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Complementarity and Uncertainty in Quantum Interference

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Abstract

This thesis is concerned with the notions of complementarity and uncertainty encountered in quantum mechanics. Its starting point is an assessment of how these concepts have been represented and illustrated by various writers dating back to their inception. Following the survey a coherent account of the connections and contrasts between complementarity and uncertainty is developed in the context of Mach-Zehnder interferometry. The effect on the interference pattern contrast of path detection via entanglement with a probe system, is explored and a joint unsharp measurement scheme of the complementary pairs, path and interference, described. The Mach-Zehnder set-up proves sufficiently versatile to show that quantum erasure and quantitative quantum erasure constitute instances of joint unsharp measurement of complementary observables. The analysis uses the representation of observables as positive operator valued measures.

Path detection and interference observation require different experimental set-ups but can be reconciled in the simultaneous unsharp measurement and preparation. This reconciliation is expressed as an uncertainty relation however the mutually exclusive feature of complementarity is not discarded. It is possible to recover strict complementarity as a limit case of the appropriate uncertainty relation.

One motivation for this study is the effort some authors have made in trying to express the founding features of quantum mechanics in the form of a hierarchy of significance. Here it is shown that complementarity and uncertainty have separate identities but are not completely independent of each other. Consequently, establishing a hierarchy of these features *within* the present formalism of quantum mechanics

is not possible.

During and resulting from the preparation of this thesis, the following papers have been published:

P. Busch, C. Shilladay. *Uncertainty reconciles complementarity with joint measurability*, Physical Review A **68**, 034102, 1-4 (2003).

P. Busch, C. Shilladay. *Complementarity and uncertainty in Mach-Zehnder interferometry and beyond*, Physics Report **435**(1), 1-31 (2006).

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Chapter 1

Complementarity and Uncertainty: An Introduction

> Referring to interference exhibited by electron 'waves', Richard Feynman wrote; "... has in it the heart of quantum mechanics. In reality, it contains the only mystery."
> *Lectures on Physics* Volume III.

Complementarity, one of the fundamental features of quantum mechanics has been and still often is discussed in terms of the Heisenberg uncertainty principle, another of the fundamental features of quantum mechanics.

In this thesis, as many other studies the experimental context into which the testing of complementarity is put is a two-path interferometer in which it is possible to determine the path taken by the quantum object. This is accomplished by introducing a probe system which interacts with the object passing through the interferometer. The interaction between the probe and quantum object can be recorded. As soon as enough information for the determination of the path taken has been acquired, the interference pattern cannot be observed. The earlier explanation for this was that the probe used for path determination has imparted random kicks to the object being investigated. Examples of this are Einstein's recoiling slit arrangement (see Bohr's paper in [66]) and Feynman's electron-light scattering scheme [29]. The mutually exclusive set-ups of observing the path of the object and an interference pattern i.e. the enforcing of the complementarity of path and interference, are presented as the position-momentum uncertainty relation in action.

However, in recent times there have been different types of experiment proposed and realized in which the back action on momentum is not the cause of the washing out of the interference pattern on path determination.

The first such proposal was made by M.O. Scully, B.-G. Englert, H. Walther (SEW) in *Quantum Optical Tests of Complementarity* published in Nature in 1991 [60]. The novel proposal of SEW initiated considerable immediate response starting with, [61], [62], [63], [75], and attracted almost 300 citations by the end of 2006.

This body of literature forms more than a debate about the controversial claims of SEW about the rôle of the uncertainty relation; authors are inspired to explore the foundations of quantum mechanics in new ways and an experiment similar to the ingenious experimental proposal by SEW was carried out by Dürr, Nonn and Rempe (DNR), [22].

In this work the proposals and the debate enlivened by SEW are placed in historical context. SEW provide a route finder to work done previously and a stimulus for subsequently proposals on the relationship between complementarity and uncertainty. It is found that this requires an exploration of the origin and propagation of the relation between complementarity and the uncertainty relation.

The experiments of SEW and DNR question the relevance of classical momentum kicks and the position-momentum uncertainty relation. They propose instead that entanglement is the explanation of the disappearance of interference in which-path experiments. In Chapter 4, section 4.3 a two path Mach-Zehnder type interferometer analogue of SEW's proposal is used to present and analyze various experiments in one common setting.

In this thesis, I will address the following questions, with the aim of clarifying long-standing misconceptions and conflations concerning complementarity, uncertainty, and entanglement.

What is the appropriate formulation and formalization of complementarity?

In Chapter 3 section 3.3.1 the introduction of the concept of complementarity to Physics by Bohr in 1928 is outlined. Initially his intention was to capture, in a broad sense, the necessity of using mutually exclusive classical descriptions in the microscopic domain in order to obtain a complete account of quantum phenomena. The notion was then developed into one that involves a relationship between certain pairs of observables: in the quantum domain there are pairs of observables for which preparation and measurement set-ups, as Bohr would say, "experimental set-ups" are mutually exclusive.

However, the formulation of complementarity did not stop there. There is some evidence that Bohr saw the necessity of quantifying the limitations of the simultaneous application of classical concepts as expressed in Heisenberg's uncertainty relation. This can be interpreted as a softening of strict complementarity to a *graded form of complementarity.*

Another recent formulation of complementarity is *preparation complementarity* encountered in Chapter 3 as *value complementarity.* Here, value complementarity is discussed in the context of accounting for the disappearance of interference when path marking is in action.

Value complementarity is a relationship between certain pairs of observables: two observables are value complementary if whenever one has a definite value, the values of the other are maximally uncertain. In the context of a two-path interferometer experiment containing a probe system which interacts with the object passing through the interferometer and a device which records the interaction, the question of how the disappearance of the interference pattern is linked to the path marking sets the following task: the representation of path observation and interference observation in terms of quantum mechanical observables. These observables will be seen to be a value complementary pair.

Complementarity as *measurement complementarity* has also been identified in reference to a pair of observables for which a sharp measurement of one of them makes any attempt at measuring the other one simultaneously or in immediate succession completely obsolete. Measurement complementarity implies the impossibility of *joint measurements* and is a special instance of von Neumann's theorem, according to which, two sharp observables are jointly measurable if, and only if, they commute.

One problem to be addressed is that of the relationship between complementarity and uncertainty in the setting of value complementarity and its relation to path-interference duality. In preparation I will review the evolution of the concept of complementarity as it is generally used in the more recent literature, specifically in its versions as preparation complementarity, i.e., the exclusivity of certain pairs of preparations, and measurement complementarity, i.e., the exclusivity of certain pairs of measurements.

What is the uncertainty principle and does it need an explanation?

Heisenberg developed the concept of uncertainty in a consideration made on quantum objects [32]. Initially, he considered what the tracks of separated water droplets made by an electron in a cloud chamber could tell us about the position and velocity of the electron.

This is developed into considering the outcome of an attempted joint measurement of position and momentum in terms of standard deviations of position and momentum observables in a Gaussian wave functions centered on the measured values. The position and momentum uncertainties in this conditional final state are then taken to represent the inaccuracies of the joint measurement. Here Heisenberg is regarding measurements as producing (approximate) eigenstates of the measured observable that would be expected to be associated with the measured value.

Heisenberg brings together two versions of uncertainty relation. First there is the familiar uncertainty for state preparations:

$$\Delta(Q, \psi)\Delta(P, \psi) \geq \frac{\hbar}{2}. \tag{1.0.1}$$

According to this separate measurements of position Q and momentum P in a state ψ have distribution widths. In Heisenberg's consideration these distribution widths are standard deviations satisfying this uncertainty relation. The other,

$$\delta q \delta p \geq \frac{\hbar}{2}. \tag{1.0.2}$$

is a trade-off relation for the inaccuracies in the joint measurements of these noncommuting observable.

A third manifestation of uncertainty can be identified in the earliest semi-classical models, designed to illustrate quantum mechanics. This is the idea that uncertainty and the uncertainty relation are a consequence of "... unavoidable, non-negligible, and uncontrollable mechanical disturbances" ([32] translated in [74]) has become part of how quantum mechanics is taught.

There have been critical investigations of the 'disturbance' idea, but the tacit identification of the position-momentum uncertainty relation with momentum kicks is still a deeply rooted conflation.

The reason for the persistence of the linking of uncertainty with kicks must be seen in the desire for causal explanation. As noted above, the recent debate has brought out a novel idea, namely, that of a quantum-mechanical, non-local kind of momentum transfer, which is worth exploring further.

A notion of a joint measurement scheme for noncommuting quantum mechanical observables was not available to Heisenberg. Hence a proper formalization and proof of the relation (1.0.2) was not possible and has been outstanding for several decades. However, he did attempt to solve the joint measurement problem, with some success, by considering sequences of measurements. The incident plane wave can be regarded as having resulted from a sharp momentum measurement. The next measurement in the sequence is an approximate or unsharp position measurement at the slit. This measurement can be seen as degrading the initial sharp momentum measurement into an unsharp momentum measurement.

A proper solution to the problem of the joint measurement of noncommuting observables requires the idea that such measurements must not be too accurate and had to wait for the introduction of unsharp observables represented as in terms of *positive operator valued measures (POVM).*

In Chapter 2 the mathematical foundation of quantum mechanics is laid out with the aim of developing a framework for representing measurement in quantum mechanics. The latter parts of this chapter extend the orthodox formalism of quantum mechanics to explore the realization of inaccurate measurements to make possible the observations of two non-commuting or incompatible quantities of a quantum system simultaneously i.e. using POVMs.

Equation 1.0.2 describes the limits of the precision available and can be found in texts on quantum mechanics. But the idea of inaccurate measurements are rarely treated in a rigorous way. A comprehensive explanation of the solution of this can be seen in [15].

Schemes which propose using unsharp observables to yield unsharp path-interference duality in photon and neutron experiments are reviewed in Chapter 3. Here (in section 3.4.2) experiments are reviewed that demonstrate that the path taken through an interferometer can be determined with a confidence of 99% and yet interference pattern with some contrast can also be observed. In Chapter 4 POVMs are used to represent path and interference observations in a two path interferometer yielding a pay-off relation between path and interference.

What is the nature of the relationship between complementarity and uncertainty?

The claims of SEW and DNR that in their experiments the disappearance of interference is not a consequence of the uncertainty relation for position and momentum amounts to the statement that the complementarity and uncertainty principles have different empirical content and consequences because they can be distinguished experimentally.

This statement stands in sharp contrast to the view, held by many, that the two principles are “more or less the same”, or two different manifestations of the same notion. It was Bohr who first considered the uncertainty relation as a “simple symbolic expression” of complementarity (as introduced by him) [5] (see Chapter 3 section 3.3.1) and he invoked the uncertainty relation to argue that interference fringes in a two-slit experiment will disappear if the conditions of the experiment are changed so as to enable path determination by measuring the recoil of the initial slit [66].

It seems that the view taken on the relationship between complementarity and uncertainty depends on the definition adopted for complementarity and possibly on the version of the uncertainty principle used. Hence, a careful look at the different formulations of the two principles will be required. Chapter 3 traces the origins of these concepts and the confusions centered around them in the early stages of the development of the interpretation of quantum mechanics.

From the formalism of quantum mechanics the existence of noncommuting pairs of observables is intimately linked to complementarity. Two (discrete) observables have a common orthonormal basis of eigenvectors if and only if they commute. Further, an observable has a definite value in a state if the probability for some value is one, i.e. if the state is an eigenstate of that observable. Therefore, noncommuting observables do not have, in general, simultaneously definite values.

(A) Thus, noncommuting observables do not admit simultaneous preparations for all of their eigenvalues nor do they admit simultaneous measurements.

If a spread of the distributions of values in each state is allowed then this can be quantified by the uncertainty relation between the observables. In the limiting case, having one observable sharp forces the other one to be maximally uncertain, value complementarity is in force as the most common contemporary formulation of complementarity;

(B) If the noncommuting observables are allowed to be unsharp then it follows that noncommutativity is no longer an obstacle to joint measurability and preparation of unsharp values is now appropriately recognized.

Some recent research papers, reviewed in Chapter 3, incorporate both features (A) and (B). This constitutes a break with the older traditions, also traced in Chapter 3, which gave preference either to the complementarity principle or the uncertainty principle. It appears that with this shift of view point a more balanced assessment has been achieved. Compared to the view that emphasized complementarity over uncertainty, the positive rôle of the uncertainty relations as enabling joint determinations and joint measurements now becomes a possibility.

The strict mutual exclusivity of (A) provides the reason for the quest for the realization of (B). In Chapter 4 it will be shown that this consideration leads to some uncertainty or trade-off relation, not necessarily involving position and momentum, which entails complementarity.

In finding ways of incorporating statements (A) and (B) above, into the formalism new and interesting questions are opened up: new versions of complementarity can be proposed, such as graded or *quantitative complementarity.* These will be found to arise in both *quantum erasure* and *quantitative quantum erasure* in the models of joint measurements proposed in Chapter 4. Here, the independent and important rôle of the measurement uncertainty relation as a necessary and sufficient condition for approximate joint measurability of value complementary observables are highlighted.

In Chapter 5 conclusions are drawn. It will be proposed that complementarity and uncertainty are related in quantum mechanics although they have separate identities and rôles.

Chapter 2

Quantum Measurement Theory: Tools and Methods

2.1 Introduction

> "Might we not be better off if we shed all pretext of making pictures of the quantum phenomenon in terms of particles and waves and the like. Why not simply establish suitable mathematical laws for the description of the observations, as Newton urged, for a branch of physics reaching maturity." E. Merzbacher, *Introduction to Quantum Mechanics*[50]

The first rôle of measurement in physics is to obtain information about a system in order to describe its present condition. The second is to enable predictions to be made about its future [56], [53], [21]. Sometimes, the measurement may reveal something of the history of the system.

A feature of classical mechanics provides a complete description in the following sense: At a particular time the position and velocity of all of the particles making up the system can be found and knowing the nature of the interactions between the particles, then, in principle it is possible to predict the future of the system with certainty.

> *"We may regard the present state of the Universe as the effects of its past and the cause of its future".* Pierre-Simon Laplace in the introduction to *Essai philosophique sur les probabilités (1814).* Laplace was a strong believer in causal determinism.

In the case of macroscopic or classical objects for which this assumption is valid, a measurement can be made sufficiently sensitive so that the interaction of the measuring device with the system being investigated is negligible or quantifiable.

However, microscopic systems do not allow this to happen. It is usually not possible to make measurements that do not at the same time disturb the system in some unpredictable way. This is central to the problem of measurement in quantum mechanics.

The measurement problem and the key features of quantum mechanics and the mathematics used to represent them are explored in text such as [37].

One of the themes central to this thesis is the notion of uncertainty in quantum mechanics. Quantifying quantum uncertainty is reviewed in Section 2.3 using the standard concepts of states and observables in quantum mechanics presented in Section 2.2. Section 2.4 extends the orthodox treatment of quantum mechanics to explore the possibility of allowing inaccurate measurements to lead to the possibility of making observations of two noncommuting or incompatible quantities of a quantum system simultaneously. This technique uses Positive Operator Value Measures (POVM). Section 2.5 explores the interaction between the quantum object and the measuring apparatus leading naturally to a set of operators identifiable as a POVM and a description of the system's state change due to measurement.

2.2 Mathematical tools.

Throughout, the usual Hilbert space formalism of quantum mechanics is employed: States are represented as (unit) vectors or as density operators and observables, initially, are represented as hermitian operators.

2.2.1 States as density operators.

The notion of a density operator, just as that of a state vector, describes a preparation procedure. The statistical properties of the ensemble of quantum systems correspond to the given preparation procedure.

A density operator, ρ, is a positive, self adjoint linear operator, such that if the set of vectors $\{\phi_k : k \in K\}$ is an orthonormal basis then $Tr(\rho) := \sum_k \langle \phi_k | \rho \phi_k \rangle = 1$.

The operator ρ has the following properties,

1. $\rho = \rho^\dagger$
2. ρ is a positive semi-definite operator, i.e. $\langle \psi | \rho \psi \rangle \geq 0 \; \forall \, |\psi\rangle \in \mathcal{H}$
3. $Tr(\rho) = 1$.

In the special case of a projection, $\rho = |\psi\rangle\langle\psi|$ is associated with any normalized state vector $|\psi\rangle$. Then the expectation value for any observable in a vector state is $\langle A \rangle_\psi = \langle \psi | A \psi \rangle$.

Now consider various pure states $|\psi_k\rangle \in \mathcal{H}$ each with its respective probability P_k. The vectors $|\psi_k\rangle \in \mathcal{H}$ are normalized but not necessarily orthogonal to each other.

$$\langle A \rangle_\rho := \sum_k P_k \langle \psi_k | A \psi_k \rangle = \sum_k P_k Tr[|\psi_k\rangle\langle\psi_k| A] = Tr(\rho A), \tag{2.2.1}$$

where $\rho = \sum_k P_k |\psi_k\rangle\langle\psi_k|$.

Example 1.

The experiments in Chapters 4 and 5 are described in the framework of a two dimensional Hilbert space, $\mathbb{C}^2$. The identity matrix together with the sigma matrices, given below, will be used as a basis, $\{\mathbf{I}, \sigma_x, \sigma_y, \sigma_z\}$, for the vector space of 2x2 hermitian matrices.

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.2.2}$$

In this case ρ will have the form $\rho_{\mathbf{n}} = \frac{1}{2}(\mathbf{I} + \mathbf{n} \cdot \overrightarrow{\sigma})$, where $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| \leq 1$.

2.2.2 Observables as self adjoint operators

Central to the formalism of quantum mechanics are self adjoint operators because,

1. they have a spectral measure,
2. their eigenvalues are real,
3. the eigenvectors corresponding to two different eigenvalues are orthogonal.
4. State vectors can be expanded as linear combinations of the eigenfunctions of a self adjoint operator. (Strictly this is only true in this form if the operator has pure point spectrum.)

Consequently, self adjoint operators can be used to represent observables so that expectation values, as defined in 2.2.1 are real numbers and the spectral representation of A, as reviewed below, enables a probability interpretation of expectation values.

Remark 1.

Suppose that an eigenvalue of an operator A is λ_k. It is possible that there are several eigenvectors that have this eigenvalue with A. In this case the eigenvalues are described as being degenerate.

Let λ_k be a $d(k)$-fold degenerate eigenvalue of A with linearly independent eigenvectors $\{|\lambda_{k1}\rangle, |\lambda_{k2}\rangle, \ldots, |\lambda_{kd(x)}\rangle\}$. This set of eigenvectors spans a linear subspace of $\mathcal{H}$ with dimension $d(k)$. For convenience the set of eigenvectors are chosen to be orthonormal so that $\langle\lambda_{km}|\lambda_{kn}\rangle = \delta_{m,n}\ m, n \in \{1, 2, \ldots, d(k)\}$ If this is done for each eigenvalue, λ_k of A, then a set of eigenvectors, $\{|\lambda_{kj}\rangle$, is obtained: $k \in \{1, 2, \ldots, K : K \leq \infty\}, j \in \{1, 2, \ldots, d(k)\}\}$. From the proof that eigenvectors with different eigenvectors are orthogonal it follows that this entire set of eigenvectors are orthonormal, $\langle\lambda_{km}|\lambda_{ln}\rangle = \delta_{km}\delta_{ln},\ m, n \in \{1, 2, \ldots, d(k)\},\ k, l \in \{1, 2, \ldots, K : K \leq \infty\}$.

Definition 1.

$P_k := \sum_{m}^{d(k)} |\psi_{km}\rangle\langle\psi_{km}|$ is the projection operator onto the subspace of eigenvectors of A with eigenvalues λ_k. It is referred to as the spectral projector and can be written $P_{A=\lambda_k}$ if it is necessary to refer to the eigenvalues of the operator.

This projector is independent of pairwise orthogonal vectors chosen to span the space of eigenvectors.

The projectors P_k and P_j are pairwise orthogonal, $P_k P_j = \delta_{kj} P_k$.

The basis expansion can now be written $|\psi\rangle = \sum_{k=1}^{K} P_k|\psi\rangle = I|\psi\rangle$ which is called the *(orthogonal) resolution of the identity.*

The operator A can be written in a form called the spectral representation of A, as $A = \sum_{k=1}^{K} \lambda_k P_k$. The proof is as follows.

Since the set of all eigenvectors of a self-adjoint operator A is an orthonormal basis set for a Hilbert space $\mathcal{H}$ any $|\psi\rangle \in \mathcal{H}$ can be expanded as $|\psi\rangle = \sum_{k=1}^{K}\sum_{m}^{d(k)} c_{km}|\lambda_{km}\rangle$ where the expansion coefficient $c_{km} \in \mathbb{C}$ are given by $c_{km} = \langle\lambda_{km}|\lambda\rangle$.

When the operator A acts upon this expansion, $A|\psi\rangle = \sum\limits_{k=1}^{K} \lambda_k \sum\limits_{m}^{d(k)} c_{km}|\psi_{km}\rangle$ which leads to,

$$A|\psi\rangle = \sum_{k=1}^{K} \lambda_k \sum_{m}^{d(k)} \langle\psi_{km}|\psi\rangle|\psi_{km}\rangle = \sum_{k=1}^{K} \lambda_k \sum_{m}^{d(k)} |\psi_{km}\rangle\langle\psi_{km}|\psi\rangle = \sum_{k=1}^{K} \lambda_k P_k|\psi\rangle.$$

2.3 Uncertainty Relations

In Chapter 1 $\Delta(Q,\psi)$ and $\Delta(P,\psi)$ were used to express the *uncertainty* or width of the distribution of separate measurements of position Q and momentum P in a state ψ. An uncertainty relation between the distribution widths of any two observables can be derived from the existing formalism, it does not need an extra postulate to be introduced.

2.3.1 The Variance of an Observable

If a measurement of an observable $A = \sum a_k P_k$ in some state ψ can be predicated to give results $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$, the *variance*, (Var), of A for a state $|\psi\rangle$ is

$$\mathrm{Var}(A,\psi) := \sum_{i=1}^{n} (a_i - \langle A\rangle)^2 p_i. \tag{2.3.1}$$

This can be rearranged to read

$$\mathrm{Var}(A,\psi) = \langle A^2\rangle_\psi - \langle A\rangle_\psi^2. \tag{2.3.2}$$

The *standard deviation* of A,

$$\Delta(A,\psi) = \sqrt{\mathrm{Var}(A,\psi)} = (\langle\psi|A^2\psi\rangle - \langle\psi|A\psi\rangle)^{\frac{1}{2}}, \tag{2.3.3}$$

is a measure of the spread of results about the expectation value of A.

Equations 2.3.2 and 2.3.3 can also be used to define the variances and standard deviations for observables with a continuous spectrum, such as position and momentum.

Example 2.

Consider a projection operator $A \in \mathbb{C}^2$ being represented by

$$A \equiv P_{\mathbf{a}} = \tfrac{1}{2}(\mathbf{I} + \mathbf{a} \cdot \overrightarrow{\sigma}) \tag{2.3.4}$$

where $\|\mathbf{a}\| = 1$ and a state by

$$\rho_{\mathbf{n}} = \tfrac{1}{2}(\mathbf{I} + \mathbf{n} \cdot \overrightarrow{\sigma}), \tag{2.3.5}$$

where $\|\mathbf{n}\| \leq 1$.

The expectation value $\langle A \rangle_{\rho_{\mathbf{n}}}$ is

$$\langle A \rangle_{\rho_{\mathbf{n}}} = \tfrac{1}{2}(1 + \mathbf{n} \cdot \mathbf{a}). \tag{2.3.6}$$

The variance of A in the state $\rho_{\mathbf{n}}$ is

$$\mathrm{Var}(A, \rho_{\mathbf{n}}) = \tfrac{1}{4}(1 - (\mathbf{n} \cdot \mathbf{a})^2). \tag{2.3.7}$$

2.3.2 The General Form of an Uncertainty Relation using Variances

An uncertainty relation in the form of an inequality can be derived in the form of the product of the variances of observables A and B. This is familiar as the Heisenberg-Robertson uncertainty relation,

$$\mathrm{Var}(A, \psi)\mathrm{Var}(B, \psi) \geq \tfrac{1}{4}\big|\langle \psi | [A, B] \psi \rangle\big|^2. \tag{2.3.8}$$

First, it is necessary to use the Cauchy-Schwarz inequality;

$$\langle A\psi|A\psi\rangle\langle B\psi|B\psi\rangle \geq \left|\langle A\psi|B\psi\rangle\right|^2 \text{ or} \tag{2.3.9}$$

$$\langle\psi|A^2\psi\rangle\langle\psi|B^2\psi\rangle \geq \left|\langle\psi|AB\psi\rangle\right|^2 \tag{2.3.10}$$

because A is self-adjoint.

The product of two self-adjoint operators AB can be written as the sum of two terms,

$$AB = \tfrac{1}{2}(AB + BA) + \tfrac{1}{2}(AB - BA) \tag{2.3.11}$$

$$AB = \tfrac{1}{2}[A, B]_+ + \tfrac{1}{2}[A, B] \tag{2.3.12}$$

where $[A, B]$ is the *commutator* of the two operators and $[A, B]_+$ is the *anticommutator*.

Hence,

$$\langle\psi|A^2\psi\rangle\langle\psi|B^2\psi\rangle \geq \left|\langle\psi|(\tfrac{1}{2}[A, B]_+ + \tfrac{1}{2}[A, B])\psi\rangle\right|^2. \tag{2.3.13}$$

This is the inequality required but, referring to equation 2.3.3, only in the special case where $\langle\psi|A\psi\rangle$ and $\langle\psi|B\psi\rangle$ vanish. In order to arrive at a more general case, equation 2.3.13 should be used with the operators $A_1 = A - \langle A\rangle I$ and $B_1 = B - \langle B\rangle I$ then

$$\mathrm{Var}(A, \psi) = \langle(A - \langle A\rangle_\psi)^2\rangle_\psi = \langle\psi|A_1^2\psi\rangle \tag{2.3.14}$$

Similarly, $\mathrm{Var}(B, \psi) = \langle\psi|B_1^2\psi\rangle$ and $[A, B] = [A_1, B_1]$ giving,

$$\mathrm{Var}(A, \psi)\mathrm{Var}(B, \psi) \geq \left|\langle\psi|(\tfrac{1}{2}[A_1, B_1]_+ + \tfrac{1}{2}[A, B])\psi\rangle\right|^2 \tag{2.3.15}$$

Because A, B, A_1 and B_1 are self-adjoint, $\langle\psi|(\tfrac{1}{2}[A_1, B_1]_+\psi\rangle$ is real and $\langle\psi|\tfrac{1}{2}[A, B]\psi\rangle$ is imaginary, hence,

$$\mathrm{Var}(A, \psi)\mathrm{Var}(B, \psi) \geq \tfrac{1}{4}\left|\langle\psi|[A_1, B_1]_+\psi\rangle\right|^2 + \tfrac{1}{4}\left|\langle\psi|[A, B]\psi\rangle\right|^2. \tag{2.3.16}$$

Equation 2.3.16 is often quoted as $\mathrm{Var}(A_1)\mathrm{Var}(B_1) \geq \frac{1}{4}\left|\langle\psi|[A,B])\psi\rangle\right|^2$ but ignoring the $\frac{1}{4}\left|\langle\psi|[A_1,B_1]_+)\psi\rangle\right|^2$ weakens the relationship.

In a paper in 1930 [68], Schrödinger reported that he had, "arrived at a slightly wider generalization than the Robertson's [uncertainty relation], which is, in fact stronger than the original Heisenberg inequality." His findings are indeed the full inequality 2.3.16, where it is noted that

$$\tfrac{1}{2}\langle[A_1,B_1]\rangle = \tfrac{1}{2}\langle AB+BA\rangle - \langle A\rangle\langle B\rangle \tag{2.3.17}$$

is the covariance between the observables A and B.

Example 3.

Here the explicit form of the uncertainty relation (equation 2.3.16) for observables represented by the Pauli spin operators is given. Consider

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3.18}$$

and recall that,

$$[\sigma_z,\sigma_x] = 2\imath\sigma_y, \quad [\sigma_z,\sigma_x]_+ = 0, \tag{2.3.19}$$

and similarly for other pairs, then

$$\begin{aligned} &\mathrm{Var}(\sigma_x,\psi)\mathrm{Var}(\sigma_z,\psi) \geq \tfrac{1}{4}|\langle\psi|[\sigma_x,\sigma_z]\psi\rangle|^2 + \tfrac{1}{4}[\langle\psi|(\sigma_x\sigma_z+\sigma_z\sigma_x)\psi\rangle - \langle\sigma_x\rangle_\psi\langle\sigma_z\rangle_\psi]^2, \\ \text{or}\quad &\mathrm{Var}(\sigma_x,\psi)\mathrm{Var}(\sigma_z,\psi) \geq \langle\sigma_y\rangle_\psi^2 + \langle\sigma_z\rangle_\psi^2\langle\sigma_x\rangle_\psi^2. \end{aligned} \tag{2.3.20}$$

Using 2.3.7 it can be seen that,

$$\mathrm{Var}(\sigma_x,\psi)\mathrm{Var}(\sigma_z,\psi) = \tfrac{1}{16}(1-n_x^2)(1-n_z^2) \tag{2.3.21}$$

If A is observable and $\|\mathbf{n}\| = 1$ then A is a projection and

$$(1-n_x^2)(1-n_z^2) = n_y^2 + n_z^2 n_x^2 \tag{2.3.22}$$

2.4 Observables as Positive Operator Valued Measures (POVM)

2.4.1 The General Idea of POVM

The formalism of quantum mechanics arose out of the need to describe mathematically the results and observations of laboratory experiments. Over the past seven decades experimental techniques have improved so that there is a continuing need for quantum mechanics to use tools from a widening area to describe new experimental evidence and to reinterpret older evidence.

In this section Positive Operator Value Measures (POVM) are introduced as a technique for describing imperfect or inaccurate measurements as well as for studying the problem of making joint measurements (observations) i.e. attempting to measure at least two properties of a quantum system simultaneously. This problem has been considered by the founders of quantum mechanics, Bohr, Heisenberg, Einstein and continues to this day.

The original interpretation of quantum mechanics, usually called the Copenhagen view, came out of thought (gedanken) experiments e.g. Young's double slit interference experiment with path detection. Path and interference observables are incompatible and it was deemed impossible to jointly detect a path and an interference pattern. One element of the Copenhagen interpretation is the claim that there is a mutual disturbance of the measurement outcomes because this would explain the mutual exclusivity of interference observation and path detection. The uncertainty relation derived above is often (wrongly) cited as the mathematical expression of this mutual disturbance.

So, the uncertainty relationship above has little to do with joint measurements of incompatible observables since the standard quantum mechanical formalism only allows a mathematical description of joint measurements of compatible observables. How can the standard formalism cope with experiments that are no longer imagined but are carried out in laboratories? It is now possible to carry out 'which-way' experiments to demonstrate the mutual disturbance caused by the joint measurement of incompatible observables.

The Copenhagen issue of complementarity, which in one common reading refers to the impossibility of joint measurement of incompatible observables makes little distinction between preparation and measurement. Both the preparation – obtaining information about the pre-measurement state, and the measurement – putting the system into a final post-measurement state, will disturb the observables.

The standard formalism uses projector valued measures corresponding to the spectral representation of self-adjoint operators. Probabilities, p_m, are represented by the expectation values of mutually commuting projection operators; $\mathrm{p}_m = \langle P_m \rangle$.

In generalizing the formalism to include POVMs the concept of an observable is extended to include a measurement procedure that can be interpreted as a joint measurement of incompatible observables. In this generalization probabilities are represented by expectation values of positive operators. In the discrete case the POVM can be described as a set of operators $\{E_k\}$.The elements of this set of operators (or effects) are not necessarily projection operators and are not necessarily commuting; hence, $E_m^2 \neq E_m = E_m^\dagger$ and possibly $E_m E_n \neq 0$.

The operators E_m, with the properties $0 \leq E_m \leq I$, $\sum\limits_m E_m = I$ called effects generate the positive operator valued measure, POVM. If the effects are mutually orthogonal $E_m E_n = 0 \; m \neq n$ then $E_m(I - E_m) = E_m \sum\limits_{n(\neq m)} E_n = \sum\limits_{n(\neq m)} E_m E_n = 0$

so $E_n = E_n^2 \; \forall n$. So, the POVM becomes a projection valued measure, PVM, the spectral measure of the standard formalism.

Having generalized the concept of a quantum mechanical observable it is possible to define a relationship between the observables describing the measurement represented by one POVM being interpreted as a non-ideal measurement of another. In doing this the notion of reality has been relaxed.

The positive operator in the range of the POVM, called an effect, represents the occurrence of a particular outcome of a measurement. The expectation value for this effect is interpreted as the probability for it occurring. However, instead of a probability of 0 or 1, which would determine the absence or presence of an effect, the probability should be in the interval $[0.5, 1]$ to determine its approximate reality or $[0, 0.5)$ its absence.

This leads to a generalized notion of properties comprising both the projection operators, which give rise to sharp measurements and the effects, called unsharp properties if they are not projections, giving generalized observables as POVMs.

Unsharp observations arise naturally in the analysis of experimental procedure, an experimenter may describe the measurement procedure as coarse grained. What type of property they represent can be determined by making reference to a known sharp observable via a relation of coarse graining or smearing as is shown below.

2.4.2 Mathematical development of POVM

Definition 2.

A Positive Operator Valued Measure, POVM, is a triple $(\Omega, \mathcal{A}, E)$ where Ω is a non-empty set of measurement outcomes; $\mathcal{A}$ is a σ-algebra of subsets of Ω.

E is an operator valued set function on $\mathcal{A}$ such that $E : \mathcal{A} \mapsto \mathcal{L}(\mathcal{H})$ where $\mathcal{L}(\mathcal{H})$

is space of bounded operators on $\mathcal{H}$. E has these properties;

1. E is positive i.e. $E(M) \geq 0$ for each M in $\mathcal{A}$;

2. $E(\varnothing) = 0$;

3. $E(\bigcup_k M_k) = \sum_k E(M_k)$ where $\{M_k\}$ is a finite or countable subset of Ω with $M_n \cap M_m = \varnothing,\ m \neq n$ ($\sigma-$additivity).

4. Normalization is usually required i.e. $E(\Omega) = I$.

Example 4.

A discrete projection valued measure and its smearing:

From 2.2.2 the spectral measure of operator is $A = \sum_k \lambda_k P_k$, giving a family of projections over all k, $\mathcal{P} = \{P_k : k \in \{1 \ldots N \leq \infty\}\}$

Using the idea that an unsharp observable can, in some cases, be obtained from a sharp one by a suitable smearing operation, [12], let us define $E_j := \sum_k w_{jk} P_k$, where w_{jk} is a stochastic matrix in which each element is positive $w_{jk} \geq 0$ and the rows sum to unity, $\sum_j w_{jk} = 1$. Then the family of effects, $E = \{E_j : j \in \{1 \ldots K \leq \infty\}\}$ constitutes a POVM.

Example 5.

A smeared version of the spectral measure of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The spectral representation has the form, $\sigma_z = \sum_{k=1,2} \lambda_k P_k^{\sigma_z}$ i.e.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}(I + \sigma_z) - \frac{1}{2}(I - \sigma_z). \tag{2.4.1}$$

$P_1 = P_+ = \frac{1}{2}(I + \sigma_z)$ and $P_2 = P_- = \frac{1}{2}(I - \sigma_z)$.

If now a smearing matrix is defined, $(g_{jk}) := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+g & 1-g \\ 1-g & 1+g \end{pmatrix}$, the smeared version of the spectral measure of σ_z is, $E_j^{\sigma_z} := \sum_{k=1,2} g_{jk} P_k$.

So, $E_1^{\sigma_z} = E_+^{\sigma_z} = \frac{1}{2}(1+g)\frac{1}{2}(I+\sigma_z) + \frac{1}{2}(1-g)\frac{1}{2}(I-\sigma_z) = \frac{1}{2}(I+g\sigma_z)$

and $E_2^{\sigma_z} = E_-^{\sigma_z} = \frac{1}{2}(I-g\sigma_z)$.

2.5 Measurement schemes: An instance of composite systems

2.5.1 Composite systems and tensor product Hilbert space

In order to observe any physical system it is made to interact with a measuring apparatus. Even if the system and the apparatus can be described by quantum mechanics, when the measurement process is over the result of the observation is recorded in a classical form; seen by the experimenters and written down or told to someone else; printed out or stored by some other part of the measuring apparatus. In other words, the apparatus remains in the domain of classical physics.

After Bohr, Copenhagenists would say that for the description of the measurement outcomes a return to the language of classical physics must be made.

When two or more quantum systems interact the final state of one of the systems may well depend upon the final states of the others. This is referred to as entanglement or non-separability: the state of the total system is not of product form. Examples where entanglement typically arises are the electron and proton in a hydrogen atom, the spatial and spin degrees of freedom of an atom passing through the non-homogeneous magnetic field of a Stern-Gerlach apparatus.

If there are state vectors representing the states of a composite system, which cannot be separated into vector states of subspaces belonging to the individual systems. How are they to be represented?

Using the Hilbert space model; if $|\psi\rangle \in \mathcal{H}_1$ and $|\phi\rangle \in \mathcal{H}_2$, it is natural to describe the system resulting from bringing the two subsystems together by a Hilbert space $\mathcal{H}$ formed from the tensor product,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{2.5.1}$$

$\mathcal{H}$ comprises all of the linear combinations of elements $|\psi\rangle \otimes |\phi\rangle$ and if ψ and ϕ are in turn linear combinations, then it is possible to bilinearly express $|\psi\rangle \otimes |\phi\rangle$ in terms of the tensor product of the components.

An orthonormal basis is formed by the tensor product of elements of the basis of $\mathcal{H}_1$ and $\mathcal{H}_2$. For finite dimensional spaces, the dimension of $\mathcal{H}$ is the product of the dimension of its components.

Consider, for example, two spin-half particles being allowed to form a composite system. Concentrating on just the spin features of the systems; the state space of the spin-half is $\mathbb{C}^2$ and has the basis $\left\{ \binom{1}{0}, \binom{0}{1} \right\}$ or $\{|1\rangle, |0\rangle\}$ in Dirac notation.

The following is then an orthonormal basis of the compound system,

$$\begin{aligned} |1\rangle|1\rangle &= (1,0,0,0)^T, \\ |0\rangle|1\rangle &= (0,1,0,0)^T, \\ |1\rangle|0\rangle &= (0,0,1,0)^T, \\ |0\rangle|0\rangle &= (0,0,0,1)^T. \end{aligned} \tag{2.5.2}$$

Since there is a list of four possible mutually orthogonal eigenvectors in this composite system every state vector can be expressed as a linear combination, a superposition, of some or all of the eigenvectors. The state space of this composite system is thus isomorphic to $\mathbb{C}^4$.

It is now possible to form the tensor product of operators, for example, $\sigma_{z,1} \otimes \sigma_{x,2}$, defined by linearity and the rule $(\sigma_{z,1} \otimes \sigma_{x,2})(|\psi\rangle|\phi\rangle) = (\sigma_{z,1}|\psi\rangle)(\sigma_{x,2}|\phi\rangle)$.

For example, $\sigma_z|1\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$, so $(\sigma_z|1\rangle)(\sigma_x|1\rangle) = |1\rangle|0\rangle = (0,0,1,0)^T$ and the same result is obtained by,

$$(\sigma_z \otimes \sigma_x)(|1\rangle|1\rangle) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.5.3}$$

In general, the mathematical entity which describes a composite of $|\psi\rangle \in \mathcal{H}_1$ and $|\phi\rangle \in \mathcal{H}_2$ is a new vector $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

2.5.2 Measurement scheme and induced POVM

Considering the interaction of a quantum system with a measuring instrument, it will now be shown how the POVM arises in the description of a measurement. A brief outline of the description of state changes due to measurements will also be given.

Consider some instrument observable such as a discrete position of a pointer on a scale. Quantum mechanical formalism would say that each of the pointer positions r_m is an eigenvalue of some self adjoint operator, P, with associated eigenvectors.

The interpretation of this reading is that when the pointer indicates r_m the system has an eigenvalue of λ_m. More generally, the relative frequency of (a finite number of) readings of r_m is (approximately) the probability of eigenvalue λ_m of the system's observable in the state before the measurement. If this condition is satisfied then the measuring scheme is a good one.

Let the system S be initially in some unknown state described by $|\psi_0\rangle$ and let the instrument pointer (belonging to apparatus A) be in some pure state described by $|\phi_0\rangle$. It is reasonable to expect that the initial reading of the pointer is zero hence the label 0. Their time evolutions up to the action of measurement are described by the state of the combined system just prior to measurement, $|\Psi_0\rangle = |\psi_0\rangle \otimes |\phi_0\rangle$.

The measurement consists of unitary evolution followed by a projection onto a state of the pointer with eigenvalues that are the readings to be recorded. During the measurement interaction the combined system will evolve under the action of a unitary operator, U, to give $|\Psi_{ev}\rangle = U|\Psi_0\rangle = U|\psi_0\rangle \otimes |\phi_0\rangle$.

The expectation value of the pointer reading is $\langle P\rangle_{\Psi_{ev}} = \langle \Psi_{ev}|I \otimes P\Psi_{ev}\rangle$, but the expectation value of the pointer reading does not give any information about the probability of the system being in a particular state so the spectral representation of operators is needed, $P = \sum_m r_m Q_m$. Where Q_m is the spectral projection operator onto the subspace of eigenvectors of P with eigenvalues r_m.

However, the pointer operator should be represented on the composite space of the system (object) and the pointer i.e. the tensor product of the object space and pointer space. The spectral representation of the pointer operator on the composite space is $\hat{P} = I_s \otimes P = \sum_m r_m I_s \otimes Q_m$ where I_s is the identity operator on the system space.

As the entangled system evolves, the expectation value of $I_s \otimes Q_m$ is

$$\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} = \langle U\Psi_0|I_s \otimes Q_m|U\Psi_0\rangle.$$

This is required to be, by virtue of the probability postulate, the probability that the pointer operator, P, gives the value r_m.

Let $\{|\varphi_i\rangle\}$ be an orthonormal basis of the system space and $\{|\zeta_j\rangle\}$ be an orthonormal basis of the pointer space, then $\{|\varphi_i\rangle \otimes |\zeta_j\rangle\}$ is an orthonormal basis in the composite space so $|\Psi_0\rangle = \big||\phi_0\rangle \otimes |\phi_0\rangle\big\rangle = \sum_{i,j} c_{i,j}\big||\varphi_i\rangle \otimes |\zeta_j\rangle\big\rangle$ giving,

$$\begin{aligned}\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} = \sum_{ijkl} \big\langle\langle\phi_0| \otimes \langle\phi_0|\big||\varphi_i\rangle \otimes |\zeta_j\rangle\big\rangle\big\langle\langle\varphi_i| \otimes \langle\zeta_j|\big|U^\dagger I_s \\ \otimes Q_m U|\varphi_k\rangle \otimes |\zeta_l\rangle\big\rangle\big\langle\langle\varphi_k| \otimes \langle\zeta_l|\big||\phi_0\rangle \otimes |\phi_0\rangle\big\rangle.\end{aligned}$$

Let $L^m_{ijkl} = \big\langle\langle\varphi_i| \otimes \langle\zeta_j|\big|U^\dagger I_s \otimes Q_m U|\varphi_k\rangle \otimes |\zeta_l\rangle\big\rangle$ since this is a number.

$$\begin{aligned}\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} &= \sum_{ijkl} \big\langle\langle\phi_0| \otimes \langle\phi_0|\big||\varphi_i\rangle \otimes |\zeta_j\rangle\big\rangle L^m_{ijkl}\big\langle\langle\varphi_k| \otimes \langle\zeta_l|\big||\phi_0\rangle \otimes |\phi_0\rangle\big\rangle \\ &= \sum_{ijkl} \langle\phi_0|\varphi_i\rangle\langle\phi_0|\zeta_j\rangle L^m_{ijkl}\langle\zeta_l|\phi_0\rangle\langle\varphi_k|\phi_0\rangle.\end{aligned}$$

Now, let the number (expectation value), $M^m_{ijkl} = \langle\phi_0|\zeta_j\rangle L^m_{ijkl}\langle\zeta_l|\phi_0\rangle$

$$\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} = \sum_{ijkl} \langle\phi_0|\varphi_i\rangle M^m_{ijkl}\langle\varphi_k|\phi_0\rangle.$$

Applying separate summations.

$$\begin{aligned}\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} &= \sum_{i,k} \langle\phi_0|\varphi_i\rangle \sum_{j,l} M^m_{ijkl}\langle\varphi_k|\phi_0\rangle\langle I_s \otimes Q_m\rangle_{|\Psi_{ev}\rangle} \\ &= \sum_{i,k} \langle\phi_0|\varphi_i\rangle N^m_{ik}\langle\varphi_k|\phi_0\rangle \\ &= \langle\phi_0| \sum_{i,k} N^m_{ik}|\varphi_i\rangle\langle\varphi_k|\phi_0\rangle.\end{aligned}$$

It is now possible to make the definition $E_m := \sum_{i,k} N^m_{ik}|\varphi_i\rangle\langle\varphi_k|$ which is a linear operator.

The operator E_m is positive since the left side is positive for all ϕ_0.

To investigate the properties of this operator further, let us return to the spectral projection operator, Q_m which are used to define E_m.

$\sum_m Q_m = 1$, therefore, $\sum_m I_s \otimes Q_m = 1 \times 1 = 1$. From this it follows that $\sum\langle\theta|E_m\theta\rangle = 1$ for all $|\phi_0\rangle$ under the condition that $|\phi_0\rangle$ is normalized. From this, $\sum\langle\phi_0|E_m\phi_0\rangle = 1 = \langle\phi_0|\sum E_m\phi_0\rangle = 1$ and therefore $\sum_m E_m = 1$ is obtained.

The set of operators $\{E_m\}$ is thus the positive operator value measure or POVM encountered in Section 2.4. Each of the positive operators in the set is associated with a measurement outcome r_m and gives the probability of its occurrence for each initial object state.

A brief outline of how state changes due to measurement are described is as follows: for each outcome m and state ρ, there is a positive operator $\rho'(m;\rho)$ of trace $\mathrm{tr}[\rho'(m;\rho)]$ $=\mathrm{tr}[\rho \cdot E_m]$ by virtue of the formula:

$$\mathrm{tr}[\rho'(m;\rho) \cdot B] := \mathrm{tr}[U\rho \otimes |\phi_0\rangle\langle\phi_o|\, U^\dagger B \otimes Q_m], \tag{2.5.4}$$

which holds for any self adjoint bounded operator B [12].

2.5.3 Joint measurability

In a joint measurement of two observables F and G, one sets out to infer the values of these observables from the output readings. Thus for every pair of values of F and G there has to be a pointer value and the statistics these pointer values should reproduce the probabilities for the values of F and G in the object's input state. Thus, there should be a POVM, E, whose probabilities should be joint probabilities for the outcomes of F and G. This means that the probability distributions of F and G should be obtained as marginal distributions of the probability distribution of E.

Such a POVM, E, is called a joint observable of F and G and F and G are the marginals of E.

According to a theorem of von Neumann [72] two sharp observables have a sharp joint observable exactly when they commute. Thus two non-commuting observables cannot be sharply measured together. However, it has been found that smeared versions of two *noncommuting* sharp observables may have a joint observable. The pair $F = \{\frac{1}{2}(I \pm f\sigma_x)\}$, $G = \{\frac{1}{2}(I \pm g\sigma_z)\}$ are known [13] (see Chapter 4) to have a joint observable exactly when,

$$f^2 + g^2 \leq 1 \tag{2.5.5}$$

Thus for two POVMs to be jointly measurable their degrees of unsharpness $|f|, |g|$ must be limited by this trade-off inequality. In this case it is straightforward to to give an example of a joint observable E, assuming for simplicity $0 \leq f, g \leq 1$,

$$\begin{aligned} E_{11} &= \tfrac{1}{4}(I + f\sigma_x + g\sigma_z), \qquad & E_{21} &= \tfrac{1}{4}(I - f\sigma_x + g\sigma_z), \\ E_{12} &= \tfrac{1}{4}(I + f\sigma_x - g\sigma_z), \qquad & E_{22} &= \tfrac{1}{4}(I - f\sigma_x - g\sigma_z). \end{aligned} \tag{2.5.6}$$

Each operator $E_{k,\ell}$ is positive because the eigenvalues are, $\frac{1}{4}(1 \pm |(f,g)|) = \frac{1}{4}(1 \pm \sqrt{f^2 + g^2}) \geq 0$ due to (2.5.5). Moreover,

$$\begin{aligned} E_{11} + E_{12} &= F_1, \qquad & E_{21} + E_{22} &= F_2, \\ E_{11} + E_{21} &= G_1, \qquad & E_{12} + E_{22} &= G_2. \end{aligned} \tag{2.5.7}$$

In Sections 4.3.2 to 4.3.4 measurement implementations of similar joint observables will be given.

2.5.4 Measurement implementation of a joint observable

A particular type of measurement scheme for a joint observable is implemented by coupling the system to a probe. A measurement is carried out on the probe and another measurement on the system. This is a special case of Section 2.5.2 and is the form in which the calculations are carried out in Chapters 3 and 4.

In the models encountered in Chapter 4, Sections 4.3.2 to 4.3.4: a photon in a prepared path state is coupled to a probe system that is in a prepared pointer state. A unitary evolution takes place in an interferometer after which a joint measurement is made of a probe observable and a detector observable. The purpose of this joint measurement is to obtain information about the photon state immediately prior to the interaction with the probe and subsequent passage through the interferometer. Such information is available in the form of the output probabilities if these can be expressed in terms of the photon's input state.

Given that the initial state of the probe and the interferometer settings are fixed in each run it follows that the output probabilities are indeed the expectation values of a POVM for the photon input state.

Let $|\psi_i\rangle$ denote the input state of the photon, $|p_0\rangle$ the initial probe state and U the unitary evolution operator representing the passage through the interferometer. Then the final state of the combined system is $|\Psi_f\rangle = U|\psi_i\rangle|p_0\rangle$. On this the sharp output observable with projections $M_{k\ell} = |k\rangle\langle k| \otimes |r_\ell\rangle\langle r_\ell|$ is measured. Here $|k\rangle$, $k = 1, 2$, are the photon path eigenstates and $|r_\ell\rangle$, $\ell = 1, 2$, are the eigenstates of a probe pointer observable. If $\psi_i = \alpha|1\rangle + \beta|2\rangle$, the output probabilities are then,

$$\begin{aligned}
\langle\Psi_f|M_{k\ell}|\Psi_f\rangle &= \langle\psi_i|\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle|\psi_i\rangle \\
&= (\alpha^*\langle 1| + \beta^*\langle 2|)\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle(\alpha|1\rangle + \beta|2\rangle) \\
&= \alpha^*\alpha\langle 1|\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle|1\rangle \\
&\quad + \alpha^*\beta\langle 1|\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle|2\rangle \\
&\quad + \beta^*\alpha\langle 2|\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle|1\rangle \\
&\quad + \beta^*\beta\langle 2|\langle p_0|U^\dagger M_{k\ell} U|p_0\rangle|2\rangle.
\end{aligned} \tag{2.5.8}$$

These can be written as

$$\langle\Psi_f|M_{k\ell}|\Psi_f\rangle = \alpha^*\alpha E^{11}_{k\ell} + \alpha^*\beta E^{12}_{k\ell} + \beta^*\alpha E^{21}_{k\ell} + \beta^*\beta E^{22}_{k\ell}. \tag{2.5.9}$$

Given the positivity of $M_{k\ell}$, these numbers are non-negative and hence the the expression (2.5.9) is a quadratic form for the variables α and β. This is to say that for each k, ℓ the matrix $(E^{ij}_{k\ell})_{ij=1,2}$ is positive semi-definite and thus represents a positive operator $E_{k\ell}$ defined in the Hilbert space of the photon.

$$\langle \Psi_f | M_{k\ell} | \Psi_f \rangle = \langle \psi_i | E_{k\ell} | \psi_i \rangle \tag{2.5.10}$$

for all ψ_i. Normalization of the output probability entails $\sum\limits_{k\ell} E_{k\ell} = 1$.

Chapter 3

Complementarity and Uncertainty: A critical assessment.

"If a house be divided against itself, that house cannot stand" *St. Mark's Gospel.* Chapter 3, verse 25

The aim of this chapter is to explore the controversy surrounding the terms complementarity and uncertainty in quantum mechanics. Since the introduction of quantum mechanics there has been discussion and debate about the relationship between these features and their relative importance. Interpretations of the writings of the founding fathers have been made by researchers and teachers and their views have propagated, keeping the debates alive. Often quiet, the debate was enlivened by a 1991 paper of Scully, Englert and Walther (SEW).

3.1 Introduction

This chapter will start with a review of the influential paper published in Nature in 1991 by Scully, Englert and Walther, [60]. They proposed an atom interferometry experiment in which entanglement is used to store information about the path of an atom. The term *path marking* will be used to refer to such processes of storing path information in a probe system by way of establishing a correlation between it and the atom. It is found that the path marking results in the destruction of the interference pattern.

The fact that the interference pattern was wiped out not by classical momentum kicks, which can be associated with an uncertainty in momentum but by entanglement, led SEW to the suggestion that the principle of uncertainty is of inferior significance to that of complementarity. Hence, the latter would be of deeper significance as a fundamental feature of quantum mechanics.

In 1998 Dürr, Nonn and Rempe [22] realized an experiment similar to that proposed by SEW but in a different set-up. This, in turn, caused a grossly misleading article in a popular science journal, announcing "An end to uncertainty - Wave goodbye to the Uncertainty Principle, you don't need it anymore." [8]), as well as a rush of research papers aimed at 'rescuing uncertainty' e.g. [42]. Several authors have shown subsequently that versions of uncertainty relations between pairs of observables other than position and momentum exist which are in fact related to the complementarity of path marking and interference patterns [4], [23].

A theory of joint, albeit unsharp, measurements of non-commuting quantities has been established for some time but is not well known. However, it has been applied in the above context by Busch et al [11], [12] Bjork et al, [4] and de Muynck, [18]. Such a theory has the effect of 'softening' Bohr's strict complementarity. The construction of such joint measurements leads naturally to 'trade-off' relations for measurement imprecisions, which can be taken as one reading of Heisenberg's uncertainty principle. These connections will be explored further in Chapter 4.

Following the survey of the debates of the 1990s resulting from the Scully et al paper [60] and the Dürr et al experiment [22] will be an investigation of the historic roots of the issues that were the subject of these debates. Further, I will relate the recent discussions of complementarity and uncertainty to the findings in the works of the 1980s.

Foundational work dating back to the origins up until the early 1970s has been carefully surveyed and analyzed in the books of Max Jammer published in 1966 and 1974 [40, 41] and will be drawn upon when required.

3.2 Complementarity versus Uncertainty: The debates of the 1990s

3.2.1 Scully, Englert and Walther.

In their 1991 paper SEW offer the principle of complementarity as, "For each degree of freedom the dynamical variables are a pair of complementary observables" and less formally as "No matter how the system is prepared, there is always a measurement whose outcome is utterly unpredictable."

I will argue in Chapter 4 Section 4.3.5 that this latter statement may actually be regarded as a broad view of the uncertainty principle. The inclusion of the term 'utterly unpredictable' indicates that SEW are referring to the limit form of minimum versus maximum uncertainty. It is readily identified with one of their earlier statements, "...two observables are complementary if precise knowledge of one of them implies that all possible outcomes of measuring the other one are equally probable." This is their version of value complementarity which will be encountered in Section 3.5.1.

In spite of their noting that for historical reasons complementarity is “often superficially identified with the wave-particle duality of matter”, they themselves revert back to wave-particle duality. In support of their claim that their method of path marking is novel, we are reminded of three historical gedanken experiments that have been used to illustrate wave-particle duality; Feynman’s electron interferometer, Einstein’s recoiling slit and Heisenberg’s gamma ray microscope. They remark that,

“In the first two of these examples Heisenberg’s position momentum uncertainty relation makes it impossible to determine which hole the electron or photon passes through without at the same time disturbing the electrons (photons) enough to destroy the interference pattern”.

They do not ask whether Heisenberg’s position-momentum inequality is the proper one to use in interference experiments. As will be seen, in Sections 3.4 of this Chapter, this question had been raised and studied previously. Nor do they *specifically, in this paper* claim that complementarity must be accepted as an independent component of quantum mechanics superior to the position-momentum uncertainty principle. Although, they do describe the uncertainty relation as an obstacle to be overcome. In Chapters 4 and 5 it will be shown that complementarity is closely related to other types of uncertainty relations.

The experiment proposed by SEW is an atom interferometer with path marking which is achieved by exciting internal degrees of freedom of the atoms. The interference fringes disappear because of “the information contained in a functioning measuring apparatus” ([60], p. 111). Scully et al conclude that this disappearance originates in the entanglement between the measuring apparatus and the system being established.

Throughout their article SEW not seek to demonstrate, illustrate or refute any modern definition of complementarity. Their aim seems to be to illustrate the inadequacy of invoking the Heisenberg position-momentum uncertainty relation to support Bohr's complementarity principle. They write, "The principle of complementarity is manifest although the position-momentum uncertainty relation plays no rôle" ([60], p. 111).

In work published between 1983 and 1985 Hilgevoord and Uffink [34], [35], [36] (Section 3.4.3 did consider measures of uncertainty other than variance; $\mathrm{Var}(p)\mathrm{Var}(q) \geq (\frac{1}{2}\hbar)^2$, has been regarded as the formal expression of the uncertainty principle in quantum mechanics. Hilgevoord and Uffink show that variance is not an adequate estimation of uncertainty in the case of an experiment involving path marking and interference. As can be seen in Section 3.4.3 of this chapter, Hilgevoord and Uffink develop a new form of trade-off relation related to the fine structure and overall width of a general wave function and its Fourier transform.

Scully et al make no reference to the work of Hilgevoord and Uffink nor to the trade-off relation developed by Mittelstaedt et al ([51], 1987 (Section 3.4.2)) in their work on unsharp wave-particle duality nor indeed to Greenberger and Yasin 1988 paper on neutron interferometry with which-way detection [30] (Section 3.4.2).

In 1989 Sanders and Milburn [58] proposed a subtle arrangement in the form a Mach-Zehnder interferometer employing path marking. Their claim is that their method causes minimal disturbance, avoiding any significant exchanges of momentum and energy in the observed system. They conclude with a trade-off relation between fringe visibility (wave-like behaviour) and a measure of the presence of a photon in one arm of the interferometer (particle-like behaviour). The inference of the presence of photon is by a measurement of the signal to noise ratio in the phase shift produced in the electric field in the medium of a Kerr cell. SEW make no reference to this paper which anticipates some of their findings.

The exploration of complementarity made in 1979 by Wootters and Zurek [77] (Section 3.4.1 is cited by SEW but receives only a scant review.

Details of the experiment SEW propose follow: An atomic beam is incident upon a double slit of an interferometer. After passing through the double slit each atom can be described by the state vector,

$$\Psi_0(\mathbf{r}) = \tfrac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})]|i\rangle \tag{3.2.1}$$

where $\mathbf{r}$ is the centre-of-mass coordinate and $|i\rangle$ is the internal state of the atom. The probability density of the atoms on a screen at point $\mathbf{r}$ is given by

$$P_0(\mathbf{r}) = |\Psi(\mathbf{r})|^2 = \tfrac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + (\psi_1^*(\mathbf{r})\psi_2(\mathbf{r}) + \psi_1(\mathbf{r})\psi_2^*(\mathbf{r}))]\langle i|i\rangle \tag{3.2.2}$$

the cross terms, $(\psi_1^*\psi_2 + \psi_1\psi_2^*)$ describe the maxima and minima of the interference.

Next, they consider the situation in which the atoms in the beam are excited into a long lived Rydberg state $|a\rangle$ by a laser pulse before passing into micro-maser cavities preceding each of the slits. Once in the cavity it is possible to realize in practice that an atom will make a transition $|a\rangle \rightarrow |b\rangle$ by the spontaneous emission of a microwave photon with a probability close to unity. The state of the atom is now entangled with the maser cavity and the total state is described by

$$\Psi(\mathbf{r}) = \tfrac{1}{\sqrt{2}}[\psi_1(\mathbf{r})|1_1 0_2\rangle + \psi_2(\mathbf{r})|0_1 1_2\rangle]|b\rangle \tag{3.2.3}$$

where $|1_1 0_2\rangle(|0_1 1_2\rangle)$ represents the field state in which there is one photon in cavity 1 and none in cavity 2 (no photon in cavity 1 and one in cavity 2). The probability density at the screen is now

$$P(\mathbf{r}) = \tfrac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + \psi_1^*(\mathbf{r})\psi_2(\mathbf{r})\langle 1_1 0_2 | 0_1 1_2\rangle + \psi_1(\mathbf{r})\psi_2^*(\mathbf{r})\langle 0_1 1_2 | 1_1 0_2\rangle]\langle b|b\rangle, \tag{3.2.4}$$

but $\langle 1_1 0_2 | 0_1 1_2\rangle = \langle 0_1 1_2 | 1_1 0_2\rangle = 0$, so there are no interference terms in the following

$$P(\mathbf{r}) = \tfrac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2]. \tag{3.2.5}$$

Thus the micro-maser cavities act as *which-way* detectors only if a photon left in the cavity changes the electromagnetic field in a detectable way.

In previous examples of an interferometer with path marking such as those proposed by Einstein and Feynman it is possible to explain the washing out of the interference pattern as being due to momentum kicks; if the kicks lead to a random phase factor so that each member of ensemble has the form, $\psi_\delta = \frac{1}{\sqrt{2}}(\psi_1 + e^{\imath\delta}\psi_2)$, where δ is uniform over $[0, 2\pi]$, then the interference pattern is washed out. In fact, taking the uniform average of the position amplitude squares of the above states over all δ,

$$\begin{aligned}\langle|\psi_\delta(\mathbf{r})|^2\rangle &= \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}|\psi_\delta(\mathbf{r})|^2 d\delta \\ &= \tfrac{1}{2}|\psi_1(\mathbf{r})|^2 + \tfrac{1}{2}|\psi_2(\mathbf{r})|^2 + \tfrac{1}{4\pi}\int_0^{2\pi}(\psi_1(\mathbf{r})e^{-\imath\delta}\psi_2^*(\mathbf{r}) + \psi_1^* e^{\imath\delta}(\mathbf{r})\psi_2(\mathbf{r}))d\delta \\ &= \tfrac{1}{2}|\psi_1(\mathbf{r})|^2 + \tfrac{1}{2}|\psi_2(\mathbf{r})|^2.\end{aligned} \tag{3.2.6}$$

The final line describes a distribution with no maxima nor minima in it and is similar in form and interpretation to equation (3.2.5). However, in the latter case there will be a change in the *momentum distribution* caused by interaction between the quantum system and the probe or path marker.

The Scully et al paper prompted a debate in which Storey, Tan, Collett, Wiseman and Walls [61] analyzing double slit interferometers in general and Scully et al's in particular conclude that the loss of interference from a double slit in the presence of a which-way detector is physically caused by momentum kicks, the magnitude of which are determined by the uncertainty principle. They maintain, therefore, that the principle of complementarity is a consequence of the Heisenberg uncertainty relation and the source of the momentum kicks in the Scully et al experiment is the repeated emission and re-absorption of microwave probe photons by the atom whilst it is in the probe detector i.e. micromaser cavity.

A rebuttal came from Englert et al in Nature (1995) [62]in which they unequivocally claim that the principle of complementarity is much deeper than the uncertainty relation although they acknowledge that complementarity is frequently enforced by $\delta x \delta p \geq \hbar/2$.

Some clarification of the possible mechanism causing momentum transfer was offered by Wiseman and Harrison [75], [76], who pointed out that there are two different ways of considering random momentum kicks. One, distinctly quantum mechanical, is the convolution of the momentum wave-function of the system with a momentum *amplitude* transfer function of the probe. The other corresponds to the classical notion of a convolution of the system's momentum *probability* distribution with the momentum *probability* distribution of the probe.

So, SEW are correct in pointing out that Bohr's semi-classical picture of random kicks and Heisenberg's uncertainty relation being used to support the enforcing of complementarity does not work in general. Their claim on the first page (p. 111) that "...the actual mechanisms that enforce complementarity vary from one experimental situation to another", is justified.

Following this quotation is a brief reference to the 1979 paper of Wootters and

Zurek [77] who do offer a quantum mechanical treatment of path marking, which yields a trade-off relation, is referred to only briefly in the sentence following the one quoted. In section 3.4 of this chapter Wootters and Zurek's exploration of complementarity is reviewed. And, as will be seen in section 3.2.2, the work of Englert and Jaeger et al confirms that there are uncertainty or trade-off relations that do describe a price for relaxing the strict *either-or* of complementarity.

In 1998 Dürr, Nonn and Rempe [22], inspired by Scully et al's gedanken experiment, realized a which-way experiment in an atom interferometer. In this experiment the classical momentum transferred to the atom is too small to explain the disappearance of the interference pattern and the two ways through the interferometer are not transversely separated by much more than the initial beam collimation. So, there is no storage of positional information (p. 36 [22]).

Anticipating questions about the existence of a back action on the transverse momentum of the atom caused by the finite size of the apparatus, Dürr et al identify one source being the localization of the atom to within one wavelength of the microwave field. The corresponding back action implied by the position-momentum uncertainty relation is four orders smaller than the fringe separation (p. 36 [22])

Further they note that the atom is only localized with a precision of the order of the beam width throughout its passage through the region in which it interacts with the microwave field. The position-momentum uncertainty relation could only be invoked if the atom could be localized (p. 36 [22]). Instead it is entanglement between the which-way probe (one of the internal degrees of freedom of the atom) and the path degree of freedom of the atom that destroys the interference fringes.

DNR compare their findings with those of Wiseman et al in a paper on 'Non-local momentum transfer in welcher-weg experiments', [76], in which there is an explanation for the disappearance of the interference fringes in a double slit. DNR point out that their results are not in conflict with those of Wiseman et al because no double slit is used and no position measurement made: "Wiseman et al's results simply do not apply which leaves an open question whether the concept of quantum mechanical momentum transfer can be generalized to schemes without a mechanical double slit."

The debate with SEW and the responses made directly to SEW highlight these views: Complementarity becomes apparent when which-way information is available in a interferometer. The loss of fringes is either due to, (1) random classical phase kicks, or (2) quantum entanglement.

Each of these views represent extreme positions and it has been argued that both can be shown to be correct in [27]. In the demonstrations proposed by Einstein and Feynman mechanism (1) gives an account of the loss of interference, whereas mechanism (2) provides an explanation for a similar loss of interference in the experiments of SEW and Dürr et al.

The more general question of whether the uncertainty relation is relevant to complementarity is also raised. If the position-momentum uncertainty relation only is considered then the answer is 'not always' but it cannot be said that the principle of complementarity is more fundamental than the principle of uncertainty. In the experiments of SEW and Dürr et al they appear as independent features of quantum phenomena.

In Chapter 4 it will be shown that, in a limit, value complementarity results from some form of uncertainty relation for quantities other than momentum and position.

3.2.2 Duality relationships

These investigations have been made in the context of an analogous Mach-Zehnder interferometer experiment.

So far in this discussion the question of achieving some which-way path information at the expense of limited fringe visibility in a two way interferometer has not been considered. This regime is intermediate to those considered in 3.2.1 and it requires strict complementarity to be relaxed i.e. to become graded complementarity. This situation was investigated by Horne, Jaeger and Shimony in 1993 [38] and by Jaeger, Shimony and Vaidman in 1995 [39]. The quantitative duality relation they arrived at anticipates the one obtained in 1996 by Englert [25].

In an effort to move away from the pictures of waves and particles and their association with classical physics but to still preserve the spirit of Bohr's complementarity, Englert coins the phrase 'interferometric duality' or "simply 'duality'". His notion of 'duality' is the mutual exclusivity of observing an interference pattern and obtaining which-way information. Jaeger et al refer to their work as formulating "...interferometric complementarities", a term which finds use in Chapter 4.

To develop this relationship a quantitative measure of which-way information is needed. Englert identifies two different methods of obtaining which-way information.

In the first method, here discussed in the context of an analogous Mach-Zehnder interferometer experiment, Englert considers a two way interferometer in which the two ways through the interferometer are labeled $|+\rangle$ and $|-\rangle$, as in Fig. 3.1.

The system entering the interferometer is prepared in an initial state,

$$\rho_i^o = \tfrac{1}{2}(\mathbf{I} + \mathbf{s}_i \cdot \overrightarrow{\sigma}) = \tfrac{1}{2}(\mathbf{I} + s_{x,i}\sigma_x + s_{y,i}\sigma_y + s_{z,i}\sigma_z), \tag{3.2.7}$$

where $\mathbf{s}_i = Tr^o(\overrightarrow{\sigma}\rho_i^o)$.

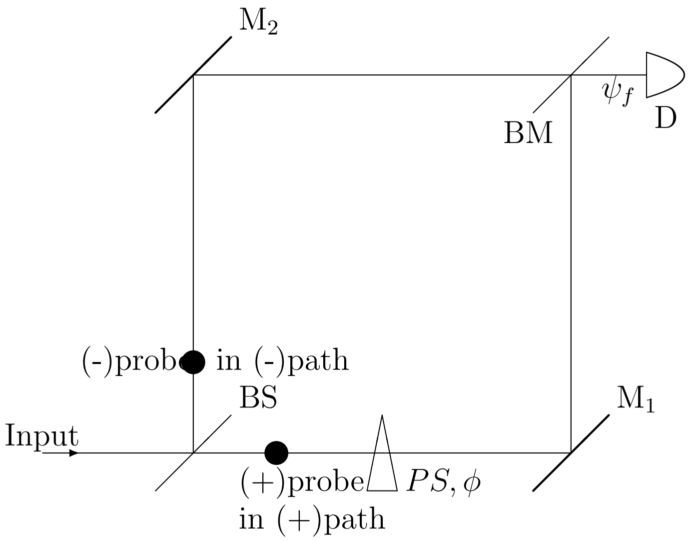

Figure 3.1: A two-way interferometer. The beam splitter BS distributes the input to the two possible paths. After the action of the phase shift at PS, the beam merger recombines the path contents and produces an output. Measurements on the output may be used either to reveal a ϕ dependent interference pattern or the path that has been taken. In a separate experiment which-way detectors (probe) are introduced

The action of the beam splitter(BS) and beam merger (BM) can be represented by,

$$\rho^o \to \exp(-\imath\tfrac{\pi}{4}\sigma_y)\rho^o \exp(\imath\tfrac{\pi}{4}\sigma_y), \tag{3.2.8}$$

and the phase shifter between BS and BM by

$$\rho^o \to \exp(-\imath\tfrac{\phi}{2}\sigma_z)\rho^o \exp(\imath\tfrac{\phi}{2}\sigma_z). \tag{3.2.9}$$

It is worth noting here there is a freedom of choice about choosing path and interference observables. Englert chooses σ_y as a path observable and σ_z as an interference observable. In Chapters 4 and 5 the reverse choice is made.

As a result of these evolutions, the initial state, ρ^o, becomes the final state,

$$\rho^o_f = \tfrac{1}{2}(\mathbf{I} + \mathbf{s}_f \cdot \overrightarrow{\sigma}), \tag{3.2.10}$$

where $\mathbf{s}_f = (-s_{x,i}, s_{x,i}\cos\phi + s_{z,i}\sin\phi, s_{y,i}\sin\phi - s_{z,i}\cos\phi)$

At the detector D the interference observable, σ_z is measured and the relative frequency with which, say, the value -1 is found reveals an interference pattern,

$$Prob(\sigma_z = -1, \phi) = Tr^o[\tfrac{1}{2}(1-\sigma_z)\rho^o_f] = \tfrac{1}{2}(1 - s_{y,i}\sin\phi + s_{z,i}\cos\phi), \tag{3.2.11}$$

giving an *a priori* fringe visibility of $V_0 = [(s_{y,i})^2 + (s_{z,i})^2]^{\frac{1}{2}}$

The probabilities for the object taking either of the two ways are

$$Prob(\sigma_z = \pm 1) = Tr^o[\tfrac{1}{2}(1 \pm \sigma_z)\exp(-\imath\tfrac{\pi}{4}\sigma_y)\rho^o_i \exp(\imath\tfrac{\pi}{4}\sigma_y)] = \tfrac{1}{2}(1 \mp s_{x,i}). \tag{3.2.12}$$

The magnitude of the difference between these probabilities gives a measure of the *a priori* which-way knowledge or *predictability*, $P = |s_{x,i}|$, of the ways through the interferometer.

Since $|\mathbf{s}_i| \leq 1$ the following relationship between fringe visibility and path predictability can be written,

$$P + V_0 \leq 1 \tag{3.2.13}$$

Greenberger et al and Jaeger et al [30, 39] derive equivalent relationships and these are in good agreement with results obtained from a neutron interferometer experiment noted by Summhammer et al [69].

If the two paths through the interferometer are not distinguished by their differing flux of particles, i.e. $P = 0$ and $V_0 = 1$, the paths could be marked by allowing information about the path to transferred to a probe. This information can be read by a suitable measurement of the probe.

Now, consider the joint probability, $p(m_i, 1)$ (or $p(m_i, 2)$) that for object i the marker observable M is found with eigenvalue m_i and that it took path 1 (or path 2). If $p(m_i, 1) \geq p(m_i, 2)$ then it is reasonable to guess that the particle took the 1 path if the outcome was m_i. This leads Englert to a term "likelihood of guessing the correct way, L_M" where

$$L_M = \sum_i max\{p(m_i, 1), p(m_i, 2)\}. \tag{3.2.14}$$

Clearly, L_M can vary between $\frac{1}{2}$ and 1 and depends upon the choice of observable M. A more convenient measure which varies between 0 and 1 can easily be constructed as $K_M = 2L_M - 1$. K_M is termed 'which-way knowledge'. At this point the path has remained indefinite so the 'knowledge' is only 'potential' for this path.

If the maximum value of this 'which-way knowledge' is measured for the best choice of M, then it is possible to quantify how much which-way information is stored for that choice of M. Here, thus is a measure of 'distinguishability',

$$D = max_M(K_M). \tag{3.2.15}$$

D is limited by the duality relation

$$D^2 + V^2 \leq 1, \tag{3.2.16}$$

where V is the fringe visibility measured on the final state after path marking is applied.

Equations 3.2.13 and 3.2.16 look similar but refer to different experimental set ups. The predictability P and the distinguishability D represent which-way information of different types. Further, equation 3.2.13 is a consequence of the initial input state of the object, phase shifter and the transparencies of the beam splitter; the beam splitter act as which-way detectors. Whereas, equation 3.2.16 results from the quantum properties of the path marking and recording in a separate (probe) system and is a stronger indication of graded complementarity in action.

3.2.3 Bjork et al on Duality and Uncertainty

In a paper of 1999 Björk et al [4] investigated complementarity and the uncertainty relations, using a version of complementarity as, "...any system has at least two properties that cannot be simultaneously known" (p. 1874 [4]). The complementary pairs being considered are are path and interference observables.

Björk et al cite an extensive list of papers in which expression of wave-particle duality are developed. Crediting their approach to that of Englert [25], the Englert duality relation is applied to a two state system and it is demonstrated that every two state system obeys a complementarity relation even before any attempt has been made to measure an observable and one complementary to it (p. 1875 [4]). In the terminology of this thesis this is an instance of preparation complementarity. This will be taken up and developed in Chapter 4.

To achieve some understanding of what operator corresponds to interference pattern visibility or contrast and path or which-way predictability, if these do in some way correspond to a measurement of an observable, Björk et al construct from the eigenstates, $|A_+\rangle$ and $|A_-\rangle$, with real eigenvalues a_+, a_-, of one operator, A, two orthonormal states

$$|B_+\rangle \equiv (|A_+\rangle + e^{\imath\delta}|A_-\rangle)/\sqrt{2} \tag{3.2.17}$$

$$\text{and} \qquad |B_-\rangle \equiv (|A_+\rangle - e^{\imath\delta}|A_-\rangle)/\sqrt{2} \tag{3.2.18}$$

and from these a complementary Hermitian operator,

$$B = b_+|B_+\rangle\langle B_+| + b_-|B_-\rangle\langle B_-|, \qquad b_+, b_- \in \mathbb{R}. \tag{3.2.19}$$

The two observables are complementary in the sense that the eigenstates of one are constructed to be equally weighted superpositions of the eigenstates of the other; this is an instance of what is referred to as value complementarity in chapter 4. Hence, Björk et al show how complementarity is a natural consequence of the superposition principle for a two-state system.

They then show that there is a direct link between the duality relation, equation 3.2.13, and the product of the variance of the complementary operators they have identified as corresponding to interference contrast and path predictability i.e. a minimum normalized uncertainty product.

$$\frac{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle}{(a_+ + a_-)(b_+ + b_-)} \geq \frac{(1-P^2)(1-V_0^2)}{16}. \tag{3.2.20}$$

Here, P and V_0 are the *a priori* path predictability and fringe visibility as used by Englert in equation 3.2.13

They conclude that,

> "We have shown how complementarity is a natural consequence of the superposition principle ... for any two-state system one can always formulate a generalized complementarity relation and that this relation typically cannot be interpreted in terms of position and momentum operators."

The duality relation 3.2.20 *implies* an uncertainty relation for a complementary pair. In Chapter 4 the reverse will be shown, namely, that duality relations are implied by suitable uncertainty relations.

3.2.4 Dürr and Rempe on duality and uncertainty

Also taking as their starting point the Englert/Jaeger et al duality relation, Dürr and Rempe [23] noted that, "none of the derivations [of the complementarity or duality relation] involves any form of uncertainty relation. It therefore seems that *[here they are quoting Englert, p. 2157 [25]]* the duality relation is logically independent of the [position-momentum] uncertainty relation".

In order to identify two observables in a two beam interferometer without which-way marking that can be used in a Heisenberg-Robertson uncertainty relation, Dürr and Rempe investigate this two dimensional set-up using the Pauli spin matrices, $\sigma_x, \sigma_y, \sigma_z$. For the object, which-way knowledge or path information is indicated by $\langle\sigma_z\rangle$. Incomplete which-way knowledge without a marker could be found by measuring the particle fluxes along the two arms of the interferometer. Interference contrast is related to $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$. (See Chapter 4.)

For a relative phase of zero between the two paths, the following relationships are identified, $\Delta\sigma_x = \sqrt{1-V^2}$, $\Delta\sigma_y = 1$, $\Delta\sigma_z = \sqrt{1-P^2}$.

Using the commutator relationship $[\sigma_j, \sigma_k] = 2\imath \sum_l \epsilon_{jkl}\sigma_l$, the uncertainty relationship for all possible pairs of the standard deviations give,

$$\sqrt{1-V^2} = \Delta\sigma_x \Delta\sigma_y \geq |\langle\sigma_z\rangle| = P, \tag{3.2.21}$$

$$\sqrt{1-P^2} = \Delta\sigma_y \Delta\sigma_z \geq |\langle\sigma_x\rangle| = V, \tag{3.2.22}$$

$$\Delta\sigma_z \Delta\sigma_x \geq |\langle\sigma_y\rangle| = 0. \tag{3.2.23}$$

Dürr and Rempe point out that the uncertainty relation they use is not, in fact, the position-momentum uncertainty relation. Further, they note that the first two equations (3.2.21, 3.2.22) are equivalent to the duality relation, equation 3.2.13, of Englert Jaeger et al.

Equation (3.2.23) is trivial. (3.2.21) and (3.2.22) contain an irrelevant observable σ_y. The approach in Chapter 4 is to give equivalent uncertainty relations which involve the measured observables only.

To clarify what Dürr and Rempe have achieved, consider a projection operator being represented by $P_{\mathbf{a}} = \frac{1}{2}(\mathbf{I} + \mathbf{a} \cdot \overrightarrow{\sigma})$, where $\|\mathbf{a}\| = 1$ and the state by $\rho_{\mathbf{n}} = \frac{1}{2}(\mathbf{I} + \mathbf{n} \cdot \overrightarrow{\sigma})$, where $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| \leq 1$.

The expectation value is $\langle P_a \rangle_{\rho_{\mathbf{n}}} = \frac{1}{2}(1 + \mathbf{n} \cdot \mathbf{a})$ and the variance of $P_{\mathbf{a}}$ in the state $\rho_{\mathbf{n}}$ is $\mathrm{Var}(P_{\mathbf{a}}, \rho_{\mathbf{n}}) = \frac{1}{4}(1 - (\mathbf{n} \cdot \mathbf{a})^2)$.

Now, using Dürr and Rempe's results: $V^2 = 1 - \Delta\sigma_x^2 = 1 - (1 - n_x^2) = n_x^2$ and $P^2 = 1 - \Delta\sigma_z^2 = n_z^2$.

So, it can be seen that because $n_x^2 + n_y^2 \leq 1$, $P^2 + V^2 \leq 1$.

Dürr and Rempe then consider an interferometer with which-way marking. In this an interaction creates an entanglement between a probe or which-way marker and the object.

In order to read out the which-way information a probe observable W will be measured. W has eigenvalues $w_1, w_2, \ldots$ and a basis of eigenstates $\{|w_1\rangle, |w_2\rangle, \ldots\}$. By letting $Prob(|\pm\rangle, w_i)$ denote the joint probability that eigenvalue w_i is found and that the particle has the path $|\pm\rangle$ and following Englert's argument that leads to equations (3.2.14) and (3.2.15) in Section 3.2.2 they show that the distinguishability, D, is the maximum value of the which way knowledge $\sum_i |Prob(+, w_i) - Prob(-, w_i)|$. Since the latter is dependent upon the choice of W, the D relies on making the best choice of W.

Their task is now to show that duality relation, equation (3.2.16), can be derived from a Heisenberg-Robertson uncertainty relation for some suitably chosen observable. To this end let,

$$\epsilon = \begin{cases} +1 & \text{if } Prob(+, w_i) \geq Prob(-, w_i) \\ -1 & \text{otherwise} \end{cases} \tag{3.2.24}$$

denote which way to bet on if the eigenstate $|w_i\rangle$ if found.

Using $Prob(+, w_i) = \langle w_i, +|\rho^{o,p}|w_i, +\rangle$, where $\rho^{o,p}$ denotes the composite object plus which-way probe system, the distinguishability can now be found from the maximum value of

$$\sum_i \epsilon(\langle w_i, +|\rho^{o,p}|w_i, +\rangle - \langle w_i, -|\rho^{o,p}|w_i, -\rangle) = \sum_i \epsilon Tr^{o,p}\{\rho^{o,p}(|w_i\rangle\langle w_i| \otimes \sigma_z)\}. \tag{3.2.25}$$

It is now possible to propose an observable $W_\epsilon = \sum_i \epsilon|w_i\rangle\langle w_i|$, in which case the which way knowledge becomes $\langle\sigma_z \otimes W_\epsilon\rangle$ and the distinguishability then becomes $D = \langle\sigma_z \otimes W_{max,\epsilon}\rangle$; the observable $W_{max,\epsilon}$ being chosen such that 'which-way knowledge' is maximized. Using the same techniques that are used to create equations (3.2.21, 3.2.22, 3.2.23), the following uncertainty relations are developed,

$$\sqrt{1-D^2} \geq \triangle(\sigma_y \otimes W_{max,\epsilon})\triangle(\sigma_z \otimes W_{max,\epsilon}) \geq |\langle\sigma_x\rangle| = V, \tag{3.2.26}$$

$$\sqrt{1-V^2} \geq \triangle\sigma_x\triangle(\sigma_y \otimes W_{max,\epsilon}) = D. \tag{3.2.27}$$

Note that the irrelevant observable σ_y is still present.

The interpretation of equation 3.2.26 is that in a which-way scheme with a path marker entangled with the object, the duality relation can be obtained from a form of Heisenberg-Robertson uncertainty relation.

Between the Börk et al and Dürr et al papers there is a marked contrast. They both start from similar positions and employing a similar rederivation of the duality relation for a two state quantum system with no marker present e.g. an object in a two way interferometer. Also, both show that there is a link between this duality relation and an uncertainty relation comprising the product of variances. Neither consider measures of uncertainty other than variances.

Björk et al consider an operator A with two eigenvalues and note “that [predictability] P in some way corresponds to measurement of A”. The second operator B that is to be used in the uncertainty relation has eigenstates that are equally weighted superpositions of eigenstates of A. From this stand point they are able to show a link that exists between the duality or complementarity relation and an uncertainty relation comprising variances.

In contrast, in their final discussion Dürr et al make the point that explanations found in textbooks for Einstein’s recoiling slit [77], [66] and Feynman’s light microscope [29] based on the position momentum uncertainty relation, the explanation offered by SEW [60] for the loss of interference in their atom interferometer and the derivation of the duality relation of Englert/Jaeger et al are based on entanglement or correlations. The calculations of Dürr and Rempe [23] and Wiseman et al [76] make use of both entanglement and *some* uncertainty relation.

Dürr and Rempe's categorization reveals a crucial point: The explanations for the loss of interference fringes involving only the uncertainty relation are (so far) limited to a few special schemes. In other words, the loss of interference cannot, in general, be explained in terms of 'classical momentum transfer'. On the other hand, explanations involving only entanglement apply to all which-way schemes known so far. This leads them to the conclusion that wave particle duality is more closely connected to entanglement than to an uncertainty relation (p. 1024).

The findings of Busch and Shilladay in [14] disagree with this conclusion. In this it is shown that entanglement can be understood as a instance of uncertainty in the context of descriptions of compound systems rather than as a separate feature. Moreover, it is to be expected that whenever an explanation of the loss of interference can be given in terms of entanglement, this can be accompanied with an explanation in terms of a form of uncertainty relation.

3.2.5 Kim and Mahler

In response to any continuing perceived attack on Heisenberg's position momentum uncertainty relation Kim and Mahler wrote a paper entitled 'Uncertainty Rescued: Bohr's complementarity for composite systems' [42]. They wrote of Heisenberg's position momentum uncertainty relation that it "is often interpreted to imply that one cannot detect for a given quantum state two conjugate observables with unlimited precision." And of Bohr's complementarity principle, that it "may be understood to mean that a state with a minimum dispersion of one observable (i.e. preparation of a respective eigenstate) implies maximum dispersion of the other (i.e. any of its eigenstates will be found with equal probability), e.g. q and p."

Kim and Mahler extend their consideration of uncertainty relations to composite systems. The uncertainty relation they use is of the general form developed in Chapter 2, Section 2.3 equation 2.3.15.

They propose that path detection is achieved by correlating (or anti-correlating) paths and marker states in this way: An initial state $|0\rangle_o|0\rangle_p$, where o and p refer to object and probe, is allowed to evolve into a final state,

$$|\Psi_f\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle_o|0\rangle_p + |1\rangle_o|1\rangle_p) \tag{3.2.28}$$

by the action of a beam splitter followed by a controlled-NOT gate (which has the effect, $|10\rangle \rightarrow |11\rangle$, $|00\rangle \rightarrow |00\rangle$), in a similar manner to the model of Zhu et al [78] analyzed in Section 4.2 (see also [13]).

Next, the total system observables are described in terms of tensor products of the individual single system Pauli operators, $\sigma_j^o \otimes \sigma_k^p$ where $j, k = x, y, z$; in particular, the covariance of σ_j^o and σ_k^p is

$$C_{\sigma_j^o,\sigma_k^p} = \langle\sigma_j^o \otimes \sigma_k^p\rangle - \langle\sigma_j^o\rangle\langle\sigma_k^p\rangle. \tag{3.2.29}$$

Since the individual system operators acting upon different systems commute, the inter-subsystem uncertainty relations are given by,

$$Var(\sigma_j^o)Var(\sigma_k^p) \geq |C_{\sigma_j^o,\sigma_k^p}|^2. \tag{3.2.30}$$

The values taken by the covariance term are $0 \leq |C_{\sigma_j^o,\sigma_k^p}| \leq 1$; $C_{\sigma_j^o,\sigma_k^p} = 0$ for product states. $|C_{\sigma_j^o,\sigma_k^p}| = 1$ implies perfect correlation or anti-correlation implying maximum ignorance of σ_j^o and σ_k^p, $\Delta\sigma_j^o = \Delta\sigma_k^p = 1$.

The preparation of an entangled state implies that each subsystem is in a non-pure state. This fact is used by [29], [61], [63], [75] to justify interpreting the resulting mixture states resulting from random kicks. However, Kim and Mahler are able to show that it is possible to remove the 'alleged' effects of these random kicks by quantum erasure.

In the erasure experiment one makes use of the fact that the state has an equivalent form,

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}}[| + x\rangle^o|+\rangle^p + | - x\rangle^o|-\rangle^p] \tag{3.2.31}$$

where $| \pm x\rangle^o = \frac{1}{\sqrt{2}}[|0\rangle^o \pm |1\rangle^o]$, $|\pm\rangle^p = \frac{1}{\sqrt{2}}[|0\rangle^p \pm |1\rangle^p]$: a similar procedure is proposed in 4.3.4.

If now measurements on the sub-ensembles associated with $|+\rangle^p$ and $|-\rangle^p$, respectively, are recorded separately, the respective interference patterns can be recovered, as proposed by SEW [60].

Kim and Mahler conclude that complementarity and uncertainty relations can be strictly related and the random classical kicks induced by a measurement process are not sufficient to explain either uncertainty or complementarity. They claim that the often-discarded mutual correlation term in the generalized uncertainty relation implies the equivalence between the set of uncertainty relations and the principle of complementarity holds in a wider sense than the historical gedanken experiments.

What Kim and Mahler do is consistent with what is done in Chapters 4 and 5: The connection between complementarity and uncertainty can be made explicit either in terms of a POVM in the Hilbert Space of the object, as in Chapters 4 and 5, or by emphasizing the correlation between object and probe.

3.2.6 Comments

In the works studied so far it is possible to identify certain, sometimes contrasting, conflations:

1. The identification of the position-momentum uncertainty relation with classical momentum kicks; this can be seen to be the case in the debate around the proposal of SEW but is avoided by Dürr *et al.*

2. In many writings (discussed by Jammer [40]), complementarity is seen as a synonym for uncertainty. This is in contrast to what SEW, DNR and others are saying, namely, that complementarity is more fundamental than uncertainty.

3. There is insufficient distinction between versions of complementarity, in the sense of strict exclusivity and in the sense of graded exclusivity with a trade-off relation. In the mid-1990 there was a shift of focus from the original, strict mutual exclusivity position, e.g. SEW (1991) still understood complementarity as value complementarity but Englert [25] clearly discusses quantitative complementarity.

4. There is also insufficient distinction between the roles of complementarity and uncertainty for preparations and for measurements, respectively. In section 3.5 of this chapter preparation and measurement forms of complementarity are discussed as well as the different forms of uncertainty which can be identified (see Sec. 3.5.3).

5. Path marking and its erasure are in a mutually exclusive (complementary) relationship; however if the path marking could be made imperfect or unsharp then

this holds the possibility of measuring jointly an unsharp path and an unsharp interference observable. In this, *quantitative quantum erasure*, case the experiment provides unsharp information about both mutually exclusive observables simultaneously.

Both erasure and quantitative erasure appear naturally out of the Mach-Zehnder analogue of SEW's experiment in Chapter 4.

The next task is to attempt to trace the origins of these conflations.

3.3 The Origins; the influence of the pioneers' thinking

3.3.1 The origin of complementarity and its meaning according to Bohr

In the world of classical physics there appears to be different sets of phenomena which are described as either wave behaviour or particle behaviour. However, it is possible to apply these pictures to the same phenomenon. The two models are distinct from each other but not mutually exclusive.

Consider water waves and sound waves. These 'undulatory motions of a medium' are the collective behaviour of particles. There is no conflict, classical physicists used the terms waves and particles together, each picture capturing different aspects of the phenomenon.

When considering the nature of light, the debate between the advocates of the wave model of light proposed by Robert Hooke and Christian Huygens and the corpuscular model proposed by Isaac Newton raged from the middle of the seventeenth century to the beginning of the nineteenth when the methods of measurement became sufficiently refined to allow *experimentum crucis.* It was Thomas Young who achieved a resolution when he used wave theory to explain coloured fringes produced by thin films and later performed the first double-slit interference experiment. Young had discovered the principle of interference.

Working independently, a few years later, Augustin Fresnel also discovered the principle of interference. Temporarily, light was regarded as fundamentally a wave phenomenon. However, in a different sense to how particles appear in the theory of water waves, the electromagnetic or optical wave theory allowed for a limit, called the geometric optic limit, described by rays which could be viewed as particle trajectories. In this sense, Newton's model was justified as an independent, approximate description.

The language used by physicists to model the problems raised by experimental observations pictures the facts to give an intuitive understanding. For example, the picture we have of light as a propagating wave uses the language of waves to predict and explain a vast range of phenomena involving light. The laws of light propagation are enshrined in classical physics as the theories of optics and electromagnetism. However, some of the phenomena produced by light, such as the photoelectric effect, seem to call for corpuscular theory.

December 14, 1900 is often regarded as the "…birthday of quantum theory". "…At the meeting of the German Physical Society, December 14 1900, Planck read his historic paper 'On the theory of the energy distribution law of the normal spectrum'", in which he presented a model requiring the microscopic constituents of a black-body to have a discrete energy spectrum. This discreteness was to be characterized by the 'universal constant h'. "His findings were destined to change the course of theoretical physics"([41], pp. 21-22).

In 1905, to model the interaction of light with matter in the photoelectric effect, Einstein was compelled to proposed light quanta. This was counterintuitive because a model of how they produce an interference pattern could not be constructed. The model for light of Hooke, Huygens and Maxwell had become distorted to include something as extensible as a wave and as localized as a particle and as such gives no intuition as to the thing itself.

And, in 1912 William Bragg wrote of the paradoxical results from on the one hand x-rays exhibiting wave like behaviour in diffraction experiments and on the other hand particles like behaviour when ionizing a gas, "The problem becomes . . . not to decide between two theories of x-rays, but to find . . . one theory which possesses the capacity of both" [40].

In parallel developments during the years 1905-1927 the concept of *matter waves* was used and developed in several studies: Bohr, starting in 1912, found it convenient to describe electrons in orbit about a nucleus forming standing waves.

De Broglie is credited with the term 'matter waves' in his hypothesis of wave-like behaviour being associating with matter. His recollection are recorded in [17].

> 'After long reflection in solitude and meditation, I suddenly had the idea, during the year 1923, that the discovery made by Einstein in 1905 should be extended to all material particles and notably to electrons.'

De Broglie concept of matter waves was essential for Schrödinger's formulation of quantum mechanics which appeared in a series of papers deriving the hydrogen spectrum [67].

In 1927 Davisson and Germer demonstrated electron diffraction by a crystal [16].

Over a period which can be traced to be between 1925 and 1927 ([41], p. 345), Bohr came to accept that where there are two mutually exclusive descriptions both needed to give a complete picture of the same object such as light or an electron, then they must stand in a certain relationship to one another to avoid any inconsistencies. Bohr coined the term complementarity to capture this relationship and introduced it to the scientific world on September 16, 1927 at a conference in Como [5].

It is worthwhile to consider the presentations of complementarity that Bohr comes up with in his 1928 and 1935 papers. These several versions are referred to by Jammer ([40], p. 87) as Pauli's version of the Copenhagen complementarity interpretation.

The quantum postulate-the origins of quantum mechanics:

This states that individuality of quantum interactions, bringing about uncontrollable change in observations. Planck's quantum of action, $\hbar$, decides the scale of the actions at which those changes become non-negligible.

([5], p. 580) *...quantum postulate, which attributes to any process an essential discontinuity, or rather individuality, completely foreign to the classical theories and symbolized by the Planck's quantum of action.*

([5], p. 580) *... This postulate implies a renunciation as regards the causal space-time co-ordination of atomic processes.*

([6], p. 697)*...the finite interaction between object and measuring agencies conditioned by the very existence of the quantum of action entails - because of the impossibility of controlling the reaction of the object on the measuring instrument if these are to serve their purpose - the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality.*

The mutual exclusion of definition and observation:

([5], p. 580)... *The very nature of quantum theory thus forces us to regard the space-time co-ordination and the claim of the causality, the union of which characterizes the classical theories as complementary but exclusive features of the description, symbolizing the idealization of the observation and definition.*

This is the first occurrence of term *complementarity*.

([5], p. 580)... *in the description of atomic phenomena, the quantum postulate presents us with the task of developing a 'complementarity' theory the consistency of which can be judged only by weighing the possibilities of definition and observation.*

([5], p. 587) *According to the quantum postulate any observation regarding the behaviour of the electron in the atom will be accompanied by a change in the state of the atom.... The complementary nature of the description [of the 'orbit' of the electron in the atom], however, appears particularly in that the use of observations concerning the behaviour of particles in the atom rests on the possibility of neglecting, during the process of observation, the interaction between the particles, thus regarding them as free.*

([5], p. 589) ... *it might be said that the concepts of stationary states and individual transition processes within their proper field of application possess just as much or as little reality as the very idea of individual particles. In both cases we are concerned with a demand of causality complementary to the space-time description. The adequate description of which is limited only by the restricted possibilities of definition and of observation.*

The necessity for complementary modes of description:

The union of space-time coordination (provided by observation) and causal description (associated with conservation laws for energy-momentum in isolated systems), characterizes classical physics but is rendered impossible in quantum physics.

([5], p. 580) *. . . the definition of the state of a physical system, as ordinarily understood, claims the elimination of all external disturbances. But in that case, according to the quantum postulate, any observation will be impossible and above all the concepts of space and time lose their immediate sense. On the other hand if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible and there can be no question of causality in the ordinary sense of the word.*

([5], p. 581)*. . . the possibility of identifying the velocity of the particle with the group velocity indicates the field of application of space-time picture of the quantum theory. Here the complementary character of the description appears since the use of wave-groups is necessarily accompanied by a lack of sharpness in the definition of period and wavelength and hence also in the definition of period and of the corresponding energy and momentum . . .*

The exclusive nature of experimental set-ups:

([5], p. 581) *. . . The two views of the nature of the constituents of light are . . . to be considered as different attempts at an interpretation of experimental evidence in which the limitation of the classical concepts is expressed in complementary ways.*

([6], p. 699) Bohr proposes an analogue of the EPR experiment comprising a rigid diaphragm with two parallel slits. Through each slit one particle with a given initial momentum passes independently of the other. If the momentum of the diaphragm is determined before and after the passage of the particles then the sum of the component of the momenta of the particles perpendicular to the slits can be found. The difference of their initial positions in the same direction can also be found. In this arrangement a subsequent single measurement, either of the momentum or of the position of one of the particles will automatically determine the position and momentum of the second without interfering with the second particle.

Like the above simple case of the choice between the experimental procedure suited for the prediction of the position or the momentum of a single particle which has passed through a slit in a diaphragm, we are, in the "freedom of choice" offered by the last arrangement [i.e. whether we want to determine position or momentum by a process which does not directly interfere with the particle concerned], just concerned with a discrimination between different experimental procedures which allow the unambiguous use of complementary classical concepts.

([6], p. 700)*...it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities, which provides room for new physical laws, the coexistence of which might at first sight appear irreconcilable with the basis principles of science. It is this entirely new situation as regards the description of physical phenomena, that the notion of complementarity aims at characterizing.*

Complementary pairs of physical quantities:

Examples are path and interference or position and momentum, which are both necessary as part of the description.

([5], p. 580)*... As regards light, its propagation in space and time is adequately expressed by electromagnetic theory. Especially the interference phenomena in vacuo and the optical properties of material are completely governed by the wave theory superposition principle. Nevertheless, the conservation of energy and momentum during interaction between radiation and matter ... This situation would seem clearly to indicate the impossibility of a causal space-time description of the light phenomena.*

([6], p. 697)*... the mutually exclusive character of any unambiguous use in quantum theory of the concepts of position and momentum,...*

No independent reality for object or measurement device alone:

Any phenomenon is determined through the whole of object (atomic system) and measuring device.

([5], p. 580)*... the quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected. Accordingly, an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation.*

([6], p. 699)*... the renunciation in each experimental arrangement of the one or the other of two aspects of the description of physical phenomena - the combination of which characterizes the method of classical physics, and which therefore in this sense may be considered as complementary to one another, - depends essentially on the possibility, in the field of quantum theory, of accurately controlling the reaction of the object on the measuring instruments, i.e. the transfer of momentum in the case of position measurement and the displacement in the case of momentum measurements.*

Uncertainty relation demonstrating consistency of complementarity interpretation with quantum formalism:

They quantify the limitations for the simultaneous application of two complementary classical descriptions.

([5], p. 581)*... [Heisenberg] has stressed the peculiar reciprocal uncertainty which affects all measurements of atomic quantities ... the complementary nature of description appearing in this uncertainty is unavoidable.*

([5], p. 582)*... The limitation in the classical concepts expressed through [the Heisenberg uncertainty] relation is ... closely connected with the limited validity of classical mechanics, which in the wave theory of matter corresponds to geometrical optics in which the propagation of waves is depicted through 'rays'. Only in this limit can energy and momentum be unambiguously defined on the basis of space-time pictures.*

([5], p. 582)*... In the language of relativity theory, the content of [the Heisenberg uncertainty] relation may be summarized in the statement that according to quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum vectors associated with individuals. This circumstance may be regarded as a simple symbolic expression for the complementary nature of the space-time description and the claims of causality. At the same time, however, the general character of this relation makes it possible to a certain extent to reconcile the conservation laws with the space-time co-ordination of observation, the idea of a coincidence of well defined events in a space-time point being replaced by that of unsharply defined individuals within finite space-time regions.*

Here Bohr acknowledges that Heisenberg's uncertainty relation does more than express complementarity. It seems that he is allowing a softening of complementarity into what is referred to in this study as graded complementarity.

([5], p. 584)*. . . it follows from the above consideration [of illustrations of Heisenberg's uncertainty relation - gamma ray microscope] that the measurement of the positional co-ordinates of a particle is accompanied not only by a finite change in the dynamical variables but also the fixation of its position means a complete rupture in the causal description of its dynamical behaviour, while the determination of its momentum always implies a gap in the knowledge of its spatial propagation. Just this situation brings out most strikingly the complementary character of the description of atomic phenomena which appears as an inevitable consequence of the contrast between the quantum postulate and the distinction between object and agency of measurement, inherent in our very idea of observation.*

This is a clear formulation of the notion that complementarity resolves the contradiction between the quantum postulate (non-separability) and the idea of observation.

People have found it difficult if not impossible to fit all of Bohr's statements on complementarity into a coherent account because they cover such a wide scope. In trying to provide a way of analyzing quantum phenomena in terms of classical concepts he left ways open for misinterpretations. For example, Einstein [66] never overcame his misgivings about "...Bohr's principle of complementarity, the sharp formulation of which, moreover, I have been unable to achieve despite much effort which I have expended on it."

3.3.2 The Origin and Meaning of the uncertainty principle

The notion of uncertainty was introduced in Heisenberg's seminal paper of 1927 [32]. In this he did not write of an uncertainty principle, his intention was to make clear that the uncertainty relation he had developed was fundamental in an understanding of quantum mechanics. He saw it as a way of expressing the physical manifestation of the canonical commutation relation for conjugate pairs of quantities, $\mathbf{QP} - \mathbf{PQ} = \imath\hbar\mathbf{I}$: he wrote,

> "It is shown that canonically conjugate quantities can be determined with a characteristic inaccuracy".

He considered that these algebraic expressions should form the basis of the formalism of quantum mechanics. In this he was following Einstein's dictum that it was the mathematical theory that determined what could be observed.

In the 1927 paper [32] and the 1930 Chicago lecture notes [33] of Heisenberg, it is possible to identify three conceptually distinct types of uncertainty relation. Each one can be interpreted as a manifestation of the principle of uncertainty as outlined in Chapter 1 page 5.

His investigation, initially, highlights three features of quantum phenomena: the discrete nature of quantum objects, the discontinuity of quantum processes and the disturbance through measurement. Heisenberg (and Bohr, section 3.3.1) realized that the measurement process in quantum mechanics raised the question of how the act of observation affects the system being observed. He writes, [32],"...a definite experiment can never give exact information on all quantum theoretical quantities. Rather, it divides physical quantities into 'known' and 'unknown' (or more or less accurately known) quantities in a way characteristic of the experiment in question."

In the illustration of the 'disturbance through measurement' Heisenberg uses the language of his training in classical physics and developments in the interaction between waves and matter such as, the photoelectric equation and the Compton effect and the matter waves of de Broglie, to propose *the gamma ray microscope* thought experiment. In this experiment a gamma ray, classical microscope is used to locate the position of an electron, "At the instant when the position [of the electron] is determined - therefore, at the moment when the photon is scattered by the electron - the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed - that is, the more exact the determination of the position."

Here is a particle, the object system, the electron, suffering an exchange of momentum with a quantum system, the probe, the photon and receiving from it uncertainty in its subsequent properties.

In the final part of his paper Heisenberg wrote this: "Because all experiments are subject to the laws of quantum mechanics, [and therefore to the position-momentum uncertainty relation], it follows that quantum mechanics establishes the final failure of causality." Nevertheless, Heisenberg and others after him tried to identify a mechanical cause for the validity of the uncertainty relation. The attempt to link the uncertainty relation to momentum kicks was immediately criticized by Bohr who pointed out that the argument of kicks did not explain the momentum uncertainty and considered the quantum nature of the measurement probe as a relevant factor enforcing the uncertainty relation in measurements. Heisenberg expressed his agreement with this in the 'Addition in Proof' in [32].

3.3.3 Conflations around Complementarity And Uncertainty

Recalling the comments in Section 3.2.6: The notions of complementarity and uncertainty can be shown to be linked in certain formulations of quantum mechanics.

However, a selective interpretation of the quotations in section 3.3.1 and of the models used by Bohr and Heisenberg have lead to the conflation of complementarity and uncertainty. Random momentum kicks are the mechanism often pointed out as enforcing complementarity.

Bohr and Heisenberg

Heisenberg was led to accept, after discussions with Bohr, that complementarity is a fundamental feature of quantum mechanics that highlights a phenomenon of the microscopic, physical world. Complementarity is manifested in wave-particle duality and wave-particle duality must be accepted as an essential component of the interpretation of the theory. In a 'Addition to proof' to [32] he wrote,

> "...the uncertainty in our observation does not arise exclusively from the occurrence of discontinuities but is tied directly to the demand that we ascribe equal validity to the quite different experiments which show up in the corpuscular theory on one hand and in the wave theory on the other."

Bohr also pointed out the classical nature of the microscope in Heisenberg's gamma-ray microscope thought experiment. Again, in the 'Addition to Proof' of [32] Heisenberg writes,

> "...[Bohr has brought to my attention] ...In the use of an idealized gamma-ray microscope for example, the necessary divergence of the bundle of rays must be taken into account. This has as one consequence that in the observation of the position of the electron, the direction of the Compton recoil is only known with a spread which leads to [the uncertainty relation]."

This awareness was included in his 1929 Chicago lectures (published as [33] in 1930) where he pointed out that, "It is characteristic of the foregoing discussion that simultaneous use is made of the deductions from the corpuscular and wave theories of light, for, on the one hand, we speak of resolving power, and, on the other hand of photons and recoils resulting from their collision with the particle under consideration."

While Heisenberg thus acknowledged complementarity, Bohr, in turn, also ascribes an independent rôle to the uncertainty relation. In his 1927 Lake Como lecture Bohr indicated that it was possible to conceive of a 'softening' of the strict mutual exclusivity in the simultaneous use of complementary terms but that there is a price to pay. This being expressed in Heisenberg's uncertainty relation ([5], p. 582) (see Section 3.3.1).

So, in two of the seminal papers on the interpretation of quantum mechanics, we see Heisenberg introducing uncertainty and endorsing complementarity manifested in wave-particle duality and Bohr introducing complementarity and describing Heisenberg's uncertainty relation as the condition for weakening strict mutual exclusivity.

In spite of this endorsement of each others ideas there was a major conceptual difference between Bohr and Heisenberg. There was no disagreement between them about what had been observed in experiments nor about the mathematical formalism. They also agreed that any interpretation of quantum phenomena must be based on the use of classical language. Heisenberg summarized Section 1 of his 1927 paper thus (p. 68, [32]), "All concepts which can be used in classical theory for the description of mechanical processes can also be defined exactly for atomic processes in analogy to the classical concepts." And, Bohr wrote near the start of his 1928 paper, (p. 580, [5]), "…our interpretation of the experimental material rests essentially upon classical concepts."

Heisenberg's theme in the main part of his paper is concerned with how the theory determines what can be observed, i.e. position, time, momentum, energy, and describing how these "observations are correlated" ([32], final paragraph). Where the language of particles gives the clearest description, this is what he uses, similarly for the language of waves. Both modes of description are used independently of the other. In his Chicago lectures of 1929 [33] he points out clearly the *equivalence* of the particle picture and the wave picture.

Bohr insists that both modes of description *must* be used. It is the discontinuity in changing from one mode of description to a complementary one which gives rise to uncertainty in what is being observed. Heisenberg (reluctantly) accepts that it is the dual aspect of the photons that enforces the validity of the uncertainty relation; whereas Bohr speaks of change of mode of description, Heisenberg's measurement examples demonstrate how the results of one measurements (eg showing fairly sharp momentum) are invalidated or changed by a subsequent measurement of position. So, there is not just a change of description but the choice of or transition to a new wave function or state vector to reflect the change of knowledge about the system.

In an article published posthumously in a collection of papers commemorating the fiftieth anniversary of the formulation of the uncertainty principle (opening article in [57], 1977) Heisenberg recalls the conceptual departure points with Bohr however he points to an area of agreement thus,

> "Still we were not able to get complete clarity; but we understood that the well defined experimental situation somehow played an important rôle ..."

Bohr and Einstein

Einstein's first challenge to Bohr's interpretation was presented at a conference in Solvay (1927). Einstein proposed that one should be able to obtain information on which slit a particle passed through while retaining an interference pattern.

Using the Heisenberg position-momentum uncertainty relation Bohr showed that if the plate carrying the double slit was regarded as a quantum mechanical object, the uncertainty in measuring its momentum is exactly equal to the reciprocal of the number of fringes per unit length on the screen in the far field. Any momentum determination sufficiently accurate to decide the position of the particle at the double slits involves a position uncertainty of those double slits which is the same order of magnitude as the fringe spacings of the interference pattern thus washing out the interference pattern and any momentum information it contains. (Bohr wrote of this in [66] in a report on his discussions with Einstein.)

In 1935 Einstein, Podolsky and Rosen asked, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" [24]. They presented a thought experiment suggesting that position and momentum should, simultaneously, be elements of reality. Since quantum mechanics cannot account for position and momentum simultaneously, as embodied in the position-momentum uncertainty relation, they argued that quantum mechanics was incomplete.

Again Bohr's reply [6] applied wave-particle duality and the position-momentum uncertainty principle to this, "simple case of a particle passing through a slit in a diaphragm". The reasoning leading from the quantum formalism to the disappearance of the interference fringes proceeds as follows: "The width of the slit ... may be taken as the uncertainty in the position of the particle relative to the diaphragm" then "...

it is simple to see from the de Broglie relation between momentum and wavelength that the uncertainty in momentum is correlated to [the width of the slit] by means of Heisenberg's general principle $\Delta p \Delta q \sim h$ which in the quantum formalism is a direct consequence of the commutation relation for any pair of conjugate variables." ([6] p. 697)

Returning to the paper by Scully et al [60] published in 1991 (see Section 3.2): in it the 'disturbance doctrine' (a phrase used by Brown and Redhead [7]) is described as a model dating back to the origins of quantum mechanics which can be shown to be inadequate in the light of their scheme. SEW's proposal is that the principle of complementarity may be manifest although the position-momentum uncertainty relationship driven by classical or semi-classical momentum kicks plays no rôle. They are led to write, "...we have conceived a *welcher Weg* detector which does not fall prey to the position-momentum uncertainty relation." They even speak of the uncertainty relation being an obstacle reminiscent of the view taken by EPR [24], who presented the uncertainty relation as a limitation of the quantum mechanical description of physical reality. EPR hoped to circumvent this limitation in their thought experiment.

Thus was revived a polemic about the question of a hierarchy of quantum principles: Is complementary more fundamental than uncertainty?

Using the Heisenberg's uncertainty relation Bohr showed that if the plate carrying the double slit was regarded as a quantum mechanical object one might conclude that the uncertainty relation provided the foundation for complementarity. This seems to be a widely accepted view. However, Bohr's considerations, like those of Heisenberg, were based on a very informal use of quantum structures and outlines of semi-classical thought experiments, without providing a rigorous quantum mechanical treatment. In addition, it should be borne in mind that any number of examples of experiments, thought or realized, can only serve as illustration of a fact but not as a conclusive demonstration of its general validity. There is thus scope for doubt over the validity or generality of the claim that the uncertainty relation constitutes the sole mechanism for enforcing complementarity. This is the purpose of Chapters 4 and 5, to put the discussion on a more rigorous basis and to clarify the question of their being a hierarchy of quantum principles.

Textbook survey

If some features of quantum mechanics have continued to be a source of misconceptions and the cause of confused debate, how are the particular modes of thinking perpetuated? Science has the reputation of dealing with objective reality however as their training progresses scientists become part of a community which is bounded by common views or approaches. A brief and by no means exhaustive nor comprehensive survey of text books spanning some seven decades identifies the support, the authorities and the early influence for some commonly held ideas.

It is possible to identify three distinct approaches towards presenting complementarity and uncertainty in the teaching of quantum mechanics:

(1) The first group regards complementarity as the most important feature of quantum mechanics

In the quantum mechanics text of Julian Schwinger (edited by one of the authors of SEW), the existence of mutually exclusive properties is described as the essence of quantum mechanics; by contrast, the importance of the uncertainty principle is played down: "... we will prefer to speak of Heisenberg's uncertainty *relation*." ([59], p.110).

In a similar way, Asher Peres ([56], p.149) introduces Bohr's complementarity principle in the version of 1935, formulated in Bohr's reply to the Einstein-Podolsky-Rosen paper; Peres paraphrases it as follows: "Its meaning is that some types of predictions are possible while others are not, because they are related to mutually incompatible tests." The index of Peres's book has only this one page reference to the term *complementarity principle*. Likewise, there is only one page reference for the term *uncertainty principle*: in the index, on page 445, the reader is referred to page 445 (!) for this term. In this way Peres removes from quantum uncertainty the status of a principle while there are numerous occurrences and applications of uncertainty *relations* in his book.

(2) The second group avoids using the term complementarity at all, emphasizing uncertainty instead.

Ballentine [1] emphasizes the statistical nature of quantum mechanics and the non-reproducibility of quantum events being due to an indeterminacy in the preparation and measurement but gives no reference to complementarity or credit to Bohr for this concept.

Volume 3 of the Feynman Lectures on Physics ([29], Section 1.1) gives great prominence to the phenomenon of interference of electron waves, or more generally, the wave-particle duality. Feynman says about this phenomenon that it, "... has in it the heart of quantum mechanics. In reality, it contains the only mystery." However, he does not mention complementarity.

In this volume readers are introduced to the position-momentum uncertainty principle being enforced by disturbance, "...when we look for certain phenomenon we cannot help but disturb in a certain minimum way and the disturbance is necessary for the consistency of the viewpoint", to explain the disappearance of interference on path detection in a two slit experiment.

Here it is worth noting that 'disturbance' can be understood in different sense, as momentum kicks, as entanglement, or as collapse, or, in the most general sense, as any form of state change enforced by a measurement.

Similarly, Sudbery ([71], p.25) describes briefly the wave-particle duality and the mutual exclusion of arrangements for wavelike phenomena and particle-like phenomena, and proceeds to quote the uncertainty principle as "a quantitative statement of these ideas."

Three founding fathers who had difficulty in accepting complementarity were Schrödinger, Dirac and von Neumann. W. Moore in *Schrödinger: Life and Thought* ([52], p.228) writes of Schrödinger, "[After long conversations with Bohr about atomic processes] Schrödinger recognizes the necessity of admitting both waves and particles but he never devised a comprehensive interpretation of quantum phenomena to rival the Copenhagen orthodoxy. He was content to remain unbeliever."

And Dirac ([20], p.1) writes, "Heisenberg's principle of uncertainty shows clearly the limitations in the possibility of simultaneous assigning numerical values, for any particular state, to two non-commuting observables, when these observables are a canonical coordinate and momentum, and provides a plain illustration of how observations in quantum mechanics may be incompatible."

He does not mention complementarity nor wave-particle duality however in illustrating uncertainty he writes "...in the limit when either the canonical coordinate or momentum is completely determined, the other is completely undetermined". A similar presentation of complementarity is made in chapter 4 as value complementarity.

J. von Neumann. [72] gives a detailed analysis of a Heisenberg type microscope experiment but does not mention complementarity nor wave particle duality.

(3) The third group embraces both terms, presenting complementarity and uncertainty as being co-existent features of quantum mechanics.

It will be no surprise that Heisenberg [33] gives an account which includes the rôle of disturbance: "...There are no infinitesimals by the aid of which an observation might be made without appreciable perturbation..."; complementarity is described as: "...the incompatibility of space-time description and laws of causality of atomic processes..."; wave-particle duality as follows: "...wave and corpuscular pictures possess equal approximate validity..." and uncertainty; "...indeterminateness of the picture of the process is a direct result of the indeterminateness of the concept of 'observation' - what objects are part of the observed system and what are part of the observer's apparatus."

W. Pauli, *General Principles of Quantum Mechanics* ([55], pp. 1-7) also follows this line: "...Due to the indeterminacy in the property of the system prepared in a specific manner (i.e. in a definite state of the system), every experiment for measuring the property concerned destroys (at least partly) the influence of a prior knowledge of the system on the (possibly statistical) statements about the result of a future measurement."

A. Messiah, *Quantum Mechanics Vol 1* ([49], Ch. 1), introduces a mutually exclusive form of complementarity, "evidences obtained under different experimental conditions cannot be comprehended within a single picture ...they must be regarded as complementary in the sense that only the totality of the observational results exhausts the possible types of information about the objects of microscopic physics." But he does not leave students with a mutually exclusive picture, a graded form of complementarity with a trade-off is described, "... The description of the physical properties of microscopic objects in classical language requires complementary variables; the accuracy in one member of the pair cannot be improved without a corresponding loss in the accuracy of the other member."

L.I. Schiff, *Quantum Mechanics* ([70], p.8) tries to illustrate a link between uncertainty and complementarity, "In order to understand the implications of the uncertainty principle in more physical terms, Bohr introduced the complementary principle ... elements that complement each other to make up a complete classical description are actually mutually exclusive ... the physical apparatus available has such properties that more precise measurements than those indicated by the uncertainty principle cannot be made."

Comments.

This brief survey reflects a certain unwillingness among many authors to assign equal weight or importance to the principles of complementarity and uncertainty. The ones who show reluctance to use complementarity may feel that it is ill defined and so steer their readers in more well defined areas of quantum mechanics. It is one of the aims of this study to show that there are clear formulations of complementarity.

The authors who show a preference for complementarity over uncertainty perceive that the uncertainty principle is a consequence of the formalism and nothing more. On the other hand complementarity describes the mutually exclusivity found in atomic physics. The rôle of the uncertainty relation is to quantify this as a trade-off and can therefore, according to this point of view, be regarded as secondary.

I hope to make clear it that complementarity can be regarded as a limiting case of uncertainty however, uncertainty can offer somewhat more. If joint measurements are considered instead of the strict no-go of complementarity, uncertainty can be shown to play an independent rôle.

Finally, in my trawl through textbooks I found this plea from E. Merzbacher, *Introduction to Quantum Mechanics*" ([50, **?**]: "Might we not be better off if we shed all pretext of making pictures of the quantum phenomenon in terms of particles and waves and the like. Why not simply establish suitable mathematical laws for the description of the observations, as Newton urged for a branch of physics reaching maturity."

3.4 Explorations of measures of uncertainty, uncertainty relations and complementarity in the 1980s

The myth that the concept of complementarity has been scrutinized and formalized only recently is neatly captured by Luis [48];

> "...this remarkable effort has been mainly developed in the past decade".

In many papers in the 1990s, schemes were proposed, some of which were implemented, to test the principle of complementarity or more specifically to investigate decoherence, in the sense of the disappearance of interference in interferometers with which-way detectors. In some cases there was a specific agenda to link decoherence and hence complementarity with position-momentum uncertainty and random momentum transfer or complementarity with entanglement or complementarity with superposition. At the same time much of the relevant work on the foundations of quantum mechanics done in the 1960s-1980s was ignored.

Lack of awareness of this work perpetuated the belief that within the principal features of quantum mechanics there exists a hierarchy of dependency or of importance. Many authors, unaware of the possible generalized formalizations of quantum mechanics, seemed to pay attention only to the position-momentum uncertainty and mutually exclusive wave-particle duality.

Following is a review of a selection of papers from the 1980s. If these papers had been appreciated and developed, articles claiming 'the death of uncertainty' might have been avoided.

3.4.1 Wootters and Zurek: demonstration of the coexistence of path and interference.

Wootters and Zurek ([77] 1979) made a full quantum mechanical model analysis of the thought experiment of Einstein which comprised a Young's double slit interference demonstration in which the plate carrying the double slit is suspended on a weak spring. They show that there is the possibility of the coexistence of path information and an interference pattern and that the position-momentum uncertainty relation is not sufficient to substantiate Bohr's claim that determination of path will completely wash out the interference pattern. This was also confirmed in a different approach by Hilgevoord and Uffink, these will then be discussed in later subsections (3.4.3).

Wootters and Zurek use Heisenberg's uncertainty principle to study the question, "To what extent is the interference pattern smeared out if we insist on determining the path of each photon with a given accuracy?"

They analyzed a situation in which the plate containing the first single slit is represented both in the position x and wave-vector k by a Gaussian function. When the plate receives a kick from the photon it can either be stopped and its position measured or its momentum be measured, as Einstein proposed. Each of these schemes gives a way of subdividing the original ensemble of photons either according to the measured position of the plate or according to its measured momentum. In the first case each sub-ensemble produces a perfect but differently shifted interference pattern and no path information. In the second case each sub-ensemble produces a smeared out interference pattern but also gives some information about the photons' paths.

Sub-ensembles of photons are considered, specifically those which on leaving the single-slit plate can be associated with a definite measured momentum of the plate, k. For a plate momentum eigenstate, $|k\rangle$ each photon in the associated sub-ensemble is correlated to a biased superposition of path states, in the sense that one path is very much more likely than the other, e.g. $|k\rangle \otimes (\sqrt{(p_1(k))}|slit_1\rangle + \sqrt{(p_2(k))}|slit_2\rangle)$ where $p_j(k)$ is the probability of the photon passing through slit $j = 1, 2$ and $p_1(k)+p_2(k)) = 1$. By considering increasing values of k, making the ratio of $p_1(k)) : p_2(k))$ sufficiently large, it would be possible to make one path more likely than the other. However, the superposition of paths still exists and gives rise to an interference pattern.

Wootters and Zurek found that, "In Einstein's version of the double slit experiment a surprisingly strong interference pattern can be retained by not insisting on a 100% reliable determination of the slit through which each photon passes." (p. 473) If the ratio of the probabilities of a photon going through either slit is 99:1 the interference pattern is not destroyed. Under this conditioning they found that the maximum to minimum ratio of the intensities of the interference pattern is approximately $\frac{3}{2}$. Despite the fact that one can predict with 99% certainty the paths of the photons, they still exhibit strong wave-like properties. These values are compatible with the duality relation (3.2.13) of section 3.2.2 as $P = .99 - .01 = .98$, $V = (1.5-1)/(1.5+1) = .2$, and so $P^2 + V^2 = 1$.

It is tempting to make statements such as 99 particles out of a 100 go through one slit but this is misleading because the paths are indeterminate. Any determination of which 99 particles go through a slit will result in the obliteration of the interference pattern.

Of complementarity Wootters and Zurek write,

> "The complementarity principle does not prevent photons from behaving once as waves and once as particles. It only states that the same photon should not reveal this 'split personality' in the same experiment." (p. 476)

Their results appear to show that it is possible to replace the mutual exclusivity of strict complementarity with a graded version, governed by appropriate trade-off relation. However, the limiting cases illustrate that there is no violation of strict complementarity. If there is no determination of the path of the photon, then $p_1(k) = p_2(k)$ and the result is a perfect interference pattern. In the other limiting case where the path of the photon is determined completely i.e. $p_j(k) = 1$ for one of $j = 1, 2$, the superposition of paths does not exist and no interference pattern is observed.

As illustrated on page 66, Bohr anticipated the findings of Wootters and Zurek with his "unsharpely defined individuals in finite space-time regions". It seems remarkable that despite Bohr's anticipation and Heisenberg's joint measurement schemes, unsharp wave-particle duality, as discovered by Wootters and Zurek and further investigated by Mittelstaedt et al [51] and by Greenberger and Yasin [30] and others since the 1980s, was perceived as surprising and intriguing.

The contribution of this paper was recognized by Englert, [25], 3.2.2, Jaeger et al, [39], and by Greenberger and Yasin [30]. Englert credits Wootters and Zurek specifically with the approach that turns which-way knowledge into a number; Jaeger et al make an important general point about the development and understanding of quantum mechanics when they write, "Wootters and Zurek . . . initiated the study of arrangements intermediate between the two extremes considered by Bohr". Furthermore, Wootters and Zurek's method, which makes crucial use of entanglement, anticipated a new set of ideas introduced later under the heading of 'quantitative quantum erasure' [65], [60], [26]. This can be seen if what was done by Wootters and Zurek is summarized: choosing a sub-ensemble of photons entangled with a particular momentum state of the first, single slit screen, in effect marks, unsharply, the path taken by just those objects in the sub-ensemble. The interference pattern contrast is degraded by requiring more precision in the path information i.e. by selecting a sub-ensemble of photons entangled with a higher momentum state of the screen.

Wootters and Zurek did more than analyze Einstein's thought experiment; they proposed an interference-path detector that could be implemented while recognizing the practical difficulties of detecting the momentum transfer between a photon and the single slit plate.

They proposed replacing the detector screen with a stack of thin, non-transparent, double-sided photon detectors to be placed parallel to the optical axis of the system. The thin edge of each of the detector plates is placed at the maxima and minima of the interference pattern. Photons from the upper (lower) slit will be detected on the upper (lower) surfaces of the detector stack.

Inspired by the analysis of Wootters and Zurek, but choosing to modify an optical discrimination method of path detection proposed by Wheeler [73], Bartell [3] proposed two more methods of observing path and interference. One observes a Fresnel interference pattern with path detection at either side of the pattern. The other has orthogonal polarizers over each slit; path detection and interference are recorded by a rotatable polarization analyzer and detector. Both are designed to show the limiting cases of either path or clear interference and the continuity of situations in which there is partial path detection and a recognizable interference pattern.

3.4.2 Mittelstaedt, Prieur and Schieder, and Greenberger and Yasin

Mittelstaedt, Prieur and Schieder ([51], 1987) proposed and realized a two path experiment in which simultaneously the path of a photon and interference pattern can be measured in an approximate sense, in the sense of unsharp observables.

Their set up is a Mach-Zehnder interferometer in which the transparency of the final beam splitter mirror, together with the contrast of the interference pattern gives some information about the photon's path. When beam splitters are set to give an effective transparency of nearly 0.5 the photons were prepared so as to give an interference pattern with almost maximum contrast. However, from the small departure of the effective transparency from 0.5 it is possible to calculate a small amount of path information.

With an effective beam splitter transparency of nearly 1 the photons have a high value of particle property and yet a very low contrast interference pattern can be observed.

Mittelstaedt et al explore the trade-off between path knowledge and interference contrast using entropy or information theoretic considerations much in the manner of Wootters and Zurek [77] and Deutsch [19].

However, Mittelstaedt et al show no awareness of the trade-off relation $P^2+V^2 \leq 1$ where P is path predictability and V is interference pattern contrast. Nevertheless their findings can be interpreted as a demonstration of the relationship (3.4.3) discovered by Greenberger and Yasin [30]. ([30] was published some months after [51].)

In 1988 Greenberger and Yasin [30] showed the possibility of simultaneous wave and particle behaviour in neutron interferometry. In its design their experiment is analogous to a double slit experiment in which one slit is wider than the other making one path more likely than the other. This is achieved by placing an absorber in one of the Bragg scattered beams.

Their model predicts that if 99% of the neutrons in one path are absorbed, the contrast of the interference pattern would be about 20% indicating that a superposition of paths was still present. These values are the same as those discussed by Wootters and Zurek.

In the first development, Greenberger and Yasin assume that the two beams are coherent. If after passing through the interferometer one beam has an amplitude a and the other has an amplitude b the recombined beams can be written as $\psi = (a\exp(\imath k_x x) + b\exp(\imath\phi)\exp(-\imath k_x x))\exp(\imath k_z z)$ where k_x and k_z are determined by the scattering conditions and ϕ is the phase difference between the beams. The intensity of the interference pattern is given by $|\psi|^2 = a^2 + b^2 + 2ab\cos(2k_x x + \phi)$ and the contrast in the pattern is $V = 2ab/(a^2+b^2)$. Now letting $a = R\cos\beta$ and $b = R\sin\beta$ allows the contrast to be written,

$$V = \sin 2\beta. \tag{3.4.1}$$

The strategy adopted for measuring the path information is to guess or predict that all the particles will be in the most intense beam. Then the success of this prediction is compared to that obtained if the observer has no knowledge of the beams' populations (probability of being in beam 1 or 2 is 0.5).

Suppose that $|a|^2 > |b|^2$ then guessing that that all of the particles will be in the 'a' beam will be correct a fraction, $|a|^2/(|a|^2 + |b|^2)$, of the guesses. Comparing this fraction with the 'no knowledge probability' gives a (normalized) measure of the particle nature, P, of the beam.

$$P = \frac{|a|^2/(|a|^2 + |b|^2) - 0.5}{0.5} = \cos 2\beta \tag{3.4.2}$$

Jaeger et al, [39], section 3.2.2 describe this concept of path distinguishability as "...the same as ours but in a restricted range of preparations ...".

Greenberger and Yasin thus obtained the relationship between path knowledge and interference in the form,

$$P^2 + V^2 = 1. \tag{3.4.3}$$

The relation is controlled by a single parameter, β for a pure state. Thus interference pattern versus path predictability is not governed by an mutual exclusivity but by a trade-off relationship that varies continuously from full wave behavior to full particle behaviour.

3.4.3 Hilgevoord and Uffink

During the 1980s Hilgevoord and Uffink [34], [35], [36],investigated uncertainty relations in quantum mechanics. They examined the rôle and limitations of standard deviations in quantum mechanical uncertainty relations and in conjunction with the principle of complementarity.

"Standard deviations as commonly used in the Heisenberg uncertainty relation are not always a reasonable estimate of the overall width of a wave function. They tend to give an overestimate of this width so that the lower bound in the Heisenberg uncertainty relation does not imply a lower bound on the widths of the position and momentum wave functions."

Their aim is to find measures other than variance which are perhaps better suited to describe the width of a wavefunction, in particular those wave functions which can be used to describe a double slit experiment. In this set-up (Fig. 3.4.3) a wave function, $\psi(q)$, with a characteristic overall width, W_ψ, is required to describe the passage of a particle through the slit. The fine structure of the waveform $\psi(q)$ is referred to as the mean peak width, w_ψ.

$\psi(q)$ has a Fourier transform, $\phi(p)$. This, represented as an interference pattern in the far field, also has a well recognized overall width, W_ϕ (a single slit diffraction envelope) and the fine detail of the maxima and minima having a mean peak width w_ϕ.

The overall width, W, which is an exterior measure of the overall extension of the wave packet, is the smallest finite interval width on q for which

$$\int_{q_0-W/2}^{q_0+W/2} |\psi(q)|^2 dq = N \qquad \text{for some} \quad q_0 \in \mathbb{R} \tag{3.4.4}$$

If N is chosen to be close to one, W is the width of the smallest interval inside of which is the main part of the total probability.

For a wave function $\psi(q)$ the mean peak width or an interior measure of the fine structure of a wave packet, w, can be understood in this way: considering

$$\left| \int \psi^*(q)\psi(q-w)\, dq \right| = M, \tag{3.4.5}$$

w is the minimum width for which a chosen value of M can be achieved, M is chosen to be close to but smaller than one.

If the system is prepared in a state $\psi(q)$ and a measurement of its position is made. The probability that this yields a value q_0 is the same as the probability of yielding a value of $(q_0 - w)$ if the system is prepared in a state $\psi(q - w)$ i.e. all values of the outcome of a position measurement are (almost) equally probable in the interval w. No peaks or fine detail will be observed in this interval.

Having to choose suitable values of $N, M \in (0, 1)$ seems to introduce an arbitrariness in quantifying the spread and fine structure of a wave-function however Hilgevoord and Uffink found a trade-off relationship, valid for all Fourier pairs, between the overall width, W_ψ of $\psi(q)$ and the mean peak width, w_ϕ, of its Fourier transform $\phi(p)$. (In the three papers of Hilgevoord and Uffink they let $\hbar = 1$. The same convention will be adopted in this review of their work.)

$$W_\psi w_\phi \geq 2[2(2N - M + 1)/N]^{\frac{1}{2}} \qquad \text{if}\, M \leq 2N - 1. \tag{3.4.6}$$

One must have $N > \frac{1}{2}$ in order to satisfy the constraint $1 \leq 2N - M$ under which trade-off relation 3.4.6 is valid.

If it is accepted that complementarity as the existence of pairs of observables such that if one is certain the other is completely uncertain and vice-versa i.e. value complementarity then relation (3.4.6) looks like a graded complementarity trade-off relation rather than an uncertainty relation claimed for it by Hilgevoord and Uffink. Uncertainty is present in an observable when the given state is not an eigenstate or a mixture of eigenstates of the observable.

Turning to the double slit experiment in which the two slits have a width $2a$ and

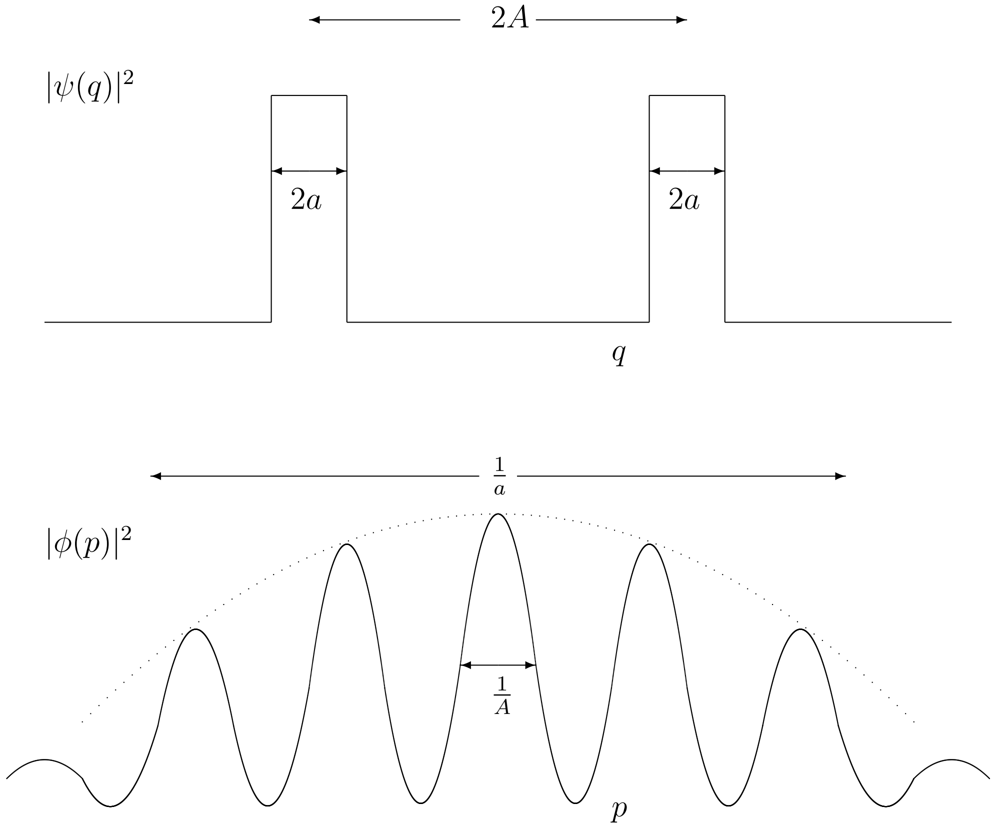

Figure 3.2: The probability density of position and momentum at the screen for a double-slit experiment.

are separated by $2A$, $A \gg a$. The appropriate wave-functions are,

$$\psi(q) = \begin{cases} (4a)^{-\frac{1}{2}} & \text{for } A - a < |q| < A + a \\ 0 & \text{elsewhere} \end{cases} \tag{3.4.7}$$

and its Fourier transform

$$\phi(p) = \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \cos(Ap)\frac{\sin(ap)}{ap}. \tag{3.4.8}$$

The probability density of momentum, $|\phi(p)|^2$, has the same shape as the interference pattern intensity.

In these two functions, two different widths can be identified; the overall width of the double slits, $\psi(q)$ is $W_\psi \approx A$ and it is this which determines the mean peak width or fringe separation of $\psi(p)$, $w_\phi \approx \frac{1}{A}$. If the slit width increases the fringe separation decreases.

The fine structure measure or mean peak width of the slits is of the order of a, $w_\psi \approx a$ and there is an inverse relationship between this and the overall width of the diffraction pattern, $W_\phi \approx \frac{1}{a}$.

Thus a trade-off relationship exists between the overall width of the slits and the fringe separation and also between the slit width and the overall width of the interference pattern.

If the position-momentum uncertainty relation, $\Delta p \Delta q \geq \frac{\hbar}{2}$ is considered for this double slit arrangement the following is found; from equation 3.4.7 $\Delta q \approx A$ but from equation (3.4.8) $\Delta p \to \infty$. Thus standard deviation is not an adequate measure of width for a general expression of uncertainty in this double slit experiment. So, an attempt at answering whether it is possible to distinguish through which slit a object passes cannot be based on the position-momentum uncertainty relation for standard deviations.

Hilgevoord and Uffink turn there attention to Einstein's question as to whether it possible to determine through which slit each of the particles passed and also preserve some interference pattern (see sections 3.3.3 and 3.4.1). Which way the particle went is decided by the recoil of some part of the apparatus. Wootters and Zurek chose to measure the recoil of the first single slit whereas Einstein suggested measuring the recoil of the double slit. Hilgevoord and Uffink choose to measure the recoil of the detector screen in the far field assigning to it a momentum wave-function, $\chi(p)$ and its Fourier transform the position wave-function, $\tilde{\chi}(q)$.

The difference in momentum of the screen caused by the impact of a particle from one slit or the other is of the order Ap_0/l where $2A$ is the slit separation, l is the distance from the screen to the double slits and p_0 is the momentum of the particles arriving at the double slits.

This recoil could be measured if the initial momentum of the screen is known with an uncertainty $\delta p \lesssim Ap_0/l$. If standard deviation is taken as the measure of uncertainty then the standard deviation of $\chi(p)$ is $\Delta p \lesssim Ap_0/l$. It follows from the position-momentum uncertainty relation that the standard deviation of $\tilde{\chi}(q)$ is $\Delta q \gtrsim l/Ap_0$.

Does this imply that the interference pattern is washed out? If the wave-function $\tilde{\chi}(q)$ is considered, from the definition of standard deviation, $(\Delta q)^2 = \int q^2|\tilde{\chi}(q)|^2 dq - \left(\int q|\tilde{\chi}(q)|^2 dq\right)^2$, even if the wave-function is concentrated at $q = q_0$, if it has 'long tails' these will dominate the standard deviation even though they contain only a small fraction of the total probability. If this causes the standard deviation to be very large then it cannot be inferred from this that the interference pattern disappears.

Hilgevoord and Uffink return to their relationship (3.4.6) to show that Bohr, in his consideration of the overall width of the momentum and position distribution, was correct but for the wrong reason. They require that the overall width of $\chi(p)$ is $W_\chi \lesssim Ap_0/l$ for $N \lesssim 1$ ($M \lesssim 1$). For these choices of N and M, the trade-off relation, (3.4.6) entail that $w_{\tilde{\chi}(q)} \gtrsim l/Ap_0$. From the definition of the mean peak-width all values of q in the interval l/Ap_0 are approximately equally probable and there can be no peaks or fine detail in the interval, hence the interference pattern disappears.

Hilgevoord and Uffink have shown that the usual uncertainty relation for standard deviations for momentum and position is *not* sufficient to entail complementarity in the sense of decoherence being forced by obtaining path knowledge. They have given another subtle form of uncertainty relation which does entail complementarity.

3.5 Modern formulations of complementarity and uncertainty

It is possible to trace the development of the formalization of complementarity back several years. Value complementarity was discussed by Schwinger in the 1960s [59] and by Kraus in 1987 [43]. The term value complementarity was coined by Lahti and Busch (page 105 [12]). Probabilistic complementarity and measurement complementarity were discussed by Lahti in the 1980s as reviewed in [12] pp. 104-108.

Nevertheless, in some papers and texts reviewed here, the meaning of complementarity, in the sense of both of a pair of complementary observables being necessary for complete description, has become overshadowed by the restriction provided by the mutually exclusivity in complementarity. Emphasis has fallen on the inability to assign sharp values to both observables simultaneously. This view extends to experimental setups such as the detection of path and interference in the same experiment.

The positive, mutual completion, aspect, which can be found in Bohr's writing (page 61) indicates that, initially, the positive aspect was the primary meaning of complementarity. This meaning has now been subsumed into the notion of uncertainty.

However, within the restriction of mutually exclusive setups it is possible to identify three distinct forms of complementarity: one has implications for the possibility of state preparation, another refers to the possibility of joint measurement and the third to the possibility of sequential measurements. These distinctions allow the investigator to choose where to put the Heisenberg cut between preparation (the object system) and the measurement system (the probe).

Complementarity can be understood to be a relationship that identifies certain pairs of observables including but not restricted to, canonically conjugate pairs observables. There is obviously considerable freedom in formalizing, and thereby specifying, the broad idea of complementarity. A variety of more specific formalizations are reviewed in [12].

Here a summary is given of forms complementarity and uncertainty which are instrumental for subsequent considerations.

3.5.1 Preparation complementarity

The most widely accepted form of preparation complementarity (e.g., [59, 43, 60, 47]) is the following one that will be referred to as value complementarity (following [12]): two observables are *value complementary* if whenever one has a definite value, the values of the other are maximally uncertain. A value of observable A is definite if it occurs with probability equal to one in a measurement of A, and the values of observable B are maximally uncertain if they occur with equal probabilities in a measurement of B.

Formally, the value complementarity of two observables A, B with discrete non-degenerate spectra and associated eigenbasis systems $\{\psi_k : k = 1, \ldots, n\}, \{\phi_\ell : \ell = 1, \ldots, n\}$ amounts to the statement that any two eigenstates have constant overlap, that is, the numbers $|\langle\psi_k|\phi_\ell\rangle|$ are independent of k, ℓ. Pairs of orthonormal basis systems with this property are called mutually unbiased. Examples of which are, the basis systems of the eigenvectors of σ_x and σ_z, $\{|+, x\rangle, |-, x\rangle\}$ and $\{|+, z\rangle, |-, z\rangle\}$

3.5.2 Measurement complementarity

Measurement complementarity can be specified to refer to a pair of observables A, B for which a sharp measurement of one of them makes any attempt at measuring the other one simultaneously or in immediate succession completely obsolete. The first form of measurement complementarity, the impossibility of joint measurements, is a special instance of von Neumann's theorem [72], according to which two observables are jointly measurable if, and only if, they commute. In a more specific, stronger form, the measurement complementarity of two observables A, B can be expressed as an exclusion relation for the quantum operations describing the state changes due to measurements of A and B (see [12], section IV 2.3).

Measurement complementarity in the latter case of sequential measurements will be taken to mean that due to the effect of a repeat measurement of A, the B measurement will not recover any additional information about the object (input) state immediately prior to the A measurement. If A and B are value complementary observables with mutually unbiased eigenbases the observable B' measured after a repeatable measurement of A is actually trivial in that its statistics carries no information about the system state prior to the A measurement. This is illustrated in the case of a repeatable measurement of σ_z followed by any measurement of σ_x. Using the notations of Chapter 2 section 2.5, the final state of the object and apparatus after a repeatable measurement coupling is,

$$\begin{aligned} |\Psi_f\rangle &= U|\varphi\rangle \otimes |\psi_0\rangle \\ &= \langle +, z|\varphi\rangle|+, z\rangle \otimes |\psi_+\rangle + \langle -, z|\varphi\rangle|-, z\rangle \otimes |\psi_-\rangle \end{aligned} \tag{3.5.1}$$

Where U is the unitary coupling operator, φ and ψ_0 are the initial states of object and apparatus, ψ_+ and ψ_- are the pointer states, and $|+, z\rangle$ and $|-, z\rangle$ are the eigenstates of σ_z associated with the eigenvalues +1 and -1, respectively.

Let $|\pm x\rangle$ denote the corresponding eigenstates of σ_x, then the probability of the outcomes of a subsequent measurement of σ_x are,

$$prob(\sigma_x = \pm, |\Psi_f\rangle) = \langle \Psi_f|(|\pm, x\rangle\langle \pm, x| \otimes I)\Psi_f\rangle = 0.5 \tag{3.5.2}$$

This can be written as $\langle \varphi|B_\pm\varphi\rangle = 0.5$ for all states φ, form which it follows that $B_\pm = \frac{1}{2}I$

Measurement complementarity is thus seen to reflect the fact that in quantum mechanics, every non-trivial measurement must alter the system's state, at least in the case of some of the input states. There can be no information gain without 'disturbance' in quantum mechanics. As will be discussed in the subsection 3.5.3, the related issue of a measurement of one variable disturbing the distributions of the other, noncommuting variable has been highlighted by Heisenberg by means of his famous thought experiments illustrating the uncertainty relation.

Value complementarity and the two versions of measurement complementarity (for joint or sequential measurements) will be seen at work in the discussion of the

MZ interferometry experiments; if the path observable is represented by σ_z, any of its associated interference observables, represented by $\cos\xi\,\sigma_x + \sin\xi\,\sigma_y$, is a complementary partner.

3.5.3 Three varieties of uncertainty

It is evident that Heisenberg considered measurements to produce (approximate) eigenstates of the measured observable corresponding to the measured value, and a careful reading of Heisenberg's 1927 [32] paper and his 1930 Chicago lecture notes [33] shows that he has in fact distinguished three variants of uncertainty relations.

Heisenberg brings together two versions of uncertainty relations: the uncertainty relation $\Delta(Q,\psi)\Delta(P,\psi) \geq \frac{\hbar}{2}$ for state preparations, according to which separate measurements of position and momentum have distributions with widths (standard deviations) satisfying this uncertainty relation; and a trade-off relation $\delta q \delta p \geq \frac{\hbar}{2}$ for the inaccuracies of joint measurements of these noncommuting observables.

Heisenberg did not have at his disposal a precise quantum mechanical notion of joint measurement of noncommuting observables, as POVMs were not available until several decades later to describe unsharp or approximate measurements. He does grapple with the notion of joint unsharp measurement and comes close to a solution by considering sequences of measurements. For example, he considers the diffraction of a matter wave at a slit and shows that if the particle's momentum was initially sharp, this precision of definition of momentum becomes degraded during the passage through the slit which effects an approximate localization of the particle. Considered as a sequence of a sharp momentum measurement followed immediately by an approximate position measurement, the outcome of the sharp momentum determination is thus seen to be modified into an unsharp momentum determination, due to the "disturbing" influence of the approximate position determination. The resulting inaccuracies in the definitions of position and momentum are shown to satisfy an uncertainty relation.

This version of uncertainty relation can be interpreted as a disturbance-accuracy trade-off relation for sequences of measurements. It has been carefully discussed in the context of interference experiments by Pauli in his 1933 review [55]. In the form described here, the disturbance of the distribution of an observable B through a measurement of A can be measured in terms of the variance of B in the state immediately after the (nonselective) A measurement operation, which is to be compared to the (near) zero variance of B in an initial (near) eigenstate. More generally, the disturbance of the distribution of B during a measurement of A should be described by some measure of the difference between the distributions of B before and after the A measurement. Interestingly, rigorous and general formulations of such disturbance uncertainty relations have been investigated only rather late (e.g., [43, 54]).

3.6 Discussion and Summary

This chapter has explored the controversy highlighted by SEW in their proposition that in the foundations of quantum mechanics there is a hierarchy, with complementarity and entanglement being more important than uncertainty.

The context in which SEW choose to test their proposition is an interference experiment with path detection. Subsequently, authors who support their conclusions such as DNR and their detractors such as Kim and Mahler also use the same experimental context. It is the quantum interference setting that the relationship between complementarity, entanglement and uncertainty are investigated in Chapter 4.

Some years after the first impact of SEW's work it was the interference with path detection setting as a testing ground for quantum mechanics which led Englert, Jaeger et al, Dürr et al and Bjork et al to find trade-off relationships between path information and interference clarity. The same experimental setting was used by Wootters and Zurek, Mittelstaedt et al and Greenberger et al some years before SEW.

The view, which SEW tried to establish, that there is a hierarchy in founding features of quantum mechanics cannot be valid. An examination of Bohr's writings showed that he meant the concept of complementarity to be broader than it has come to be used and taught in subsequent times. Similarly, Heisenberg realized that the concept of uncertainty contained more than the often quoted position-momentum uncertainty relation.

It is proposed in this work that the term *uncertainty principle* refers to the broad statement that there are pairs of observables for which there is a trade-off relationship in the degrees of sharpness of the preparation or measurement of their values, such that a simultaneous or sequential determination of the values requires a nonzero amount of unsharpness (latitude, inaccuracy, disturbance). There are a variety of measures of uncertainty, inaccuracy, and disturbance with which such trade-off relations can be formulated, usually in the form of inequalities.

In the framework of quantum mechanics, the uncertainty principle is satisfied due to the existence of noncommuting pairs of observables, and the respective trade-off relations reflect the extent of the noncommutativity.

The *complementarity principle* is the statement that there are pairs of observables which stand in the relationship of complementarity. As can be seen above, this is satisfied in quantum mechanics for observables whose basis systems of eigenvectors are mutually unbiased. Thus it may be concluded that the 'principle' of complementarity, as formalized here, is a consequence of the quantum mechanical formalism. There seems to be no need to speak of a complementarity *principle*, unless one sets out to use such a principle in a more general framework to deduce quantum mechanics. Here the complementarity principle will be regarded as a description of one remarkable non-classical feature of quantum mechanics. Similarly the *uncertainty principle* will be regarded as a description of another such feature of quantum mechanics.

The concepts of complementarity and uncertainty highlight the following aspects of quantum mechanics: (1) the impossibility of assigning simultaneously sharp values to certain pairs of noncommuting observables, be it by preparation or measurement; however, (2) there is the possibility of simultaneously assigning unsharp values to such observables.

The formalizations of the features (1) and (2) reviewed in the preceding subsections, which are those most commonly used in the recent research literature, have clearly identified (1) as an expression of the idea of complementarity and (2) as the essence of the uncertainty. With a terminological shift, a more balanced view could be achieved compared to the view that emphasized complementarity over uncertainty. The positive rôle of the uncertainty relations as enabling joint determinations and joint measurements would be highlighted more prominently and even if it turned out that the uncertainty statement, (2) entails the complementarity statement, (1), in limited conditions, this would not deny the significance of the strict mutual exclusivity of sharp value assignments. It is this strict mutual exclusivity which is the motivation for the search for simultaneous but unsharp value assignments.

Irrespective of the particular terminological preference, formalizing the respective statements (1) and (2) has opened up new and interesting questions: (1) and (2) have become claims that can or cannot be proven as consequences of the theory, and it becomes possible to study the logical relationships between these statements.

If we contrast a statement of value complementarity viz. (3) two observable are value complementary if whenever one of them has a sharp value, the value of the other is completely indeterminate, with statement of uncertainty (2) above then (3) can be seen as a limiting case of (2). A specific example of this is in the duality relations (found in Section 3.2.2). These describe a trade-off between path detection and interference contrast. The limiting case of these relationships is (4) wave-particle duality as described by Bohr (section 3.3.1)

A resolution of the controversy created by promotion of a hierarchy of foundational features of quantum mechanics may be resolved ('the joining of the house divided') if the whole of Bohr's views, laid down in the quote given earlier, repeated here, is taken fully into account;

> *. . . according to quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum vectors associated with individuals. This circumstance may be regarded as a simple symbolic expression for the complementary nature of the space-time description and the claims of causality. At the same time, however, the general character of this relation makes it possible to a* <u>*certain extent*</u> *to reconcile the conservation laws with the space-time co-ordination of observation, the idea of a coincidence of well defined events*

> *in a space-time point being replaced by that of unsharply defined individuals within finite space-time regions.* (My underlining.)

In other words, graded or quantitative complementarity is taken into account as a distinct possibility. Indeed it is necessary for the joint, unsharp measurement of observables which under strict complementarity are mutually exclusive. Path detection and interference contrast are two such quantities but if a degree of unsharpness is allowed in their measurement, then these measurements are related by a trade-off or duality relation (detailed in Section 3.2.2). One of the tasks attempted in Chapter 4 is to relate this duality relation to an uncertainty relation which uses familiar measures of uncertainty, i.e. variances.

Chapter 4

Complementarity and Uncertainty in Two-Path Interferometry

4.1 Introduction

"Quantum phenomena do not occur in Hilbert spaces. They occur in laboratories."

Asher Peres. (1993), *Quantum Theory: Concepts and Methods*, [56] p.373.

The main aim of this chapter is to study the rôles and relative significance of uncertainty and entanglement in the explanation of complementary quantum phenomena such as the mutual exclusivity of path marking and interference detection in a two path interferometer.

The independent significance of the uncertainty principle lies partially in the rôle played by its measurement version. This will be brought out clearly in the following studies of joint measurement schemes for path and interference observables or similar pairs of complementary quantities. Some of these new schemes arose as a result of the analysis of *quantum erasure* (section 4.3.3) and *quantitative quantum erasure* (section 4.3.4.

The entanglement between a quantum object and a path-marking and recording probe system plays a crucial rôle in *quantum erasure*, a process conceived by Scully and Drühl in 1982 [65]. In an experiment in which the whole ensemble of objects does not reveal interference as a result of the path marking and recording process, entanglement can be used to select a 'quantum-erasure' sub-ensemble that does show interference. If the path marking and recording process is incomplete then it has been shown to be possible to select sub-ensembles with unsharp path information but which retain, albeit with lowered contrast, interference; such a process is referred to as *quantitative quantum erasure* [26].

It will be shown that quantitative erasure is an instance of joint measurement. This is natural if erasure and quantitative erasure are considered as conditional or selective observations; conditional probabilities presuppose joint probabilities.

Section 4.2 is derived from [13] which was written in response to a paper of Zhu et al [78], in which they demonstrated an analogue of path-interference duality in a nuclear spin experiment. Here a modification of their scheme is proposed yielding a joint measurement scheme for unsharp path and interference observables. A new trade-off relation is deduced between the degrees of unsharpness (or contrasts) of the unsharp path and interference observables measured together; this is an instance of a joint-measurement uncertainty relation, but at the same time it expresses measurement complementarity as the mutual exclusion of sharp joint measurements of these observables. Furthermore, a model-independent connection is demonstrated between the joint measurability of two complementary path and interference (or spin 1/2) observables and a trade-off relation between their respective contrasts.

Section 4.3 comprises an edited version of an analysis of a Mach-Zehnder interferometer (MZI) analogue of the SEW experiment presented in [14]. The MZI setup provides in one common framework the following experiments:

(1) use of the interferometer for path detection;

(2) an interference detection setup;

(3) introduction of path marking via entanglement with a probe system;

(4) quantum erasure;

(5) quantitative quantum erasure.

In contrast to the treatment of Zhu et al's experiment (Section 4.2) the MZI setup provides a more direct analogue of the various experiments, as proposed by SEW and those following them. In these an entangled state is achieved before the first beam splitter.

Further, the setups in (3), (4) and (5) give rise to new instances of joint measurement schemes for path and interference observables. In each case measurement complementarity can be recovered as a limit if the sharpness of one observable is made perfect.

All of the experiments introduced and analyzed in Chapter 4 form the basis for a systematic account of the interconnection between complementarity and uncertainty in Chapter 5.

4.2 Complementarity between path and interference observables in joint measurements

4.2.1 Nuclear spin analog of path-interference duality via entanglement

In two-path interferometer experiments interference is exhibited by measuring a population of the final state of the object dependent on some phase factor. The path taken by the object is determined in a (simultaneous) measurement of another marker or probe state [58]. Zhu et al investigate the possibility of a 'path-interference' duality utilizing the internal states of a quantum object or a bulk ensemble, specifically an ensemble of nuclear spins.

The joint measurement scheme proposed here can be realized as an extension of the experiment of Zhu et al [78]. In their experiment Zhu et al use nuclear magnet resonance (NMR) to entangle the spin states of the ^{13}C nucleus with those of the ^{1}H nucleus in a chloroform molecule, $^{13}CHCl_3$. The ^{13}C nucleus is taken to be the object system (o), while the ^{1}H nucleus serves as the 'path' detection. The unitary operations used to transform the initial state $|\Psi_i\rangle$ into the final state $|\Psi_f\rangle$ are realized in this experiment by application of electromagnetic pulse sequences with appropriate frequencies or field gradients.

The analog of a 'path' observable is now the spin component $s_3^{(o)}$ of the ^{13}C nucleus. The interference pattern is revealed as the relative frequency of a particular outcome of a suitable measurement that is sensitive to the relative phase between the 'path' eigenstates in the prepared input state.

An ensemble of the $^{13}CHCl_3$ molecules is prepared with the two spin-1/2 nuclei in the initial state $|0\rangle_o|0\rangle_p$, where o and p refer to the object and probe. The states $|0\rangle, |1\rangle$ denote the eigenstates of the object's and probe's spin component σ_z, respectively.

An operation, $R_1^o(\vartheta) = \begin{pmatrix} \alpha(\vartheta) & -\beta(\vartheta) \\ \beta(\vartheta) & \alpha(\vartheta) \end{pmatrix}$ on the ^{13}C nucleus (the object) transforms $|0\rangle_o|0\rangle_p$ into an intermediate state

$$R_1^o(\vartheta)|0\rangle_o|0\rangle_p \to |\Psi_i\rangle = [\alpha(\vartheta)|0\rangle_o + \beta(\vartheta)|1\rangle_o]|0\rangle_p =: \psi_i \otimes |0\rangle_p \qquad (4.2.1)$$

where $|\alpha(\vartheta)|^2 + |\beta(\vartheta)|^2 = 1$. The final state is achieved after a unitary operation

$$U^o(\phi) = \begin{pmatrix} 1 & \exp(\imath\phi) \\ -\exp(-\imath\phi) & 1 \end{pmatrix},$$

$$U^o(\phi) : |\Psi_i\rangle \rightarrow |\Psi\rangle = \tfrac{1}{\sqrt{2}}[(\alpha(\vartheta) + \beta(\vartheta)e^{\imath\phi})|0\rangle_o + (\beta(\vartheta) - \alpha(\vartheta)e^{-\imath\phi})|1\rangle_o]|0\rangle_p. \quad (4.2.2)$$

Since the the state of the ^{1}H nucleus (the probe) remains unchanged, $|0\rangle_p$, throughout, the two 'paths' along which the ^{13}C nucleus evolves into the final state $|0\rangle_o$ or $|1\rangle_o$ through the intermediate state are indistinguishable. In terms used earlier in this thesis: the state is a superposition of the two 'path' eigenstates.

The probability of finding the ^{13}C nucleus in the final state $|0\rangle_o$ is

$$\tfrac{1}{2}[1 + 2\alpha(\vartheta)\beta(\vartheta)\cos\phi], \quad (4.2.3)$$

and for $|1\rangle_o$ it is

$$\tfrac{1}{2}[1 - 2\alpha(\vartheta)\beta(\vartheta)\cos\phi]. \quad (4.2.4)$$

Repeating the experiment for different values of ϕ will show that the probabilities depend on ϕ i.e. an interference pattern is is obtained.

In a further experiment Zhu et al produce another intermediate state

$$|\Psi_{ie}\rangle = \alpha(\vartheta)|00\rangle + \beta(\vartheta)|11\rangle \quad (4.2.5)$$

from the initial state by following the operation $R_1^o(\vartheta)$ by a controlled-NOT gate CN^{op} which has the effect of flipping ^{1}H into the same state as ^{13}C if they are in different states e.g.

$$|10\rangle \rightarrow |11\rangle \quad (4.2.6)$$

otherwise

$$|00\rangle \rightarrow |00\rangle. \quad (4.2.7)$$

The unitary evolution, $U^o(\phi)|0\rangle_o = \frac{1}{\sqrt{2}}(|0\rangle + e^{\imath\phi})$, and $U^o(\phi)|1\rangle_o = \frac{1}{\sqrt{2}}(e^{\imath\phi} + |0\rangle)$, produces the final state,

$$
\begin{aligned}
|\Psi_f\rangle \; = \; & \tfrac{1}{\sqrt{2}}[\alpha(\vartheta)|0\rangle_o|0\rangle_p + \beta(\vartheta)e^{\imath\phi}|0\rangle_o|1\rangle_p \\
& -\alpha(\vartheta)e^{-\imath\phi}|1\rangle_o|0\rangle_p + \beta(\vartheta)|1\rangle_o|1\rangle_p. \qquad (4.2.8)
\end{aligned}
$$

It is now clear that 'path' marking has been achieved because the part of the superposition containing $|11\rangle$ or $|01\rangle$ must have evolved from $|11\rangle$. Similarly the part of the superposition containing $|10\rangle$ or $|00\rangle$ must have evolved from $|00\rangle$.

The probability of finding the ^{13}C nucleus in the final state $|0\rangle_o$ is

$$\tfrac{1}{2}|\alpha(\vartheta)|^2 + \tfrac{1}{2}|\beta(\vartheta)|^2 = \tfrac{1}{2}. \qquad (4.2.9)$$

This being a constant independent of ϕ means that the interference fringes are washed out which is to be expected since this experiment displays sharp path information.

However, Zhu et al claim that measuring the 'coherence', $\alpha(\vartheta)\beta(\vartheta)\sin\phi$, between the final states $|00\rangle$ and $|01\rangle$ will reveal interference fringes because of the dependence of the 'coherence' on ϕ thus allowing the observation of sharp path information and good contrast interference fringes in a single experiment.

4.2.2 Applying the most general measurement model.

The original formulation of quantum mechanics does not allow the description of joint measurements of incompatible observables such as interference and path. However, this has become possible in the general formalism of positive operator valued measure discussed in Chapter 2 sections 2.5.3 and 2.5.4. Here the aim is to write the outcome probabilities in terms of the input state $|\psi_i\rangle$. This will give a POVM

$\{F_{++}, F_{+-}, F_{-+}, F_{--}\}$. The form will be found, for example, of F_{++} which when applied to the input state will produce the same outcome probabilities as the projection $P_{o+} \otimes P_{p+}$ acting on the final state, Ψ_f, i.e. $\langle \psi_i | F_{++} \psi_i \rangle = \langle \Psi_f | P_{o+} \otimes P_{p+} \Psi_f \rangle$.

Using these projection operators, $P_{o\pm} \otimes P_{p\pm}$ on the final state, Ψ_f (equation 4.2.8), gives,

$$\begin{aligned} &\left\langle \Psi_f | P_{o\pm} \otimes P_{p\pm} \Psi_f \right\rangle = \\ &\quad \left(\tfrac{1}{\sqrt{2}} \alpha^* \langle 00| - \tfrac{1}{\sqrt{2}} \alpha^* e^{i\phi} \langle 10| + \tfrac{1}{\sqrt{2}} \beta^* e^{-i\phi} \langle 01| + \tfrac{1}{\sqrt{2}} \beta^* \langle 11| \right) \dots \\ &\quad \dots (P_{o\pm} \otimes P_{p\pm}) \dots \\ &\quad \dots \left(\tfrac{1}{\sqrt{2}} \alpha |00\rangle - \tfrac{1}{\sqrt{2}} \alpha e^{-i\phi} |10\rangle + \tfrac{1}{\sqrt{2}} \beta e^{i\phi} |01\rangle + \tfrac{1}{\sqrt{2}} \beta |11\rangle \right). \end{aligned} \tag{4.2.10}$$

Here it is appropriate to be reminded that any projection operator in $\mathbb{C}^2$ can be expanded in terms of the Pauli spin matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and the identity operator, $P_{o\pm} = \frac{1}{2}(I \pm \mathbf{o} \cdot \sigma)$, where $\mathbf{o} = (\sin\theta^o \cos\phi^o, \sin\theta^o \sin\phi^o, \cos\theta^o)$ is the unit vector of the Poincaré sphere defining a point on the surface with polar angle θ^o and azimuth angle ϕ^o.

From this point the action of P_{p+} and P_{o+} will be explored; the actions of P_{p-} and P_{o-} will be explored later.

$$P_{o+} = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta^o & \sin\theta^o e^{-\imath\phi^o} \\ \sin\theta^o e^{\imath\phi^o} & 1 - \cos\theta^o \end{pmatrix} =: \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix}. \tag{4.2.11}$$

Similarly for, p, the path marker system,

$$P_{p+} = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta^p & \sin\theta^p e^{-\imath\phi^p} \\ \sin\theta^p e^{\imath\phi^p} & 1 - \cos\theta^p \end{pmatrix} =: \begin{pmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix}. \tag{4.2.12}$$

$\langle \Psi_f | P_{o+} \otimes P_{p+} \Psi_f \rangle$ generates the following terms with Q_{00} in common,

$$\tfrac{1}{2} \alpha^* \alpha (R_{00} + R_{11} - e^{-\imath\phi} R_{01} - e^{\imath\phi} R_{10}) Q_{00}. \tag{4.2.13}$$

Substituting R_{jk} and Q_{00} and $\alpha = \langle\psi_i|0\rangle$ gives,

$$\tfrac{1}{4}\alpha^*\alpha(1-\sin\theta^o\cos(\phi+\phi^o))(1+\cos\theta^p) =: \tfrac{1}{4}\langle\psi_i|0\rangle\langle 0|\psi_i\rangle(1+\mathrm{A})(1+\mathrm{B}). \quad (4.2.14)$$

where ψ_i is the input state of o and $\mathrm{A} = -\sin\theta^o\cos(\phi^o+\phi)$ and $\mathrm{B} = \cos\theta^p$.

Collecting together terms with Q_{01} in common;

$$\begin{aligned}
&\tfrac{1}{2}\alpha^*\beta(e^{\imath\phi}R_{01}+R_{11}-e^{2\imath\phi}R_{10}-e^{-\imath\phi}R_{11})Q_{01}\\
&\qquad=\tfrac{1}{4}\alpha^*\beta e^{\imath(\phi-\phi^p)}\sin\theta^p(\cos\theta^o-\imath\sin\theta^o\sin(\phi+\phi^o) \qquad (4.2.15)\\
&\qquad=:\tfrac{1}{4}\langle\psi_i|1\rangle\langle 0|\psi_i\rangle(\mathrm{W}+\imath\mathrm{X})(\mathrm{Y}-\imath\mathrm{Z}).
\end{aligned}$$

where $\mathrm{W} = \cos(\phi-\phi^p)$, $\mathrm{X} = \sin(\phi-\phi^p)$, $\mathrm{Y} = \cos\theta^o\sin\theta^p$ and $\mathrm{Z} = \sin\theta^o\sin(\phi^o+\phi)\sin\theta^p$.

The following terms have Q_{10} in common;

$$\begin{aligned}
&\tfrac{1}{2}\beta^*\alpha(e^{-\imath\phi}R_{00}+e^{-\imath\phi}R_{11}-e^{-2\imath\phi}R_{01}-R_{10})Q_{10}\\
&\qquad=\tfrac{1}{4}\beta^*\alpha e^{-\imath(\phi-\phi^p)}(\cos\theta^o+\imath\sin\theta^o\sin(\phi+\phi^o))\sin\theta^p \qquad (4.2.16)\\
&\qquad=:\tfrac{1}{4}\langle\psi_i|1\rangle\langle 0|\psi_i\rangle(\mathrm{W}-\imath\mathrm{X})(\mathrm{Y}+\imath\mathrm{Z}).
\end{aligned}$$

And the following terms have Q_{11} in common,

$$\begin{aligned}
&\tfrac{1}{2}\beta^*\beta(R_{00}+R_{11}+e^{-\imath\phi}R_{01}+e^{\imath\phi}R_{10})Q_{11}\\
&\qquad=\tfrac{1}{4}\beta^*\beta(1+\sin\theta^o\cos(\phi+\phi^o))(1-\cos\theta^p) \qquad (4.2.17)\\
&\qquad=:\tfrac{1}{4}\langle\psi_i|1\rangle\langle 1|\psi_i\rangle(1-\mathrm{A})(1-\mathrm{B}).
\end{aligned}$$

So, this effect, F_{++}, associated with outcomes for the $+\mathbf{o}$ and $+\mathbf{p}$ directions, is

$$\mathrm{F}_{++} = \frac{1}{4}\begin{pmatrix} (1+\mathrm{A})(1+\mathrm{B}) & (\mathrm{W}+\imath\mathrm{X})(\mathrm{Y}-\imath\mathrm{Z}) \\ (\mathrm{W}-\imath\mathrm{X})(\mathrm{Y}+\imath\mathrm{Z}) & (1-\mathrm{A})(1-\mathrm{B}). \end{pmatrix}. \qquad (4.2.18)$$

In terms of the Pauli matrices this becomes

$$\mathrm{F}_{++} = \tfrac{1}{4}((1+\mathrm{AB})I + (\mathrm{WY}+\mathrm{XZ})\sigma_x + (-\mathrm{XY}+\mathrm{WZ})\sigma_y + (\mathrm{A}+\mathrm{B})\sigma_z), \quad (4.2.19)$$

or

$$F_{++} = \tfrac{1}{4}((1 + AB)I + N_1\sigma_x + N_2\sigma_y + (A + B)\sigma_z). \tag{4.2.20}$$

Here a summary of the abbreviations may be useful;

$$\begin{aligned}
A &= -\sin\theta^o \cos(\phi^o + \phi),\\
B &= \cos\theta^p,\\
W &= \cos(\phi - \phi^p),\\
X &= \sin(\phi - \phi^p),\\
Y &= \cos\theta^o \sin\theta^p,\\
Z &= \sin\theta^o \sin(\phi^o + \phi)\sin\theta^p\\
N_x &= (WY + XZ)\\
&= (\cos(\phi - \phi^p)\cos\theta^o \sin\theta^p + \sin(\phi - \phi^p)\sin\theta^o \sin(\phi^o + \phi)\sin\theta^p)\\
N_y &= (-XY + WZ)\\
&= (-\sin(\phi - \phi^p)\cos\theta^o \sin\theta^p + \cos(\phi - \phi^p)\sin\theta^o \sin(\phi^o + \phi)\sin\theta^p).
\end{aligned}$$

So far F_{++} has been considered. F_{++} is associated with outcomes corresponding to $+\mathbf{o}$ and $+\mathbf{p}$. If the effect F_{+-} is to be found, which is associated with outcomes corresponding to the $+\mathbf{o}$ and $-\mathbf{p}$ directions, then $\theta^p \rightarrow \pi - \theta^p$ and $\phi^p \rightarrow \pi + \phi^p$. Applying these gives,

$$\begin{aligned}
F_{+-} =& \tfrac{1}{4}((1 - AB)I - (WY + XZ)\sigma_x - (-XY + WZ)\sigma_y + (A - B)\sigma_z), \text{ or}\\
F_{+-} =& \tfrac{1}{4}((1 - AB)I - N_x\sigma_x - N_y\sigma_y + (A - B)\sigma_z).
\end{aligned} \tag{4.2.21}$$

Similarly, the effect, F_{-+}, will be found, corresponding to the $-\mathbf{o}$ and $\mathbf{p}$ directions, $\theta^o \rightarrow \pi - \theta^o$ and $\phi^o \rightarrow \pi + \phi^o$ giving,

$$\begin{aligned}
F_{-+} =& \tfrac{1}{4}((1 - AB)I - (WY + XZ)\sigma_x - (-XY + WZ)\sigma_y - (A - B)\sigma_z), \text{ or}\\
F_{-+} =& \tfrac{1}{4}((1 - AB)I - N_1\sigma_x - N_2\sigma_y - (A - B)\sigma_z).
\end{aligned} \tag{4.2.22}$$

Finally, the effect, F_{--}, will be found, associated with the $-\mathbf{o}$ and $-\mathbf{p}$ directions, $\theta^o \to \pi - \theta^o$, $\phi^o \to \pi + \phi^o$, $\theta^p \to \pi - \theta^p$ and $\phi^p \to \pi + \phi^p$ giving,

$$\begin{aligned}\mathrm{F}_{--} =& \tfrac{1}{4}((1+\mathrm{AB})I + (\mathrm{WY}+\mathrm{XZ})\sigma_x + (-\mathrm{XY}+\mathrm{WZ})\sigma_y - (\mathrm{A}+\mathrm{B})\sigma_z), \text{ or}\\ \mathrm{F}_{--} =& \tfrac{1}{4}((1+\mathrm{AB})I + \mathrm{N}_1\sigma_x + \mathrm{N}_2\sigma_y - (\mathrm{A}+\mathrm{B})\sigma_z).\end{aligned} \tag{4.2.23}$$

These four effects can be grouped into two pairs in three different ways, and summing the two elements of each pair gives a new pair of effects adding up to I. In this way one obtains three marginal POVMs:

$$\begin{aligned}
\mathrm{F}_+^{(1)} &= \mathrm{F}_{++} + \mathrm{F}_{+-} = \tfrac{1}{2}(I + \mathrm{A}\sigma_z),\ \ \mathrm{A} = -\sin\theta^o\cos(\phi^o+\phi);\\
\mathrm{F}_-^{(1)} &= \mathrm{F}_{-+} + \mathrm{F}_{--} = \tfrac{1}{2}(I - \mathrm{A}\sigma_z);\\
\mathrm{F}_+^{(2)} &= \mathrm{F}_{++} + \mathrm{F}_{-+} = \tfrac{1}{2}(I + \mathrm{B}\sigma_z),\ \ \mathrm{B} = \cos\theta^p;\\
\mathrm{F}_-^{(2)} &= \mathrm{F}_{+-} + \mathrm{F}_{--} = \tfrac{1}{2}(I - \mathrm{B}\sigma_z);\\
\mathrm{F}_+^{(3)} &= \mathrm{F}_{++} + \mathrm{F}_{--} = \tfrac{1}{2}\left((1+\mathrm{AB})I + \mathrm{N_x}\sigma_x + \mathrm{N_y}\sigma_y\right);\\
\mathrm{F}_-^{(3)} &= \mathrm{F}_{+-} + \mathrm{F}_{-+} = \tfrac{1}{2}\left((1-\mathrm{AB})I - \mathrm{N_x}\sigma_x - \mathrm{N_y}\sigma_y\right).
\end{aligned} \tag{4.2.24}$$

The first two marginal POVMs are smeared versions of the sharp observable $\sigma_z^{(o)}$ since their effects are combinations of the spectral projections of $\sigma_z^{(o)}$. The third POVM is a smeared version of an observable complementary to $\sigma_z^{(o)}$; its spectral decomposition is in terms of the spectral measure of $\sigma_{\mathbf{n}}^{(o)}$, with $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$, $\mathbf{N} = (\mathrm{N_x}, \mathrm{N_y}, 0)$, and $\mathbf{n}$ perpendicular to the vector $(0,0,1)$ associated with $\sigma_z^{(o)}$.

The joint measurement scheme developed here will now be discussed.

4.2.3 Discussion of the joint measurement scheme

Using this measurement scheme as illustrated in Figure 4.1 it is possible to compare the marginal probabilities with the probabilities obtained in a sharp measurement.

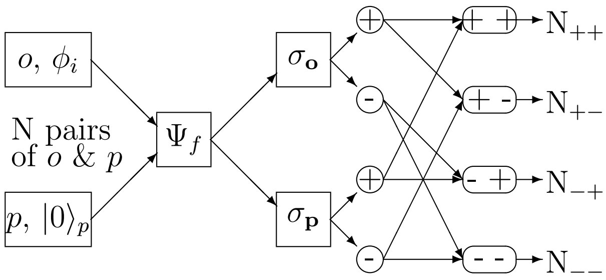

Figure 4.1: N pairs of object o and probe p systems are entangled into the state ψ_i (Equation 4.2.8). Measurements of spin are made separately on the object, o and the probe, p and pairs of outcomes counted. In an ideal experiment the frequencies N_{++}, N_{+-}, N_{-+}, N_{--} will add up to N.

For example,

$$\begin{aligned} \langle\psi_i|F_+^{(1)}\psi_i\rangle &= \tfrac{1}{2}\Big(1 + A\langle\psi_i|\sigma_z^{(o)}\psi_i\rangle\Big) \\ &= \tfrac{1}{2}\left(1 + A\langle\psi_i|P_+^{\sigma_z}\psi_i\rangle - A\langle\psi_i|P_-^{\sigma_z}\psi_i\rangle\right) \\ &= A\langle\psi_i|P_+^{\sigma_z}\psi_i\rangle + \tfrac{1}{2}(1-A) \end{aligned} \tag{4.2.25}$$

where $P_+^{\sigma_z}$, $P_-^{\sigma_z} = I - P_+^{\sigma_z}$ are the spectral projections of $\sigma_z^{(o)}$. The number

$$\langle\psi_i|F_+^{(1)}\psi_i\rangle = \langle\psi_i|F_{++}\psi_i\rangle + \langle\psi_i|F_{+-}\psi_i\rangle \approx (N_{++} + N_{+-})/N \tag{4.2.26}$$

is the probability of a + outcome in an unsharp measurement of $\sigma_z^{(o)}$, represented here by the marginal POVM $\{F_+^{(1)}, F_-^{(1)}\}$. Eq. 4.2.25 can be solved for the probability $\langle\psi_i|P_+^{\sigma_z}\psi_i\rangle$ of a + outcome in a sharp measurement of $\sigma_z^{(o)}$. In this way the frequencies N_{++} and N_{+-} obtained in this scheme can be used to reconstruct this sharp measurement probability.

Similarly:

$$\begin{aligned} \langle\psi_i|F_+^{(3)}\psi_i\rangle &= \tfrac{1}{2}(1 + AB) + \tfrac{1}{2}\langle\psi_i|\mathbf{N}\cdot\sigma\,\psi_i\rangle \\ &= \tfrac{1}{2}(1 + AB - |\mathbf{N}|) + |\mathbf{N}|\langle\psi_i|P_+^{\mathbf{n}}\psi_i\rangle. \end{aligned} \tag{4.2.27}$$

Here, the fact that the spectral projections of $\sigma_{\mathbf{n}}^{(o)}$ satisfy $\mathrm{P}_{+}^{\mathbf{n}} + \mathrm{P}_{-}^{\mathbf{n}} = I$, has been used.

Again it is seen that estimation of the third marginal probability (by way of collecting the counts $\mathrm{N}_{++} + \mathrm{N}_{--}$) allows one to reconstruct the probability for a + outcome of a sharp measurement of $\sigma_{\mathbf{n}}^{(o)}$, where this observable is complementary to $\sigma_z^{(o)}$. Thus, obtained is a simultaneous determination of the probabilities for two complementary observables from the statistics of a single experiment.

4.2.4 Looking for a complementarity relationship

One can see Bohr's principle of complementarity at work in the present joint measurement scheme. Consider the case in which $\mathrm{F}_{\pm}^{(3)}$ becomes a projection, hence a sharp observable complementary to $\sigma_z^{(o)}$; that is: ${\mathrm{N_x}}^2 + {\mathrm{N_y}}^2$=1. In this case,

$$
{\mathrm{N_x}}^2 + {\mathrm{N_y}}^2 = \sin^2\theta^p[\cos^2\theta^o + \sin^2\theta^o\sin^2(\phi^o + \phi)] = 1,
$$

hence $\sin^2\theta^p$=1 and $\sin^2(\phi^o + \phi)$=1 or $\sin^2\theta^p$=1 and $\cos^2\theta^o$=1. Either case implies that $\mathrm{A}^2 = \sin^2\theta^o\cos^2(\phi^o + \phi)$=0 and $\mathrm{B}^2 = \cos^2\theta^p$=0 i.e. AB=0. So, demanding that $\mathrm{F}_{\pm}^{(3)}$ is a sharp observable, means that there is no information about $\sigma_z^{(o)}$, as the first two marginal POVMs become *trivial*: $\mathrm{F}_{\pm}^{(1)} = \mathrm{F}_{\pm}^{(2)} = \frac{1}{2}I$.

Conversely, requiring that $\mathrm{F}_{\pm}^{(2)}$ is a sharp observable, then $|\mathrm{B}| = |\cos\theta^p| = 1$ and $\mathrm{N_x} = \mathrm{N_y} = 0$, giving $\mathrm{F}_{\pm}^{(3)} = \frac{1}{2}(1 \pm |\mathrm{A}|)I$. Similarly, if $\mathrm{F}_{\pm}^{(1)}$ is a sharp observable, then $\mathrm{F}_{\pm}^{(3)} = \frac{1}{2}(1 \pm |\mathrm{B}|)I$. Hence, for this joint measurement scheme, marginal observables are demonstrating *measurement complementarity*, in the sense that, in the limit where one marginal becomes sharp the other becomes a trivial observable.

It is possible to go further and give a quantitative relationship expressing a form of measurement complementarity (see Chapter 3, Section 3.5.2).

A natural measure for the quality, or *contrast* C, of the smeared marginal observables given above is the difference between the largest and smallest eigenvalues of each of the effects $\mathrm{F}_{\pm}^{(j)}$. This quantity can be determined as the greatest possible probability for the associated outcome, minus the smallest possible probability. In the case of a projection, that number is 1-0=1. Thus, the closer the contrast of one of the above marginal POVMs is to 1, the closer its effects are to projections.

The eigenvalues of both effects of the first marginal POVM, $\mathrm{F}_{\pm}^{(1)}$ are $\frac{1}{2}(1 \pm \mathrm{A})$ and those of the second marginal, $\mathrm{F}_{\pm}^{(2)}$, are $\frac{1}{2}(1 \pm \mathrm{B})$. The effect $\mathrm{F}_{+}^{(3)}$ of the third marginal has the eigenvalues $\frac{1}{2}(1 + \mathrm{AB} \pm ({\mathrm{N_x}}^2 + {\mathrm{N_y}}^2)^{\frac{1}{2}})$. Those of the other effect, $\mathrm{F}_{-}^{(3)}$, of the third marginal are $= \frac{1}{2}(1 - \mathrm{AB} \pm ({\mathrm{N_x}}^2 + {\mathrm{N_y}}^2)^{\frac{1}{2}})$.

Thus, for the contrasts of each of the marginals $\mathrm{F}^{(1)}$, $\mathrm{F}^{(2)}$, and $\mathrm{F}^{(3)}$ the values $\mathrm{C}_1 = |\mathrm{A}|$, $\mathrm{C}_2 = |\mathrm{B}|$, and $\mathrm{C}_3 = ({\mathrm{N_x}}^2 + {\mathrm{N_y}}^2)^{\frac{1}{2}}$, are obtained, respectively. After some manipulation it can be seen that,

$$\begin{aligned}
({\mathrm{N_x}}^2 + {\mathrm{N_y}}^2) &= (\mathrm{W^2Y^2 + X^2Z^2 + 2WYXZ}) + (\mathrm{X^2Y^2 + W^2Z^2 - 2WYXZ}) \\
&= \mathrm{W^2Y^2 + X^2Z^2 + X^2Y^2 + W^2Z^2} \\
&= (\mathrm{W^2 + X^2})(\mathrm{Y^2 + Z^2}) = (\mathrm{Y^2 + Z^2}) \\
&= \cos^2\theta^o \sin^2\theta^p + \sin^2\theta^o \sin^2(\phi^o + \phi)\sin^2\theta^p \qquad (4.2.28) \\
&= (\cos^2\theta^o + \sin^2\theta^o(1 - \cos^2(\phi^o + \phi))\sin^2\theta^p \\
&= \left(1 - \sin^2\theta^o \cos^2(\phi^o + \phi)\right)\left(1 - \cos^2\theta^p\right) \\
&= (1 - \mathrm{A}^2)(1 - \mathrm{B}^2),
\end{aligned}$$

that is,

$$\mathrm{C}_3^2 = \left(1 - \mathrm{C}_1^2\right)\left(1 - \mathrm{C}_2^2\right). \tag{4.2.29}$$

Here, equation (4.2.29) is a state independent relationship between two unsharp $\sigma_z^{(o)}$ path observables and the third marginal observable, $\mathrm{F}^{(3)}$ an unsharp interference observable. It says that there is a trade-off: the better the contrast of the latter observable, the worse the contrast of the two $\sigma_z^{(o)}$ marginals must be; and vice versa.

One can take $(1 - \mathrm{C}_i^2)$ as a measure of the intrinsic unsharpness, or fuzziness of the POVM. This becomes evident if one calculates the variance of, say, the POVM $\mathrm{F}^{(1)}$, considered as a smeared version of the observable σ_z which has values ± 1. The variance is defined as $\mathrm{Var}_\rho(\mathrm{F}) = \int (t - \bar{t})^2 \, d\langle \mathrm{F}_t \rangle_\rho$. For discrete measures $\bar{t} = \langle \mathrm{F}_+^{(1)} \rangle_\rho - \langle \mathrm{F}_-^{(1)} \rangle_\rho = \mathrm{A} r_z$, and the values to be distributed are ± 1 so,

$$\mathrm{Var}_\rho(\mathrm{F}^{(1)}) = (1 - \bar{t})^2 \langle \mathrm{F}_+^{(1)} \rangle_\rho + (-1 - \bar{t})^2 \langle \mathrm{F}_-^{(1)} \rangle_\rho; \tag{4.2.30}$$

and recalling that for some general state $\rho_{\mathbf{r}} = \frac{1}{2}(\mathbf{I} + \mathbf{r} \cdot \overrightarrow{\sigma})$, $\langle \mathrm{F}_\pm^{(1)} \rangle_\rho = \frac{1}{2}(1 \pm \mathrm{A} r_z)$ and $\mathrm{Var}_\rho(\sigma_z) = 1 - r_z^2$, gives

$$\mathrm{Var}_\rho(\mathrm{F}^{(1)}) = 1 - \mathrm{A}^2 r_z^2 = \mathrm{Var}_\rho(\sigma_z) + (1 - \mathrm{A}^2)(1 - \mathrm{Var}_\rho(\sigma_z)). \tag{4.2.31}$$

The minimum of the variance in equation (4.2.30) over all ρ is obtained at eigenstates of σ_z, where one obtains the value $(1 - \mathrm{A}^2)$ (as $r_z = \pm 1$). This minimal spread of outcomes reflects the intrinsic fuzziness of the measurement, that is, the uncertainty about the actual value prior to measurement.

Equation (4.2.29) implies the following trade-off relations for the complementary *pairs* of marginals:

$$\left(\mathrm{N_x}^2 + \mathrm{N_y}^2\right) + \mathrm{A}^2 \leq 1, \quad \left(\mathrm{N_x}^2 + \mathrm{N_y}^2\right) + \mathrm{B}^2 \leq 1. \tag{4.2.32}$$

Which in terms of the contrasts are,

$$\mathrm{C}_3^2 \leq 1 - \mathrm{C}_1^2, \quad \mathrm{C}_3^2 \leq 1 - \mathrm{C}_2^2, \tag{4.2.33}$$

or, introducing the *unsharpness* $\mathrm{U}_i = 1 - \mathrm{C}_i^2$,

$$\mathrm{U}_1 + \mathrm{U}_3 \geq 1, \quad \mathrm{U}_2 + \mathrm{U}_3 \geq 1. \tag{4.2.34}$$

It appears as if these relations are consequences of the fact that the observables $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ are jointly measurable. Similar results were obtained in different joint measurement models, and with other measures of unsharpness, in [9, 18, 47]. In the present case, it is possible to go one step further and show that conditions of the form of eqs. (4.2.33) and (4.2.34) are in fact *necessary and sufficient* conditions for the joint unsharp measurability of complementary (qubit/spin-$\frac{1}{2}$) observables.

In [10], P. Busch formulated necessary and sufficient conditions for unsharp spin-$\frac{1}{2}$ observables to be jointly measurable (i.e. their effects occur in the range of a common POVM). A pair of two valued POVMs, $\{F_{\pm}^{(1)} = \frac{1}{2}(I \pm \mathbf{a} \cdot \sigma)\}$ and $\left\{F_{\pm}^{(3)} = (I \pm \mathbf{b} \cdot \sigma)/2\right\}$ is jointly measurable exactly when $|\mathbf{a}+\mathbf{b}| + |\mathbf{a}-\mathbf{b}| \leq 2$. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, the two POVMs represent unsharp versions of complementary observables. In this case the coexistence condition assumes the form,

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 \leq 1. \tag{4.2.35}$$

This is indeed equivalent to the trade-off relationships of eq. (4.2.32) deduced from the model where $\mathbf{a} = (0, 0, -\mathrm{A})$ or $\mathbf{a} = (0, 0, \mathrm{B})$, and $\mathbf{b} = (\mathrm{N_x}, \mathrm{N_y}, 0)$.

If $\mathrm{B} = 0$ is put into the model, the trade-off relation reduces to $(\mathrm{N_x}^2 + \mathrm{N_y}^2) + \mathrm{A}^2 = 1$. Hence this relation, taken as an expression of complementarity and deduced from the model, is equivalent to the joint measurability condition in the special case where $\mathrm{B} = 0$ (or similarly where $\mathrm{A} = 0$).

4.2.5 Conclusion

Zhu et al claim, without proof, to have obtained perfect population and coherence data from the statistics of a single run of an experiment, and thereby they seem to suggest that complementarity is not unconditionally valid but only subject to some qualifications. Analysis shows that the joint unsharp measurement of complementary observables is well consistent with complementarity.

It has been shown here that the statistics obtained allow one to reconstruct the probabilities for sharp measurements of two spin components of o, namely, σ_z and $\sigma_{\mathbf{n}}$. This means that the present experimental scheme constitutes an effective state determination procedure in that two out of three parameters characterizing any quantum state can be measured. In particular, this confirms that the population and coherence data (as defined in [78]) can indeed be recovered from such a joint measurement scheme. It is worth noting that there is a price for this feat: the joint measurement can neither be repeatable (it does not produce eigenstates of the outcomes), nor is it predictable: even when an eigenstate is put in, the outcomes will scatter; this is the necessary inaccuracy of the unsharp joint measurement.

The inequality $V_0^2 + P^2 \leq 1$, equation 3.2.13 (Chapter 3, Section 3.2.2) bears resemblance in form to equation(4.2.33) but their significance is fundamentally different: the quantities V_0, P refer to the object state and to mutually exclusive measurement of sharp path and interference observables; a generalization to arbitrary pairs of unsharp path and interference observables is straightforward [4]. By contrast, equation 4.2.33 describes a state-independent relation between specific pairs of unsharp path and interference observables, namely those which are jointly measurable. The former relation is an expression of complementarity for preparations while the latter reflects measurement complementarity and at the same time enables joint measurability.

To conclude: it has been shown that for complementary pairs of 'qubit' observables, the very possibility of making joint unsharp measurements is logically connected with the measurement version of Heisenberg's uncertainty principle, which is discussed in Chapter 3, section 3.5.3.

This connection is illustrated in a new realizable joint measurement scheme, the analysis of which gives rise to a novel trade-off relation for the contrasts of the measured unsharp observables. The contrast measures introduced above enable an operational interpretation of a joint measurability condition found earlier by P. Busch [10].

Finally, the measurement scheme discussed here provides an illustration of the way in which the use of entanglement (between the object and a probe) can lead to more powerful state determination procedures.

4.3 Complementarity and Uncertainty in Mach-Zehnder Interferometry

4.3.1 Path marking and quantum erasure in an atom interferometer

The first part of this section is a brief revisit to the SEW treatment of two path interference covered in Chapter 3 section 3.2.1. This will serve as an introduction to the second theme of their paper namely quantum erasure.

SEW consider the situation in which atoms, prepared in an excited state $|a\rangle$ and propagating in a superposition of states corresponding to two collimated beam paths, arrive singly at micro-maser cavities preceding each of the double slits([60], Fig. 3). Once in the cavity, the atoms will make a transition $|a\rangle \rightarrow |b\rangle$, spontaneously emitting a microwave photon. The state of the atom plus field changes from

$$\Psi_0(\mathbf{r}) = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})]|0_1 0_2\rangle|a\rangle \tag{4.3.1}$$

to the entangled state

$$\Psi(\mathbf{r}) = \tfrac{1}{\sqrt{2}}[\psi_1(\mathbf{r})|1_1 0_2\rangle + \psi_2(\mathbf{r})|0_1 1_2\rangle]|b\rangle \tag{4.3.2}$$

where $|1_1 0_2\rangle(|0_1 1_2\rangle)$ represents the field state in which there is one photon in cavity 1 and none in cavity 2 (no photon in cavity 1 and one in cavity 2). Thus, the micromaser cavities act as *which-way* detectors only if a photon left in the cavity changes the electromagnetic field in a detectable way.

SEW next considered the possibility of recovering coherence and thus the interference pattern while removing or *erasing* the path information left in the microwave cavity detectors.

To model this they propose a modified arrangement whereby the two cavities are separated by a shutter-detector combination ([60], Fig. 5a). This allows for the radiation either to be confined to the upper or lower cavity when the shutters are closed or for the radiation to be absorbed by a detector behind each shutter when it is opened. In the latter case the path marking information can be said to have been erased.

In the quantum erasure experiment one makes use of the fact that the state (4.3.2) has an equivalent form,

$$\Psi(\mathbf{r}) = \tfrac{1}{\sqrt{2}}[\psi_+(\mathbf{r})|+\rangle + \psi_-(\mathbf{r})|-\rangle]|b\rangle \tag{4.3.3}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}[|1_1 0_2\rangle \pm |0_1 1_2\rangle]$, $\psi_\pm = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) \pm \psi_2(\mathbf{r})]$.

To display the effects of the quantum erasure the experimental procedure is as follows: after an atom arrives on the screen, the shutter is opened and the state of the erasure detector behind the shutters, which has changed from $|d\rangle$ to $|e\rangle$ is recorded. In half the observations it will be left in an excited state indicating that there has been a photon in one of the cavities which has been absorbed. In the remaining cases there is no detection. Thus,

$$\frac{1}{\sqrt{2}}[\psi_+|+\rangle + \psi_-|-\rangle]|b\rangle|d\rangle \longrightarrow \frac{1}{\sqrt{2}}[\psi_+|0_1 0_2\rangle|b\rangle|e\rangle + \psi_-|-\rangle|b\rangle|d\rangle]. \tag{4.3.4}$$

As seen in equation (4.3.3) the symmetric atom state, $|\psi_+\rangle$, is coupled with the symmetric cavity field state, i.e. the atom will be left in the translational state $|\psi_+\rangle$ if the field is registered to be in the state $|+\rangle$. The probability density of those atoms will show the maxima and minima (fringes) of an interference pattern, $P_+(\mathbf{r}) = |\psi_+(\mathbf{r})|^2 = P_0(\mathbf{r})$ (Chapter 3 equation 3.2.2).

Atoms arriving at the screen for which there is no corresponding signal from the quantum erasure detector, i.e. the field state is detected to be $|-\rangle$ so that the atom is left in the translational state $|\psi_-\rangle$, will display anti-fringes, $P_-(\mathbf{r}) = |\psi_-(\mathbf{r})|^2$.

If all the events are counted, irrespective of the quantum erasure detector state, the distribution is,

$$\frac{1}{2}P_+(\mathbf{r}) + \frac{1}{2}P_-(\mathbf{r}) = \frac{1}{2}[|\psi_1|^2 + |\psi_2|^2] = P(\mathbf{r}). \tag{4.3.5}$$

The maxima of one pattern overlap the minima of the other one, washing out the fringes.

This consideration shows that in the entangled state, $\Psi(\mathbf{r})$ (equation (4.3.2)), the information about path as well as interference is fully available. Choosing to measure the path marking basis states, $|1_1 0_2\rangle, |0_1 1_2\rangle$, of the probe field yields which way information. Measuring instead the field states $|+\rangle, |-\rangle$ allows the recovery of interference fringes or anti-fringes, respectively. The two options are mutually exclusive (an instance of measurement complementarity); in the first case it is the interference information which is lost whereas in the second case it is path information which is lost.

It should be noted that this situation is analogous to the Einstein, Podolsky, Rosen (EPR) experiment (in Bohm's version for spin $\frac{1}{2}$ pairs) which also makes use of an entangled state with more than one Schmidt decomposition as in Eqs. (4.3.2) and (4.3.3).

4.3.2 A Mach-Zehnder interferometer analogue of SEW's experiment

Consider a special case of the MZI in Fig. 4.2, first with no path marking and no phase shifter activated ($\delta = 0$). The two possible input states from I_1, I_2 will be represented by orthogonal unit vectors $|1\rangle, |2\rangle$, of a two dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^2$, which will be interpreted as path eigenstates. When a photon entering via I_1 (represented by a "path" state $|1\rangle$) arrives at the beam splitter BS_1 it is changed to an equally weighted superposition and similarly for an input state $|2\rangle$.

$$|1\rangle \to= \tfrac{1}{\sqrt{2}}[|1\rangle + \imath|2\rangle], \qquad |2\rangle \to= \tfrac{1}{\sqrt{2}}[\imath|1\rangle + |2\rangle] \tag{4.3.6}$$

Arriving at detector D_1 will be a component of the path $I_1BS_1M_1$ reflected by BS_2 carrying a total phase shift of $\frac{\pi}{2}$ from M_1 plus $\frac{\pi}{2}$ from BS_2 and a component of the path $I_1BS_1M_2$ transmitted by BS_2 also carrying a total phase shift of $\frac{\pi}{2}$ from BS_1 plus $\frac{\pi}{2}$ from M_2. Hence, detector D_1 will register an output as these two are in phase.

Arriving at detector D_2 will be a component of the path $I_1BS_1M_1$ transmitted by BS_2 carrying a total phase shift of $\frac{\pi}{2}$ from M_1 and a component of the path $I_1BS_1M_2$ reflected by BS_2 carrying a total phase shift of $\frac{\pi}{2}$ from BS_1 plus $\frac{\pi}{2}$ from M_2 plus $\frac{\pi}{2}$ from BS_2. Hence, detector D_2 will register no output as these two are π out of phase. So, if $I_1BS_1M_1BS_2D_1$ is the path represented by $|1\rangle$, any measurement of the output

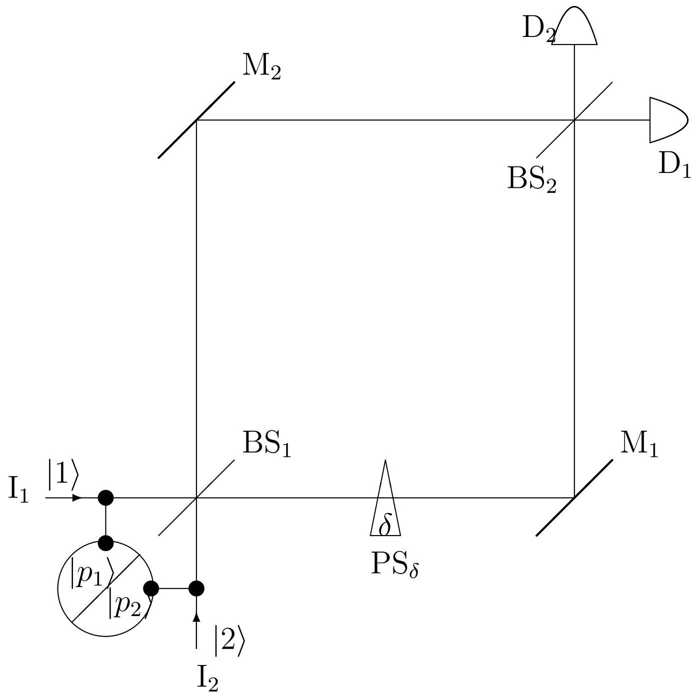

Figure 4.2: A Mach-Zehnder interferometer with path marking and phase shifter.

of D$_1$ has associated with it projector $|1\rangle\langle 1|$. This will be identified with the spectral projection of the Pauli operator σ_z associated with the eigenvalue 1, $|1\rangle\langle 1| = \frac{1}{2}(I+\sigma_z)$.

Similarly, I$_2$BS$_1$M$_2$BS$_2$D$_2$ corresponds to $|2\rangle$ and any measurement of the output of D$_2$ has associated with it projector $|2\rangle\langle 2| = \frac{1}{2}(I - \sigma_z)$.

Next, a MZI with path marking before the beam splitter BS$_1$ will be considered.

This will be implemented by introducing a probe system which interacts with the photons. The probe is represented by a two dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^2$, with path marking pointer states $|p_1\rangle$ and $|p_2\rangle$ where $|p_1\rangle$ marks $|1\rangle$ and $|p_2\rangle$ marks $|2\rangle$. A phase shifter, δ in one path after BS$_1$ completes the analogy with the SEW experiment.

The inputs from I$_1$, I$_2$ without marking (the probe is in a neutral state $|p_0\rangle$) can be represented by

$$\psi_i \to \psi_i \otimes |p_0\rangle = (\alpha|1\rangle + \beta|2\rangle) \otimes |p_0\rangle. \tag{4.3.7}$$

When the path marking is turned on the state arriving at BS_1 is

$$\psi_i \otimes |p_0\rangle \to \alpha|1\rangle \otimes |p_1\rangle + \beta|2\rangle \otimes |p_2\rangle. \tag{4.3.8}$$

The final state, after beam splitter BS_2 is:

$$\begin{aligned}\Psi_f^\delta &= \tfrac{1}{2}\alpha[(-e^{\imath\delta} - 1)|1\rangle + \imath(e^{\imath\delta} - 1)|2\rangle] \otimes |p_1\rangle \\ &\quad + \tfrac{1}{2}\beta[\imath(-e^{\imath\delta} + 1)|1\rangle - (1 + e^{\imath\delta})|2\rangle] \otimes |p_2\rangle.\end{aligned} \tag{4.3.9}$$

A variety of possible experiments will now be discussed.

Path detection in outputs D_1, D_2

The simplest case of this MZI is with no path marking, $|p_1\rangle = |p_2\rangle = |p_0\rangle$ and no phase shift, $\delta = 0$, analogous to a double slit interferometer (SEW) with no path marking and the far field detector placed centrally; as explained above the output state is,

$$\Psi_f^o = -(\alpha|1\rangle + \beta|2\rangle) \otimes |p_0\rangle = -\psi_i \otimes |p_0\rangle. \tag{4.3.10}$$

Observing the output of detectors D_1, D_2 with no path marking is represented by the PVM, $M_1 = |1\rangle\langle 1| \otimes I$, $M_2 = |2\rangle\langle 2| \otimes I$.

The probabilities for an output at D_1 and D_2 are,

$$\begin{aligned}\langle \Psi_f^o|M_1|\Psi_f^o\rangle &= \langle \psi_i|1\rangle\langle 1|\psi_i\rangle = |\alpha|^2 \\ \langle \Psi_f^o|M_2|\Psi_f^o\rangle &= \langle \psi_i|2\rangle\langle 2|\psi_i\rangle = |\beta|^2.\end{aligned} \tag{4.3.11}$$

The input observable measured by this experiment has a POVM $\{E_1^0,\ E_2^0\}$ defined by,

$$\langle \Psi_f^o|M_k|\Psi_f^o\rangle = \langle \psi_i|E_k^o|\psi_i\rangle \tag{4.3.12}$$

for all ψ_i and $k = 1, 2$ It follows that E_1^0, E_2^0 are the projection

$$E_1^0 = |1\rangle\langle 1| = \tfrac{1}{2}(I + \sigma_z), \quad E_2^0 = |2\rangle\langle 2| = \tfrac{1}{2}(I - \sigma_z). \tag{4.3.13}$$

This reproduces the discussion of path detection connected with Figure 4.2: If $|\psi_i\rangle = |1\rangle$ then the probabilities of detection at D_1 and D_2 are $\langle 1|E_1^0|1\rangle = \langle 1|\frac{1}{2}(I+\sigma_z)|1\rangle = 1$ and $\langle 1|E_2^0|1\rangle = \langle 1|\frac{1}{2}(I - \sigma_z)|1\rangle = 0$, i.e. the photon will be registered in D_1 with certainty, the path is completely determined. No photon is registered on detector D_2.

A similar consideration can be applied to $|\psi_i\rangle = |2\rangle$.

The probabilities for the detection events at D_1, D_2 are thus expectation values of E_1^0, E_2^0 in the input state ψ_i. E_1^0 and E_2^0 form a PVM representing a sharp path observable which can be identified with σ_z.

Interference detection at D_1, D_2

An interference measurement will now be considered. In a double slit interferometer both slits would be open and a detector placed at the first minimum. If $\delta = \frac{\pi}{2}$ is chosen, the output state is,

$$\Psi_f^{\frac{\pi}{2}} = \tfrac{(1+\imath)}{\sqrt{2}} \left[-\alpha \tfrac{1}{\sqrt{2}}(|1\rangle + |2\rangle) + \beta \tfrac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \right] \otimes |p_0\rangle. \tag{4.3.14}$$

If a measurement of $M_k = |1\rangle\langle 1| \otimes I$, $k = 1, 2$ is now applied by observing the output of D_k the expectation is,

$$\begin{aligned} \langle \Psi_f^{\frac{\pi}{2}} | M_1 | \Psi_f^{\frac{\pi}{2}} \rangle &= \tfrac{1}{2}(|\alpha|^2 - \beta^*\alpha - \alpha^*\beta + |\beta|^2) = \langle \psi_i | E_1^{\frac{\pi}{2}} | \psi_i \rangle, \\ \langle \Psi_f^{\frac{\pi}{2}} | M_2 | \Psi_f^{\frac{\pi}{2}} \rangle &= \tfrac{1}{2}(|\alpha|^2 + \beta^*\alpha + \alpha^*\beta + |\beta|^2) = \langle \psi_i | E_2^{\frac{\pi}{2}} | \psi_i \rangle, \end{aligned} \tag{4.3.15}$$

where $E_1^{\frac{\pi}{2}}$ and $E_1^{\frac{\pi}{2}}$ are

$$E_1^{\frac{\pi}{2}} = \tfrac{1}{2}(I - \sigma_x), \quad E_2^{\frac{\pi}{2}} = \tfrac{1}{2}(I + \sigma_x). \tag{4.3.16}$$

Following customary practice, an observable with the form $\cos\delta\,\sigma_x + \sin\delta\,\sigma_y$, $0 \le \delta < \pi$, will be considered as an interference observable, given that the path is represented by σ_z. Interference observables are singled out by the condition that the interference contrast can assume its maximum possible value. In this case their eigenstates give equal probabilities of $\frac{1}{2}$ to the path projections $|1\rangle\langle 1|, |2\rangle\langle 2|$, thus fulfilling the condition of value complementarity.

In the present experiment, the measured input observable is defined by the projections of eq. (4.3.16); these are the spectral projections of the operator σ_x (or $-\sigma_x$), which is indeed an interference observable.

The path marking set up

Now consider the case where $|p_1\rangle$ and $|p_2\rangle$ are mutually orthogonal , $\langle p_1|p_2\rangle = 0$. This is a MZI analogy of SEW's path marking experiment. The influence of the path marking on the outputs of the detectors can be found using each of the four measurement projections of path $|k\rangle$ jointly with marker $|p_\ell\rangle$, $M'_{k\ell} = |k\rangle\langle k| \otimes |p_\ell\rangle\langle p_\ell|$, $k, \ell = 1, 2$ e.g.

$$\langle \Psi_f^\delta | M'_{11} | \Psi_f^\delta \rangle = |\tfrac{1}{2}\alpha(e^{\imath\delta} + 1)|^2 = \tfrac{1}{4}|\alpha|^2(1 + \cos\delta). \tag{4.3.17}$$

The incoming state being given by equation 4.3.8.

Then using $\langle \Psi_f^\delta | M'_{k\ell} | \Psi_f^\delta \rangle = \langle \psi_i | E'_{k\ell} | \psi_i \rangle$ the POVM associated with the input states can be found, (in short, input observable.

$$\begin{aligned} E'_{11} &= \tfrac{1}{2}(I + \sigma_z)\cos^2\tfrac{\delta}{2}, \qquad & E'_{21} &= \tfrac{1}{2}(I + \sigma_z)\sin^2\tfrac{\delta}{2}, \\ E'_{12} &= \tfrac{1}{2}(I - \sigma_z)\sin^2\tfrac{\delta}{2}, \qquad & E'_{22} &= \tfrac{1}{2}(I - \sigma_z)\cos^2\tfrac{\delta}{2}. \end{aligned} \tag{4.3.18}$$

These effects are all fractions of path projections.

The marginal input POVM associated with the detectors D_1, D_2 is

$$\begin{aligned} F_1' &= E_{11}' + E_{12}' = \tfrac{1}{2}(I + \cos\delta\sigma_z), \\ F_2' &= E_{22}' + E_{21}' = \tfrac{1}{2}(I - \cos\delta\sigma_z). \end{aligned} \tag{4.3.19}$$

This POVM represents a path observable smeared by $\cos\delta$. The unsharpness inherent in the detector marginal is reflected in the non-zero probability of detector D_2 firing even if the input state is a path eigenstate $|1\rangle$. Here the effect of the perfect path marking interaction is seen: irrespective of the phase parameter value, the MZI does not detect any interference effects. When $\delta = 0$, the POVM $\{F_1', F_2'\}$ becomes a sharp path observable and when the MZI is set to observe interference, $\delta = \frac{\pi}{2}$, this POVM is reduced to being trivial, $F_1' = \frac{1}{2}I = F_2'$, giving no path nor interference information. This is in line with the prediction of SEW: path marking results in the interference pattern being washed out. In addition it is found that after the path marking interaction, all the detectors are able to “see” is a “shadow” of the path information provided by the path marker.

The path marker marginals are,

$$\begin{aligned} G_1' &= E_{11}' + E_{21}' = \tfrac{1}{2}(I + \sigma_z), \\ G_2' &= E_{22}' + E_{12}' = \tfrac{1}{2}(I - \sigma_z). \end{aligned} \tag{4.3.20}$$

These are sharp path observables under all δ.

It is possible to define a third marginal,

$$\begin{aligned} H_1' &= E_{11}' + E_{22}' = \cos^2\tfrac{\delta}{2}\, I, \\ H_2' &= E_{12}' + E_{21}' = \sin^2\tfrac{\delta}{2}\, I, \end{aligned} \tag{4.3.21}$$

which in the present experiment turns out to be trivial.

4.3.3 Quantum erasure

In the previous experiment, each path was correlated with one of two orthogonal marker states. A new set of pointer states, which are superpositions of the two orthogonal path marker states, can now be considered,

$$|q_1\rangle = \tfrac{1}{\sqrt{2}}(|p_1\rangle + e^{\imath\gamma}|p_2\rangle), \quad |q_2\rangle = \tfrac{1}{\sqrt{2}}(|p_1\rangle - e^{\imath\gamma}|p_2\rangle). \tag{4.3.22}$$

Observing these symmetric states involves outputs for which both $|p_1\rangle$ and $|p_2\rangle$ are equally likely, so no information about the path is recorded.

The final state (4.3.9) in terms of the new pointer states is

$$\begin{aligned}
\Psi_f^{\delta,\gamma} \quad &= \tfrac{1}{2\sqrt{2}}\left[\left(-\alpha(1+e^{\imath\delta}) + \imath e^{-\imath\gamma}\beta(1-e^{\imath\delta})\right)|1\rangle\right.\\
&\left. + \left(-\imath\alpha(1-e^{\imath\delta}) - e^{-\imath\gamma}\beta(1+e^{\imath\delta})\right)|2\rangle\right] \otimes |q_1\rangle \\
&+ \tfrac{1}{2\sqrt{2}}\left[\left(-\alpha(1+e^{\imath\delta}) - \imath e^{-\imath\gamma}\beta(1-e^{\imath\delta})\right)|1\rangle\right.\\
&\left. + \left(-\imath\alpha(1-e^{\imath\delta}) + e^{-\imath\gamma}\beta(1+e^{\imath\delta})\right)|2\rangle\right] \otimes |q_2\rangle.
\end{aligned} \tag{4.3.23}$$

As before, the four joint probabilities for the marker and detector outputs, are defined as the expectations of the projections, $M''_{k\ell} = |k\rangle\langle k| \otimes |q_\ell\rangle\langle q_\ell|$, $k, \ell = 1, 2$.

The associated input POVM is determined via the relation $\langle\Psi_f^{\delta,\gamma}|M''_{k\ell}|\Psi_f^{\delta,\gamma}\rangle = \langle\psi_i|E''_{k\ell}|\psi_i\rangle$ (here also the input state is the one reached after path marking, equation 4.3.8):

$$\begin{aligned}
E''_{11} &= \tfrac{1}{4}(I - \sin\delta\cos\gamma\sigma_x - \sin\delta\sin\gamma\sigma_y + \cos\delta\sigma_z) = \tfrac{1}{4}(I - \mathbf{n}''\cdot\sigma),\\
E''_{21} &= \tfrac{1}{4}(I + \sin\delta\cos\gamma\sigma_x + \sin\delta\sin\gamma\sigma_y - \cos\delta\sigma_z) = \tfrac{1}{4}(I + \mathbf{n}''\cdot\sigma),\\
E''_{12} &= \tfrac{1}{4}(I + \sin\delta\cos\gamma\sigma_x + \sin\delta\sin\gamma\sigma_y + \cos\delta\sigma_z) = \tfrac{1}{4}(I + \mathbf{m}''\cdot\sigma),\\
E''_{22} &= \tfrac{1}{4}(I - \sin\delta\cos\gamma\sigma_x - \sin\delta\sin\gamma\sigma_y - \cos\delta\sigma_z) = \tfrac{1}{4}(I - \mathbf{m}''\cdot\sigma).
\end{aligned} \tag{4.3.24}$$

Here the unit vectors are introduced,

$$\mathbf{n}'' = (\sin\delta\cos\gamma, \sin\delta\sin\gamma, -\cos\delta) \quad \mathbf{m}'' = (\sin\delta\cos\gamma, \sin\delta\sin\gamma, \cos\delta). \tag{4.3.25}$$

The marginal POVM associated with the detector outputs is obtained by summing over both probe outputs:

$$\begin{aligned} F_1'' &= E_{11}'' + E_{12}'' = \tfrac{1}{2}(I + \cos\delta\sigma_z), \\ F_2'' &= E_{21}'' + E_{22}'' = \tfrac{1}{2}(I - \cos\delta\sigma_z). \end{aligned} \tag{4.3.26}$$

This is a smeared path observable. The marginal POVM associated with the probe outputs is obtained by summing over both detection outputs:

$$\begin{aligned} G_1'' &= E_{11}'' + E_{21}'' = \tfrac{1}{2}I, \\ G_2'' &= E_{12}'' + E_{22}'' = \tfrac{1}{2}I. \end{aligned} \tag{4.3.27}$$

This is a trivial observable, it provides no information about the input state ψ_i.

The fact that the detector POVM is a smeared path observable and the probe POVM is trivial can be understood as follows. The entanglement between probe and photon is devised to establish a strict correlation between the path states $|1\rangle$, $|2\rangle$ and the pointer states $|p_1\rangle$, $|p_2\rangle$, for any photon input state ψ_i. This correlation information is not accessible by measuring a probe observable with the eigenstates $|q_1\rangle$, $|q_2\rangle$ because these are equal weight superpositions of the path marker states. Further, the reduced state of the photon after the coupling has been established is a mixture of the path states, so that any phase relation between these states has been erased. Accordingly, the detector outputs cannot detect any interference indicative of coherence between the path input states, and the only information left about the input is path information.

A third marginal input POVM is defined as follows:

$$H_1'' = E_{11}'' + E_{22}'' = \tfrac{1}{2}(I - \sin\delta\cos\gamma\sigma_x - \sin\delta\sin\gamma\sigma_y),$$
$$H_2'' = E_{12}'' + E_{21}'' = \tfrac{1}{2}(I + \sin\delta\cos\gamma\sigma_x + \sin\delta\sin\gamma\sigma_y). \tag{4.3.28}$$

This is a smeared interference observable, the unsharpness being determined by the term $\sin\delta$ and the direction, of the associated Poincaré sphere vectors being given by $\pm(\cos\gamma, \sin\gamma, 0)$. By varying γ from 0 to 2π, all possible interference observables can be realized.

The quantum erasure scheme presented here constitutes a joint unsharp measurement of path and interference observables as represented by the marginal POVMs $\{F_1'', F_2''\}$ and $\{H_1'', H_2''\}$.

Also for $\delta = \frac{\pi}{2}$, all four effects $E_{k\ell}$ are fractions of spectral projections of a sharp interference observable; the marginal $\{H_1'', H_2''\}$ becomes a sharp interference observable and the marginal $\{F_1'', F_2''\}$ becomes a trivial observable. Here the observation of SEW have been recovered i.e. the detector statistics conditional on the probe output readings display perfect interference with perfect visibility. In fact, somewhat more has been found: independently of the photon input state, the conditional probabilities for detections at D_1, D_2 given a probe recording of $|q_1\rangle$ (say) are

$$\begin{aligned} Prob(D_1|q_1) &= \langle\psi_i|E_{11}''\psi_i\rangle/\langle\psi_i|G_1''\psi_i\rangle \\ &= \langle\psi_i|\tfrac{1}{2}(I - \cos\gamma\sigma_x - \sin\gamma\sigma_y)\psi_i\rangle, \\ Prob(D_2|q_1) &= \langle\psi_i|E_{21}''\psi_i\rangle/\langle\psi_i|G_1''\psi_i\rangle \\ &= \langle\psi_i|\tfrac{1}{2}(I + \cos\gamma\sigma_x + \sin\gamma\sigma_y)\psi_i\rangle. \end{aligned} \tag{4.3.29}$$

For $\gamma = 0$ and the input state $\psi_i = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, this gives $Prob(D_1|q_1) = 0$ and $Prob(D_2|q_1) = 1$. This corresponds to the case of perfect interference antifringes. Similarly, the detector probabilities conditional on $|q_2\rangle$ and the above input eigenstate of σ_x are 1, and 0 for D_1 and D_2, respectively. These are characteristic of interference fringes.

This situation is a consequence of the fact that for the above input and $\delta = \frac{\pi}{2}$, $\gamma = 0$, the total output state is an EPR state, analogous to the state described in the SEW quantum erasure setup of Section 2:

$$\begin{aligned} \Psi_f^{\frac{\pi}{2}} &= -\tfrac{1}{\sqrt{2}}(|1\rangle \otimes |p_1\rangle + |2\rangle \otimes |p_2\rangle) \\ &= -\tfrac{1}{\sqrt{2}}(1+\imath)\tfrac{1}{\sqrt{2}}[|-,x\rangle \otimes |q_1\rangle + |+,x\rangle \otimes |q_2\rangle]. \end{aligned} \tag{4.3.30}$$

where $|\pm, x\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$.

4.3.4 Quantitative quantum erasure

The two possible experimental options discussed in the preceding subsections, of path marking and quantum erasure are mutually exclusive in that they require settings and operations that cannot be performed at the same time: for path determination, the probe eigenstates $|p_1\rangle, |p_2\rangle$ must be read out, while for the recovery of interference it is necessary to record the detector outputs conditional on the probe output states $|q_1\rangle, |q_2\rangle$. Erasure was achieved by choosing $\delta = \frac{\pi}{2}$, which led to the POVM $\{E''_{k\ell}\}$ being constituted of (fractions of) spectral projections of an interference observable. Accordingly, the only non-trivial marginal is the sharp interference observable $\{H''_1, H''_2\}$.

If, however, the interferometric parameter δ is varied between 0 and $\frac{\pi}{2}$, then the POVM $\{E''_{k\ell}\}$ is a joint observable for an unsharp path and an unsharp interference observable. In this case the experiment provides simultaneous information about these noncommuting quantities. In the limit of $\delta = 0$, the interference marginal $\{H''_1, H''_2\}$ becomes trivial and the path marginal $\{F''_1, F''_2\}$ becomes sharp.

The possibility of obtaining some joint information about both observables, path and interference, can also be achieved by modifying the path marking coupling in such a way that the correlation between the paths and the probe indicator observable is not perfect. This has been described as *quantitative quantum erasure* [e.g., [26]]. Here it is shown that quantitative quantum erasure is an instance of a joint unsharp measurement.

Consider the path marking interaction to be of the same form as before, eq. (4.3.9), but specifying the marker states $|p_1\rangle, |p_2\rangle$ to be nonorthogonal. Their associated Poincaré sphere vectors will be chosen to be tilted by an angle θ away from the $\pm z$ axis towards the positive x axis, respectively. Choose as pointer states $|q_1\rangle, |q_2\rangle$ equal to the up and down eigenstates of σ_z and define,

$$\begin{aligned} |p_1\rangle &= \cos\tfrac{\theta}{2}|q_1\rangle + \sin\tfrac{\theta}{2}|q_2\rangle, \\ |p_2\rangle &= \sin\tfrac{\theta}{2}|q_1\rangle + \cos\tfrac{\theta}{2}|q_2\rangle. \end{aligned} \tag{4.3.31}$$

The final state after the path marking interaction is

$$\begin{aligned} \Psi_f^{\delta,\theta} = & \big[\big(-\tfrac{\alpha}{2}\cos\tfrac{\theta}{2}(1+e^{i\delta}) + i\tfrac{\beta}{2}\sin\tfrac{\theta}{2}(1-e^{i\delta})\big)\ |1\rangle \\ & + \big(-i\tfrac{\alpha}{2}\cos\tfrac{\theta}{2}(1-e^{i\delta}) - \tfrac{\beta}{2}\sin\tfrac{\theta}{2}(1+e^{i\delta})\big)\ |2\rangle\big] \otimes |q_1\rangle \\ & + \big[\big(-\tfrac{\alpha}{2}\sin\tfrac{\theta}{2}(1+e^{i\delta}) + i\tfrac{\beta}{2}\cos\tfrac{\theta}{2}(1-e^{i\delta})\big)\ |1\rangle \\ & + \big(-i\tfrac{\alpha}{2}\sin\tfrac{\theta}{2}(1-e^{i\delta}) - \tfrac{\beta}{2}\cos\tfrac{\theta}{2}(1+e^{i\delta})\big)\ |2\rangle\big] \otimes |q_2\rangle. \end{aligned} \tag{4.3.32}$$

Now it is possible to determine the input effects $E'''_{k\ell}$ associated with the output projections $M'''_{k\ell} = |k\rangle\langle k| \otimes |q_\ell\rangle\langle q_\ell|$ via $\langle\Psi_f^{\delta,\theta}|M'''_{k\ell}|\Psi_f^{\delta,\theta}\rangle = \langle\psi_i|E'''_{k\ell}|\psi_i\rangle$ (again, the incoming state is the one reached after path marking, equation 4.3.8),

$$\begin{aligned}
E'''_{11} &= \tfrac{1}{4}[I(1+\cos\theta\cos\delta) - \sin\delta\sin\theta\sigma_x + (\cos\delta+\cos\theta)\sigma_z] \\
&= \tfrac{1}{4}[I(1+\cos\theta\cos\delta) + \mathbf{m}'''\cdot\sigma] \\
&= \tfrac{1}{2}(1+\cos\theta\cos\delta)[\tfrac{1}{2}(I+\hat{\mathbf{m}}'''\cdot\sigma)] \\
E'''_{21} &= \tfrac{1}{4}[I(1-\cos\theta\cos\delta) + \sin\delta\sin\theta\sigma_x - (\cos\delta-\cos\theta)\sigma_z] \\
&= \tfrac{1}{4}[I(1-\cos\theta\cos\delta) - \mathbf{n}'''\cdot\sigma] \\
&= \tfrac{1}{2}(1-\cos\theta\cos\delta)[\tfrac{1}{2}(I-\hat{\mathbf{n}}'''\cdot\sigma)] \\
E'''_{12} &= \tfrac{1}{4}[I(1-\cos\theta\cos\delta) - \sin\delta\sin\theta\sigma_x + (\cos\delta-\cos\theta)\sigma_z] \\
&= \tfrac{1}{4}[I(1-\cos\theta\cos\delta) + \mathbf{n}'''\cdot\sigma] \\
&= \tfrac{1}{2}(1-\cos\theta\cos\delta)[\tfrac{1}{2}(I+\hat{\mathbf{n}}'''\cdot\sigma)] \\
E'''_{22} &= \tfrac{1}{4}[I(1+\cos\theta\cos\delta) + \sin\delta\sin\theta\sigma_x - (\cos\delta+\cos\theta)\sigma_z] \\
&= \tfrac{1}{4}[I(1+\cos\theta\cos\delta) - \mathbf{m}'''\cdot\sigma] \\
&= \tfrac{1}{2}(1+\cos\theta\cos\delta)[\tfrac{1}{2}(I-\hat{\mathbf{m}}'''\cdot\sigma)]
\end{aligned} \tag{4.3.33}$$

where

$$\begin{aligned}
\mathbf{m}''' &= (-\sin\delta\sin\theta, 0, (\cos\delta+\cos\theta)), \\
\mathbf{n}''' &= (-\sin\delta\sin\theta, 0, (\cos\delta-\cos\theta))
\end{aligned} \tag{4.3.34}$$

and $\hat{\mathbf{m}}'''$, $\hat{\mathbf{n}}'''$ are unit the vectors

$$\begin{aligned}
\hat{\mathbf{m}}''' &= \mathbf{m}'''/(1+\cos\theta\cos\delta) \\
\hat{\mathbf{n}}''' &= \mathbf{n}'''/(1-\cos\theta\cos\delta)
\end{aligned} \tag{4.3.35}$$

Note that the four effects are multiples of projections.

The three marginal POVMs are determined as before:

$$\begin{aligned}
F_1''' &= E_{11}''' + E_{12}''' = \tfrac{1}{2}(I - \sin\delta\sin\theta\sigma_x + \cos\delta\sigma_z)\\
F_2''' &= E_{22}''' + E_{21}''' = \tfrac{1}{2}(I + \sin\delta\sin\theta\sigma_x - \cos\delta\sigma_z)\\
G_1''' &= E_{11}''' + E_{21}''' = \tfrac{1}{2}(I + \cos\theta\sigma_z)\\
G_2''' &= E_{22}''' + E_{12}''' = \tfrac{1}{2}(I - \cos\theta\sigma_z)\\
H_1''' &= E_{11}''' + E_{22}''' = \tfrac{1}{2}I(1 + \cos\theta\cos\delta)\\
H_2''' &= E_{12}''' + E_{21}''' = \tfrac{1}{2}I(1 - \cos\theta\cos\delta)
\end{aligned} \tag{4.3.36}$$

For $\delta = \frac{\pi}{2}$, the first marginal POVM (corresponding to the detector statistics) becomes an unsharp interference observable, while the second marginal POVM (corresponding to the probe output statistics) is an unsharp path observable. In both cases the unsharpness is determined by the parameter θ.

4.3.5 Manifestations of complementarity in MZI.

The sequence of experiments in section 4.3.2 is a demonstration of complementarity in different guises. In the first two experiments path detection (page 124) and interference detection (page 125) are mutually exclusive because this requires settings of the parameter δ which cannot be realized in the same experiment, namely, $\delta = 0$ for path (σ_z) measurement and $\delta = \frac{\pi}{2}$ for interference (σ_x) measurement, respectively. Here is an instance of the complementarity of measurement setups or measurement complementarity: these two noncommuting sharp observables do not admit any joint measurement.

These experiments are also found to confirm preparation complementarity, defined in Chapter 3, Section 3.5.1: Recalling that sending a path eigenstate $|1\rangle$ or $|2\rangle$ into the MZI setup to observe interference leads to the probability $\langle 1|E_1^{\frac{\pi}{2}}|1\rangle = \langle 1|E_2^{\frac{\pi}{2}}|1\rangle = \frac{1}{2}$, the value of the interference observable is maximally uncertain. And, an interference eigenstate, $|\pm x\rangle$, is fed into the MZI setup to measure path ($\delta = 0$), the path observable is uncertain $\langle \pm x|E_1^0|\pm x\rangle = \langle \pm x|E_2^0|\pm x\rangle = \frac{1}{2}$.

Value complementarity, defined in Chapter 3, Section 3.5.1, is quoted by SEW but they do not explain why it can be used to explain duality. Nevertheless, value complementarity can be used to explain the disappearance of interference fringes in the MZI resulting from path marking. Once perfect correlation between the path states and the marker states is established in the entangled state (4.3.9), the reduced state of the photon is a mixture of the path eigenstates $|1\rangle$ and $|2\rangle$. In each of these, the path is definite, and therefore, in accordance with value complementarity, the outcomes of a subsequent interference measurement are equally probable: no interference fringes show up. Indeed, this remains true for any mixture of path eigenstates.

This account in terms of preparation complementarity views the path marking interaction as part of a preparation process. An alternative explanation is possible in terms of measurement complementarity.

In the experiment of section 4.3.2, where sharp path marking is followed by the interference set up with $\delta = \frac{\pi}{2}$, it was found that the path measurement interaction leads to a complete loss of interference information detectable in D_1, D_2. All that the detectors can "see" is path information, irrespective of the value of δ (Eq. (4.3.19)).

If the sharp path marking is relaxed into unsharp path marking, section 4.3.4, setting the interferometer with $\delta = \frac{\pi}{2}$ defines an unsharp interference observable, which is jointly measured with the path that can be recorded at the path marker.

It is found that the less accurate the path marking is set by making $\cos\theta$ in

$G'''_{1,2} = \frac{1}{2}(I \pm \cos\theta\,\sigma_z)$ smaller, the sharper will be the interference measurement as $\sin\theta$ in $F'''_{1,2} = \frac{1}{2}(I \mp \sin\theta\,\sigma_x)$ becomes larger.

Here it can be seen that measurement complementarity defined in Chapter 3, Section 3.5.2 follows in the limits of making the path marginal or the interference marginal perfectly sharp, rendering the other trivial.

A similar analysis applies to the quantum erasure setup (Section 4.3.3): if $0 < \delta < \frac{\pi}{2}$, this setup realizes a joint measurement of the POVMs $\{F''_{1,}, F''_2\}$ and $\{H''_{1,}, H''_2\}$ which are unsharp path and interference observables.

Consider the case of $\delta = \frac{\pi}{2}$, where $F''_{1,2} = \frac{1}{2}I$ and $H''_{1,2} = \frac{1}{2}(I \mp \cos\gamma\sigma_x \mp \sin\gamma\sigma_y)$. Here is a sharp interference measurement and no path measurement. With $\delta = 0$, $F''_{1,2} = \frac{1}{2}(I \pm \sigma_z)$ and $H''_1 = \frac{1}{2}I$, so that there is a sharp path measurement and no interference. These two limit cases of a joint measurement scheme illustrate once more measurement complementarity as a relation of strict mutual exclusivity.

4.4 Discussion

At the end of Chapter 3 several different formulations were given of complementarity and uncertainty relations referring to the limitations and possibilities of preparing and measuring simultaneously, sharp and unsharp observables. These formulations have all been applied here to n.m.r. and MZI analogue versions of an interferometer with a path marking option. An important feature of the MZI analogue of the two-slit experiment is that the path marking is done without random momentum kicks. In fact, the coupling for the path marking process was arranged so as to constitute a non-demolition measurement in that if the input is a path eigenstate, say $|1\rangle$ then the total state after being correlated with a path marker is $|1\rangle|p_1\rangle$. Thus, the system has remained undisturbed, it is still in the original path eigenstate.

Chapter 5

Complementarity and Uncertainty: A Coherent Account

5.1 Introduction

"I wonder to myself a lot:

Now is it true or is it not,

That what is which and which is what?"

A.A. Milne *Winnie-the-Pooh* Chapter 7 (1926)

In Section 4.3.5 a study is made of how different versions of complementarity are made manifest in MZI. Preparation complementarity can be demonstrated because path detection and interference detection require that the MZI be setup in different, mutually exclusive ways. Or in the context of measurement complementarity; in a joint measurement scheme it is possible to show that going to the limit of perfect sharpness of one marginal renders the other marginal trivial; thus, the sharp measurement of one observable and any nontrivial measurement of the other observable are mutually exclusive.

Value complementarity can also be used to explain the disappearance of interference fringes in the MZI resulting from path marking. If the path is definite then the outcomes of a subsequent interference measurement are equally probable i.e. the interference fringes are washed out.

5.2 Value complementarity from preparation uncertainty relations

Recalling, from Chapter 3, Section 3.5.1, that a pair of value complementary observables A, B is characterized by the condition that in each of the eigenstates of A, all eigenvalues of B are equally likely to occur as outcomes of a B measurement; and vice versa. It is now shown that for qubit observables such as those occurring in the interferometry measurements discussed in previous sections, the value complementarity property can always be obtained as a consequence of some suitable uncertainty relation for the observables in question.

In what follows general states are represented as density operators $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$, $|\mathbf{r}| \leq 1$. Consider two observables represented by σ_x and σ_z. For these observables we recall from Chapter 2 2.3, the usual uncertainty relation in terms of variances reads,

$$\mathrm{Var}(\sigma_x)\mathrm{Var}(\sigma_z) \geq \tfrac{1}{4}|\langle[\sigma_x, \sigma_z]\rangle|^2 + \tfrac{1}{4}[\langle\sigma_x\sigma_z + \sigma_z\sigma_x\rangle - 2\langle\sigma_x\rangle\langle\sigma_z\rangle]^2. \tag{5.2.1}$$

Recalling Example 3 on page 19 and Equation 2.3.22; $(1-n_x^2)(1-n_z^2) = n_y^2 + n_z^2 n_x^2$

Using solely this inequality one cannot recover value complementarity without further information on the terms of the left hand side. But using the algebraic and spectral properties of the Pauli operators, one finds the right hand side of Eq. (5.2.1) to be equal to $\langle\sigma_y\rangle^2 + \langle\sigma_x\rangle^2\langle\sigma_z\rangle^2$. Then one can argue as follows: if the path is definite, that is, if $\rho = |\psi\rangle\langle\psi|$ with ψ an eigenstate of σ_z, the left hand side of the uncertainty relation (5.2.1) is zero, and therefore the terms on the right hand side must vanish, too.

Example 3, recalled above, can be used to illustrate this conclusion: Since $n_x^2 + n_y^2 + n_z^2 = 1$ if $n_z^2 = 1$ then $n_x^2 = n_y^2 = 0$. The terms on the righthand side of Equation 5.2.1 must vanish.

Thus as $\langle\sigma_z\rangle = 1$, then $\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0$. Since the eigenvalues of these quantities are ± 1, it follows that these observables have a uniform distribution. By symmetry, σ_z is uniformly distributed if σ_x has a definite value.

5.3 Quantitative duality relations are uncertainty relations

In the debates of the 1990s over complementarity in the context of interferometry and which-path experiments (Chapter 3, Section 3.2), the meaning of the term wave particle duality has gradually shifted away from a relation of strict exclusion of path determination and interference observation (in the same setup) to the broader idea of a continuous trade-off between approximate path determination and approximate interference determination. These discussions were eventually linked with earlier theoretical and experimental work of the 1980s on simultaneous but imperfect path determination and interference observation, as reviewed in chapter 3 section 3.2.4 and [23].

In Chapter 3 sections 3.2.2, 3.2.3 and 3.2.4, trade-off relations of the form $P^2 + V^2 \leq 1$ were presented as characterizations of the duality between path predictability and interference visibility. It has been debated whether the associated quantities are connected with uncertainties, and it has been shown that the associated trade-off relations are related in various ways to some forms of uncertainty relations; in the MZI context, reference is made, in particular, to the work of Björk *et al* [4], Dürr and Rempe [23], and Luis [47].

Using familiar measures of uncertainty, a stronger result will now be obtained. A simple demonstration is given, that in the context of MZ interferometry experiments, quantitative duality relations are indeed equivalent to the uncertainty relation for an appropriate pair of associated observables.

Any state ρ can be represented by a matrix of the following form in the basis of eigenvectors of σ_z:

$$\begin{aligned} &\rho = \begin{pmatrix} w_+ & re^{-i\theta} \\ re^{i\theta} & w_- \end{pmatrix}, \\ &w_\pm \geq 0,\ w_+ + w_- = 1, \\ &0 \leq r \leq \sqrt{w_+ w_-},\ 0 \leq \theta < 2\pi. \end{aligned} \tag{5.3.1}$$

The *path contrast* of ρ will be defined as

$$C_P = C_P(\rho) := |\mathrm{prob}(\sigma_z = +1, \rho) - \mathrm{prob}(\sigma_z = -1, \rho)| = |w_+ - w_-|. \tag{5.3.2}$$

This is identical to the predictability P entering the duality relation $P^2 + V^2 \leq 1$. Similarly defining the interference contrast of ρ as

$$C_I = C_I(\rho) = |\mathrm{prob}(\sigma_x = +1, \rho) - \mathrm{prob}(\sigma_x = -1, \rho)| = 2r\cos\theta. \tag{5.3.3}$$

With the specification $\theta = 0$, or with an alternative choice of interference observable, this reduces to the visibility $V = (I_{max} - I_{min})/(I_{max} + I_{min})$. Using $r^2 \leq w_+ w_-$, the following duality relation is easily obtained, much in the same way as in section 3.2.2.

$$C_P^2 + C_I^2 = w_+^2 + w_-^2 - 2w_+w_- + 4r^2\cos^2\theta \leq 1. \tag{5.3.4}$$

Observing that

$$C_P^2 = \langle\sigma_z\rangle^2 = 1 - \mathrm{Var}(\sigma_z),$$
$$C_I^2 = \langle\sigma_x\rangle^2 = 1 - \mathrm{Var}(\sigma_x), \tag{5.3.5}$$

then the above duality inequality can be equivalently expressed as,

$$\mathrm{Var}(\sigma_z) + \mathrm{Var}(\sigma_x) = 2 - (C_P^2 + C_I^2) \geq 1. \tag{5.3.6}$$

Again, Example 3 (page 19) can be used to support this:

$$\mathrm{Var}(\sigma_x) + \mathrm{Var}(\sigma_z) = (1 - n_x^2) + (1 - n_z^2) = 2 - (n_x^2 + n_z^2) = 1 + n_y^2 \geq 1 \tag{5.3.7}$$

Thus, the present duality relation is equivalent to a form of uncertainty trade-off relation. As before, value complementarity is again entailed as a limit case.

It will now be shown that this last inequality is actually a direct consequence of the uncertainty relation (5.2.1). As noted there that relation can be rewritten as follows:

$$(1 - \langle\sigma_z\rangle^2)(1 - \langle\sigma_x\rangle^2) \geq \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2\langle\sigma_x\rangle^2. \tag{5.3.8}$$

This, in turn, is equivalent to

$$1 - \langle\sigma_z\rangle^2 - \langle\sigma_x\rangle^2 \geq \langle\sigma_y\rangle^2, \qquad \text{that is, to} \tag{5.3.9}$$
$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 \leq 1. \tag{5.3.10}$$

Using again $\mathrm{Var}(\sigma_z) = 1 - \langle\sigma_z\rangle^2$, etc, it is seen that the uncertainty relation (5.2.1) is actually equivalent to the following inequality:

$$\mathrm{Var}(\sigma_x) + \mathrm{Var}(\sigma_y) + \mathrm{Var}(\sigma_z) \geq 2. \tag{5.3.11}$$

Using $\mathrm{Var}(\sigma_y) \leq 1$ equation 5.3.6 follows.

For the pure state in Example 3 (page 19), $\mathrm{Var}(\sigma_x) + \mathrm{Var}(\sigma_y) + \mathrm{Var}(\sigma_z) = (1 - n_x^2) + (1 - n_y^2) + (1 - n_z^2) = 3 - (n_x^2 + n_y^2 + n_z^2) = 2$.

It is worth noting that besides σ_x, the operator σ_y also constitutes an interference observable with respect to the path σ_z. Thus, substituting $Var(\sigma_z) = 1 - C_P^2$, $\mathrm{Var}(\sigma_x) = 1 - C_I^2 \equiv 1 - C_{I,x}^2$, and a similar term $\mathrm{Var}(\sigma_y) = 1 - C_{I,y}^2$, gives a generalized and sharpened duality, or rather, "tripality", relation:

$$C_P^2 + C_{I,x}^2 + C_{I,y}^2 \leq 1. \tag{5.3.12}$$

Thus the full uncertainty relation for σ_x, σ_z, including the commutator and covariance terms, is equivalent to the additive triple trade-off relation for the variances of $\sigma_x, \sigma_y, \sigma_z$ as well as this new "tripality" relation for three mutually complementary observables.

5.4 Measurement complementarity from measurement inaccuracy relations

The measurement schemes of subsections 4.3.3 and 4.3.4 were found to constitute joint measurements of unsharp path and interference observables of the form $F = \{F_{1,2} = \frac{1}{2}(I \pm f\sigma_x)\}$ and $G = \{G_{1,2} = \frac{1}{2}(I \pm g\sigma_z)\}$. For instance, in equation (4.3.36), setting $\delta = -\frac{\pi}{2}$, gives $f = \sin\theta$ and $g = \cos\theta$, so that $f^2 + g^2 = 1$. This is an instance of the following general result: POVMs F, G of the above form are jointly measurable if and only if the following trade-off inequality holds (see Chapter 2, Section 2.5.3):

$$f^2 + g^2 \leq 1. \tag{5.4.1}$$

For a general state $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$, $|\mathbf{r}| \leq 1$, the contrasts of the distributions of F and G are defined thus,

$$\begin{aligned} C_F(\rho) &= |Tr[\rho F_1] - Tr[\rho F_2]| = |f r_x|, & (5.4.2) \\ C_G(\rho) &= |Tr[\rho G_1] - Tr[\rho G_2]| = |g r_y|. & (5.4.3) \end{aligned}$$

The contrasts of the POVMs F, G are the respective maximal contrasts over all states ρ:

$$C_F = |f|, \quad C_G = |g|. \tag{5.4.4}$$

These quantities measure the degree of unsharpness,

$$U_F := 1 - C_F^2 = 1 - f^2, \quad U_G := 1 - C_G^2 = 1 - g^2, \tag{5.4.5}$$

in the POVMs F, G. The unsharpness of F can also be defined as the minimum variance of the distribution of F for all states ρ. Using the development used for equations 4.2.30 and 4.2.31

$$\begin{aligned} Var_\rho(\sigma_x) &= 1 - r_x^2 \\ \text{and } Var_\rho(F) &= 1 - f^2 r_x^2 \\ &= U_F + f^2 - f^2 r_x = U_F + f^2 Var_\rho(\sigma_x) \geq U_F. \end{aligned} \tag{5.4.6}$$

Taking the minimum over all ρ gives

$$Var_{min}(F) = U_F. \tag{5.4.7}$$

The above joint measurability criterion can be written in terms of the degrees of unsharpness:

$$U_F + U_G = 2 - f^2 - g^2 \geq 1. \tag{5.4.8}$$

This inequality is an uncertainty trade-off relation which must be satisfied if the two noncommuting unsharp path and interference observables F and G are to be jointly measurable. Here is an instance of Heisenberg's uncertainty principle for the inaccuracies which are necessarily present in joint measurements. As far as I am aware, this is one [13] of two cases in which an inaccuracy relation has been proven as a necessary condition for joint measurability. The other example is the case of position and momentum, and the corresponding uncertainty relation for joint measurements is reviewed in [15].

Finally, it is worth noting that the variances of the marginals F, G in a joint measurement satisfy the uncertainty relation

$$
\begin{aligned}
Var_\rho(F) + Var_\rho(G) &= (1 - f^2 r_x^2) + (1 - g^2 r_x^2) = 2 - (f^2 + g^2) r_x^2 \\
&\geq 2 - (f_1^2 + g^2) \geq 1 \\
\text{and} \quad U_F + U_G &= 2 - (f^2 + g^2) \\
\text{hence} \quad Var_\rho(F) + Var_\rho(G) &\geq U_F + U_G \geq 1
\end{aligned}
\tag{5.4.9}
$$

Measurement complementarity is obtained as a limiting case for a pair F, G which are jointly measurable: if it is stipulated that one marginal, say F, becomes sharp, $U_F = 0$, or $|f| = 1$, then the other marginal, G, becomes a trivial POVM, $g = 0$, $G_{1,2} = \frac{1}{2}I$. Thus, if the path F is measured sharply, any attempt at obtaining information on interference will fail as the only unsharp interference observable G that can be measured jointly with F is trivial.

5.5 Discussion

Complementarity in quantum mechanics describes the limitations of the joint definition and measurement of certain pairs of noncommuting observables.

The first view of complementarity given by Bohr exposed the problems of using classical descriptions to obtain a complete account of quantum phenomena. Bohr proposed that it was necessary to use familiar classical descriptions but in a limited sense because for quantum phenomena, certain pairs of classical descriptions apply to mutually exclusive domains.

This mutual exclusivity was for many years after the appearance of complementarity, the interpretation of this feature of quantum mechanics. Support for this in the formalism was seen in the impossibility of preparing or measuring the values of two non-commuting observables simultaneously. However, it can be seen in Bohr's earliest writing on complementarity that the notion of mutual exclusivity needed to be broadened to include the simultaneous application of mutually exclusive descriptions. This step opens up the notion of graded complementarity to allow trade-off relations between complementary (mutually exclusive) observables. In the models explored in Chapter 4 noncommutativity is not an obstacle to joint measurability provided that unsharp observables are allowed.

In illustrating complementarity Bohr referred to the importance of experimental set-ups. A complete experiment will have two phases; an initial preparation phase in which an ensemble of quantum objects interacts with a system which prepares them to be in a particular state or set of states. Following this there will be a measurement scheme designed to reveal the (eigen)value of some observable. It is necessary to apply the concept of complementarity to both phases of the experiment.

In this thesis the following understanding of complementarity has been developed in a precise form. If one observable has a sharp value the other is completely unsharp; strict preparation complementarity is in force. Furthermore, measurement complementarity is a way of accounting for the results of joint measurement schemes. The joint measurement schemes in Chapter 4 have within them strict complementarity. Indeed, in the limit of their application, when one of the observables becomes sharp, no information is obtained about the other one, strict measurement complementarity is recovered: non-commuting observables remain mutually exclusive. Thus the strict form remains one aspect of a proper formulation of complementarity.

Next, several claims and views about complementarity and uncertainty have been investigated in this thesis. Of particular concern have been the findings of Scully, Englert and Walther and Zhu et al and the question of the loss of interference in path marking experiments and the relationship between complementarity and uncertainty in this type of experiment.

The interference experiments with path marking proposed by SEW and of Zhu et al are conveniently modeled by a two-arm interferometer. In Chapter 5 it has been shown that preparation complementarity, in the form of value complementarity, and measurement complementarity for path and interference observables can be associated with appropriate uncertainty relations for preparations and joint measurements, respectively.

In an any scheme involving joint observables a graded form of complementarity is manifest. The limitation imposed by this form of complementarity are described by a duality relation comprising the two observables. It has been shown that in the context of the experiments considered here, these duality relations can be described in the form of an uncertainty relation.

Complementarity remains one of the fundamental features of quantum mechanics as does uncertainty. As confirmed by this thesis, the two concepts are connected but one cannot be used to define the other nor are they in a hierarchical relationship.

It is worth noting that the connections between complementarity and uncertainty exhibited and made precise in this work have so far been shown to apply in the specific context of which-path experiments, which are described in the framework of a two-dimensional Hilbert space. It will be interesting to extend the results obtained here to more general situations.

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