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Half-plane partial slip contact problems with a constant normal load subject to a shear force and differential bulk tension

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Abstract

This article provides a new form of solution to half-plane contact problems in partial slip where a normal load has been applied, held constant and is subsequently loaded with both a shear force and differential bulk tension. It uses a formulation where a displacement correction is made to the fully stuck solution. An approximate solution is used to study an isolated contact edge, which employs an asymptotic solution to each edge of the contact. A comparison of the approximation with the exact solution is given to show the range of loading where the asymptotic solution gives good results.

1. Introduction

Incomplete contacts between elastic components subject to time-varying loads are prone to regions of slip, even when the net shear force is less than that needed to cause rigid body motion (sliding). They are of practical interest as many contacts of this class exist in most mechanical assemblies. They provide frictional damping, which is beneficial, but also cause surface damage, leading to the nucleation of 'fretting' fatigue cracks. The most noteworthy example, capable of idealisation by half-plane theory, is the dovetail root of fan blades in gas turbines. This contact is of interest as fans are becoming bigger, and there is a move to lighter but low-friction composite materials.

There are two methods by which surface shear tractions may be excited within the contact, which give rise to zones of slip: one is through a shear force, the other is a differential bulk tension. The first solution for cyclically applied shear force and bulk tension to a Hertzian contact, with application to fretting fatigue tests, was by Nowell and Hills [1]. They solved the partial slip problem numerically by forming an integral equation in terms of the 'corrective' shear traction needed to the slipping solution. Using the Ciavarella [2] - Jäger [3] theorem, Ciavarella and Macina [4] solved the problem of pure bulk tension analytically for a Hertzian contact. Building on this, Vázquez et al. [5] obtained an exact solution when both bulk tension and shear are present with slip zones of opposite sign, also for a Hertzian contact.

In this paper, we develop a method that allows us to solve problems with the above loading cases for a general half-plane contact geometry whereas, many of the existing solutions are restricted to Hertzian contacts. Instead of a traction correction to the slipping solution, the method uses a displacement correction to the fully stuck solution. The mathematical framework for finding the corrective displacement and the slip stick boundary is given in the following section.

2. Mathematical framework

In contacts that may be modeled as half-planes the normal traction falls to zero at the edges in a square root manner. Therefore, the edges are the first points to slip when subjected to a tangential shear force, Q, and/or differential bulk

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tension. We define the differential bulk tension as $\sigma = \sigma_1 - \sigma_2$ where σ_1 is the tension in the top body and σ_2 in the bottom body. The size of the contact is independent of any tangential loads (we assume Dundurs parameter $\beta = 0$ [6]) and we let its half width be given by a. We want both the extent of slip and the slip displacement as the stick zone recedes, due to both a shear force and bulk tension. Figure 1 shows a possible configuration for the slip stick boundaries.

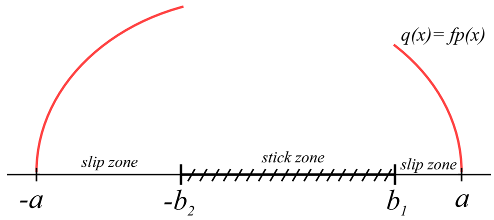

Figure 1: Zones of stick and slip at the contact interface

We start by inhibiting slip at all points on the contact interface and find the corresponding shear traction. To introduce slip we add glide dislocations [7] in the regions where slip is expected. The dislocations induce a shear traction over the entire contact. In the areas which experience slip, this traction must equal the difference between the slipping, $q_{sl}(x)$, and fully stuck shear traction, $q_{st}(x)$,

$$q_c(x) = q_{sl}(x) - q_{st}(x); \quad x \in \textit{slip zones}. \tag{1}$$

A single glide dislocation b_x at a point c on the contact interface gives rise to

$$q(x) = \frac{E^*}{2\pi} \frac{b_x(c)}{\sqrt{a^2 - x^2}} \frac{\sqrt{a^2 - c^2}}{c - x}. \tag{2}$$

If we introduce a distribution of dislocations, we can write down the following integral equation,

$$\frac{E^*}{2\pi\sqrt{a^2 - x^2}} \left[\int_{b_1}^{a} \frac{\sqrt{a^2 - \xi^2} B_x(\xi)}{\xi - x} d\xi + \int_{-a}^{-b_2} \frac{\sqrt{a^2 - \xi^2} B_x(\xi)}{\xi - x} d\xi \right] = q_c(x) \quad x \in \textit{slip zones}, \tag{3}$$

where the unknown function $B_x(x) = \frac{db_x}{dx}$ is the dislocation density. The dislocation density, $B_x(x)$, is the same as the slip displacement gradient or relative surface strain. The slipping traction $q_{sl} = \pm f p(x)$ where f is the coefficient of friction and $p(x)$ is the contact pressure. The size of the slip zones is given by b_1 and b_2. In Equation 3 and hereafter, a dash through the integral sign indicates that the integral is evaluated in a Cauchy Principal Value sense. The integral equation may be inverted analytically (Appendix A), leading to the following solution,

$$B_x(x) = \frac{-2}{E^*\pi} \sqrt{(x - b_1)(x + b_2)} \left[\int_{b_1}^{a} \frac{q_c(\xi) d\xi}{\sqrt{(\xi - b_1)(\xi + b_2)}(\xi - x)} - \int_{-a}^{-b_2} \frac{q_c(\xi) d\xi}{\sqrt{(\xi - b_1)(\xi + b_2)}(\xi - x)} \right] \quad b_1 \le x \le a, \tag{4}$$

$$B_x(x) = \frac{2}{E^*\pi} \sqrt{(x - b_1)(x + b_2)} \left[\int_{b_1}^{a} \frac{q_c(\xi) d\xi}{\sqrt{(\xi - b_1)(\xi + b_2)}(\xi - x)} - \int_{-a}^{-b_2} \frac{q_c(\xi) d\xi}{\sqrt{(\xi - b_1)(\xi + b_2)}(\xi - x)} \right] \quad -a \le x \le -b_2. \tag{5}$$

The two side conditions generated from the inversion determine the slip-stick boundaries, and are

$$\int_{-a}^{-b_2} \frac{q_c(s) ds}{\sqrt{(s - b_1)(s + b_2)}} - \int_{b_1}^{a} \frac{q_c(s) ds}{\sqrt{(s - b_1)(s + b_2)}} = 0, \tag{6}$$

$$\int_{-a}^{-b_2} \frac{q_c(s)sds}{\sqrt{(s-b_1)(s+b_2)}} - \int_{b_1}^{a} \frac{q_c(s)sds}{\sqrt{(s-b_1)(s+b_2)}} = 0. \tag{7}$$

3. Constant normal load

When the normal load (denoted by P) is held constant, the size of the contact does not change and so the behaviour over a load cycle is relatively easier to study. For fully stuck half-plane contacts, the shear traction is given by

$$q_{st}(x) = \frac{Q}{\pi\sqrt{a^2-x^2}} + \frac{\sigma x}{4\sqrt{a^2-x^2}} \qquad |x| \leq a, \tag{8}$$

where the first term is generated by a transverse shear load ($|Q| < fP$) and the second term from a differential bulk tension. This result depends only on the size of the contact, the exact geometry does not matter. If the contact is in partial slip, the direction of slip is dependent upon the relative mix of the tangential loading. Slip in the two edge zones will occur in the same direction if the shear force is dominant (Figure 2a), but in opposite directions if the differential bulk tension is dominant (Figure 2b).

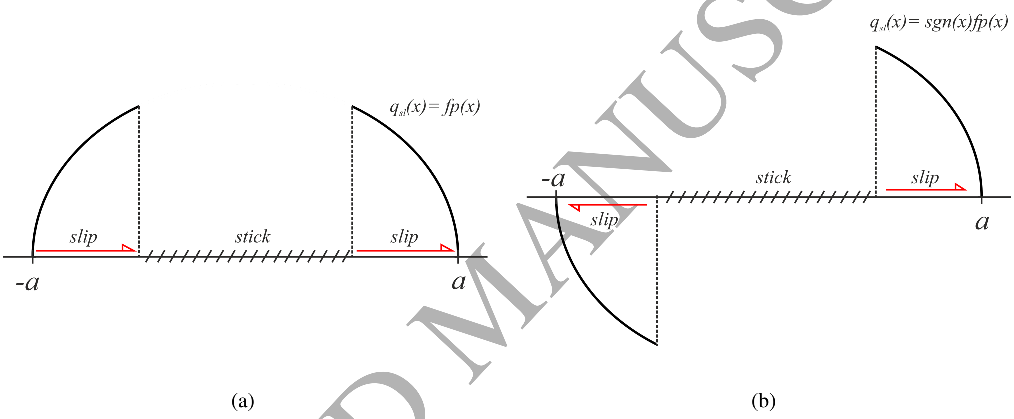

Figure 2: (a) The sliding shear traction - slip zones of the same sign; and (b) The slipping shear traction - slip zones of opposite sign (positive σ)

When the normal load is held constant, we know $q_{st}(x)$ (Equation 8) together with the contact pressure distribution, $p(x)$. These may be substituted into the integrals in Equations 4 and 5 to find the dislocation density or displacement gradient. Further details are given in Appendix B. The fully stuck traction, $q_{st}(x)$, does not affect the dislocation density but does affect the slip-stick boundary. We are left with only the contribution from the slipping traction which is a scaled normal traction,

$$B_x(x) = \frac{-2f}{E^*\pi}\sqrt{(x-b_1)(x+b_2)}\left[\pm\int_{b_1}^{a} \frac{p(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)} \mp \int_{-a}^{-b_2} \frac{p(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)}\right] \quad b_1 \leq x \leq a, \tag{9}$$

$$B_x(x) = \frac{2f}{E^*\pi}\sqrt{(x-b_1)(x+b_2)}\left[\pm\int_{b_1}^{a} \frac{p(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)} \mp \int_{-a}^{-b_2} \frac{p(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)}\right] \quad -a \leq x \leq -b_2. \tag{10}$$

The signs chosen depend on the signs and relative magnitudes of the shear force and differential bulk tension. The side conditions are given by,

$$\pm\int_{-a}^{-b_2} \frac{fp(s)ds}{\sqrt{(s-b_1)(s+b_2)}} \mp \int_{b_1}^{a} \frac{fp(s)ds}{\sqrt{(s-b_1)(s+b_2)}} = \pi\frac{\sigma}{4}, \tag{11}$$

$$\pm \int_{-a}^{-b_2} \frac{fp(s)sds}{\sqrt{(s-b_1)(s+b_2)}} \mp \int_{b_1}^{a} \frac{fp(s)sds}{\sqrt{(s-b_1)(s+b_2)}} = \frac{\pi\sigma(b_1-b_2)}{8} - Q. \tag{12}$$

3.1. Shear force only

We now look at the simple case where only one tangential load is applied. In the absence of any bulk tension, we can make a number of simplifications. The slip zones at either end of the contact will be of the same size and slip is in the same direction, therefore $b_1 = b_2 = b$. This means,

$$q_c(x) = fp(x) - \frac{Q}{\pi\sqrt{a^2-x^2}} \quad b \le |x| \le a. \tag{13}$$

The slip displacements are an even function of x so that the dislocation density is odd, and therefore $B_x(x) = -B_x(-x)$, giving

$$B_x(x) = \frac{4f}{E^*\pi}\sqrt{x^2-b^2}\int_b^a \frac{\xi p(\xi)d\xi}{\sqrt{\xi^2-b^2}(\xi^2-x^2)} \quad b \le |x| \le a \tag{14}$$

The first side condition (Equation 11) is automatically satisfied, the second may be written as

$$\int_b^a \frac{sp(s)ds}{\sqrt{s^2-b^2}} = \frac{Q}{2f} \tag{15}$$

3.2. Bulk tension only

Analogously, in the absence of a shear force, we can make a number of simplifications. The slip zones at either end of the contact will be of the same size but slip is in opposite directions and therefore, $b_1 = b_2 = b$. This allows us to say

$$q_c(x) = sgn(x)fp(x) - \frac{\sigma x}{4\sqrt{a^2-x^2}} \quad b \le |x| \le a. \tag{16}$$

The slip displacements are an odd function of x so that the dislocation density is even, and therefore $B_x(x) = B_x(-x)$, giving

$$B_x(x) = \frac{4f}{E^*\pi}\sqrt{x^2-b^2}\int_b^a \frac{xp(\xi)d\xi}{\sqrt{\xi^2-b^2}(\xi^2-x^2)} \quad b \le |x| \le a \tag{17}$$

The first side condition is no longer automatically satisfied, and demands that

$$\int_b^a \frac{p(s)ds}{\sqrt{s^2-b^2}} = \frac{\sigma\pi}{8f}. \tag{18}$$

On the other hand, the second side condition is now automatically satisfied (Equation 12).

4. Application to a Hertzian contact

So far, our results have been geometry independent. We consider a Hertzian contact to illustrate the proposed method for a particular geometry to show that we get results consistent with existing theory. The normal traction distribution for a Hertzian contact is given by,

$$p(x) = \frac{2P}{\pi a}\sqrt{1-\left(\frac{x}{a}\right)^2} = p_0\sqrt{1-\left(\frac{x}{a}\right)^2}. \tag{19}$$

This may be substituted into Equations 11 and 12 to find the extent of the stick zone in the general case of combined shear and bulk tension loading. If the slip directions are the same (positive Q and σ),

$$\frac{b_1}{a} = \sqrt{1 - \frac{Q}{fP} + \frac{\sigma}{4fp_0}}; \quad \frac{b_2}{a} = \sqrt{1 - \frac{Q}{fP} - \frac{\sigma}{4fp_0}}. \tag{20}$$

We can also find the strains developed in the slip regions by substitution of the contact pressure into Equation 9 and Equation 10,

$$B_x(x) = Sign(x)\frac{2fp_0}{aE^*}\sqrt{(x-b_1)(x+b_2)} \quad x \in slip\ zones. \tag{21}$$

The limits of this solution can be found when the slip zone length becomes zero (either $b_2 \to -a$ or $b_1 \to a$). To avoid this and for the slip zones to be of the same sign,

$$\frac{\sigma}{fp_0} \leq 4(1 - \sqrt{1 - Q/fP}). \tag{22}$$

When this condition is violated, the slip zones are of opposite sign, and the following two conditions must be satisfied (positive Q and σ),

$$\begin{aligned} &\frac{-2}{\sqrt{(1+\beta_1)(1+\beta_2)}}\Bigg[(1+\beta_1)(1+\beta_2)E\left(\frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\right) + (1+\beta_2)\Big(\frac{\beta_1-\beta_2}{2}+2\Big)K\left(\frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\right) \\ &+(\beta_1-\beta_2)\Pi\Big(\frac{\beta_2-1}{\beta_2+1}, \frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\Big) - \frac{\beta_1^2-\beta_2^2}{2}\Pi\Big(\frac{1-\beta_1}{1+\beta_2}, \frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\Big)\Bigg] = \frac{\pi\sigma}{4fp_0} \end{aligned} \tag{23}$$

where $\beta_1 = \frac{b_1}{a}$ and $\beta_2 = \frac{b_2}{a}$, and

$$\begin{aligned} &\frac{-3a}{2\sqrt{(1+\beta_1)(1+\beta_2)}}\Bigg[-(1+\beta_2)(1+\beta_1)(\beta_1-\beta_2)E\Big(\frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\Big) \\ &-(1+\beta_2)\Big(\Big(\frac{\beta_1}{3}+2\Big)\beta_2 - \frac{\beta_2^2}{2} - \frac{\beta_1^2}{2} - \frac{2\beta_1}{3} - \frac{2}{3}\Big)K\Big(\frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\Big) \\ &-\Big(\frac{\beta_1+\beta_2}{2}\Big)\Big(\beta_1^2 - \frac{2\beta_1\beta_2}{3} + \beta_2^2 - \frac{4}{3}\Big)\Pi\Big(\frac{1-\beta_1}{\beta_2+1}, \frac{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}{(1+\beta_1)(1+\beta_2)}\Big)\Bigg] \\ &= \frac{\pi\sigma(b_1-b_2)}{8fp_0} - \frac{Q}{fp_0}. \end{aligned} \tag{24}$$

Where, $K(\gamma)$, $E(\gamma)$ and $\Pi(n,\gamma)$ are elliptic integrals of the first, second and third kind. The results are shown in Figure 3 where the dotted lines give the boundary between shear and bulk dominated behaviour at either end of the contact. To find the corresponding values of b_2 we just need to flip the y-axis, i.e. replace σ/fp_0 with $-\sigma/fp_0$.

5. Asymptotic approximation

We focus our attention on the slip zone at one end of the contact and introduce an asymptotic solution. The simplified geometry is shown in Figure 4, which is similar to a semi-infinite crack, with the exception that the interface does not support a tension, and that the symmetrical solution for loading (the contact pressure) falls smoothly to zero as the contact edge is approached.

We consider only one edge of the contact, such that both $a, b \to \infty$ but $a - b = d$ remains finite. The resulting problem is to derive the shear traction due to a distribution of glide dislocations close to one edge of an adhered semi-infinite contact.

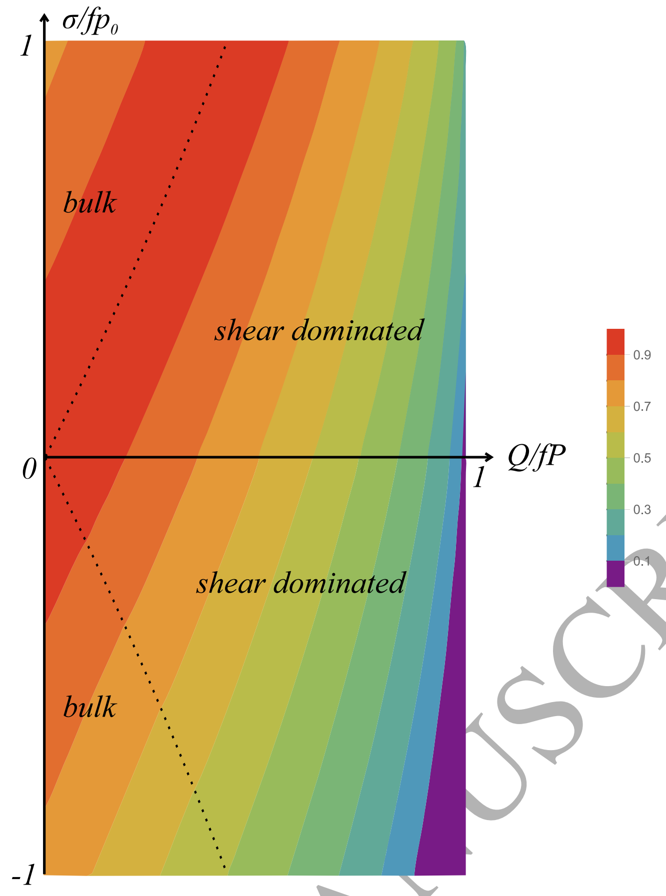

Figure 3: Contours of b_1/a

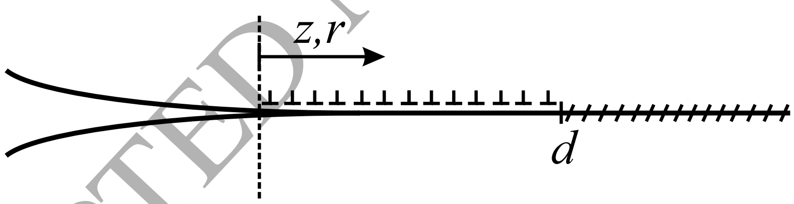

Figure 4: Dislocations at one edge of a half-plane contact

5.1. Glide dislocations on the centreline of a semi-infinite crack

The solution for a glide dislocation near the edge of a semi-infinite adhered contact is equivalent to one close to the tip of a semi-infinite crack and may therefore be found readily. If we move the origin to the left-hand contact edge and make the substitutions $x = -a + z$, $\xi = -a + r$, a distribution of glide dislocations (of density $B_x(r)$) will produce the shear traction

$$q(z) = \frac{E^*}{2\pi\sqrt{z}} \int_0^d \frac{\sqrt{r}B_x(r)}{r - z} dr, \tag{25}$$

where d is the extent of slip. The dislocations must provide a corrective traction such that the shear traction in the slip zone is equal to $f p(z)$, which leads to the following integral equation

$$q_c(z) = \frac{E^*}{2\pi\sqrt{z}} \int_0^d \frac{\sqrt{r}B_x(r)}{r - z} dr = f p(z) - q_{st}(z), \quad 0 \leq z \leq d. \tag{26}$$

We now take a single-term description of the near-edge tractions and set $p(z) = K_N\sqrt{z}$ and $q(z) = \frac{K_T}{\sqrt{z}}$ [8] where,

$$K_N = \frac{1}{\pi}\sqrt{\frac{2}{a}}\frac{dP}{da} \quad K_T = \frac{\sigma}{4}\sqrt{\frac{a}{2}} \pm \frac{Q}{\pi\sqrt{2a}}. \tag{27}$$

Equations 25, 26 and 27 then become

$$fK_N z - K_T = \frac{E^*}{2\pi}\int_0^d \frac{\sqrt{r}B_x(r)}{r-z}dr, \quad 0 \le z \le d, \tag{28}$$

Which may then be inverted assuming that the solution is bounded both ends. We find that the dislocation distribution and the size of the slip zone are given by

$$B_x(z) = \frac{fK_N}{E^*}\sqrt{d-z}; \quad d = \frac{2K_T}{fK_N}. \tag{29}$$

5.2. Limits of the approximation

To find the limits of the asymptotic approximation, we compare the work done against friction for a Hertzian contact when subject to either a monotonically increasing shear force or differential bulk tension, since there is an exact solution to the problem. For a given slip zone length (b/a), the work done is slightly different for the two problems because the slip displacements are greater when induced by the exertion of bulk tension. For the case when the slip zone is 10% of the contact half width $(b/a = 0.9)$, the work done against friction, $W(z/d)$ is shown in Figure 5. We also include the work done calculated using the asymptotic approximation in Figure 5. It is apparent that the asymptotic approximation always yields an underestimate of the work done if the slip is induced by pure shear and, conversely, always yields an overestimate when induced by pure bulk tension. Any combination of shear and bulk loading will result in an energy dissipation that lies between the two extremes; thus the pure loading cases are the extremes of error between the finite solution and the asymptotic approximation.

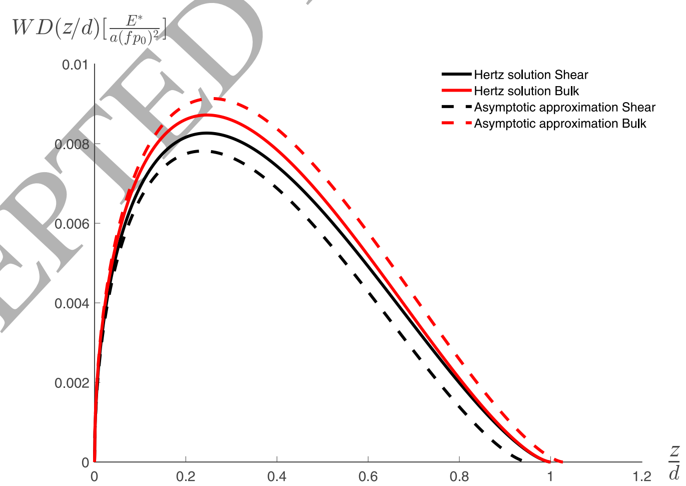

Figure 5: Energy dissipation at one edge of a Hertzian contact $(b/a = 0.9, d/a = 0.1)$

In Figure 6, we plot for a given b/a, the fractional error in the total energy dissipated (the integral of $W(z/d)$ over the region $0 < z/d < 1$) in the slip zone using an asymptotic form compared with the full solution. The ratio $b/a = 0.9$ gives rise to a 10% error in the energy dissipated.

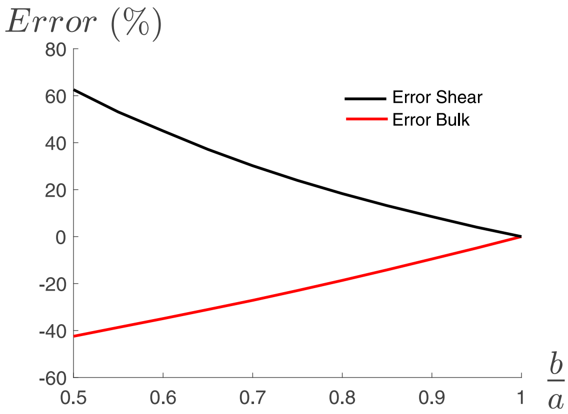

Figure 6: Fractional error in energy from the asymptotic form

6. Conclusion

In this paper, we have shown that it is possible to treat any combination of shear force and bulk tension when the normal load is held constant by starting with the fully stuck solution and then introducing a corrective slip displacement. It can be applied to any contact that may be idealised as two partially bonded half-planes. The advantage of the method is the automatic preservation of any locked in slip displacements - which makes it easier to solve problems where the stick zone is not at the centre of the contact. We also derived an asymptotic approximation that is much simpler to use both numerically and theoretically. We showed that, when the slip zone is less than 10% of the overall contact size, the approximation gives a good estimate of the energy dissipation.

References

[1] D. Nowell and D.A. Hills. Mechanics of fretting fatigue tests. *International Journal of Mechanical Sciences*, 29(5):355 – 365, 1987.

[2] M. Ciavarella. The generalized Cattaneo partial slip plane contact problem. i - theory. *Int. J. Solids Struct.*, 35(18):2349 – 2362, 1998.

[3] J. Jäger. A new principle in contact mechanics. *J. Tribol.*, 120(4):677 – 684, 1998.

[4] M. Ciavarella and G. Macina. New results for the fretting-induced stress concentration on hertzian and flat rounded contacts. *Int. J. Mech. Sci.*, 45:449– 467, 2003.

[5] Jesús Vázquez, Carlos Navarro, and Jaime Domínguez. Analytical solution for a cylindrical contact with reverse slip. *The Journal of Strain Analysis for Engineering Design*, 48(3):189–197, 2013.

[6] J. Dundurs. Discussion: edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. *ASME Journal of Applied Mechanics*, 36(3):650 – 652, 1969.

[7] D.A. Hills, P.A. Kelly, D.N. Dai, and A.M. Korsunsky. *Solution of crack problems: The distributed dislocation method*. Kluwer, 1996.

[8] D. A. Hills, R. M. N Fleury, and D. Dini. Partial slip incomplete contacts under constant normal load and subject to periodic loading. *Int. J. Mech. Sci.*, 108-109:115 – 121, 2016.

[9] F. D. Gakhov. *Boundary Value Problems*. Pergamon, 1966.

Appendix A. Inversion of the integral equation

We seek to invert the following singular integral equation in the two slip zones,

$$q_c(x) = \frac{E^*}{2\pi\sqrt{a^2-x^2}}\left[\int_{b_1}^{a}\frac{\sqrt{a^2-\xi^2}B_x(\xi)d\xi}{\xi - x} + \int_{-a}^{-b_2}\frac{\sqrt{a^2-\xi^2}B_x(\xi)d\xi}{\xi - x}\right], \quad x \in (-a, -b_2) \cup (b_1, a). \tag{A.1}$$

By scaling variables using the contact size,

$$x = at, \quad \xi = as, \quad B_x(as) = F(s), \quad \frac{2\sqrt{1-t^2}}{E^*} q_c(at) = Q(t) \tag{A.2}$$

we reduce (A.1) to

$$Q(t) = \frac{1}{\pi}\left[\int_{-1}^{-\beta} \frac{\sqrt{1-s^2}}{s-t} F(s) ds + \int_{\alpha}^{1} \frac{\sqrt{1-s^2}}{s-t} F(s) ds\right], \quad t \in (-1, -\beta) \cup (\alpha, 1) \tag{A.3}$$

where $\alpha = b_1/a$, $\beta = b_2/a$. The problem is now to invert the singular integral equation

$$Q(t) = \frac{1}{\pi}\int_L \frac{\phi(s)}{s-t} ds \tag{A.4}$$

where $\phi(s) = \sqrt{1-s^2}F(s)$ and L is the union of disjoint contours $(-1, -\beta)$, $(\alpha, 1)$. By extending the contour L to a closed contour C, we can reformulate (A.4) as a Riemann-Hilbert problem with discontinuous coefficients for $\phi(z)$, which is analytic except on C. We can solve this using the method of Gakhov [9], which reduces the problem to one with continuous coefficients by considering functions with the appropriate local behaviour at the end points of L. There are thus different solutions depending on which behaviour for $\phi(z)$ is most physically relevant; here we consider two, the solution unbounded at all ends of L and the solution bounded at all ends of L.

Appendix A.1. Unbounded at all ends

If we seek a solution that is unbounded at $\pm 1, \alpha, -\beta$, the Riemann-Hilbert problem has index 2, and the appropriate inversion gives

$$\phi(t) = \frac{1}{\pi\sqrt{(1-t^2)(t-\alpha)(t+\beta)}}\left[\int_{-1}^{-\beta} \frac{\sqrt{(1-s^2)(s-\alpha)(s+\beta)}}{s-t} Q(s) ds - \int_{\alpha}^{1} \frac{\sqrt{(1-s^2)(s-\alpha)(s+\beta)}}{s-t} Q(s) ds + \rho(t)\right], \tag{A.5}$$

for $t \in (\alpha, 1)$, where $\rho(t)$ is an arbitrary polynomial of maximum degree 2. In general, we must use physical constraints to determine $\rho(t)$. After returning to the unscaled variables, we find that the dislocation density is given by

$$\begin{aligned}\sqrt{1-\frac{x^2}{a^2}} B_x(x) = \frac{-1}{\pi\sqrt{\left(1-\frac{x^2}{a^2}\right)\left(\frac{(x-b_1)(x+b_2)}{a^2}\right)}}&\left[\int_{b_1}^{a} \frac{\sqrt{(1-\frac{\xi^2}{a^2})(\frac{(\xi-b_1)(\xi-b_2)}{a^2})}}{\xi - x}\left(\frac{2}{E^*}\sqrt{1-\frac{\xi^2}{a^2}} q_c(\xi)\right) d\xi\right.\\ &\left. - \int_{-a}^{-b_2} \frac{\sqrt{(1-\frac{\xi^2}{a^2})(\frac{(\xi-b_1)(\xi-b_2)}{a^2})}}{\xi - x}\left(\frac{2}{E^*}\sqrt{1-\frac{\xi^2}{a^2}} q_c(\xi)\right) d\xi + \rho(x)\right]\end{aligned} \tag{A.6}$$

for $x \in (b_1, a)$. A similar expression can be found for $x \in (-a, -b_2)$, but for brevity is omitted here.

Appendix A.2. Bounded at all ends

In the context of this paper, the more physically-relevant regime is that in which the function $\phi(z)$ is bounded at $\pm 1, \alpha, -\beta$. In this regime, the Riemann-Hilbert problem has index -2 and the appropriate solution is given by

$$\phi(t) = \frac{-\sqrt{(t-\alpha)(t+\beta)(1-t^2)}}{\pi}\left[\int_{\alpha}^{1} \frac{Q(s)}{\sqrt{(1-s^2)(s-\alpha)(s+\beta)}(s-t)} ds - \int_{-1}^{-\beta} \frac{Q(s)}{\sqrt{(1-s^2)(s-\alpha)(s+\beta)}(s-t)} ds\right] \tag{A.7}$$

for $t \in (\alpha, 1)$. Therefore, returning to unscaled variables, we find that the dislocation density for $x \in (b_1, a)$ is

$$B_x(x) = \frac{-2}{E^*\pi}\sqrt{(x-b_1)(x+b_2)}\left[\int_{b_1}^{a} \frac{q_c(\xi)}{\sqrt{(\xi-b_1)(\xi+b_2)}} \frac{d\xi}{(\xi-x)} - \int_{-a}^{-b_2} \frac{q_c(\xi)}{\sqrt{(\xi-b_1)(\xi+b_2)}} \frac{d\xi}{(\xi-x)}\right]. \tag{A.8}$$

Similarly for $x \in (-a, -b_2)$, the dislocation density is given by

$$B_x(x) = \frac{2}{E^*\pi}\sqrt{(x-b_1)(x+b_2)}\left[\int_{b_1}^{a}\frac{q_c(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)} - \int_{-a}^{-b_2}\frac{q_c(\xi)d\xi}{\sqrt{(\xi-b_1)(\xi+b_2)}(\xi-x)}\right]. \tag{A.9}$$

Since the index of the Riemann-Hilbert problem is negative, we must also enforce two consistency conditions, which are given by

$$\int_{-a}^{-b_2}\frac{q_c(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} - \int_{b_1}^{a}\frac{q_c(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} = 0, \tag{A.10}$$

for $j = 1, 2$. In general, for a given $q_c(x)$, (A.10) will allow us to determine expressions for b_1 and b_2, which we discuss in more detail in Appendix B.

Appendix B. Integral evaluation for a fully stuck contact with a constant normal load

The function $q_c(x)$ is made up of three terms

$$q_c(x) = \frac{Q}{\pi}q_c^1(x) + \frac{\sigma}{4}q_c^2(x) + q_c^3(x) = -\frac{Q}{\pi\sqrt{a^2-x^2}} - \frac{\sigma x}{4\sqrt{a^2-x^2}} + fp(x), \tag{B.1}$$

where the first two terms are general and the final term is specific to the geometry being considered. We aim to evaluate the integral constraints (A.10), by using contour integration to evaluate each integral for q_c^i, $i = 1, 2, 3$. In order to facilitate this, throughout we shall consider the branch of the function $\sqrt{z^2-a^2}$ that is real, positive and continuous across the real axis for $x > a$ and the branch of the function $\sqrt{(z-b_1)(z-b_2)}$ that is real, positive and continuous across the real axis for $x > b_1$. We can calculate the two general parts of the consistency conditions directly. Evaluating (A.10) considering only the first term, $q_c^1(x)$, we find that

$$\int_{-a}^{-b_2}\frac{q_c^1(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} - \int_{b_1}^{a}\frac{q_c^1(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} = \begin{cases} 0 & \text{for} \quad j = 1, \\ -\pi & \text{for} \quad j = 2. \end{cases} \tag{B.2}$$

Similarly, evaluating (A.10) considering only the second term, $q_c^2(x)$, we find that

$$\int_{-a}^{-b_2}\frac{q_c^2(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} - \int_{b_1}^{a}\frac{q_c^2(s)s^{j-1}ds}{\sqrt{(s-b_1)(s+b_2)}} = \begin{cases} -\pi & \text{for} \quad j = 1, \\ -\dfrac{(b_1-b_2)\pi}{2} & \text{for} \quad j = 2. \end{cases} \tag{B.3}$$

Appendix B.1. Side condition for a Hertzian contact

Finally, we shall consider the geometry-specific part of the consistency conditions (A.10) for the particular example of a Hertzian contact,

$$q_c^3(x) = fp(x) = p_0\sqrt{1-\left(\frac{x}{a}\right)^2}. \tag{B.4}$$

Thus, we need to consider an integral of the form

$$I = \int_{-a}^{-b_2}\frac{s^{j-1}\sqrt{a^2-s^2}ds}{\sqrt{(s-b_1)(s+b_2)}} - \int_{b_1}^{a}\frac{s^{j-1}\sqrt{a^2-s^2}ds}{\sqrt{(s-b_1)(s+b_2)}}, \tag{B.5}$$

which can also be evaluated using contour integration, giving

$$I = \begin{cases} \dfrac{(b_1-b_2)\pi}{2} & \text{for} \quad j = 1, \\ \left(\dfrac{3}{8}(b_2^2+b_1^2) - \dfrac{b_1 b_2}{4} - \dfrac{a^2}{2}\right)\pi & \text{for} \quad j = 2. \end{cases} \tag{B.6}$$