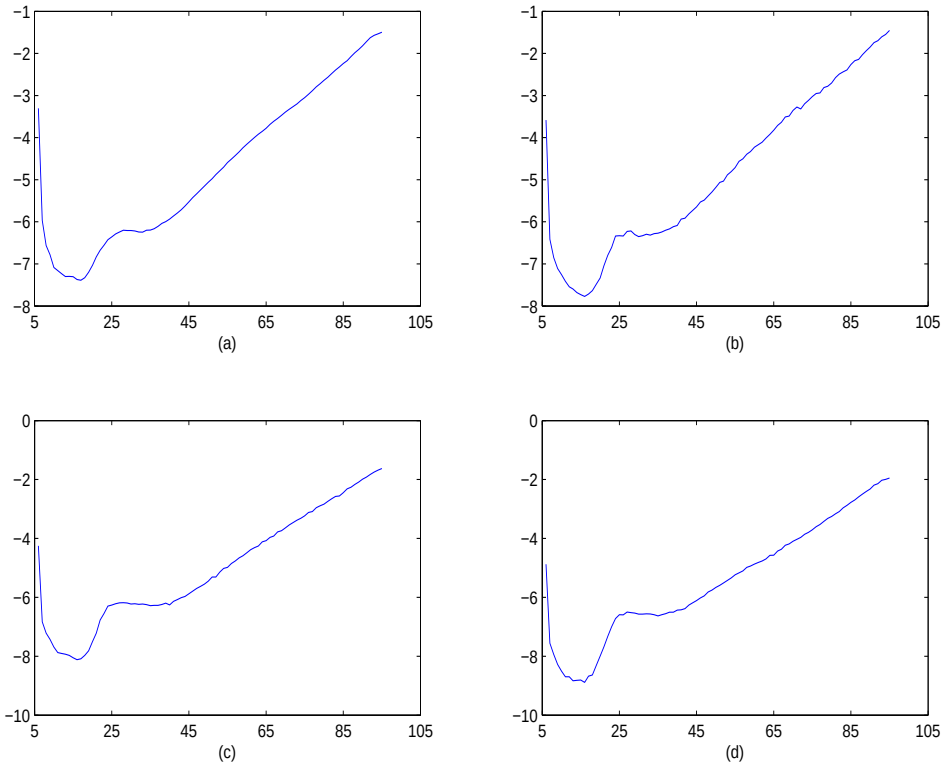

FIGURE 4. Logarithm of mortality for US males during the years (a) 1950, (b) 1965, (c) 1980, and (d) 2005.

As with all stochastic mortality models, the mortality model proposed above has an identifiability problem, meaning that different parameterizations could lead to identical values for $\ln(m_{x,t})$. However, this can be resolved by setting identifiability constraints. As the model has the same time series structure to that of the Plat (2009), following an approach of Cairns *et al.* (2009, model M6), we have

(1) $\sum_{c=c_0}^{c=c_l} \gamma_c = 0$
(2) $\sum_{c=c_0}^{c=c_l} c\gamma_c = 0$
(3) $\sum_t \kappa_t^3 = 0$

where c_0 and c_1 are the earliest and latest year of birth to which a cohort effect is fitted, and $c = t - x$. These constraints are the same as for the Plat(2009) model as is the rationale behind the choice of the constraints.

Fitting methodology — The original method by which to fit such a stochastic model was to use SVD as used in Lee and Carter (1992). Brouhns *et al.* (2002) described an alternative fitting methodology for the Lee-Carter model in which the number of deaths $D_{x,t}$ is modeled

as a Poisson distribution with parameter $(E_{x,t}m_{x,t})$ where $m_{x,t}$ is the mortality rate we are estimating. The main advantage of the Brouhns *et al.* (2002) approach over the SVD approach is that it accounts for the heteroscedasticity of the mortality data for different ages. Indeed this method has been used more commonly, see for example Renshaw and Haberman (2003, 2006) and Cairns *et al.* (2009), Plat (2009). We adopt this approach and model the number of deaths by $D_{x,t} \approx \text{Poisson}(E_{x,t}m_{x,t})$. The parameters of model (10) are estimated by maximizing the log-likelihood function[7]:

$$L(\phi; D, E) = \sum_{x,t} D_{x,t} \ln[E_{x,t}m_{x,t}(\phi)] - E_{x,t}m_{x,t}(\phi) - \ln(D_{x,t}!). \tag{11}$$

Besides estimates for a_x, the fitting procedure described above leads to time series of estimations of κ_t^1, κ_t^2, κ_t^3, and γ_{t-x}. After fitting the model we take the fitted values for the time series and fit suitable ARIMA-processes.

4.2. **Comparison of fit quality with existing models.** To evaluate whether the proposed model fits historical data well, we fit the model to the data sets described in Section 3. We also fit the model to the three different age ranges, 5-89, 20-89 and 50-89 to show the flexibility of the proposed model. The fitting quality for each of the countries, using a MAPE and BIC measure are presented in table 4.

TABLE 4. MAPE and BIC results for model M10

Country	MAPE			BIC		
	5-89	20-89	50-89	5-89	20-89	50-89
GB	5.88	2.77	1.29	-28326	-21887	-13148
E&W	6.05	2.79	1.37	-27549	-21634	-13074
SCO	11.88	2.79	2.76	-18586	-15768	-10096
US	4.14	2.31	1.35	-50487	-29998	-18727
NL	6.11	2.67	1.99	-19586	-16729	-10601
AUS	8.14	2.65	2.48	-22466	-18327	-10956
NZ	13.65	2.69	5.39	-17272	-14714	-9433

When a wider age range is used the logarithm of mortality is no longer relatively linear. However, when comparing the results with table 1 (Excluding the results from model M2,the Renshaw-Haberman model, because of robustness problems) we see that this non-linearity is captured adequately by the quadratic lower age effect in the proposed model. Across all countries considered in this paper the proposed model fits the data better than the previous best fitting

[7]We used an adaptation of the R-code of the software package "Lifemetrics" which is an open source toolkit for measuring and managing longevity and mortality risk, designed by J.P. Morgan, see http://www.lifemetrics.com and http:// www.r-project.org/.

models. Looking at the results when compared with table 2 the performance of the model is still very good when compared with the leading stochastic models of mortality. Comparisons with the results of table 3 show that the model still outperforms the existing stochastic models for the age range 50-89. As the improved specification has been done within a 4 factor framework this model has a similar structure to the previously best performing model on a fitting measure, namely the Plat model. Thus the model remains relatively parsimonious and this is reflected in the BIC measures in table 4 when compared with tables 6, 7, and 8 in the appendix.

The goodness of fit of stochastic mortality models can be evaluated by analyzing residuals of the models, Dowd *et al.* (2010a) applied the t-test, variance ratio test, and the Jarque-Bera test among others to six stochastic mortality models (M1, M2, M3, M5, M6 and M7) using the English and Welsh male mortality data. We carried out similar tests for model M10 using US and GB data. Results[8] of these tests show that the proposed M10 model performs adequately when compared to those in the Dowd *et al.* (2010a).

Fitting the ARIMA processes — In the remainder of this subsection, we focus on the populations of GB and US males and on fitting to the age range 5-89. After fitting the model to the population data the next step is to select and fit suitable ARIMA-process to the time series' of κ_t^1, κ_t^2, κ_t^3, and γ_{t-x}. The fitted parameters κ_t^1, κ_t^2, κ_t^3, and γ_{t-x} for GB males are given in figure 5 and for US males are given in figure 7. The estimates for the α_x parameters are given in figure 6 and figure 8. The figures shows that the pattern of the important parameter κ_t^1 is well-behaved. The patterns of the other parameters all reveal some autoregressive behavior. Since the factor κ_t^1 drives a significant part of the uncertainty in mortality rates, its relatively regular behavior (for this particular dataset) will also show in the relatively narrow confidence intervals.

The parameters for the Plat model are plotted in the appendix as figures 11, 12, 13 and 14 for comparison purposes. They show that the qualitative characteristics of the parameters κ_t^1, κ_t^2, κ_t^3, and γ_{t-x} remain unchanged with the more general model specification.

It is commonly assumed that the time series driving the dynamics, namely κ_t^1 should be fitted with an ARIMA(0,1,0) time series. For the other parameters, which show some autoregressive behavior, we have fit them with ARIMA(1,0,0) processes as in Plat (2009). It is also commonly assumed (see Renshaw and Haberman (2006), CMI (2007) and Cairns *et al.* (2011)) that the process for γ_{t-x} is independent of the other processes, so the parameters of this process

[8]The results are available upon request.

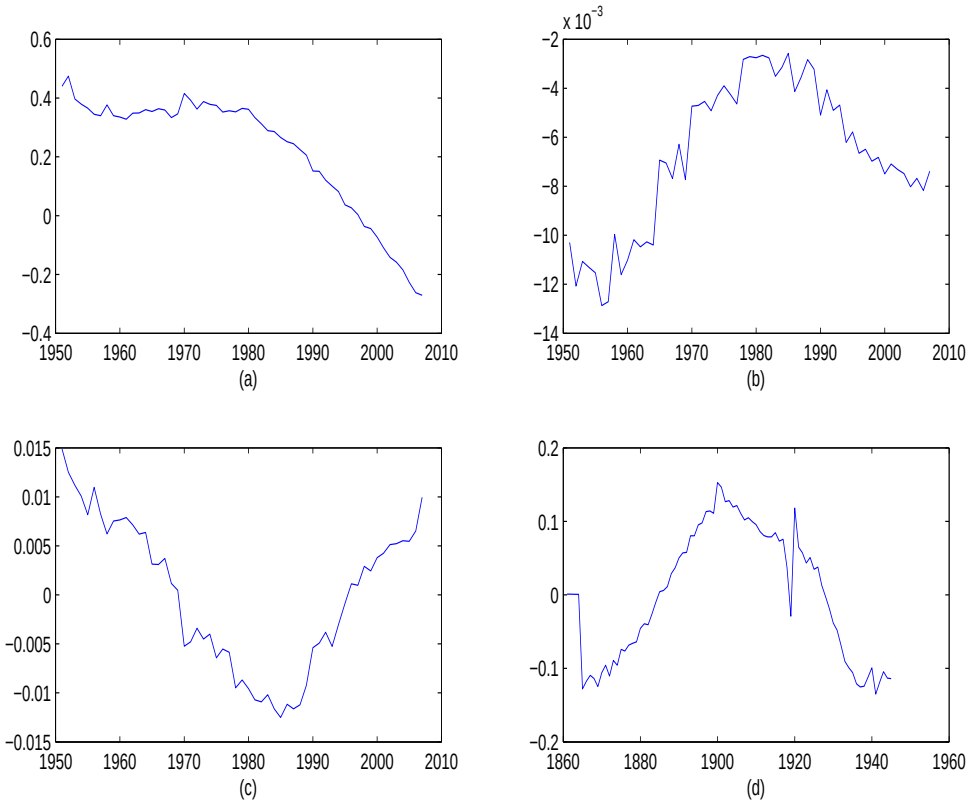

FIGURE 5. Estimated values of (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} based on GB males aged 5-89 between years 1950 and 2006.

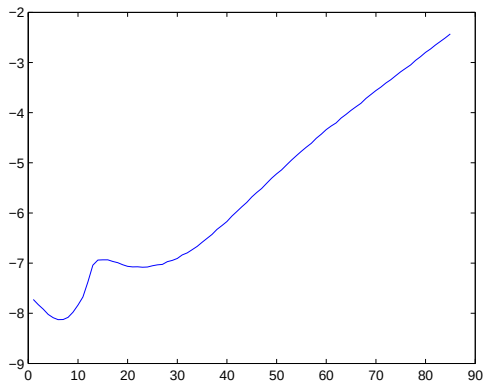

FIGURE 6. Estimated values of a_x based on GB males aged 5-89 between years 1950 and 2006.

can be fitted independently using Ordinary Least Squares. The other processes can be fitted simultaneously using Seemingly Unrelated Regression.

4.3. **Forecasting.** This section shows the simulation results and results of robustness tests for the proposed mortality model.

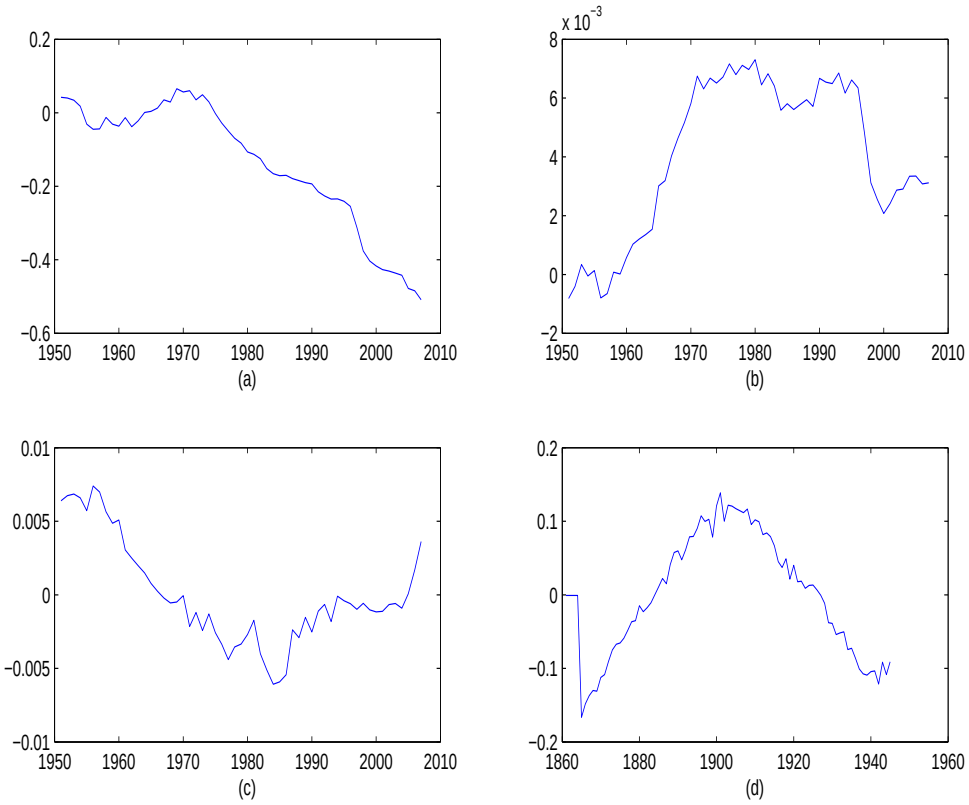

FIGURE 7. Estimated values of (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} based on US males aged 5-89 between years 1950 and 2006.

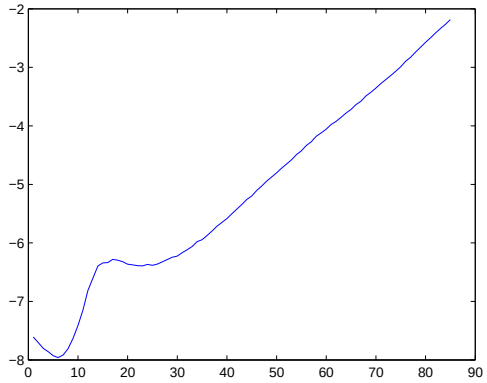

FIGURE 8. Estimated values of a_x based on US males aged 5-89 between years 1950 and 2006.

Using the fitted ARIMA processes and the fitted values for a_x and γ_{t-x} (see figures 5, 6, 7 and 8[9]), future mortality rate scenarios can be constructed using Monte Carlo simulation. Figures 9 and 10 show simulation results for ages 15, 35, 55 and 75 for GB males and US males.

[9]The fitted values for a_x and γ_{t-x} for England & Wales, Scotland, Netherlands, Australia and New Zealand are available in the appendix in figures 15 and 16.

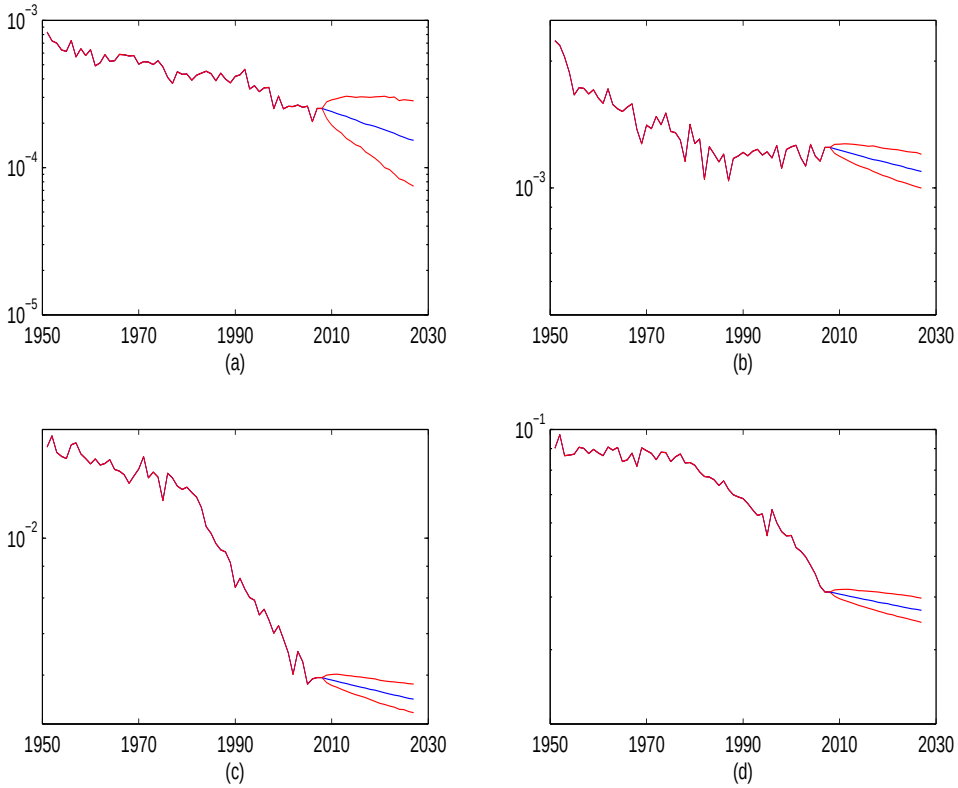

FIGURE 9. Log mortality rates from 1950-2006 followed by forecasting results 2006 - 2026 (mean and 95% confidence intervals) for ages (a) 15, (b) 35, (c) 55, and (d) 75 for GB males.

For higher ages, the widths of the confidence intervals are broadly similar as the models of Plat (2009) and Cairns *et al.* (2011) confirming the results are biologically plausible. The results for younger ages (15 and 35) also seem plausible, where the observed historical variability is reflected in the wider confidence intervals.

Recall that some models suffer from a lack of robustness, for instance the Renshaw-Haberman model is not robust for changes in range of years. The model proposed in this paper is tested for robustness by fitting the model to data from 1975-2006. In doing this we are looking to observe that the qualitative characteristics of the fitted parameters have not changed because the fitting period is different. We are not looking to show that the trend direction is unchanged, or that the actual forecasts are unchanged. It is a characteristic of these sorts of models that the forecasted trend will to an extent be dependent on the period over which the model has been fitted to the data. Given that it is likely the trend forecast will be different when fit to the period 1975-2006 compared to 1950-2006, it is inevitable, for all models, that the simulation results will be somewhat different.

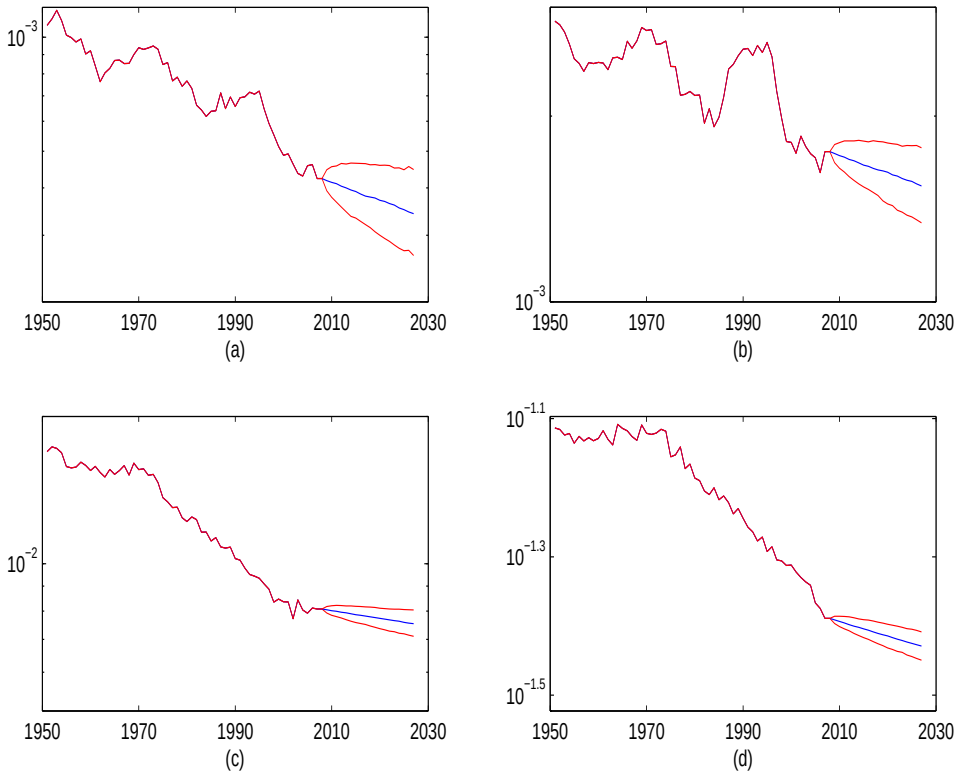

FIGURE 10. Log mortality rates from 1950-2006 followed by forecasting results 2006 - 2026 (mean and 95% confidence intervals) for ages (a) 15, (b) 35, (c) 55, and (d) 75 for US males.

Figures 17 and 18 in the Appendix plot the fitted parameters for GB and US data from 1975-2006. The illustrations show that the estimated parameters do not show significantly different qualitative characteristics when fitted to a different data set. The conclusion is that the proposed model is robust for the fitting periods given above.

Furthermore, backtesting (as in Dowd *et al.* 2010) of the model has been carried out, meaning that the model is fitted to historical data, 1950-1999 in this case, and the forecast results are compared with the actual observations for the period 2000-2006. The results are illustrated in figure 19 in the Appendix where we can see that the proposed model performs adequately.

We have shown so far that the proposed model produces plausible results and they seem robust. Plat (2009) came to the same conclusion for model M9 and Cairns *et al.* (2011) came to the same conclusion for the models of Currie (2006) and Cairns *et al.* (2006, 2009), M7. The models of Cairns *et al.* (2006, 2009) are designed for higher ages, so will not produce plausible results for lower ages. Compared to those models the proposed model has the advantage that it does produce plausible results for a full age range.

Compared to the model of Currie (2006) the proposed model has the advantage that it has a non-trivial correlation structure. This is important because often insurers and pension funds have different type of exposures for younger or middle ages (term insurance, pre-retirement spouse option) than for higher ages (pensions, annuities). Aggregating these different types of exposures can only be done sufficiently if the model has a non-trivial correlation structure. Assuming an almost perfect correlation between ages, as in the Currie (2006) model, will possibly lead to an overstatement of the diversification benefits that arise when aggregating these exposures. Compared to the model of Plat (2009) the proposed model produces plausible forecasts for the lower age range (below age 20) for which the Plat model was not designed.

5. CONCLUSIONS

In this paper we identify and address a limitation of the Plat (2009) model and previous stochastic mortality models. This limitation is in the inability of existing models to adequately fit mortality rates at the lower ages due to the non-linear dynamics at the lower ages, the so called "lifestyle" mortality profile. We believe that it is important to be able to factor in such mortality rates into a single mortality model because of the cumulative nature of mortality and from a demographic viewpoint it is clearly important to be able to model and forecast mortality rates at all ages. The proposed model has the additional flexibility to fit to the mortality rates of a wider age range, 5-89. In particular, the model captures the non-linear profile of mortality at lower ages. We show that the model has a better fit for the range of countries considered in this study. We have also shown that the model does not lose any of the benefits of the previous stochastic models.

The results of this analysis have exposed the weakness of previous models when trying to fit to non-linear features of the data and shows that a more non-linear flexibility is needed to capture the mortality profile, particularly at lower ages. To develop this area further we now need to address the "lifestyle" factors affecting mortality rates in this age range. These may be affected by policy, social, environmental and economic pressures suggesting that a future approach may be to model the underlying causes rather than by trend forecasting.

Appendix: Additional Tables and Figures

TABLE 5. The names of stochastic mortality models

Name	Model and Name
M1	Lee and Carter (1992)
	$\ln(m_{x,t}) = a_x + b_x\kappa_t + \epsilon_{x,t}$
M2	Renshaw and Haberman (2006)
	$\ln(m_{x,t}) = a_x + b_x^1\kappa_t + b_x^2\gamma_{t-x} + \epsilon_{x,t}$
M3	Currie (2006)
	$\ln(m_{x,t}) = a_x + \kappa_t + \gamma_{t-x} + \epsilon_{x,t}$
M5	Cairns *et al.* (2006)
	$\text{logit}(q_{x,t}) = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \epsilon_{x,t}$
M6	Cairns *et al.* (2009) with cohort effect
	$\text{logit}(q_{x,t}) = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \gamma_{t-x} + \epsilon_{x,t}$
M7	Cairns *et al.* (2009) with cohort and quadratic age effect
	$\text{logit}(q_{x,t}) = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \kappa_t^3((x - \bar{x})^2 - \sigma_x^2) + \gamma_{t-x} + \epsilon_{x,t}$
M9	Plat (2009)
	$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3(\bar{x} - x)^+ + \gamma_{t-x} + \epsilon_{x,t}$
M10	Quadratic effect model
	$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3\big((\bar{x} - x)^+ + [(\bar{x} - x)^+]^2\big) + \gamma_{t-x} + \epsilon_{x,t}$

Note: The model M4 and M8 are not included in our analysis. The M4 is a P-splines model developed in Currie (2006), it is of a structurally different nature to the remaining stochastic models. The M8 in Cairns *et al.* (2009) with diminishing cohort effect is a modification of the M5, it was primarily designed for ages over and above 50. The M10 is the model that we propose in this paper.

TABLE 6. The BIC for the model fit to ages 5-89

Model	M1	M2	M3	M5	M6	M7	M9
GB	-38228	**-28854**	-34181	-150856	-96992	-57365	-33640
E&W	-36686	**-28315**	-32984	-136658	-88876	-53211	-32295
SCO	-20960	**-20633**	-20787	-29413	-25752	-22671	-20998
US	-72612	**-43997**	-69820	-552628	-271679	-258323	-66989
NL	-24914	**-22122**	-22516	-55568	-37711	-26912	-22178
AUS	-24340	**-23217**	-25594	-82648	-44443	-29848	-25692
NZ	**-17842**	-18288	-18154	-31360	-23208	-22149	-18284

TABLE 7. The BIC for the model fit to ages 20-89

Model	M1	M2	M3	M5	M6	M7	M9
GB	-33926	**-24684**	-27031	-91937	-58540	-39770	-24921
E&W	-32516	**-24236**	-26558	-83889	-54516	-37511	-24551
SCO	-18326	-17904	**-17575**	-22881	-19994	-18897	-17689
US	-64565	**-37863**	-56350	-368252	-143067	-97548	-43425
NL	-20928	-19012	-18980	-32420	-26601	-21424	**-18778**
AUS	-20833	**-19909**	-21449	-57681	-30301	-23740	-20697
NZ	**-15282**	-15714	-15484	-22794	-18177	-16394	-15818

TABLE 8. The BIC for the model fit to ages 50-89

Model	M1	M2	M3	M5	M6	M7	M9
GB	-20246	-15074	-16493	-26013	-15756	-14874	**-14834**
E&W	-19591	-14891	-16301	-25094	-15524	-14817	**-14759**
SCO	-11394	-11247	**-11050**	-11675	-11111	-11146	-11259
US	-29981	**-20581**	-22653	-35419	-27749	-22562	-21598
NL	-13344	-11902	-11910	-13009	**-11684**	-11786	-11909
AUS	-12337	**-12187**	-12864	-12798	-12324	-12306	-12278
NZ	**-9534**	-9873	-9801	-9698	-9847	-9993	-9984

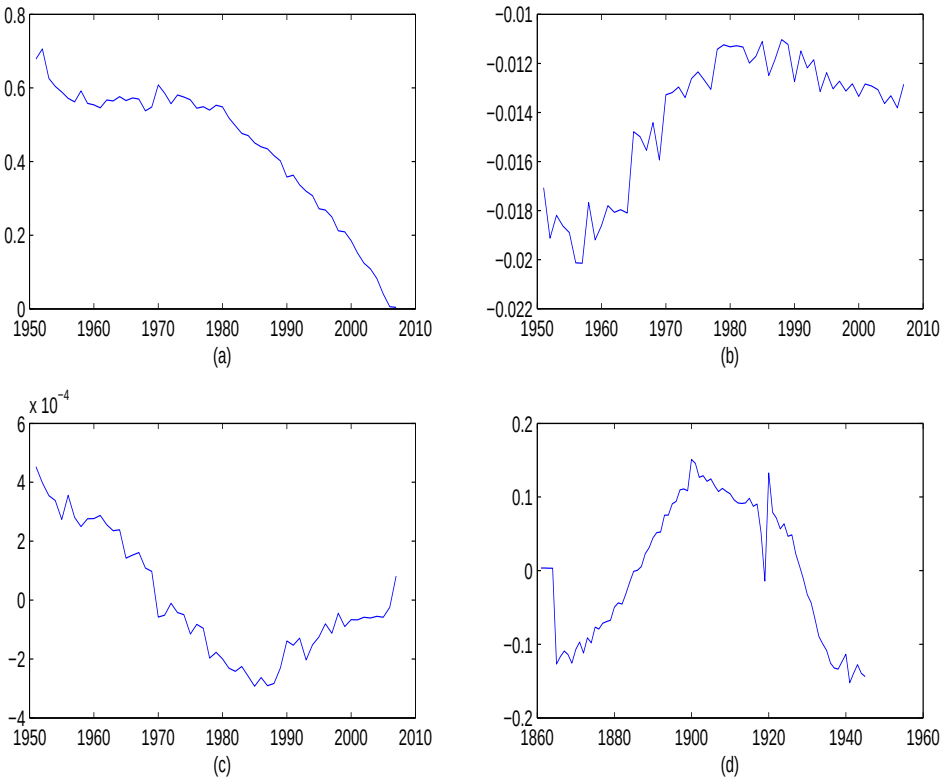

FIGURE 11. Estimated values of (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} based on GB males aged 5-89 between years 1950 and 2006 for the Plat model.

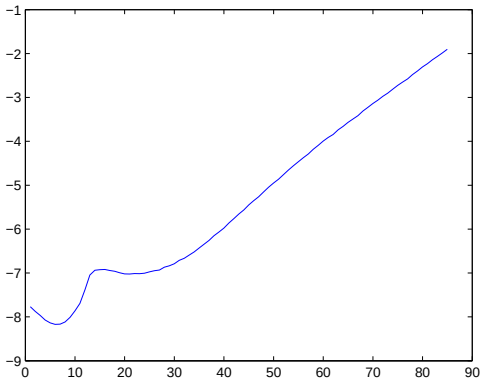

FIGURE 12. Estimated values of a_x based on GB males aged 5-89 between years 1950 and 2006 for the Plat model.

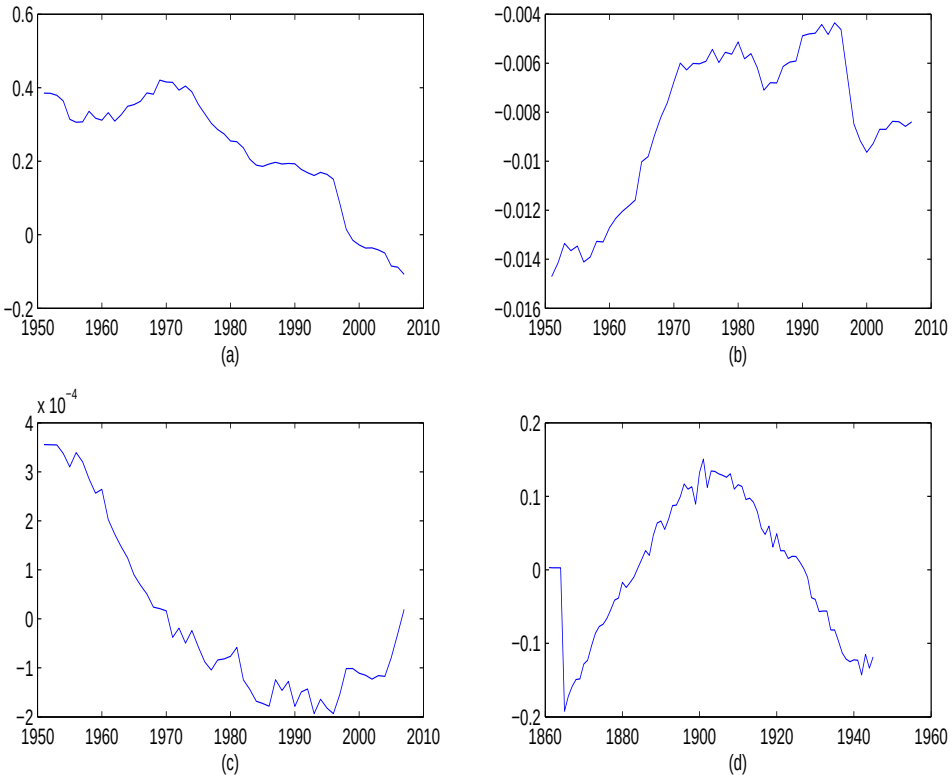

FIGURE 13. Estimated values of (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} based on US males aged 5-89 between years 1950 and 2006 for the Plat model.

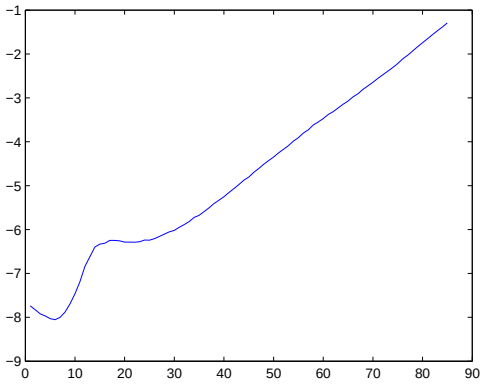

FIGURE 14. Estimated values of a_x based on US males aged 5-89 between years 1950 and 2006 for the Plat model.

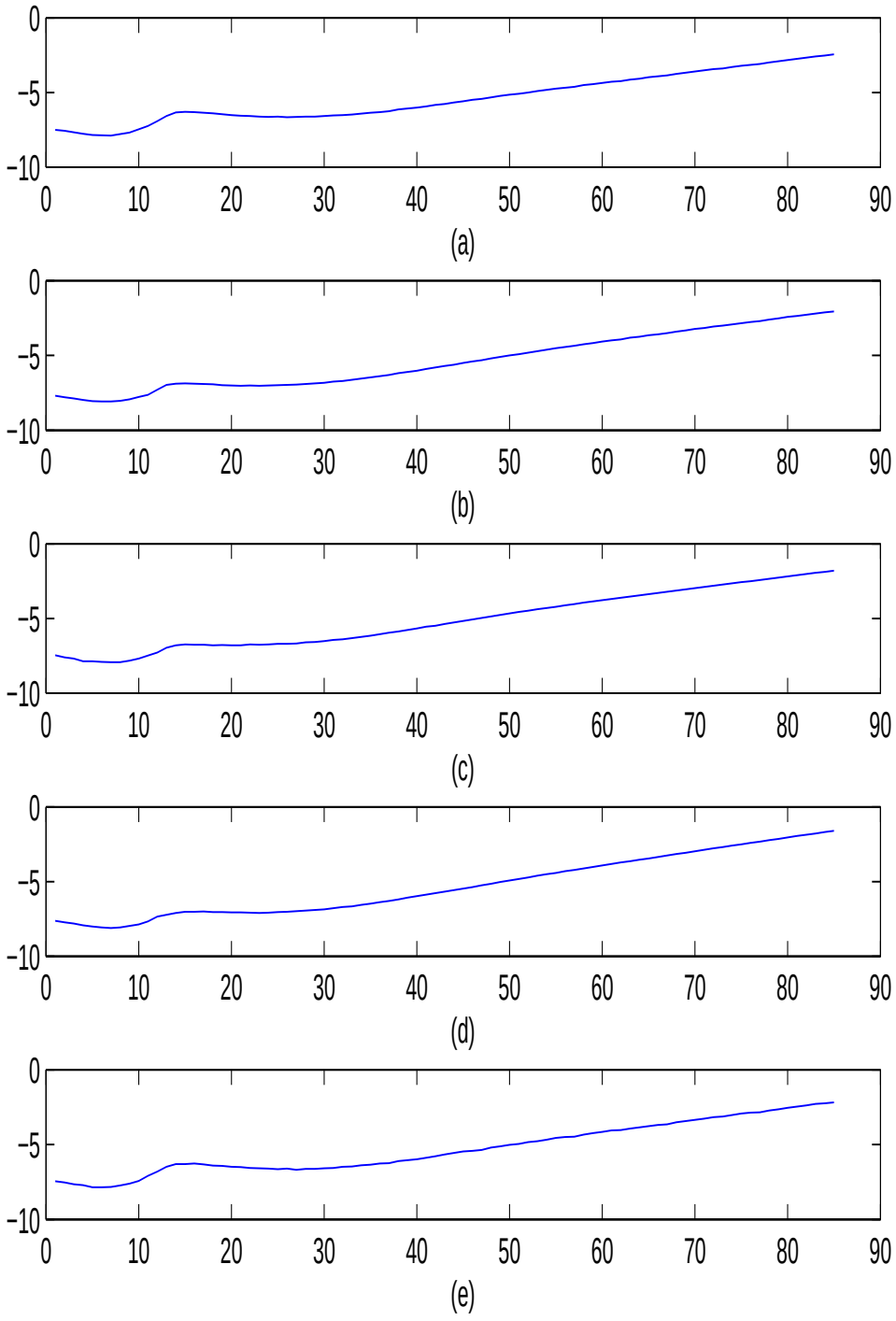

FIGURE 15. Estimated values of a_x based on ages 5-89 for countries (a) Australia, (b) England and Wales, (c) Scotland, (d) New Zealand and (e) Netherlands.

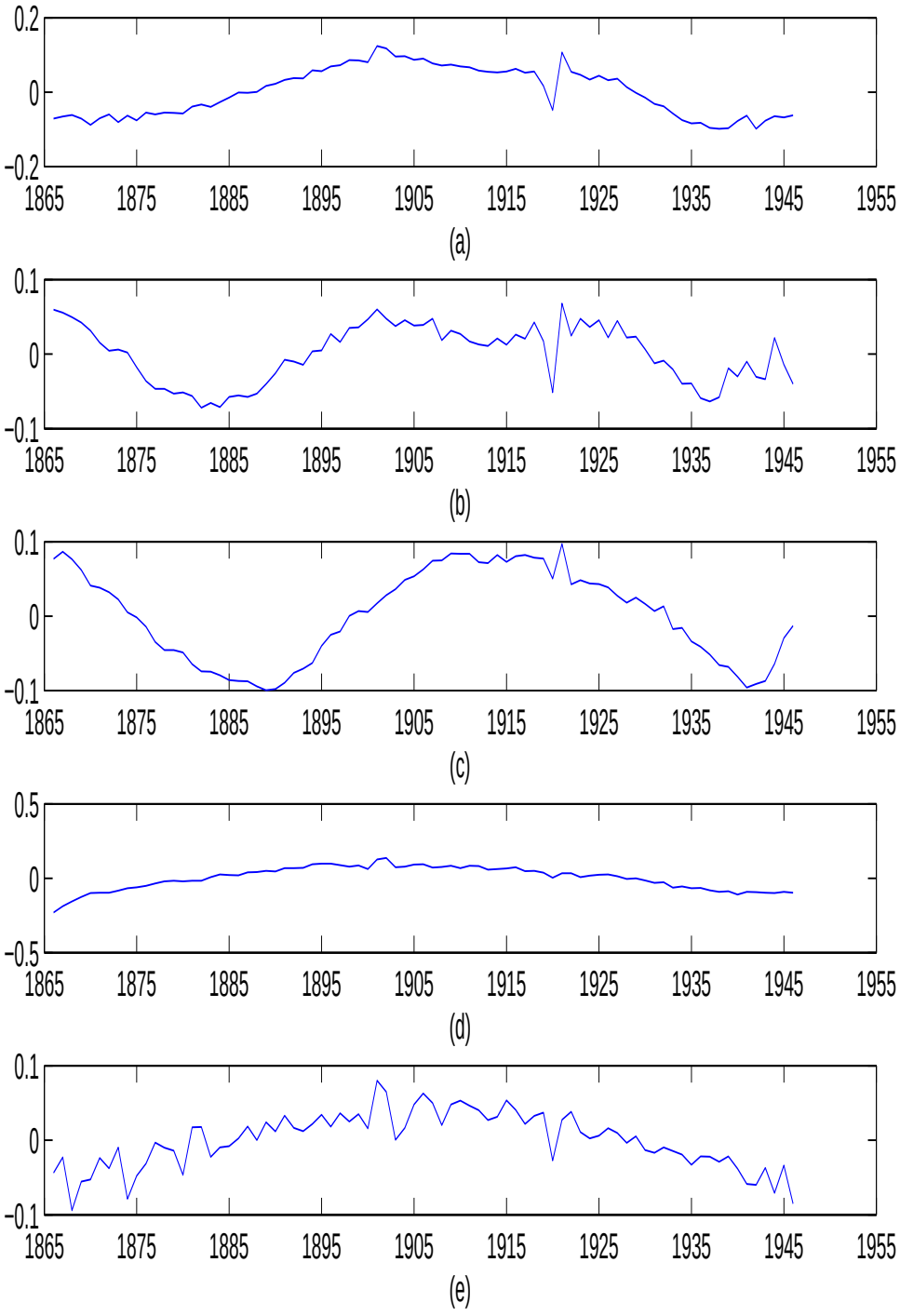

FIGURE 16. Estimated values of γ_{t-x} based on year of birth 1865-1955 for countries (a) England and Wales, (b) Scotland, (c) Netherlands, (d) Australia and (e) New Zealand.

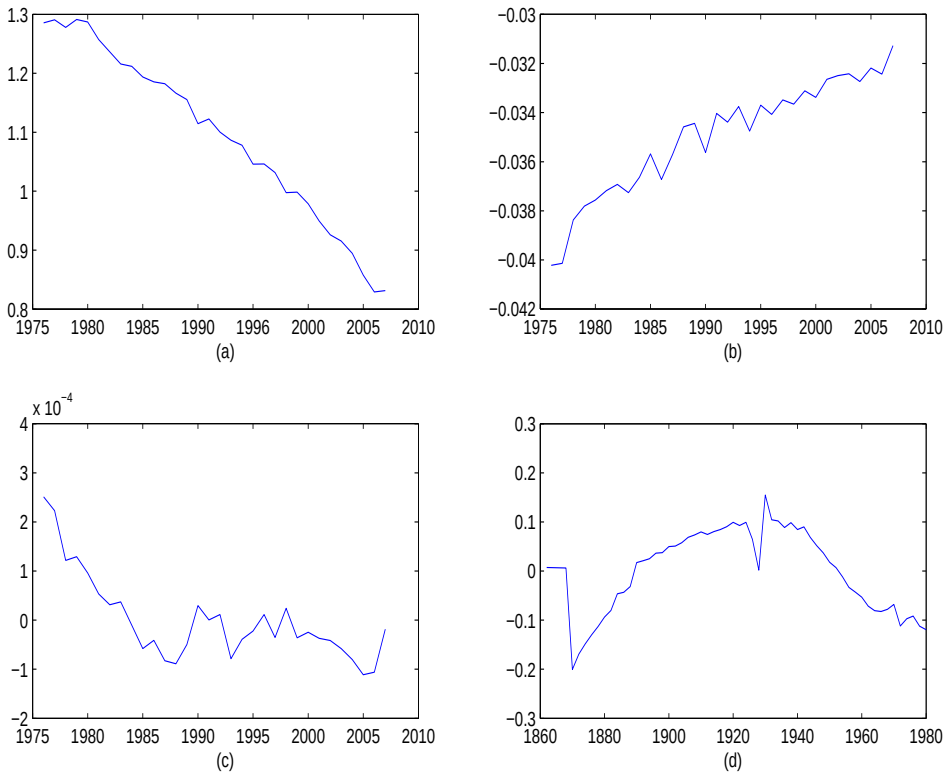

FIGURE 17. GB fitted parameters (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} with data from 1975-2006.

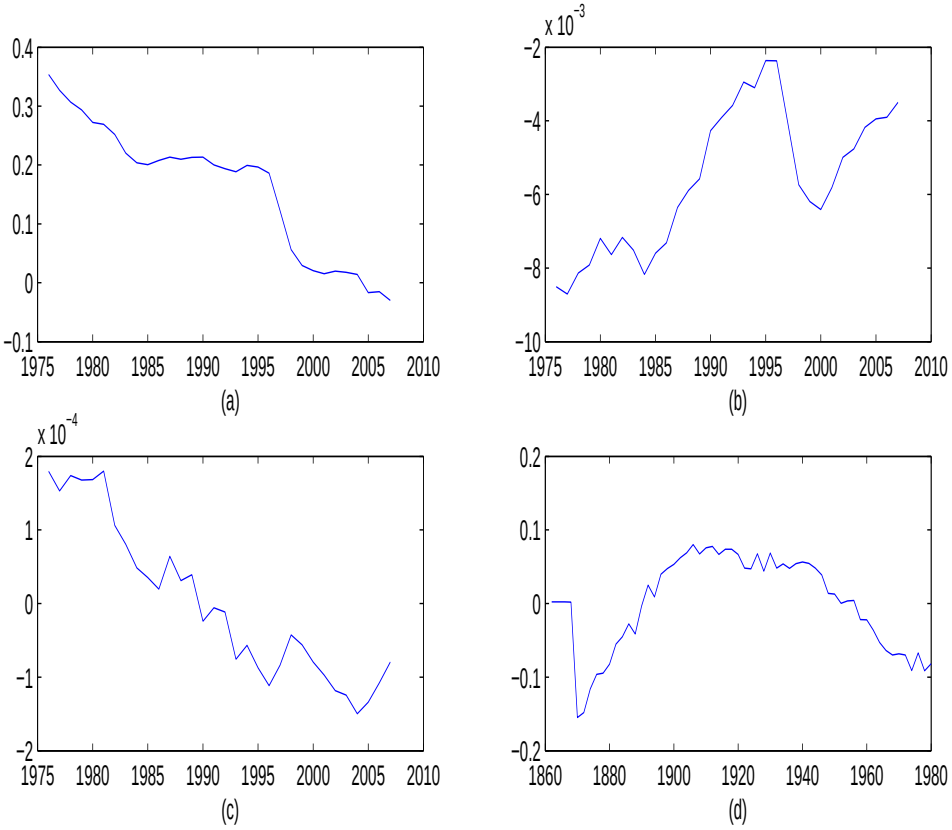

FIGURE 18. US fitted parameters (a) κ_t^1, (b) κ_t^2, (c) κ_t^3, and (d) γ_{t-x} with data from 1975-2006.

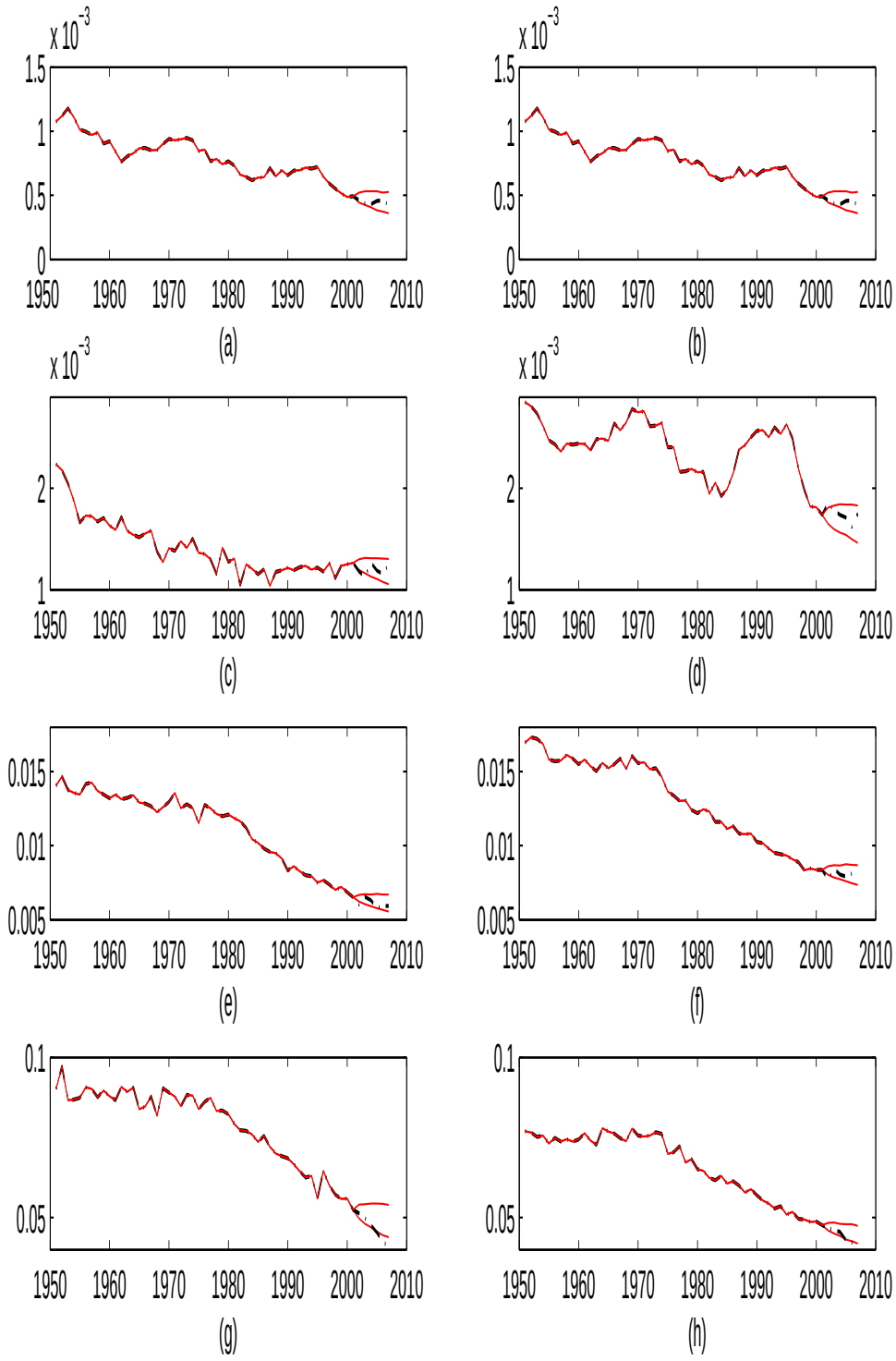

FIGURE 19. Log mortality rates from 1950 - 2006 plotted with 95% confidence intervals from 2000-2006 based on fitting from 1950-2000. Plots show ages (a) and (b) 15, (c) and (d) 35, (e) and (f) 55, and (g) and (h) 75 for countries GB and US respectively.

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