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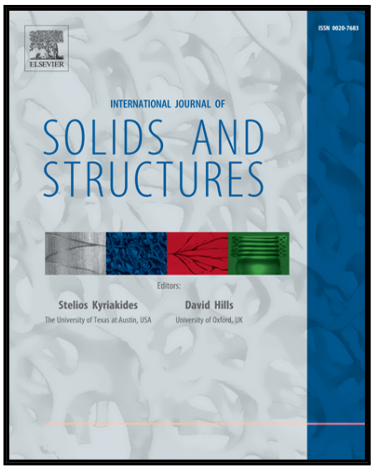

Solution of half-plane contact problems by distributing climb dislocations

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Solution of half-plane contact problems by distributing climb dislocations

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April 26, 2018

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Abstract

We derive a novel method for addressing half-plane contact problems by using distributions of climb dislocations over the contact patch to balance the relative overlapping profile of the contacting bodies. Using this technique, we are able to forgo inverting a singular integral equation, simplifying the analysis. We illustrate the method by deriving the associated contact pressures, external forces and applied moments for several examples of both symmetric and non-symmetric indenters.

1 Introduction

The standard way to solve contact problems is to write down a formulation based on the surface displacements caused by a unit normal force applied to the surface, and then to use this as a Green's function so that the bodies may be made to conform over a region (the contact patch) while the contact pressure is in equilibrium with the external normal load [1]. If the contact initially conforms, for example in the case of a complete contact, an alternative strategy is to start with a domain which is the combination of the two bodies, and to analyse the internal state of stress subject to the applied loads as if the interface was continuous and everywhere adhered. If both (a) the direct traction across the interface line is everywhere negative, and (b) the ratio of the magnitude of the shear to direct traction is everywhere less than the coefficient of friction then the solution already found is, indeed, correct. However, if there are local regions of violations of these conditions - for example if there are regions where the traction ratio exceeds the coefficient of friction, then by using the solution of a glide edge dislocation in the combined bodies it is easy to insert extra tangential displacements to represent the effects of slip and to restore the Coulomb (for example) friction condition. Equally, if there are also regions of interfacial tension, climb edge dislocations may be inserted to relieve the tension and permit separation. One property of an edge dislocation in plane form is that it is Volterra in character and so one does not need to be worried about the line of the path cut which was used to form it: it may always be assumed to lie along the contact interface.

Here a form of this idea is applied to *incomplete* contacts. We assume that the contacting bodies are sufficiently large to be approximated by half planes, and that they are made from the same material so that the effects of direct and shear loading are uncoupled. We solve the normal contact problem alone (we have recently looked at the corresponding tangential loading/slip problem [2, 3]). Suppose that we bring together the two half-planes and bond them over a region $[-b\ a]$, which will become the contact patch. We now insert climb dislocations within that interval whose Burgers vectors sum to form the profile of the bodies actually brought into contact but are of opposite sign. Provided that there are no regions of tension within the putative contact, nor regions of interpenetration external to it (that is, the Signorini conditions hold), we will have solved the contact problem correctly.

2 Dislocation solution

Two elastically similar half-planes ($y > 0$ and $y < 0$), each having plane strain elastic modulus E^*, are bonded together over the interval $[-b, a]$, where $a, b > 0$. In the *absence* of deformation, there is a small overlap between the two bodies denoted by $h(x) = h_1(x) - h_2(x)$ where a subscript 1 denotes the front face profile of the upper body and a subscript 2 corresponds to the lower body.

Deformation of the contacting bodies induces a relative normal surface displacement, $v(x) = v_1(x) - v_2(x)$ that must satisfy

$$v'(x) + h'(x) = 0 \quad \text{for} \quad -b < x < a.$$

The interfacial pressure needed to achieve intimate contact, $p(x)$, is given by

$$\frac{dv}{dx} = \frac{2}{\pi E^*} \int_{-b}^{a} \frac{p(\xi)\, d\xi}{\xi - x}, \tag{1}$$

for $-b < x < a$. As the extent of contact is determined by the Signorini inequalities, a 'bounded-both-ends' solution is needed to this equation. It is given by

$$p(x) = -\frac{E^* \sqrt{(a-x)(x+b)}}{2\pi} \int_{-b}^{a} \frac{v'(\xi) d\xi}{\sqrt{(a-\xi)(\xi+b)}\,(\xi - x)}, \tag{2}$$

where the consistency condition from the inversion is

$$0 = \int_{-b}^{a} \frac{v'(\xi)}{\sqrt{(a-\xi)(\xi+b)}} d\xi. \tag{3}$$

Now let us suppose that the surface irregularity is simply an edge dislocation of magnitude b_y and located at point $c \in (-b, a)$ so that $v'(x) = -b_y(c)\delta(x - c)$ where $\delta(\cdot)$ is Dirac's delta function. The contact pressure developed by such a dislocation is therefore given by

$$p(x) = \frac{E^* \sqrt{(a-x)(x+b)}}{2\pi} \int_{-b}^{a} \frac{b_y(c)\delta(\xi - c)\, d\xi}{\sqrt{(a-\xi)(\xi+b)}\,(\xi - x)} = \frac{E^* b_y(c)}{2\pi(c-x)} \sqrt{\frac{(a-x)(x+b)}{(a-c)(c+b)}}.$$

Thus, if we represent the overlap $h'(x)$ as a distribution of these dislocations of density $B_y(x) = db_y/dx$, we produce a contact pressure distribution given by

$$p(x) = \frac{E^* \sqrt{(a-x)(x+b)}}{2\pi} \int_{-b}^{a} \frac{B_y(\xi)\, d\xi}{\sqrt{(a-\xi)(\xi+b)}\,(\xi - x)}.$$

3 Symmetric contact problems

Firstly, we shall restrict ourselves to symmetric problems in which $a = b$. We note that, for a symmetric problem, the consistency condition (3) is trivially satisfied since $h'(\xi)$ and hence $v'(\xi)$ are odd functions. Therefore, in order to determine the contact half-width, a, we must specify equilibrium between the normal force and the contact pressure.

3.1 Example – Hertzian contact

As an example of a familiar problem, consider the contact of two cylinders of relative radius of curvature R. The amount of material which overlaps in the contact, $h(x)$, is given by

$$x^2 = -h^2 + 2Rh \simeq 2Rh, \; h \ll R$$

and thus

$$\frac{dh}{dx} \simeq \frac{x}{R}.$$

The contact pressure distribution is therefore given by

$$p(x) = \frac{E^*\sqrt{a^2-x^2}}{2\pi R} \fint_{-a}^{a} \frac{\xi d\xi}{\sqrt{a^2-\xi^2}(\xi - x)} = \frac{E^*\sqrt{a^2-x^2}}{2R}. \tag{4}$$

For equilibrium with an external force, P, we must have

$$P = \frac{E^*}{2R}\int_{-a}^{a} \sqrt{a^2-x^2}dx = \frac{E^*\pi a^2}{4R}, \tag{5}$$

so that we may write the solution in the conventional form (see Barber [1])

$$p(x) = \frac{2P}{\pi a^2}\sqrt{a^2-x^2}.$$

3.2 More general symmetric contacts

Suppose we write down the relative overlapping profiles of the contacting bodies as a power series polynomial, so the slope at any point may also be written down in the form

$$\frac{dh}{dx} = \sum_m A_{2m+1}x^{2m+1}, \tag{6}$$

where we only seek odd terms in the gradient profile as the punch is symmetric. There seems no reason why we should not be able to use such a general profile form, subject to the Signorini conditions on the solution, which may only be checked *a posteriori*. We also reiterate that the solution (2)–(3) is completely general: we are not restricted to indenters of the form (6). We have chosen this form for purely illustrative purposes.

We are interested in the contact pressure distribution together with the normal force, calculated by the integral of the contact pressure over the contact patch (cf. (5)). In order to calculate these quantities, we must consider principal value integrals of the form

$$I_n = \fint_{-a}^{a} \frac{\xi^n d\xi}{\sqrt{a^2-\xi^2}(\xi - x)}. \tag{7}$$

We can evaluate these integrals by contour integration and present the details in appendix A. Since for symmetric problems $n = 2m+1$ is odd, we find

$$I_{2m+1} = \pi \sum_{l=0}^{m} x^{2l}a^{2(m-l)}(-1)^{m-l}\binom{-\frac{1}{2}}{m-l}. \tag{8}$$

3.2.1 Normal force

In order to calculate the contact half-width a, we must ensure that the total contact pressure is in equilibrium with the applied normal force P, so that

$$P = \int_{-a}^{a} p(x)dx.$$

For a general indenter of the form (6), such an expression must be solved numerically for a. However, we can explicitly calculate the contribution of each term of the series, which we present here.

The contact pressure associated with the term $A_{2m+1}x^{2m+1}$ is found to be

$$p_{2m+1}(x) = \frac{A_{2m+1}E^*}{2}\sqrt{a^2-x^2}\sum_{l=0}^{m} x^{2l}a^{2(m-l)}(-1)^{m-l}\binom{-\frac{1}{2}}{m-l}, \tag{9}$$

for $m \in \mathbb{N}$. In practice, we will only be interested in relatively small values of $2m+1$, so it is informative to present these explicitly. Upon calculating the coefficients in (9) we find

$$p_1(x) = \frac{A_1 E^*}{2}\sqrt{a^2 - x^2}, \tag{10}$$

$$p_3(x) = \frac{A_3 E^*}{2}\sqrt{a^2 - x^2}\left(x^2 + \frac{a^2}{2}\right), \tag{11}$$

$$p_5(x) = \frac{A_5 E^*}{2}\sqrt{a^2 - x^2}\left(x^4 + \frac{x^2 a^2}{2} + \frac{3a^4}{8}\right), \tag{12}$$

with the first of these relating directly to the cylinder example in §3.1 with $A_1 = 1/R$.

For equilibrium with an external force, P, we must have

$$P = \sum_m P_{2m+1}, \quad \text{where} \quad P_{2m+1} = \int_{-a}^{a} p_{2m+1}(x)dx.$$

Utilising (9), we can calculate the contribution from the term $A_{2m+1}x^{2m+1}$, finding

$$P_{2m+1} = \frac{A_{2m+1}E^* \pi a^{2(m+1)}}{4}\sum_{l=0}^{m}\frac{(-1)^{m-l}\Gamma(l+\frac{1}{2})}{\Gamma(m+1-l)\Gamma(l-m+\frac{1}{2})\Gamma(2+l)}. \tag{13}$$

For small powers, the corresponding values for P_{2m+1} are

$$P_1 = \frac{A_1 E^* \pi a^2}{4},\ P_3 = \frac{3A_3 E^* \pi a^4}{16},\ P_5 = \frac{5A_5 E^* \pi a^6}{32}, \tag{14}$$

where the first expression again confirms the calculation for a Hertzian cylinder in §3.1 provided that $A_1 = 1/R$.

We plot the contact pressure scaled by the normal force for $m = 0, 1, 2$ in figure 1 with $a = 1$. As m increases, which, for $|x| < 1$ corresponds to a flatter profile in the gradient function $h'(x)$, the contact pressure at $x = 0$ reduces, with two peaks forming closer to the edges of the contact. The magnitude of this peak increases as m increases.

4 Non-symmetric problems

As an example of the flexibility of this procedure, we shall now consider several non-symmetric problems in which $a \neq b$.

4.1 Two adjoined cylinders

Firstly, we shall consider an extension to the classical Hertzian cylinder example, in which two cylinders of different radii of curvature are adjoined at $x = 0$. For such an indenter, it is straightforward to see that

$$h'(x) = \begin{cases} \dfrac{x}{R_1} & \text{for } 0 < x < a, \\ \dfrac{x}{R_2} & \text{for } -b < x < 0, \end{cases}$$

where R_1, R_2 are the respective relative radii of curvature of each part of the indenter.

We can utilise the consistency condition (3) to find the ratio $\beta = b/a$ as a function of $R = R_1/R_2$. A straightforward integration leads to

$$\pi(\beta - 1)\left(\frac{1+R}{1-R}\right) = 4\sqrt{\beta} + 2(\beta - 1)\arcsin\left(\frac{\beta - 1}{\beta + 1}\right). \tag{15}$$

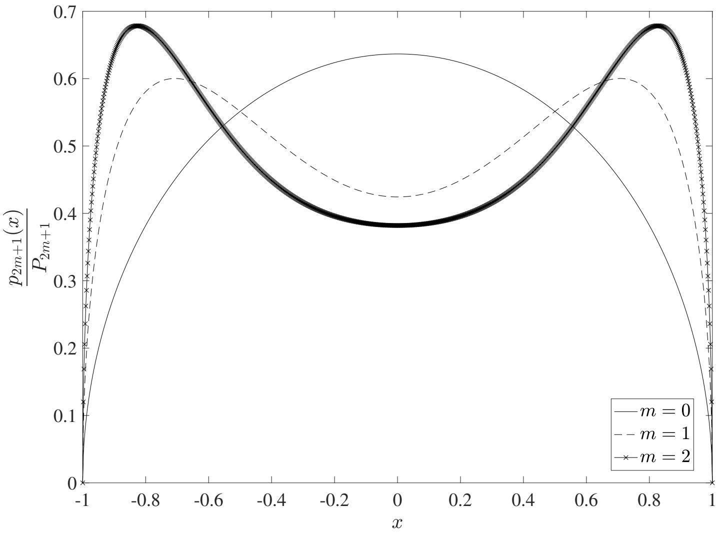

Figure 1: The first three odd contact pressures scaled by their respective normal forces. In this plot $a = 1$.

We are therefore able to find $\beta(R)$ by solving (15) numerically. We start from the known solution $\beta(1) = 1$ and increase/decrease R incrementally, solving the resulting nonlinear equation using Newton's method. We plot the results in figure 2. It is clear that if we fix the relative radius of curvature for the right-hand side of the indenter (i.e. $x > 0$), then as we increase the relative radius of curvature of the left-hand side of the indenter so that R decreases, β increases and thus $b > a$ so that the contact patch is larger to the left of the adjoining line. Similarly, if we reduce the relative radius of curvature of the left-hand side so that R increases, naturally β decreases, with $b < a$ so that the contact patch is now larger to the right of the adjoining line.

After some algebraic manipulation, we are also able to calculate the contact pressure, $p(x)$, by evaluating the principal value integral (2) finding that

$$\begin{aligned} p(x) =& \frac{E^*}{4R_1}(R+1)\sqrt{(a-x)(x+b)} + \frac{E^*}{2\pi R_1}(R-1)\sqrt{(a-x)(x+b)}\arcsin\left(\frac{b-a}{b+a}\right) \\ &+ \frac{E^* x}{4\pi R_1}(R-1)\log\left(\frac{(2a-x)b+ax-2\sqrt{ab}\sqrt{(a-x)(x+b)}}{(2a-x)b+ax+2\sqrt{ab}\sqrt{(a-x)(x+b)}}\right), \end{aligned} \tag{16}$$

for $-b < x < a$. We note that if we set $R_1 = R_2$, we immediately see from (15) that $\beta = 1$, so that $a = b$, and therefore (16) reduces to the standard Hertzian solution for a cylinder, (4).

Since we only have one relation between a, b given by (15), we need a to supply a second condition to solve for the contact width explicitly. Therefore, as in the symmetric case, we again enforce equilibrium between the contact pressure and a normal force, P. Using the known form of the contact pressure (16), we find that this condition becomes

$$P = \frac{E^*\pi}{16R_1}(a+b)^2\left[\frac{1}{2}(1+R) + \frac{1}{\pi}(R-1)\arcsin\left(\frac{b-a}{b+a}\right)\right] + \frac{E^*}{8R_1}(R-1)\sqrt{ab}(b-a). \tag{17}$$

The external moment, M, that must be supplied to prevent any rotation of the indenter is given by

$$M = \int_{-b}^{a} xp(x)dx,$$

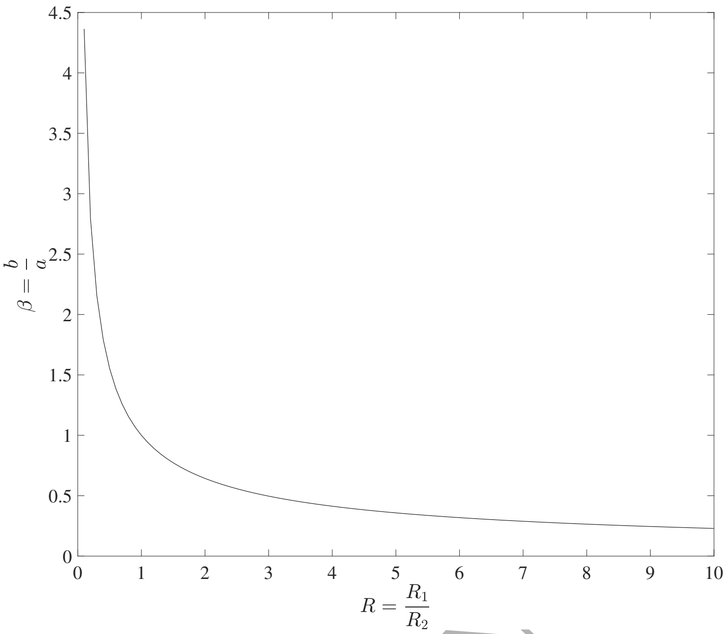

Figure 2: The evolution of $\beta = b/a$ as a function of the ratio of the relative radii of curvatures of the adjoined cylinders, R, as given by (15). Note that for $R < 1$, the left-hand side ($x < 0$) of the indenter has a larger relative radius of curvature than the right-hand side ($x > 0$) and vice versa for $R > 1$. Note that $\beta = 1$ returns the Hertzian contact problem.

which can be evaluated explicitly, obtaining

$$M = \frac{-E^*\pi}{32R_1}(b-a)(a+b)^2\left[\frac{1}{2}(1+R)+\frac{1}{\pi}(R-1)\arcsin\left(\frac{b-a}{b+a}\right)\right] - \frac{E^*}{48R_1}(R-1)\sqrt{ab}(3b^2-2ab+3a^2). \tag{18}$$

We plot both P and M as functions of the ratio R in figure 3. Again, we note that if $R = 1$, we get, as expected, zero applied moment and the corresponding normal force for the Hertzian cylinder given in (5).

4.2 More general non-symmetric contacts

Suppose we proceed as in §3.2 and attempt to write down the relative overlapping profiles of the contacting bodies as a power series polynomial, so the slope at any point can be expressed in the form

$$\frac{dh}{dx} = \sum_n A_n x^n, \tag{19}$$

where we now include both odd and even terms in the expansion. The consistency condition (3) is readily evaluated term-by-term to give

$$0 = \pi\sum_{l=0}^{n}\binom{-\frac{1}{2}}{l}\binom{-\frac{1}{2}}{n-l}(-1)^l\beta^{n-l}. \tag{20}$$

In order to calculate the contact pressure, we are now interested in a generalisation of (7) to the non-symmetric contact patch:

$$J_n = \int_{-b}^{a}\frac{\xi^n d\xi}{\sqrt{(a-\xi)(\xi+b)}\,(\xi-x)}, \tag{21}$$

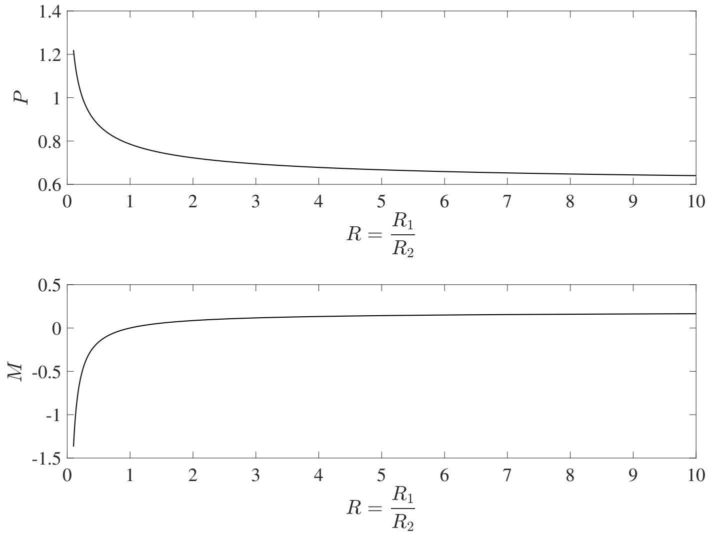

Figure 3: The normal force and first moment as a function of R. For the purposes of illustration we have simply taken $E^*, R_1, a = 1$ in this figure. Note that the first moment vanishes in the case of equal cylinders as expected.

where $n \in \mathbb{N}$. These principal value integrals are evaluated explicitly in appendix A and we therefore find the contribution to the contact pressure of the n^{th}-term of the series (19) to be

$$p_n(x) = \frac{A_n E^*}{2}\sqrt{(a-x)(x+b)}\sum_{l=0}^{n-1}\sum_{j=0}^{n-1-l}\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{n-1-l-j}x^l a^j b^{n-1-l-j}(-1)^j. \tag{22}$$

As previously, we shall only in general be interested in the first few powers of n, so we present the corresponding contact pressures explicitly here:

$$p_1(x) = \frac{A_1 E^*}{2}\sqrt{(a-x)(x+b)}, \tag{23}$$

$$p_2(x) = \frac{A_2 E^*}{2}\sqrt{(a-x)(x+b)}\left(x - \frac{(b-a)}{2}\right), \tag{24}$$

$$p_3(x) = \frac{A_3 E^*}{2}\sqrt{(a-x)(x+b)}\left(x^2 - \frac{(b-a)}{2}x + \frac{3}{8}\left(b^2 - \frac{2}{3}ab + a^2\right)\right), \tag{25}$$

We note that if $a = b$, (23), (25) reduce to their symmetric counterparts (10), (11).

As previously, we need to supplement (20) by enforcing equilibrium between the applied normal force, P, and the total contact pressure. The contribution of the n^{th}-term of the series (19) to the normal force is given by

$$P_n = \frac{A_n E^* \pi}{2}\sum_{l=0}^{n-1}\sum_{j=0}^{n-1-l}\sum_{m=0}^{l+2}\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{n-1-l-j}\binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{l+2-m}a^{j+m}b^{n+1-m-j}(-1)^{j+m+1}, \tag{26}$$

Finally, the contribution of the n^{th}-term of the power series to the total applied moment required to prevent rotation of the indenter is given by

$$M_n = \frac{A_n E^* \pi}{2}\sum_{l=0}^{n-1}\sum_{j=0}^{n-1-l}\sum_{m=0}^{l+3}\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{n-1-l-j}\binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{l+3-m}a^{j+m}b^{n+2-m-j}(-1)^{j+m+1}. \tag{27}$$

Again, for reference, we report the first few values of the normal force and applied moment here:

$$P_1 = \frac{A_1 E^* \pi}{16}(a+b)^2, \qquad M_1 = \frac{A_1 E^* \pi}{32}(a-b)(a+b)^2, \tag{28}$$

$$P_2 = \frac{A_2 E^* \pi}{16}(a-b)(a+b)^2, \qquad M_2 = \frac{A_2 E^* \pi}{256}(9a^2 - 14ab + 9b^2)(a+b)^2, \tag{29}$$

$$P_3 = \frac{3A_3 E^* \pi}{256}(5a^2 - 6ab + 5b^2)(a+b)^2, \quad M_3 = \frac{3A_3 E^* \pi}{256}(a-b)(3a^2 - 2ab + 3b^2)(a+b)^2. \tag{30}$$

4.3 Unbounded domains

When one of the ends of the contact patch is taken to be large enough that the contact may be modelled as semi-infinite, say without loss of generality that $b = -\infty$, the method as described in §2 is valid provided that $h'(x)$ decays sufficiently rapidly as $x \to -\infty$ in order for the integrals exist. Specifically, (3) requires that $h'(x)$ is $o(1)$ as $x \to \infty$, else there is a non-integrable singularity in the integrand for large ξ. It is worth noting however that, when the integrals exist, in such semi-infinite contacts, the consistency condition will explicitly give a value for the remaining endpoint a. This is in contrast to the finite contact problems we have considered, in which we must enforce equilibrium with the applied normal force to determine a and b.

5 Summary

We present novel way of solving half-plane contact problems by using distributions of climb dislocations, present over the contact, whose integral forms the relative profile of the contacting bodies but of opposite sign. This has some attractions in that (a) it is not necessary to invert a singular integral equation, and (b) it is perfectly possible to write down the contact pressure distributions and resultant external force and moment in a simple closed form. We have demonstrated this for both symmetric and non-symmetric contacts.

6 Acknowledgements

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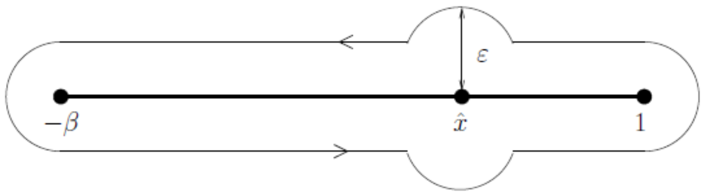

Figure 4: The contour Γ oriented anticlockwise about the branch cut of the square root. The contour consists of arcs of a circle centred at $\hat{x}$, the singular point of the principal value integral (21), arcs of circles around the endpoints of the branch cut at $\hat{\xi} = -\beta, 1$ and straight line segments on the top and bottom of the cut.

A Evaluation of the contact pressure for a general polynomial

After writing the relative overlapping profiles on the contacting bodies as a power series, we are led to consider the principal value integral J_n defined in (21) — a generalisation of the symmetric case (7). By rescaling $\xi = a\hat{\xi}$, $x = a\hat{x}$, we can remove the size of the contact patch a from the integral, finding that

$$J_n = a^{n-1}K_n = a^{n-1} \fint_{-\beta}^{1} \frac{\hat{\xi}^n d\hat{\xi}}{\sqrt{(1-\hat{\xi})(\hat{\xi}+\beta)}(\hat{\xi}-\hat{x})}. \tag{31}$$

We can evaluate K_n using contour integration. Let $\hat{\zeta} = \hat{\xi} + \mathrm{i}\hat{\eta}$ and consider

$$L = \int_{\Gamma} \frac{\hat{\zeta}^n d\hat{\zeta}}{\sqrt{(\hat{\zeta}-1)(\hat{\zeta}+\beta)}(\hat{\zeta}-\hat{x})}, \tag{32}$$

where Γ is the contour depicted in figure 4. The branch cut for the square root is taken along $-\beta < \hat{\xi} < 1, \hat{\eta} = 0$ with the square root positive and real on $\hat{\xi} > 1, \hat{\eta} = 0$. In the limit in which $\varepsilon \to 0$ and Γ approaches the cut, it is straightforward to show that

$$K = 2\mathrm{i}J_n.$$

Since the integrand of L is analytic away from the branch cut, by the deformation theorem the integral is equivalent to that around a circle of radius R centred at the origin, where $R \gg 1$. Therefore, by considering the asymptotic expansion of the integrand for large $|\hat{\zeta}|$, the only contribution to K is from the $\hat{\zeta}^{-1}$-term in the expansion. With the choice of branch above, the asymptotic behaviour of the integrand is found to be

$$\frac{\hat{\zeta}^n}{\sqrt{(\hat{\zeta}-1)(\hat{\zeta}+\beta)}(\hat{\zeta}-\hat{x})} = \hat{\zeta}^{n-2}\left(\sum_{k=0}^{\infty}\frac{\hat{x}^k}{\hat{\zeta}^k}\right)\left(\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{(-1)^k}{\hat{\zeta}^k}\right)\left(\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(\frac{\beta}{\hat{\zeta}}\right)^k\right) \quad \text{as} \quad |\hat{\zeta}| \to \infty.$$

Multiplying the power series, we deduce that

$$\frac{\hat{\zeta}^n}{\sqrt{(\hat{\zeta}-1)(\hat{\zeta}+\beta)}(\hat{\zeta}-\hat{x})} = \hat{\zeta}^{n-2}\sum_{k=0}^{\infty}\frac{d_k}{\hat{\zeta}^k} \quad \text{as} \quad |\hat{\zeta}| \to \infty, \tag{33}$$

where

$$d_k = \sum_{l=0}^{k}\sum_{j=0}^{k-l}\hat{x}^l\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{k-l-j}(-1)^j\beta^{k-l-j}.$$

Whence

$$L = 2\mathrm{i}K_n = 2\pi\mathrm{i}d_{n-1} \implies K_n = \pi d_{n-1}.$$

Therefore, returning to original variables, we conclude that

$$J_n = \pi \sum_{l=0}^{n-1} \sum_{j=0}^{n-1-l} a^{n-1-l} x^l \binom{-\frac{1}{2}}{j} \binom{-\frac{1}{2}}{n-1-l-j} (-1)^j \beta^{n-1-l-j}. \tag{34}$$