Extending the Mossakovskii method to contacts supporting a moment

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Abstract

In this article, we extend the Mossakovskii approach to half-plane contacts supporting a moment. Since the method relies on approximating the punch geometry by a series of flat punches, we choose the load path in (P, M)-space that fixes the body tilt, which allows us to reduce the standard Cauchy singular integral formulation to a non-symmetric Abel integral equation. We use the formulation to derive simple expressions for the applied normal force and necessary applied moment as functions of the contact extent and indenter tilt, while also deriving the coefficients of the square-root terms in the contact pressure expansion at the edges of the contact. These results are analysed in detail for two specific examples: the tilted wedge and the tilted flat-and-rounded punch. We conclude by briefly discussing the equivalent tangential problem when an applied shear force and differential bulk tensions are present.

Keywords: Half-plane theory, B. Contact mechanics, C. Asymptotic analysis

1. Introduction

Half-plane theory is a well-established model for considerations of contact problems relevant to large-scale industrial applications. The contacting bodies are assumed to be sufficiently large that local to the contact, a half-plane idealisation derived from the Flamant solution for the stress field due to a line force at the apex of a wedge is a reasonable approximation [4, 5]. In the limit in which the contacting bodies are elastically similar, the problems for the normal and tangential displacement gradients — gradients to circumvent the problem of the unknown rigid-body terms in the formulation — decouple. For the purposes of this analysis, we shall concentrate on the normal displacement gradients primarily, although much of what we discuss will be applicable to the tangential problem as well, and we return to this topic later.

Consider the formulation in figure 1 where the (x, y)-plane has been chosen so that the origin is at the minimum of the upper body, which has profile $y = g(x)$. We define the relative normal displacement of the contacting bodies as $v = v_1(x) - v_2(x)$, where a subscript 1 denotes the upper body and a subscript 2 denotes the lower body. Then, for elastically-similar bodies, the relative normal displacement gradient is related to the contact

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pressure $p(x)$ by the following integral equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi}\int_{-b}^{a} \frac{p(s)}{s-x}\,\mathrm{d}s, \tag{1}$$

where $(-b, a)$ is the contact patch, κ is Kolosov's constant and μ is the modulus of rigidity. When $-b < x < a$, (1) is interpreted in the Cauchy principal value sense, otherwise it is regular. Within the contact set, we know that

$$v'(x) + g'(x) = 0 \quad \text{for} \quad -b < x < a, \tag{2}$$

where a prime indicates differentiation with respect to argument, so that, for a given geometry, one can invert (1) to find the contact pressure following the methodology of, for example, [10]. In particular, since in an incomplete contact the contact pressure is bounded at the ends of the contact patch, a consistency condition relating a and b must be satisfied for a solution to exist, namely

$$0 = \int_{-b}^{a} \frac{v'(s)}{\sqrt{(a-s)(s+b)}}\,\mathrm{d}s. \tag{3}$$

Over recent years, there has been much consideration of (1) in partial slip problems, see for example [6, 7, 13, 16, 18] and references therein. Although there are notable exceptions such as [3, 8], the singular integral formulation can prove analytically and numerically tricky to handle. An alternative formulation of the problem devised by Mossakovskii [17] and extended by [11, 14, 20] utilises the known contact pressure induced by a flat punch to approximate the upper body by an infinite series of flat punches, which reduces the singular integral to an Abel integral equation, which can be inverted. However, in each of these studies, the consideration has been for symmetric problems (i.e. $b = a$), which precludes the presence of a moment. In many industrial applications, for example in oil wellheads or the dovetail joint of a turbine blade, the presence of an applied moment is inevitable.

In this study, we adapt the Mossakovskii method to problems with an applied moment. In §2, we shall formulate the problem and derive the corresponding non-symmetric Abel integral formulation. We shall use this formulation to derive general results for the applied normal force and applied moment, as well as the behaviour of the contact pressure at the ends of the contact patch in §3. We move on to discuss the solution for two specific examples in §4, the tilted wedge and the tilted flat-and-rounded punch. In §5, we return to the tangential problem and consider applications of the method there, before concluding with a summary and discussion in §6.

2. Problem formulation

We consider the problem depicted in figure 1, in which a body with profile $y = g(x)$ is brought into contact with an elastically-similar half-plane under an applied normal force, P. We shall assume at the outset that the geometry is known and fixed as P varies, so that there is no relative rotation of the body as we change P. Nevertheless, for non-symmetric geometries about the origin, an applied moment, M, is necessary to maintain the contact in this orientation. The geometrical constraint is necessary for the Mossakovskii method to

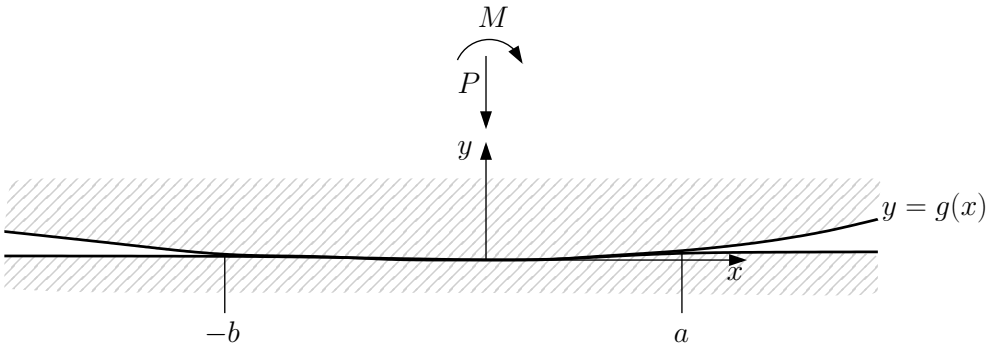

Figure 1: A large, almost flat body $y = g(x)$ is pressed into an elastically-similar half-space with an applied normal force, P, and an applied moment, M. The contact patch extends over $(-b, a)$.

apply, else in general we run into issues with the minimum of the punch changing as M, P vary and the punch rotates. Hence, we restrict ourselves to a particular load path in (P, M) space that fixes the geometry, and as is well-known, the final results for, for example, the contact pressure, are the same as had we taken a different load path. Thus our assumption is not overly restrictive. We shall discuss this further in §3.2.

We know that the contact pressure in the contact patch $-b < x < a$ is related to the relative normal displacement gradient by (1) and that to maintain vertical equilibrium, we must have

$$P = \int_{-b}^{a} p(s)\,\mathrm{d}s. \tag{4}$$

If we take P and the indenter geometry $g(x)$ as known, then we can view (3) and (4) as a pair of equations for a, b. Moreover, these equations are uncoupled: in particular, we shall view (3) as an equation that gives b as a function of a for a specific geometry. Once this has been solved, then (4) gives a as a function of P. For the rest of this analysis, we shall assume that $b(a)$ is known and that as P increases, both a and $b(a)$ increase (which is reasonable provided the body is convex).

Given these results, we can adapt the methodology of [11] to contacts supporting a moment by considering the non-symmetric flat punch in figure 2. It is straightforward to show that ~~substituting~~ the pressure distribution $m(x, a)$, where

$$m(x,a) = \begin{cases} \dfrac{1}{\sqrt{(a-x)(x+b(a))}} & \text{for } -b(a) < x < a \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

corresponds to that of a flat punch by substituting into (1), which gives

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi}h(x,a) = \frac{\kappa+1}{2\mu\pi}\begin{cases} 0 & \text{for } -b(a) < x < a, \\ \dfrac{-\mathrm{sgn}(x)\pi}{\sqrt{(x-a)(x+b)}} & \text{otherwise} \end{cases}. \tag{6}$$

The Mossakovskii idea is then to replace the indenter by an infinite superposition of flat punches, so that

$$p(x,a) = \int_0^a F(s)m(x,s)\,\mathrm{d}s, \tag{7}$$

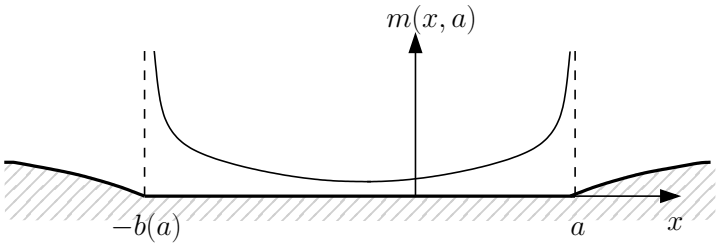

Figure 2: The complete contact of a flat punch extending over $(-b(a), a)$ and the associated contact pressure $m(x, a)$ given by (5).

where the aim is to relate the unknown function $F(s)$ to the indenter geometry. By the previous analysis, we see that

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi}\int_0^a F(s)h(x,s)\,\mathrm{d}s \tag{8}$$

for all $-b(a) < x < a$. Hence, by (2), we must have

$$\frac{2\mu g'(x)}{\kappa+1} = \int_0^x \frac{F(s)}{\sqrt{(x-s)(x+b(s))}}\,\mathrm{d}s \text{ for } 0 < x < a, \tag{9}$$

and we obtain our solution provided that $b(a)$ has been found from (3). Note that, without loss of generality, we could equivalently solve for $F(s)$ on $(-b(a), 0)$. The integral equation (9) is a non-symmetric Abel integral equation for $F(s)$, which is, unfortunately, not readily analytically inverted in general, unlike in the symmetric problem, [11]. It is however relatively straightforward to deal with numerically, particularly as we have removed the singularity associated with the standard inversion procedure, [10]. We discuss such a procedure for a specific example in §4.2 and Appendix B.

Once we have found $F(s)$, the contact pressure is then given by

$$p(x,a) = \begin{cases} \displaystyle\int_x^a \frac{F(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } 0 < x < a, \\ \displaystyle\int_{b^{-1}(-x)}^a \frac{F(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } -b(a) < x < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where $b^{-1}(\cdot)$ denotes the inverse of $b(\cdot)$. To elucidate how the lower limit of the second integral has been obtained, we recall that since we have assumed $b(\cdot)$ is an increasing function, then for all $-b(a) < x < 0$, there exists a unique $s \in (0, a)$ such that $b(s) = -x$. Then, for $s < b^{-1}(-x)$, by (5) the contribution of $m(x, s)$ in the integrand is necessarily zero.

3. General results

3.1. Applied normal force

We recall the vertical equilibrium condition (4). Upon substituting (7) into (4), we have

$$P = \int_0^a F(s)\int_{-b(a)}^a m(x,s)\,\mathrm{d}x\mathrm{d}s = \int_0^a F(s)\int_{-b(s)}^s \frac{1}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}x\mathrm{d}s. \tag{11}$$

Thus, changing variables in the inner integral by setting

$$x = \frac{s - b(s)}{2} + \frac{(s + b(s))}{2} X \tag{12}$$

yields

$$P = \pi \int_0^a F(s)\,\mathrm{d}s \text{ or, equivalently, } F(a) = \frac{1}{\pi}\frac{\mathrm{d}P}{\mathrm{d}a}, \tag{13}$$

as in the symmetric case, [11]. Thus, we can replace the unknown function $F(\cdot)$ by $\pi^{-1}P'(\cdot)$ everywhere it appears in the analysis, provided that the initial condition $P(0) = 0$ is applied where necessary (i.e., there is no contact unless a normal force is supplied). We do so henceforth, and in particular, the Abel equation (9) becomes

$$\frac{2\mu\pi g'(x)}{\kappa + 1} = \int_0^x \frac{P'(s)}{\sqrt{(x - s)(x + b(s))}}\,\mathrm{d}s \text{ for } 0 < x < a, \tag{14}$$

subject to $P(0) = 0$, where a prime denotes differentiation. We note that the inversion of (14) will give $P(s)$ for $0 < s < a$.

3.2. Applied moment

To find the applied moment, M, necessary to sustain the contact, we must have

$$M = \int_{-b(a)}^a sp(s)\,\mathrm{d}s. \tag{15}$$

After performing a similar analysis, we find that

$$M = \frac{1}{2}\int_0^a (s - b(s))P'(s)\,\mathrm{d}s = \frac{1}{2}\left[P(a)(a - b(a)) - \int_0^a P(s)(1 - b'(s))\,\mathrm{d}s\right], \tag{16}$$

where we have integrated by parts to achieve the final equality.

Hence, (14) and (16) give P and M as functions of the indenter geometry $g(x)$. As we alluded to at the start of §2, the methodology has relied on the geometry being maintained as P varies, so that we have chosen the particular load path in (P, M) space where M is given by (16). While this is a constraint necessary for the approach to work, it is not necessarily over-restrictive, as the results are load-path independent.

Hence, an algorithm for using the method for a general punch profile is as follows. Suppose we are given a symmetric indenter $y = h(x)$ that we tilt by an angle of α to the horizontal. If we denote the tilted indenter shape by $y = g(x; \alpha)$ (the semicolon indicating that α is acting as a parameter, not a variable), then (14) and (16) can be used to find the P and M necessary to sustain a contact over $(-b(a), a)$ with tilt α. We can then repeat the process varying the value of α, which populates the space $P(a, \alpha)$ and $M(a, \alpha)$. One can then consider a general loading curve in (P, M)-space and find the resulting values of α. We will illustrate this in the simple case of a tilted wedge in §4.1. Naturally, there may also be applications for problems in which the constancy of the tilted geometry is required, and this methodology thus gives the necessary applied moment (16) to provide this as P changes.

3.3. Application to asymptotic approaches

Finally, we note that there has been a lot of recent work in the field of asymptotic methods for partial slip problems in which the stress-fields local to the edges of contact are crucial in understanding the onset of fretting fatigue in large industrial machinery, [1, 9, 12]. The idea is to consider the problem locally where the geometry may be simpler — for example, a flat-and-rounded punch may be taken as Hertzian near the edges of the contact — and use knowledge of the local form of the contact pressure (and shear traction $q(x)$) to set up a problem with suitably-chosen far-field bulk tensions and/or shear forces to obtain slip-zones of the same size. The simpler geometry is then easier to study in laboratory prototypes and numerical studies. Such methods may also be of use in problems where the slip zones have opposite signs and hence, the Ciavarella-Jäger theorem no longer applies, [2].

In these cases, the punch geometry will be known (perhaps from finite element solutions or otherwise) for general input conditions P, M and applied shear forces and bulk tensions (usually denoted Q and σ). Given the geometry, one can use the Mossakovskii method to extract quickly the local coefficient of the contact pressure (and shear tractions, see §5) at the contact edges, $K_{n,a}$ and $K_{n,b}$, without the difficulties associated with extracting them numerically. It is straightforward to perform an asymptotic expansion of the integral (7) as x approaches the edges of the contact to deduce

$$p = \frac{2}{\pi}\frac{P'(a)}{\sqrt{a+b(a)}}\sqrt{a-x} + o(\sqrt{a-x}) \text{ as } x \to a^-, \text{ so that } K_{n,a} = \frac{2}{\pi}\frac{P'(a)}{\sqrt{a+b(a)}}, \tag{17}$$

while

$$p = \frac{2}{\pi}\frac{P'(a)}{b'(a)\sqrt{a+b(a)}}\sqrt{b(a)+x} + o(\sqrt{b(a)+x}) \text{ as } x \to -b(a)^+, \text{ so that } K_{n,b} = \frac{2}{\pi}\frac{P'(a)}{b'(a)\sqrt{a+b(a)}}, \tag{18}$$

where we have relegated the details to Appendix A. In particular, we note that $K_{n,b} = K_{n,a}/b'(a)$.

4. Application to specific geometries

4.1. The tilted wedge

Consider a large wedge of half-angle $\pi/2 - \phi$, where $0 < \phi \ll 1$, that is tilted at an angle $\alpha < \phi$ measured clockwise from the unrotated state, see figure 3. Since we are assuming that the wedge half-angle is large enough that we can reasonably use a half-space approximation for the contact, the wedge profile is approximated by

$$g(x) = \begin{cases} \Delta - (\phi+\alpha)x & \text{for} \quad x < 0 \\ \Delta + (\phi-\alpha)x & \text{for} \quad x > 0 \end{cases}, \tag{19}$$

where Δ is a rigid-body translation.

By the geometry of the problem, we expect there to be a similarity solution, so that $b = \gamma a$ for some $\gamma \geq 0$. Upon evaluating (3), we find

$$\gamma = \left(1 - \sin\left(\frac{\pi\alpha}{2\phi}\right)\right)\left(1 + \sin\left(\frac{\pi\alpha}{2\phi}\right)\right)^{-1}, \tag{20}$$

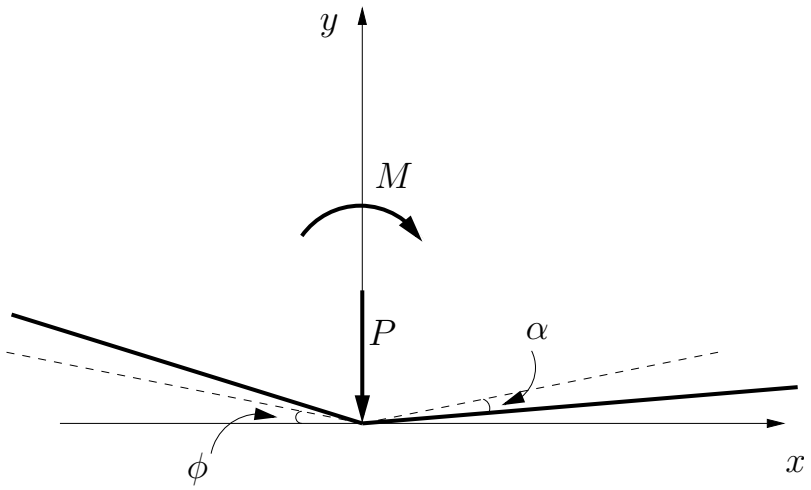

Figure 3: A large, almost flat tilted wedge of half-angle ϕ and tilt angle α. The normal force P is applied through the wedge apex (i.e. at the coordinate origin), along with a moment M about the y-axis.

as in [19].

Similarly, seeking a solution of the form $P(a) = P_0 a$, we can invert (14) to find that

$$P_0 = \frac{4\mu\phi}{\kappa+1}\sqrt{\gamma}, \tag{21}$$

which we note is again consistent with [19]. Note that the corresponding contact pressure is given by

$$p(x) = \frac{-4\mu\phi}{\pi(\kappa+1)} \log\left|\frac{\sqrt{1/\gamma} - \sqrt{(a-x)/(x+\gamma a)}}{\sqrt{1/\gamma} + \sqrt{(a-x)/(x+\gamma a)}}\right|. \tag{22}$$

Finally, we can utilise (16) to show that

$$M(a) = \frac{P_0}{4}(1-\gamma)a^2, \tag{23}$$

is the moment necessary to maintain such a contact. Noting that there is a sign error in equation (27) of [19], one can in fact also show that (23) is consistent with their result once the corrected algebra is performed.

We can now manipulate (21), (23) to find the dependence of a and α on P, M. Hence if P, M are given, we can find α from

$$\frac{(\kappa+1)}{16\mu\phi}\frac{P^2}{M} = \frac{\sqrt{\gamma}}{1-\gamma}, \tag{24}$$

which, utilising (20), yields

$$\alpha = \frac{2\phi}{\pi}\arcsin\left(\frac{8\mu\phi M}{\sqrt{64\mu^2\phi^2M^2 + (\kappa+1)^2P^4}}\right), \tag{25}$$

where we have assumed $\alpha > 0$ without loss of generality due to the symmetry of the indenter. Note that for fixed P, as $M \to 0$, we see that $\alpha \to 0$ as expected.

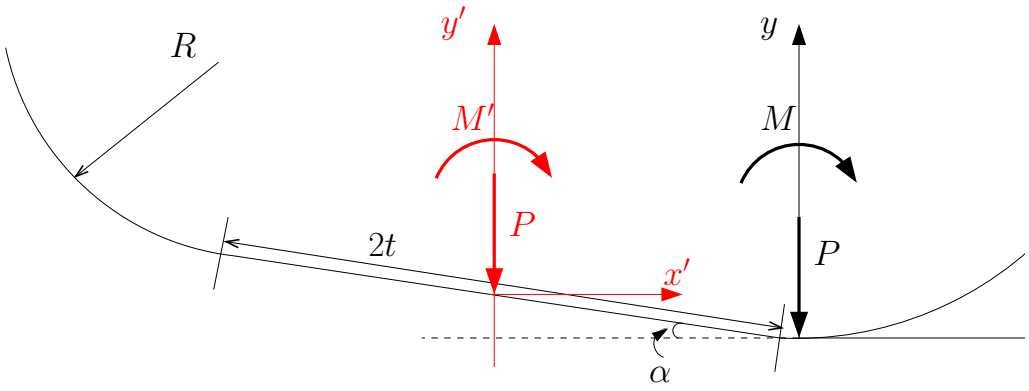

Figure 4: (Colour online.) A flat-and-rounded punch with flat section of size $2t$, rounded radius of curvature, R, and tilt angle α to the horizontal. The coordinate axes (x', y') and (x, y), and their respective associated applied normal forces and moments (P, M') and (P, M) are discussed in detail in the text.

In a similar manner, a is given as a function of P and M by

$$a = \frac{(\kappa+1)^2 P^3}{4\mu\phi} \frac{1}{-8\mu\phi M + \sqrt{64\mu^2\phi^2 M^2 + (\kappa+1)^2 P^4}}. \tag{26}$$

where the negative root has been discounted as that gives $a < 0$. We should note the limit in which $M = 0$ returns the expected relation between a and P, while the limit in which $P \to 0$ and M is finite is singular in the sense that both the numerator and denominator vanish simultaneously. Again, this is as expected, since the limit in which there is no applied normal force but merely a moment makes no sense physically in this problem.

Hence for the wedge, we are able to invert the problem to one of finding a and α when given M and P. We are able to do this so simply for a wedge since the geometry is such that as P and M change, the body rotates but the minimum remains fixed.

Finally, we note that ~~$K_{n,a}(a)$~~, the coefficient of the square-root term in the pressure expansion at the right-hand contact edge is given by (17) to be

$$K_{n,a} = \frac{8\mu\phi}{\pi(\kappa+1)} \sqrt{\frac{\gamma}{a(1+\gamma)}}. \tag{27}$$

and hence, at the left-hand contact point, by (18) we simply have

$$K_{n,b} = \frac{K_{n,a}}{\gamma}. \tag{28}$$

4.2. The tilted flat-and-rounded punch

The flat-and-rounded punch is a geometry that is of great interest in industrial applications. Here we consider a punch whose flat length is $2t$ that is rotated by a small angle α clockwise. Since the Mossakovskii approach requires the approximation of the indenter by a series of rectangular punches, we must consider the coordinate frame fixed with the minimum of the punch. Thus, if in the frame fixed with the line of symmetry of the unrotated punch,

we have

$$y' = \begin{cases} \Delta - \alpha x' + \dfrac{(x'-t)^2}{2R} & \text{for} \quad x' > t \\ \Delta - \alpha x' & \text{for} \quad -t < x' < t \\ \Delta - \alpha x' + \dfrac{(x'+t)^2}{2R} & \text{for} \quad x' < -t \end{cases}, \tag{29}$$

we make the change of variables

$$x' = R\alpha + t + x,\ y' = \Delta - \frac{\alpha^2 R}{2} - \alpha t + y, \tag{30}$$

so that the punch is now given by $y = g(x)$ where

$$g(x) = \begin{cases} -\dfrac{\alpha^2 R}{2} - \alpha x + \dfrac{(R\alpha + x)^2}{2R} & \text{for} \quad x > -R\alpha \\ -\dfrac{\alpha^2 R}{2} - \alpha x & \text{for} \quad -R\alpha - 2t < x < -R\alpha \\ -\dfrac{\alpha^2 R}{2} - \alpha x + \dfrac{(R\alpha + 2t + x)^2}{2R} & \text{for} \quad x < -R\alpha - 2t \end{cases}. \tag{31}$$

The indenter is depicted in figure 4, with the original (x', y')-frame shown in red and the shifted (x, y)-frame shown in black. We note that, in this analysis, the applied normal force and applied moment (P, M) are measured with respect to the (x, y)-frame. If we wish to relate back to problems in which these are applied at the original line of symmetry, (P, M') (depicted in red in figure 4), we must account for the origin shift in the applied moment, namely $M' = M + P(t + R\alpha)$, where M' is the moment applied at the origin in the (x', y')-plane.

For small enough normal force, only the right-hand rounded part of the punch is in contact, so that the consistency condition (3) simply gives

$$b(a) = a, \tag{32}$$

which is valid for $0 < a < a_1 = \alpha R$.

As the normal force increases further, the left-hand contact point moves onto the flat part of the punch. Hence, $b(a)$ now satisfies

$$\alpha\pi R = \sqrt{(a_1 + a)(b - a_1)} + \frac{\pi}{4}(2a_1 + a - b) + \frac{(2a_1 + a - b)}{2}\arcsin\left(\frac{2a_1 + a - b}{a + b}\right) \tag{33}$$

for $a_1 < a < a_2$, where the parameter a_2 is given by the solution to the equation

$$b(a_2) = \alpha R + 2t. \tag{34}$$

When the applied normal force increases even further and $a > a_2$, the left-hand contact edge moves onto the left-hand rounded part of the punch. Hence, $b(a)$ then satisfies

$$\begin{aligned} \alpha\pi R \;=\; & \frac{(2a_1 + a - b)}{2}\arcsin\left(\frac{2a_1 + a - b}{a + b}\right) - \frac{(2b(a_2) + a - b)}{2}\arcsin\left(\frac{2b(a_2) + a - b}{a + b}\right) + \\ & \frac{\pi}{2}(a_1 + b(a_2) + a - b) + \sqrt{(a_1 + a)(b - a_1)} - \sqrt{(a + b(a_2))(b - b(a_2))}. \end{aligned} \tag{35}$$

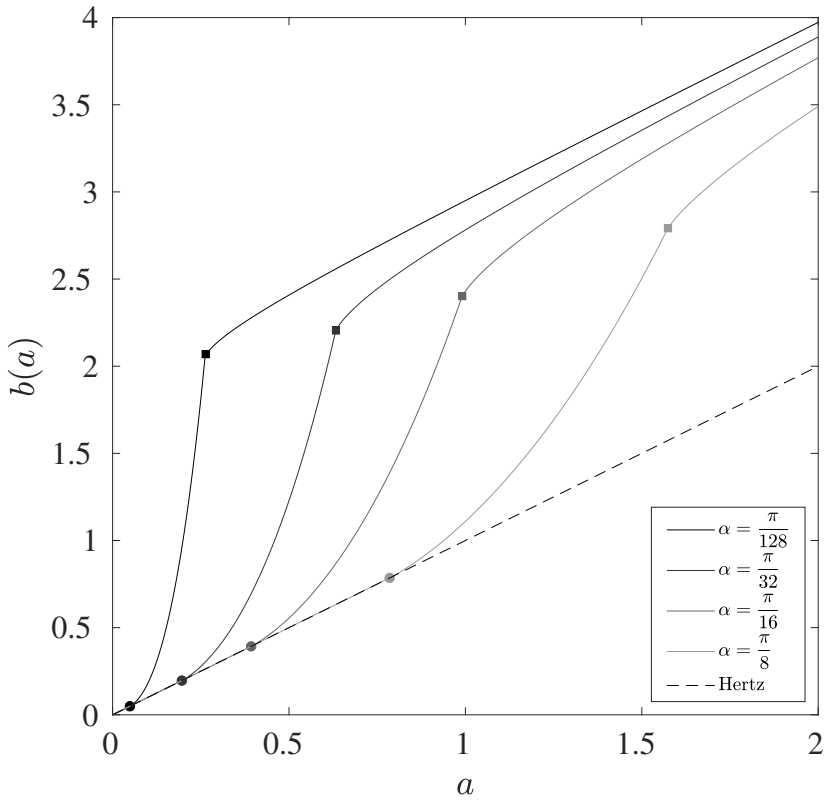

Figure 5: The left-hand contact point $b(a)$ as a function of a for a flat-and-rounded punch at various tilt angles, α. For the purposes of the plot, we have taken $t = 1$ and $R = 2$ and we have included the Hertzian case where $b(a) = a$ for reference. The filled circles indicate αR, the value of a for which $b(a)$ moves onto the flat part of the punch and the filled squares represent a_2, where $b(a)$ moves onto the left-hand rounded part of the punch.

for $a \geqslant a_2$.

We plot a profile of $b(a)$ as a function of a for different tilts in figure 5. The filled circles indicate the first transition from rounded to flat, while the filled squares indicate the second transition from flat to the other rounded part of the punch. Note that $b(a)$ cannot be differentiable at a_1 or a_2, so that $K_{n,b}$ is not defined there, see (18).

To find the applied normal force necessary to sustain such a contact, we return to (14), which we must invert for $P(s)$ where $0 < s < a$. If $0 < a < a_1$, then, since $b(a) = a$, (14) is a standard Abel integral equation, which is readily inverted to show that

$$P(s) = \frac{\pi\mu}{(\kappa+1)R} s^2, \tag{36}$$

for $0 < s < a$, which is simply the solution for a Hertzian contact [11].

If $a > a_1$, since (14) must hold for all $0 < x < a$, it is still true that $P(s)$ is given by (36) for $0 < s < a_1$. However, for $a_1 < x < a$ we have

$$\frac{2\pi\mu}{(\kappa+1)R}\sqrt{x^2 - a_1^2} = \int_{a_1}^{x} \frac{P'(s)}{\sqrt{(x-s)(x+b(s))}}\,\mathrm{d}s \text{ for } a_1 < x < a, \tag{37}$$

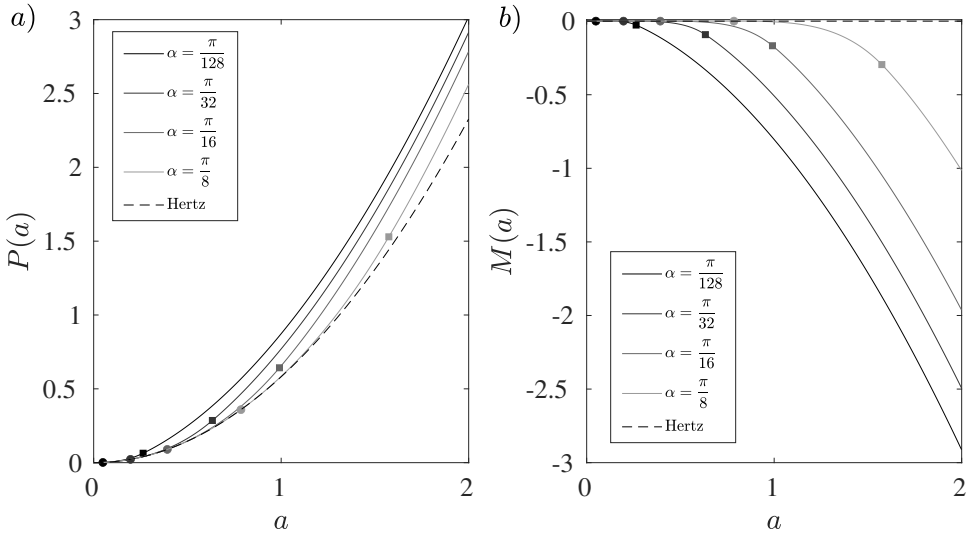

Figure 6: The (a) applied normal force and (b) applied moment necessary to sustain the contact as a function of a for a flat-and-rounded punch at various tilt angles, α. For the purposes of the plot, we have taken $t = 1$ and $R = 2$. The filled circles and squares perform the same function as in figure 5. For reference we have also included the equivalent curves for the Hertzian contact of a cylinder of radius R.

which we must invert for $P(s)$, $a_1 < s < a$. In general, we must do this numerically. Fortunately, this is a relatively straightforward process, which we describe in Appendix B. We plot the resulting curve of $P(a)$ for various tilt angles in figure 6a. Note that $P(a)$ is differentiable everywhere even though $b(a)$ is not.

Having determined $P(a)$, we can utilise (16) to calculate the applied moment necessary to maintain the tilt angle α as P varies. Again, when $0 < a < a_1$, we can evaluate this explicitly, finding that $M = 0$, as expected (as the contact is essentially Hertzian). For $a > a_1$, we see that

$$M(a) = \frac{1}{2}\left[P(a)(a - b(a)) - \int_{a_1}^{a} P(s)(1 - b'(s))\,\mathrm{d}s\right], \tag{38}$$

which we can evaluate numerically using our known solutions for $P(a)$ and $b(a)$. Note that even though $b'(s)$ is not defined when $s = a_2$, as it is defined everywhere else, we can still perform the integration in (38). We plot the results in figure 6b. We note that this moment needs to be taken in context. This is the moment about the minimum of the tilted punch that needs to be supplied to maintain the contact. We also note that this is an illustration of how we can sweep over values of α to fill out the (a, α)-space for M and P, which enables us to 'invert' the problem and seek a and α for a given (P, M) combination, as discussed in §3.2; however, we will not pursue this any further in this paper.

Finally, we plot the K_n-coefficients (17)–(18) as functions of a for various tilt angles in figure 7. For the left-hand coefficient, the change in behaviour $b(a)$ as the applied normal force increases is clear: when the left-hand contact point progresses onto the flat part of the indenter, there is a sharp increase in $b'(a)$, as evidenced in figure 5. This causes $K_{n,b}$ to decrease until $b(a)$ progresses to the second rounded part of the punch.

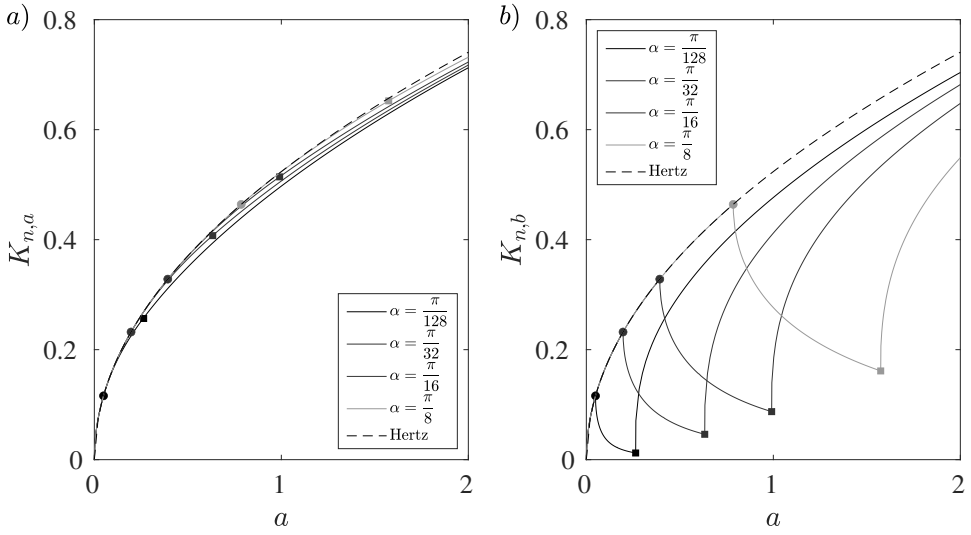

Figure 7: The (a) right-hand and (b) left-hand coefficients of the square-root expansion of the contact pressure at the contact edges as a function of a for a flat-and-rounded punch at various tilt angles, α. For the purposes of the plot, we have taken $t = 1$ and $R = 2$. The filled circles and squares perform the same function as in figure 5. For reference we have also included the equivalent curves for the Hertzian contact of a cylinder of radius R. The change in behaviour of $K_{n,b}$ is due to the rapid change in $b'(a)$ as the left-hand contact point progresses from rounded to flat (circles) and then from flat to the other round part of the punch (squares).

5. The tangential problem

To conclude, we briefly return to the tangential problem, that we have thus far set aside. As discussed in section 1, for elastically-similar materials, the normal and tangential problems decouple. Indeed, the relation between the tangential displacement gradient $u'(x) = u_1'(x) - u_2'(x)$ and the shear tractions $q(x)$ is exactly the same as (1) with v replaced by u and p replaced by q.

Hence, in the case where there is purely a shear force, Q, applied in the contact (i.e. shear tractions are not excited by differential bulk tensions), and if we assume that the normal load is a monotonically increasing function of time and that friction is large enough for slip to be prohibited along the whole of the contact patch, one can follow the same argument as [11] in the non-symmetric problem to conclude that the shear traction is given by

$$q(x,a) = \begin{cases} \displaystyle\int_x^a \frac{G(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } 0 < x < a, \\ \displaystyle\int_{b^{-1}(-x)}^a \frac{G(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } -b(a) < x < 0, \\ 0 & \text{otherwise} \end{cases}, \tag{39}$$

where

$$Q(a) = \pi \int_0^a G(s)\,\mathrm{d}s \text{ or, equivalently, } G(a) = \frac{1}{\pi}\frac{\mathrm{d}Q}{\mathrm{d}a}. \tag{40}$$

When there is a differential bulk tension $\sigma = \sigma_1 - \sigma_2$, we must introduce a second auxiliary function to replace (5) to find its influence on the shear traction. Extending the argument

of [11], we define:

$$n(x,a) = \begin{cases} \dfrac{x}{\sqrt{(a-x)(x+b(a))}} & \text{for } -b(a) < x < a \\ 0 & \text{otherwise} \end{cases}, \tag{41}$$

which, upon substituting into the tangential form of (1), induces a relative surface strain of

$$u'(x) = \frac{(\kappa+1)}{2\mu\pi} l(x,a) \text{ where } l(x,a) = \begin{cases} \pi & \text{for } -b(a) < x < a, \\ \pi\left(1 - \dfrac{|x|}{\sqrt{(a-x)(x+b(a))}}\right) & \text{otherwise} \end{cases}. \tag{42}$$

Then, assuming the bodies remain fully adhered in the contact region, we must find a function $H(\cdot)$ that satisfies

$$\frac{\partial}{\partial a}\left(\epsilon(x,a)\right) = \frac{\partial}{\partial a}\left(\frac{(\kappa+1)\sigma(a)}{8\mu} + \int_0^a H(s) l(x,s)\,\mathrm{d}s\right) \tag{43}$$

in the contact patch, where $\epsilon(x,a)$ is the relative surface strain, [11]. Thus,

$$H(a) = -\frac{1}{4}\frac{\mathrm{d}\sigma}{\mathrm{d}a}. \tag{44}$$

Therefore, in the general case, the shearing traction has a contribution from both the traction induced by the applied tangential force, Q, and a contribution from the differential bulk tension, σ, so that

$$q(x,a) = \begin{cases} \displaystyle\int_x^a \frac{G(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s + \int_x^a \frac{H(s)x}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } 0 < x < a \\ \displaystyle\int_{b^{-1}(-x)}^a \frac{G(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s + \int_{b^{-1}(-x)}^a \frac{H(s)x}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s & \text{for } -b(a) < x < 0, \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

where $G(\cdot)$ and $H(\cdot)$ are given by (40) and (44) respectively. We note that we can expand (45) at each end of the contact patch to find the coefficient of the square-root terms in the shear traction there, we find that

$$q(x,a) = \frac{2}{\pi}\frac{1}{\sqrt{a+b(a)}}\left(Q'(a) - \frac{\pi a\sigma'(a)}{4}\right)\sqrt{a-x} + o\left(\sqrt{a-x}\right) \text{ as } x \to a^-, \tag{46}$$

and

$$q(x,a) = \frac{2}{\pi b'(a)}\frac{1}{\sqrt{a+b(a)}}\left(Q'(a) + \frac{\pi b(a)\sigma'(a)}{4}\right)\sqrt{x+b(a)} + o\left(\sqrt{x+b(a)}\right) \text{ as } x \to -b(a)^+, \tag{47}$$

which, again, may have applications in asymptotic methods for studying partial-slip problems.

6. Summary and discussion

In this analysis, we have demonstrated how to extend the Mossakovskii method to non-symmetric contacts in which a moment is present. The central idea is to utilise the consistency condition from the usual inversion of the Cauchy principal value integral relating the contact pressure and the indenter geometry as a tool to find the left-hand contact point as a function of the right (note this was arbitrarily chosen, we could have chosen to find $a(b)$ as readily as $b(a)$). One can then use the Mossakovskii flat-punch superposition as in the symmetric problem to derive a non-symmetric Abel integral relating the applied normal force to the punch geometry. This is a similar idea to the use of distributions of climb dislocations as Green's functions for the contact problem as demonstrated in [15].

The Abel integral formulation allows us to derive a simple relation for the necessary applied moment to sustain the contact in terms of the the contact patch size and the applied normal force, as well as straightforward expressions for the coefficients of the square-root term in the pressure expansion local to each edge of the contact, which may be of use in asymptotic methods for studying fretting fatigue in partial-slip problems [1, 9, 12].

We have demonstrated the methodology for two particular examples. The first example was the tilted wedge, for which analytic solutions for the applied normal force, the applied moment and the pressure coefficients can be found, and were shown to match the closed-form solutions found through an alternative route by [19]. The second, more industrially-relevant problem was that of a tilted flat-and-rounded punch. After noting that the necessity of the coordinate frame being centred at the minimum of the indenter introduced three distinct regimes in the problem, we were able to derive numerical solutions for the contact patch size, the applied normal force and the pressure coefficients, as well as the moment necessary to sustain the contact.

One feature of the Mossakovskii method for a general indenter when there is a moment is that one must assume that the tilt is fixed and solve the problem, which fixes the load-path in (P, M)-space that one must take to apply the method. If we do not follow this path, the minimum of the indenter changes as it rotates, which leads to issues in the formulation, as it is a history-dependent problem in the sense that it builds up the punch geometry through a series of flat punches. If one allows the indenter to rotate and the minimum to move slightly, there is part of the punch which is now not captured by the superposition of flat punches given in (7). However, since the results are load-path independent, this is only a minor point.

We have shown that for certain geometries such as the tilted wedge, since the minimum does not change as the normal force increases, the method can be inverted to consider the more general problem: 'what are α and a for given values of M and P?'. Moreover, even for geometries that do not meet this condition, we can still use the given geometry found from full finite element simulations or analytic solutions as a starting point for the method, which is then able to determine the coefficients of the contact pressure at the contact edges very quickly.

We concluded our discussion by highlighting that, provided the contact ends both monotonically increasing in with the normal force and that friction is large enough to prohibit slip at all points, the methodology can readily be extended to the tangential problem in which an applied shear force and/or differential bulk tensions excite shear tractions within the contact.

Appendix A. Integral expansions

We shall illustrate the methodology by considering the contact pressure expansion close to the right-hand contact point. Recall that the contact pressure for $0 < x < a$ is given by

$$p(x,a) = \frac{1}{\pi}\int_x^a \frac{P'(s)}{\sqrt{(s-x)(x+b(s))}}\,\mathrm{d}s. \tag{A.1}$$

If $x = a - \delta$ where $0 < \delta \ll 1$, we set $s = a - \delta S$ in the integral, which yields

$$p = \frac{\sqrt{\delta}}{\pi}\int_0^1 \frac{P'(a-\delta S)}{\sqrt{(1-S)(a-\delta+b(a-\delta S))}}\,\mathrm{d}S. \tag{A.2}$$

We can then expand the integrand as $\delta \to 0$ as a regular perturbation, which yields

$$p = \frac{\sqrt{\delta}}{\pi}\frac{P'(a)}{\sqrt{a+b(a)}}\int_0^1 \frac{1}{\sqrt{1-S}}\,\mathrm{d}S + O\left(\delta^{3/2}\right), \tag{A.3}$$

which can be evaluated and thus gives

$$K_{n,a} = \frac{2}{\pi}\frac{P'(a)}{\sqrt{a+b(a)}}. \tag{A.4}$$

The procedure at the left-hand contact point follows a similar argument.

Appendix B. Numerical solution of the non-symmetric Abel equation for the flat-and-rounded punch

For ease of notation, we shall set $\mathcal{P}(s) = P'(s)$ throughout. We wish to invert the integral equation

$$\frac{2\pi\mu}{(\kappa+1)R}\sqrt{x^2-a_1^2} = \int_{a_1}^x \frac{P'(s)}{\sqrt{(x-s)(x+b(s))}}\,\mathrm{d}s \text{ for } a_1 < x < a \tag{B.1}$$

to find $\mathcal{P}(s)$ for $a_1 < s < a$.

We consider a finite set of points from the range of $a_1 < x < a$ over which the integral equation is valid, defined by $X_j = a_1 + j\epsilon$, where $\epsilon = (a-a_1)/N$ and $j = 1, 2, \ldots, N$. We take N to be sufficiently large so that $0 < \epsilon \ll 1$.

At the first point, $X_1 = a_1 + \epsilon$, the range of the integral in (B.1) is very small, so that the integral can be asymptotically approximated to be

$$\int_{X_1-\epsilon}^{X_1} \frac{\mathcal{P}(s)}{\sqrt{(X_1-s)(X_1+b(s))}}\,\mathrm{d}s = \frac{2\mathcal{P}(X_1)}{\sqrt{X_1+b(X_1)}}\sqrt{X_1-a_1} + O(\epsilon^{3/2}). \tag{B.2}$$

We can then simply rearrange (B.1) to find $\mathcal{P}(X_1)$.

At the next point X_2, we now know $\mathcal{P}$ for $0 < s < X_1$. On that interval we can find the interpolant of $\mathcal{P}$, which we shall denote $\mathcal{P}_{int}$. Hence, (B.1) can now be expressed as

$$\frac{2\pi\mu}{(\kappa+1)R}\sqrt{X_2^2-a_1^2} - \int_{a_1}^{X_1} \frac{\mathcal{P}_{int}(s)}{\sqrt{(X_2-s)(X_2+b(s))}}\,\mathrm{d}s = \int_{X_1}^{X_2} \frac{\mathcal{P}(s)}{\sqrt{(X_2-s)(X_2+b(s))}}\,\mathrm{d}s. \tag{B.3}$$

As before, we can then asymptotically approximate the integral on the right-hand side, finding now that

$$\int_{X_2-\epsilon}^{X_2} \frac{\mathcal{P}(s)}{\sqrt{(X_2-s)(X_2+b(s))}}\,\mathrm{d}s = \frac{2\mathcal{P}(X_2)}{\sqrt{X_2+b(X_2)}}\sqrt{X_2-X_1}+O(\epsilon^{3/2}), \tag{B.4}$$

and we then rearrange accordingly to find $\mathcal{P}(X_2)$.

This process can be iterated to find $\mathcal{P}(a_1+j\epsilon)$ for $j=1,2,\ldots,N$, and we can integrate to find $P(a_1+j\epsilon)$ utilising the known boundary condition $P(a_1)=\mu\pi a_1^2/(R(\kappa+1))$. We can then check convergence of the method by varying N.

References

[1] H. Andresen, D. A. Hills, J. R. Barber, and J. Vázquez. Frictional half-plane contact problems subject to alternating normal and shear loads and tension in the steady state. Intl. J. Solids Struct., 168:166–171, 2019.

[2] H. Andresen, D. A. Hills, and M. R. Moore. The steady state partial slip problem for half plane contacts subject to a constant normal load using glide dislocations. Eur. J. Mech.-A/Solids, 79:103868, 2020.

[3] H. Andresen, D. A. Hills, and J. Vázquez. Closed-form solutions for tilted three-part piecewise-quadratic half-plane contacts. Intl. J. Mech. Sci., 150:127–134, 2019.

[4] J. R. Barber. Elasticity. Springer, 2002.

[5] J. R. Barber. Contact Mechanics. Springer, 2018.

[6] M. Ciavarella. The generalized Cattaneo partial slip plane contact problem. I Theory. Intl. J. Soilds Struct., 35(18):2349–2362, 1998.

[7] M. Ciavarella. The generalized Cattaneo partial slip plane contact problem. II Examples. Intl. J. Solids Struct., 35(18):2363–2378, 1998.

[8] M. Ciavarella and G. Demelio. On non-symmetrical plane contacts. Intl. J. Mech. Sci., 41:1533–1550, 1999.

[9] R. M. N. Fleury, D. A. Hills, R. Ramesh, and J. R. Barber. Incomplete contacts in partial slip subject to varying normal and shear loading, and their representation by asymptotes. J. Mech. Phys. Solids, 99:178–191, 2017.

[10] F. D. Gakhov. Boundary Value Problems. Pergamon, 1966.

[11] D. A. Hills, M. Davies, and J. R. Barber. An incremental formulation for half-plane contact problems subject to varying normal load, shear, and tension. J. Strain Anal. Eng. Des., 46(6):436–443, 2011.

[12] D. A. Hills, R. M. N. Fleury, and D. Dini. Partial slip incomplete contacts under constant normal load and subject to periodic loading. Intl. J. Mech. Sci., 108:115–121, 2016.

[13] D. A. Hills, R. Ramesh, J. R. Barber, and M. R. Moore. Methods to solve half-plane partial slip contact problems. International Journal of Solids and Structures, 155:155–159, 2018.

[14] J. Jäger. A new principle in contact mechanics. ASME J. Tribol., 120:677–684, 1998.

[15] M. R. Moore and D. A. Hills. Solution of half-plane contact problems by distributing climb dislocations. Intl. J. Solids Struct., 2018.

[16] M. R. Moore, R. Ramesh, D. A. Hills, and J. R. Barber. Half-plane partial slip contact problems with a constant normal load subject to a shear force and differential bulk tension. J. Mech. Phys. Solids, 118:245–253, 2018.

[17] V. I. Mossakovskii. Application of the reciprocity theorem to the determination of the resultant forces and moments in three-dimensional contact problems. PMM, 17:477–482, 1953.

[18] R. Ramesh, J. R. Barber, and D. A. Hills. Plane incomplete contact problems subject to bulk stress with a varying normal load. Intl. J. Mech. Sci., 122:228–234, 2017.

[19] A. Sackfield, D. Dini, and D. A. Hills. The tilted shallow wedge problem. Eur. J. Mech.-A/Solids, 24(6):919–928, 2005.

[20] D. A. Spence. An eigenvalue problem for elastic contact with finite friction. In Math. Proc. Camb. Phil. Soc., volume 73, pages 249–268, 1973.