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Subordinacy and Spectral Analysis of Schrödinger Operators

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SUMMARY

Summary of Thesis submitted for Ph.D. degree

by Daphne Jane Gilbert

on

Subordinacy and Spectral Analysis of Schrödinger Operators

This thesis is concerned with the spectral analysis of Schrödinger operators with central potentials, and some related aspects of scattering theory. After an introductory discussion on the aims of the thesis and its relation to existing work, the background mathematical material required for subsequent developments is presented in Chapter II. The theory of subordinacy, which relates the absolutely continuous, singular continuous and discrete parts of the spectrum to the relative asymptotic behaviour of solutions of the radial Schrödinger equation, is established in Chapter III for the case where $L = \frac{-d^2}{dr^2} + V(r)$ is regular at 0 and limit point at infinity. In Chapter IV, it is shown that the general eigenfunction expansion theory of Weyl-Kodaira can be simplified for a Schrödinger operation in $L_2(0,\infty)$ whenever the corresponding operator on any finite interval containing the origin has singular spectrum and the potential is integrable at infinity; an incidental outcome is an extension of the theory of subordinacy to include cases where L is singular at both ends of the interval $(0,\infty)$. The simplified expansion theory enables the class of potentials for which the usual phase shift formula for the scattering operator holds to be extended in Chapter V, so as to include more singular behaviour at the origin than any previously considered. Using this result, it is shown that a Schrödinger operator exists for which the theory is asymptotically complete and the scattering amplitude is a discontinuous function of energy. Chapter VI is concerned with the inductive construction of potentials having singular continuous spectrum; there is a particular emphasis on the generation of singular continuous measures from sequences of absolutely continuous measures, and some improvements to existing results and relevant examples are presented. The thesis is concluded with a brief indication of some outstanding problems, and suggestions for further research.

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	line 6	$+\displaystyle\int_{x+\zeta}^{x+M} \dfrac{2y(\lambda-x)\tilde{\rho}(\lambda)}{((\lambda-x)^2+y^2)^2}\,d\lambda + \displaystyle\int_{(M,\infty)} \dfrac{y}{(\lambda-x)^2+y^2}\,d\tilde{\rho}(\lambda)$
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CHAPTER I

INTRODUCTION

The time independent Schrödinger equation

$$-\Delta u + Vu = \lambda u$$

is of fundamental importance to the mathematical description of those quantum mechanical systems where the potential V is independent of time. An elliptic partial differential operator of the form

$$-\Delta + V$$

acting in the Hilbert space $L_2(\mathbb{R}^3)$ is known as a Schrödinger operator.

This thesis is concerned with the qualitative spectral analysis of self-adjoint Schrödinger operators with spherically symmetric potentials; for such operators the Hilbert space may be decomposed into mutually disjoint partial wave subspaces, and the spectral analysis of a Schrödinger operator in $L_2(\mathbb{R}^3)$ may be reduced to the spectral analysis of the ordinary differential operators

$$-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \qquad r \in (0,\infty)$$

in each partial wave subspace. ([AJS] Ch.11). For convenience we shall usually assume the term $\frac{l(l+1)}{r^2}$ to be included in V(r) so that the general problem further simplifies to consideration of the ordinary differential operator

$$L = -\frac{d^2}{dr^2} + V(r) \qquad r \in (0,\infty) \qquad (1.1.1)$$

acting in $\mathcal{H} = L_2(0,\infty)$.

The associated one dimensional Schrödinger equation

$$-\frac{d^2}{dr^2} + V(r)u = \lambda u \qquad r \in (0,\infty) \qquad (1.1.2)$$

is of the Sturm-Liouville type; we shall draw on the considerable body of existing theory relating to Sturm-Liouville equations, which have widespread applications in the physical sciences, as the need arises.

It will be assumed throughout that V(r) is locally integrable on $(0,\infty)$ and that non-trivial self-adjoint extensions of the symmetric differential operator L with domain C_{00}^{∞} (the set of infinitely differentiable functions

- .. . , ∞)) exist. The condition that V(r) be locally integrable is sufficient to ensure that Weyl's limit point, limit circle classification, which is extensively used during this thesis, applies ([CL] Ch.9, §2). Note that, although $u \in L_2(0,\infty)$ cannot be in the domain of a one-dimensional Schrödinger operator unless $Lu \in L_2(0,\infty)$, it is not necessary for V(r) to be in $L_2(0,\infty)$, even locally ([KA] Ch.VI, §4.1). Methods for establishing self-adjointness for semibounded and unbounded operators are widely discussed in the literature (eg. [KA] Ch.VI, [RS II],[S1]), and apart from a short summary of some relevant aspects of operator theory in Chapter II, §4, which clarifies the role of boundary conditions, will not be further considered here.

The spectrum of a self-adjoint Schrödinger operator, or Hamiltonian, H, represents the possible energy levels of the system and is defined mathematically to be the complement in $\mathbb{R}$ of all λ for which the resolvent operator $(H-\lambda I)^{-1}$ is bounded. The methods of spectral analysis which we shall adopt fall into three distinct categories.

The first method, which we call after Glazman ([G]), the direct method, deduces properties of the spectrum from prior knowledge of the potential, and, where appropriate, of associated boundary conditions. We contribute a new method of this type through the theory of subordinacy developed in Chapter III; provided certain aspects of the asymptotic behaviour of solutions of $Lu = xu$ can be established for each real x from knowledge of the potential and boundary conditions, the nature and location of the specturm may be completely determined. Classically the direct method has been the most usual approach to the problem of identifying corresponding Hamiltonians and spectra (see eg. [T2], [G], [DS] Ch.XIII §9. G,H.); however, unlike many examples of this method, our theory of subordinacy has very general application. Where L is regular at 0 and limit point at infinity we only require that V(r) be locally integrable, and where L is singular at 0 and limit point at infinity the only additional requirement is that V(r) be integrable at infinity.

The second approach to spectral analysis is known as the inverse method l by Gel'fand and Levitan ([GL]). They established sufficient conditions which ensure that a given monotonically increasing function $\rho(\lambda)$ is the spectral function of some Schrödinger operator, and devised a method for obtaining the operator from $\rho(\lambda)$. In practise it is not easy to derive Schrödinger operators analytically from their spectral functions since the solution of integral equations is involved; however, the inverse method is invaluable as a tool for testing hypotheses and providing counter examples. In this role it will be used to clarify the nature and extent of the correlation between the asymptotic behaviour of solutions of the Schrödinger equation and the spectrum in Chapter III, and again during the proof of the existence of a Schrödinger operator where the wave operators exist and are complete, but for which the scattering amplitude is a discontinuous function of energy, in Chapter V.

The third method of spectral analysis adopted in this thesis is that of inductive construction of potentials. The starting point here is neither a given spectrum, nor a given Schrödinger operator; instead, under carefully controlled circumstances, sequences of operators with absolutely continuous spectrum on $\mathbb{R}^+ = (0,\infty)$ are chosen inductively to ensure a particular type of spectrum of the limiting operator. In general, therefore, only an overall conception of the limiting operator and of its spectrum is assumed at the outset, more precise details emerging in accordance with the constraints of the inductive construction. This method, as yet relatively undeveloped, was first used to show that a potential consisting of a sequence of "bumps" will have singular continuous spectrum on $\mathbb{R}^+$ provided the separation between the "bumps" increases sufficiently rapidly with distance ([P1]). Some related theoretical questions, with particular reference to singular continuous spectra will be considered in Chapter VI, where a new example of an inductively constructed potential will be presented for which the associated Schrödinger operator with Dirichlet boundary conditions at the origin has singular continuous spectrum.

The theory of potential scattering is inextricably linked to the spectral analysis of Schrödinger operators, and, indeed, has had a considerable influence

... ...icular, the decomposition of the spectrum into absolutely continuous, singular continuous and pure point parts is motivated by the underlying physical interpretation of the theory. Provided

(i) the spectrum of H_a is singular for some $a > 0$ where H_a is a self-adjoint operator defined by the differential expression (1.1.1) in $L_2(0,a]$,

(ii) the potential is of short range, that is,

$$V(r) = O(r^{-(1+\varepsilon)}) \quad \text{as} \quad r \to \infty$$

then the wave operators exist and are complete so that the absolutely continuous subspace $\mathcal{H}_{a.c.}(H)$ of $\mathcal{H}$ may be identified with the subspace of scattering states of H ([P4]). Condition (i) ensures that the spectrum of H is simple, which is necessary for asymptotic completeness ([AM] §I). The subset of $\mathcal{H}$ corresponding to the pure point spectrum consists of so-called bound states, that is, states which are localised in a finite neighbourhood of the origin at all times. Whether or no the singular continuous spectrum has an identifiable physical interpretation is still in some doubt although plausible suggestions supported by rigorous mathematical analysis have been made ([P1] §4); however many potentials whose mathematical form is quite simple (eg. [P1] §3) or which are of considerable physical interest (see eg. [S2]) give rise to this type of spectrum.

We shall review some aspects of scattering theory in Chapter V in the light of the theory developed in Chapters III and IV. Using the simplified eigenfunction expansion and the time-dependent formalism we show that conditions (i) and (ii) above are sufficient to ensure the validity of the usual phase shift formula for the scattering operator (cf. [GR]), and a new proof of asymptotic completeness emerges incidentially during this process. The explicit formula for the phase shift, together with our earlier analysis of the spectrum then enables us to demonstrate that discontinuity of the scattering amplitude as a function of energy can occur, even when the theory is asymptotically complete.

There is throughout this thesis a special emphasis on "pathological" singular spectra. It may be partly due to the difficulties of interpretation and analysis that such spectra received little attention during the early development of quantum mechanics. However, in more recent years, experimental and theoretical interest in disordered systems and almost periodic potentials, together with the recognition that absolutely continuous and isolated point spectra are generically absent in such cases, has led to a vigorous current literature on all types of singular spectra (eg. [AS], [BS], [MO], [P], [S2]).

Our special emphasis on singular continuous and dense singular spectra does not, however, derive from any belief in their exceptional importance. If we start from the premise that any comprehensive theory should give equal consideration to all types of spectrum, then it is inevitable that those parts of the spectrum which are comparatively less amenable to analysis should incur more labour. Also, the relative neglect until recently of certain types of singular spectra has meant that some aspects of quantum theory which directly or indirectly involve such spectra have not been fully developed. Therefore, where the results of our comprehensive approach have been used to extend or clarify the limits of some existing theories in Chapter V, it is the aspects concerning singular spectra which are most prominent because it is these that have not been fully considered before.

Where possible, we indicate the relationship between the contents of this thesis and pre-existing work at appropriate points in the text; however, in order to give some sort of overview, we shall briefly summarise some of the main features from this point of view.

With the exception of Proposition 2.24 and Theorem 2.25, much of the preliminary mathematical material assembled in Chapter II occurs in some form or another here and there in the literature. However, the proofs have for the most part been devised by the author in order to unify the material; sometimes they may be equivalent to existing proofs, sometimes they may differ. Theorem 2.25, which relates absolutely continuous and singular spectra the growth rate of the resolvent $(H-zI)^{-1}$ as z approaches the real axis, is distinct from, yet complementary to, a theorem by Gustafson and Johnson which characterises the absolutely continuous subspace of $\mathcal{H}$ in terms of the growth rate of resolvents ([GJ]).

So far as we are aware, the theory of subordinate solutions presented in Chapter III and amplified in Chapter IV §5 is wholly new. This theory provides the kind of systematic correlation between the behaviour of solutions of the Schrödinger equation and the nature of the spectrum that was assumed, erroneously, to be true by those who identified the spectrum in terms of bounded solutions (see eg. [KR] pp.71, 82, [G] §58). Later the theory of subordinacy illuminates the simplified eigenfunction expansion of Theorem 4.9, since where L is limit point at 0, the kernel of the corresponding transform is a solution of (1.1.2) which is subordinate at 0.

Some aspects of the eigenfunction expansion theory derived in Chapter IV appear to have been obtained independently in an alternative but equivalent formulation by Kac ([K1], [K2]). Since details of this work were inaccessible, we have been unable to ascertain the extent to which the results and methods of proof coincide with our own. However, in the brief summary which is available in translation, there is no mention of the surjective property of the associated isomorphism, which we prove in the Appendix, nor does the relationship between the simplified expansion and the well-established expansions which are valid when the differential expression (1.1.1) is regular at 0, (see eg. [CL] Ch.9 §3) appear to have been considered.

The results of Chapter V depend crucially on the theory developed in Chapter IV. The simplified expansion of Theorem 4.9, which is established for all operators where the potential is integrable at infinity and the spectrum of H_a is singular for some $a > 0$ (see (i) above), enables us to verify the phase shift formula for the scattering operator for a far wider class of potentials than any previously considered. Indeed, we only require that conditions (i) and (ii) above be satisfied, whereas it is usual to impose a far stronger condition at the origin, as, for example

$$V(r) = O(r^{-(2-\epsilon)}) \quad \text{as} \quad r \downarrow 0 \tag{1.1.3}$$

while retaining a comparable condition to (ii) at infinity. (eg. [GR], [KU2]). In terms of the spectrum of H_a, (1.1.3) ensures this is isolated pure point (see Ch.V, §1), whereas condition (i) permits the potential to be highly at the origin, provided only that the possibility of absorption is excluded. Although continuity of the scattering amplitude as a function of energy has been proved for many potentials (see eg. [AJS], Prop.10.13, [D], [LE]), the existence of Hamiltonians for which the scattering amplitude is a discontinuous function of energy has not, to our knowledge, been previously established.

Our final chapter centres on a theorem due to Pearson ([P1], §2, Thm.1), which we re-examine with a view to weakening or removing some of the original conditions. The theorem concerns the generation of singular continuous measures from sequences of absolutely continuous measures, and is formulated with the inductive construction of operators with singular continuous spectra in mind. By means of step function approximations, we show that the continuity conditions on the generating sequences of periodic functions can be considerably weakened and the analyticity and strictly positive lower bound conditions removed entirely. An assessment of Pearson's construction theorem by Avron and Simon ([AS] Appendix 3) also confirms, using Kakutani's theorem, that several of the original conditions are not necessary, though in matters of detail there are a number of differences between their conclusions and ours.

In §2 we use Pearson's method to establish a new class of potentials for which the spectrum is singular continuous in the interval (inf $V(r)$, sup $V(r)$), while in §4 we illustrate the generation of singular continuous measures from sequences of periodic functions by a specific example for which a surprisingly detailed analysis is possible. This type of example is not new (see [RN], §24 for a rather different presentation) although we believe that some of our detailed findings may be.

With a view to our later requirements, we shall begin by introducing some basic mathematical concepts and establishing some elementary relationships between them in the following chapter. For simplicity we shall at first suppose that the differential operator (1.1.1) is regular at 0, and we remark that the almost exclusive attention given to the limit point case at infinity stems from the fact that almost all cases of physical and mathematical interest are of this type.

CHAPTER II

MATHEMATICAL FOUNDATIONS

§1. Introduction

Let H be a self-adjoint operator arising from the time-independent Schrödinger equation

$$-\frac{d^2 u(r,\lambda)}{dr^2} + V(r)u(r,\lambda) = \lambda u(r,\lambda) \qquad r \in [0,\infty) \tag{2.1.1}$$

and a regular boundary condition at 0. In the terminology of H. Weyl ([W3]), the differential operator

$$L = -\frac{d^2}{dr^2} + V(r) \tag{2.1.2}$$

is in the limit point case at infinity and in the limit circle case at 0.

In this chapter we develop some mathematical tools that will be required in the subsequent spectral analysis of operators of this type.

It is a part of established theory that, associated with each such operator H, there exists a monotonically increasing spectral function $\rho(\lambda)$ which is unique up to an additive constant. ([CL] Ch.9, Thm.3.1). The spectrum of H is the set of points of increase of $\rho(\lambda)$, and the decomposition of the Borel-Stieltjes measure μ generated by $\rho(\lambda)$ into its discrete, singular continuous and absolutely continuous parts gives an indication of the behaviour associated with different energy levels under time evolution. Broadly speaking, the discrete spectrum represents the binding energies of the system and the absolutely continuous spectrum the energy levels at which scattering can be expected to occur. The interpretation of the singular continuous spectrum is more speculative; many authors have maintained it has no physical counterpart ([RS1] Ch.I §1.4), while others have made suggestions which have yet to be confirmed by experiment ([P1] §4). However, as we shall see in Chapter V, the study of the singular continuous spectrum has applications to situations where it does not explicitly occur, so we shall consider it as thoroughly as the other parts of the spectrum.

Using the theory developed by H. Weyl and later amplified by E.C. Titchmarsh we shall show in §3 that the spectral properties of $\rho(\lambda)$ are intimately related to the boundary behaviour of an analytic function m(z) which is defined for $\mathrm{Im}\, z > 0$ by the condition

$$u_2(r,z) + m(z)u_1(r,z) \in L_2[0,\infty) \qquad (2.1.3)$$

Here $u_1(r,z)$ and $u_2(r,z)$ are those solutions of $Lu=zu$ which satisfy the conditions

$$u_1(0,z) = -\sin\alpha \qquad u_1'(0,z) = \cos\alpha$$

$$u_2(0,z) = \cos\alpha \qquad u_2'(0,z) = \sin\alpha \qquad (2.1.4)$$

for some α in $[0,\pi)$. We shall show in Chapter III that the boundary behaviour of m(z) at each point x of the real axis is also related to the relative asymptotic behaviour of certain linearly independent solutions of the Schrödinger equation (2.1.1) at energy $\lambda = x$. Thus m(z) will act as an intermediary, enabling us to characterise the various parts of the spectrum in terms of properties of the solutions of (2.1.1).

In §4 we give a brief account of operator theory as it applies to second order linear differential equations of the Sturm-Liouville type, and indicate some relationships between $\rho(\lambda)$, m(z) and H. We also derive criteria for distinguishing the sets on which the absolutely continuous and singular spectra are concentrated in terms of the resolvent operator.

First, however, in § 2 we shall briefly summarise some relevant aspects of measure theory, and then investigate the relationship between the character of the measure μ and properties of the derivative of $\rho(\lambda)$ on measurable sets of points. We remark that our results concerning this relationship do not depend on a quantum mechanical context, but would apply equally to any increasing function that is continuous on the right, and the measure generated by it.

§2. The Spectral Measure and its Derivative

The spectral function $\rho(\lambda)$ is monotonically increasing, continuous on the right and unique up to an additive constant ([CL] Ch.9, §3). For convenience we may take $\rho(0)=0$. The right continuity of $\rho(\lambda)$ implies that $|\rho(c)|<\infty$ if $|c|<\infty$, so that if a set function μ' is defined on the algebra $\mathfrak{A}'$ of half-open intervals $(a,b]$ of $\mathbb{R}$ by

$$\mu'((a,b]) = \rho(b) - \rho(a) \qquad (2.2.1)$$

then μ' is a σ-finite measure on $\mathfrak{A}'$.

The Hahn Extension Theorem states that a σ-finite measure on an algebra $\mathfrak{A}$ may be uniquely extended to a complete measure on a σ-algebra containing $\mathfrak{A}$. (2.2.2)

Hence μ' may be extended to a complete measure μ on a σ-algebra Σ containing $\mathfrak{A}'$; we shall call μ the spectral measure associated with H. By (2.2.1) the spectral measure of bounded subsets of $\mathbb{R}$ is finite, which is a stronger property than that of σ-finiteness, and implies, in particular that μ is a regular measure ([R] Thm.2,18).

Unless otherwise stated, we shall take as the measurable sets those subsets of $\mathbb{R}$ which are Borel measurable.

A measurable function $f(\lambda)$ is then specified by the requirement that for each α in $\mathbb{R}$, $\{\lambda : f(\lambda)>\alpha\}$ be a Borel set. (2.2.3)

In the case where a σ-finite measure defined on the algebra $\mathfrak{A}'$ of half-open intervals is extended to a complete measure ι on a σ-algebra Σ_ι, we refer to the elements of Σ_ι as ι-measurable sets.

If S is any subset of $\mathbb{R}$, we denote by $\mathfrak{B}_S$ the σ-algebra of Borel subsets of S.

Let ι , κ be σ-finite measures on $\mathfrak{B}_S$.

ι is said to be absolutely continuous with respect to κ on $\mathfrak{B}_S$ if

$$\kappa(E)=0 \quad \Rightarrow \quad \iota(E)=0$$

for all E in $\mathcal{B}_S$. We write $\iota \ll \kappa$ (2.2.4)

ι and κ are said to be mutually singular on $\mathcal{B}_S$ if there exist two sets E_1 and E_2 in $\mathcal{B}_S$ such that

$$E_1 \cap E_2 = \phi \quad , \quad E_1 \cup E_2 = S$$

and $\iota(E_1) = \kappa(E_2) = 0$.

We write $\iota \perp \kappa$. (2.2.5)

By the Lebesgue Decomposition Theorem ι may be uniquely decomposed into two measures $\iota_{a.c.}$ and $\iota_{s.}$ such that

$$\iota = \iota_{a.c.} + \iota_{s.}$$

where $\iota_{a.c.} \ll \kappa$ and $\iota_{s.} \perp \kappa$.

If we take κ to be Lebesgue measure a further unique decomposition may be accomplished. Replacing ι by the spectral measure μ, we define set functions $\mu_{s.c.}$ and $\mu_{d.}$ by

$$\mu_{s.c.}(E) = \mu_{s.}(E \setminus C) \tag{2.2.6}$$

$$\mu_{d.}(E) = \mu_{s.}(E \cap C) \tag{2.2.7}$$

for all E in $\mathcal{B}_S$, where $C = \{\lambda \in S : \mu(\{\lambda\}) > 0\}$. Since $\rho(\lambda)$ can have, at most, a countable number of discontinuities, C is denumerable, from which may be deduced that $\mu_{s.c.}$ and $\mu_{d.}$ are measures. Their uniqueness follows from the uniqueness of C.

We now have the following decompositions of μ on C:

$$\mu = \mu_{a.c.} + \mu_{s.} \tag{2.2.8}$$

$$\mu = \mu_{a.c.} + \mu_{s.c.} + \mu_{d.} \tag{2.2.9}$$

$\mu_{s.c.}$ is known as a singular continuous measure, reflecting the facts that $\mu_{s.c.}$ is singular with respect to κ and $\mu_{s.c.}(\{\lambda\})=0$ for all λ in S. $\mu_{d.}$ is variously described as discrete, pure point or purely atomic and is concentrated on a denumerable set of points which have strictly positive

μ-measure.

Corresponding to the decompositions (2.2.8) and (2.2.9) are unique decompositions of $\rho(\lambda)$ into montonically increasing functions ([HS] Thm. 19.61), viz:

$$\rho(\lambda) = \rho_{a.c.}(\lambda) + \rho_{s.}(\lambda) \tag{2.2.10}$$

$$\rho(\lambda) = \rho_{a.c.}(\lambda) + \rho_{s.c.}(\lambda) + \rho_{d.}(\lambda) \tag{2.2.11}$$

where $\rho_{a.c.}(b) - \rho_{a.c.}(a) = \mu_{a.c.}((a,b])$ etc.,

and $\rho_{a.c.}(0) = \rho_{s.}(0) = \rho_{s.c.}(0) = \rho_{d.}(0)$.

The functions $\rho_{a.c.}(\lambda)$, $\rho_{s.}(\lambda)$, $\rho_{s.c.}(\lambda)$ and $\rho_{d.}(\lambda)$ are absolutely continuous, singular, singular continuous and saltus functions respectively, so that for Lebesgue almost all λ on S

$$\frac{d\rho_{s.}(\lambda)}{d\lambda} = \frac{d\rho_{s.c.}(\lambda)}{d\lambda} = \frac{d\rho_{d.}(\lambda)}{d\lambda} = 0 \tag{2.2.12}$$

where $\frac{d\rho(\lambda)}{d\lambda}$ is defined to be $\lim_{\delta \to 0} \frac{\rho(\lambda-\delta, \lambda+\delta)}{2\delta}$ whenever the limit exists.

The <u>absolutely continuous</u>, <u>singular</u>, <u>singular continuous</u> and <u>discrete spectra</u> of H are the sets of points of increase of $\rho_{a.c.}(\lambda)$, $\rho_{s.}(\lambda)$, $\rho_{s.c.}(\lambda)$ and $\rho_{d.}(\lambda)$ respectively. We shall show in §4 that this formulation is consistent with the more usual definitions in terms of resolvent operators.

Although we shall not be particularly concerned with the essential spectrum, which consists of all the non-isolated points of the spectrum of H, it may sometimes be mentioned in passing.

We shall continue to denote Lebesgue measure by κ unless otherwise stated. Let $\frac{d\mu}{d\kappa}(x)$ denote $\lim_{\kappa(I_x) \to 0} \left\{ \frac{\mu(I_x)}{\kappa(I_x)} : I_x \text{ is an interval of } \mathbb{R} \text{ containing } x \right\}$ for each x in $\mathbb{R}$ for which the limit exists. We remark that, since μ and κ are regular measures, it is immaterial whether we take I_x to range over all intervals, or just over all open, all half-open or all closed intervals containing x.

Lebesgue's Theorem states that a monotonic function possesses a finite derivative Lebesgue almost everywhere on $\mathbb{R}$. (2.2.13)

Hence $\frac{d\mu(x)}{d\kappa}$ and $\left.\frac{d\rho(\lambda)}{d\lambda}\right|_x$ exist and are finite and equal Lebesgue almost everywhere on $\mathbb{R}$; moreover, by implication, the set $S = \{x \in \mathbb{R} : \frac{d\mu(x)}{d\kappa} \text{ exists}\}$ is Lebesgue measurable. In fact, as we now show, S is a Borel set.

Let $f_n(x) = \sup_{\kappa(I_x) < \frac{1}{n}} \{ \frac{\mu(I_x)}{\kappa(I_x)} : I_x \text{ is an open interval containing } x \}$ and consider $S_\alpha = \{ x \in \mathbb{R} : f_n(x) > \alpha \}$

If $x \in S_\alpha$, there exists an open interval $C_{x,\alpha}$ containing x such that $\kappa(C_{x,\alpha}) < \frac{1}{n}$ and $\frac{\mu(C_{x,\alpha})}{\kappa(C_{x,\alpha})} > \alpha$. Clearly if x' is in $C_{x,\alpha}$ then x' is also in S_α, and so $S_\alpha = \bigcup_{x \in S_\alpha} C_{x,\alpha}$. Thus S_α is an open set, so that $f_n(x)$, and, consequently $F(x) = \lim_{n\to\infty} f_n(x)$ are Borel measurable functions.

Similarly, if

$g_n(x) = \inf_{\kappa(I_x) < \frac{1}{n}} \{ \frac{\mu(I_x)}{\kappa(I_x)} : I_x \text{ is an open interval containing } x \}$ then $G(x) = \lim_{n\to\infty} g_n(x)$ is a Borel measurable function.

Since $S = \{x \in \mathbb{R} : F(x) - G(x) = 0\}$, S is a Borel set.

We shall have occasion here and later to use the inverse method of Gel'fand and Levitan ([GL]). If a monotonically increasing function $\rho(\lambda)$ is given, the authors obtain necessary and sufficient conditions for the existence of an operator H whose spectral function is $\rho(\lambda)$. We shall make particular use of their result that if $\rho(\lambda)$ is an arbitrary increasing function on a finite interval I, then there always exists an operator H, defined as at the beginning of §1, whose spectral function equals $\rho(\lambda)$ on I.

We now prove some results which relate the rate of increase of the measure μ to measurable sets of points on which the decomposed parts of μ are concentrated.

2.1 Lemma: If S_A is a measurable subset of $\mathbb{R}$ with the property that for each x in S_A there exists $\Delta_x > 0$ such that $\frac{\mu(I_x)}{\kappa(I_x)} \leq A$ for all intervals I_x containing x with $\kappa(I_x) < \Delta_x$, then $\mu \ll \kappa$ on S_A, and $\mu(S_A) \leq A\kappa(S_A)$.

Proof:

The proof is in three stages. In (i) we show that if S_A is a closed set with the given property then $\mu(S_A) \leq 2A\kappa(S_A)$. In (ii) we extend the result of (i) to general measurable subsets of $\mathbb{R}$ and deduce that $\mu \ll \kappa$ on S_A. We use the absolute continuity of μ on S_A in (iii) to prove that $\mu(S_A) \leq A\kappa(S_A)$.

(i) Let S_A be a closed subset of $\mathbb{R}$ and let $\varepsilon > 0$ be given.

Define $S_A' = S_A \cap [0,1]$.

Since κ is a regular measure, there exists an open set S such that

$$S_A' \subset S \tag{2.2.14}$$

and $$\kappa(S) < \kappa(S_A') + \frac{\varepsilon}{2A} \tag{2.2.15}$$

For each x in S_A' we may choose $\delta_x < \frac{\Delta_x}{2}$ to satisfy

$$[x-\delta_x, x+\delta_x] \subset S \tag{2.2.16}$$

Clearly $S_A \subset \bigcup_{x \in S_A} [x-\delta_x, x+\delta_x]$, and, indeed, since S_A' is compact, there exists a minimal finite subcover $\mathcal{C}$ of S_A' by sets of the form $(x-\delta_x, x+\delta_x)$, where $x \in S_A'$.

We may write

$$\mathcal{C} = \{ U_i = (x_i - \delta_{x_i}, x_i + \delta_{x_i}), i = 1, \ldots, p \}$$

where the U_i are assumed ordered in such a way that, for each i = 1,...., p - 1,

$$x_i - \delta_{x_i} < x_{i+1} - \delta_{x_{i+1}} \tag{2.2.17}$$

The minimality of $\mathcal{C}$ ensures that no two of the left end points of the U_i's coincide, and also, as we shall now show, that for each i = 1, ...,p - 1,

$$x_i < x_{i+1} \tag{2.2.18}$$

and

$$x_i + \delta_{x_i} < x_{i+1} + \delta_{x_{i+1}} \tag{2.2.19}$$

For if (2.2.18) were false, we should have by (2.2.17)

$$\delta_{x_{i+1}} - \delta_{x_i} < x_{i+1} - x_i \leq 0$$

which implies

$$x_{i+1} + \delta_{x_{i+1}} \leq x_i + \delta_{x_i} \tag{2.2.20}$$

(2.2.17) and (2.2.20) together imply that $U_{i+1} \subset U_i$ which is impossible by the minimality of $\mathcal{C}$. Hence (2.2.20) must be false, and so (2.2.18) and (2.2.19) are proved.

We are now in a position to construct from $\mathcal{C}$ a finite cover $\mathcal{C}'$ with the following properties:

$\mathcal{C}' = \{ U_i',\ i = 1, \ldots p \}$ where each U_i' is an interval and $U_i' \subseteq U_i$ for each i = 1,...,p.

$$U_i' \cap U_j' = \phi \text{ for all } i,\ j \in \{ 1,\ldots p\} \text{ such that } i \neq j \tag{2.2.21}$$

$$\bigcup_{i=1}^{p} U_i' = \bigcup_{i=1}^{p} U_i \tag{2.2.22}$$

We shall prove that for each $U_i' \in \mathcal{C}'$

$$\mu(U_i') < 2A\kappa(U_i') \tag{2.2.23}$$

The detailed construction of $\mathcal{C}'$ is as follows:

For each $i \in \{ 1,\ldots p \}$ such that

$$\left(\bigcup_{\substack{j=1 \\ j \neq i}}^{p} U_j\right) \cap U_i = \phi$$

we set $U_i' = U_i$.

If for some $i \in \{ 2,\ldots p - 1 \}$

$$\left(\bigcup_{\substack{j=1 \\ j \neq i}}^{p} U_j\right) \cap U_i \neq \phi \tag{2.2.24}$$

then by ordering of the U_i and the minimality of $\mathcal{C}$, either $U_{i-1} \cap U_i \neq \phi$ or $U_i \cap U_{i+1} \neq \phi$ or both. If (2.2.24) holds for i = 1 or p then $U_1 \cap U_2 \neq \phi$ or $U_{p-1} \cap U_p \neq \phi$ respectively.

For each $i \in \{1, \dots p-1\}$ such that $U_i \cap U_{i+1} \neq \phi$, U_i' and U_{i+1}' are defined in such a way that the midpoint between the left hand endpoint of U_{i+1} and the right hand endpoint of U_i is the common left and right hand endpoint of U_{i+1}' and U_i' respectively.

For each $i \in \{1, \dots p-1\}$ such that $U_i \cap U_{i+1} \neq \phi$, the right and left hand endpoints of U_i and U_{i+1} respectively become the right and left hand endpoints of U_i' and U_{i+1}' respectively.

Each interval U_i' is either open or half open, subject to the general conditions (2.2.21) and (2.2.22). In order to prove (2.2.23) we first show that for each i = 1,...,p,

$$x_i \in U_i' \tag{2.2.25}$$

It is sufficient to show that for each $i \in \{1, \dots p-1\}$ for which $U_i \cap U_{i+1} \neq \phi$

$$x_i < q_i < x_{i+1} \tag{2.2.26}$$

where $q_i = \frac{1}{2}(x_i + \delta_{x_i} + x_{i+1} - \delta_{x_{i+1}})$ is the partition point between U_i' and U_{i+1}'. Since the right and left inequalities of (2.2.26) are immediate by (2.2.19) and (2.2.17) respectively, (2.2.25) is proved.

Now for each i = 1,....,p, if $m_i = \max_{y_i \in U_i'} |x_i - y_i|$

$$\begin{aligned} \mu(U_i') &\leq \mu\,[x_i - m_i, x_i + m_i] \\ &< A\,\kappa\,[x_i - m_i, x_i + m_i] \\ &< 2A\,\kappa(U_i') \end{aligned} \tag{2.2.27}$$

where we have used the hypothesis, $m_i \leq \delta_{x_i} < \frac{\Delta x_i}{2}$, and (2.2.25). Thus we have proved (2.2.23), and it is now straightforward to show that

$$\mu(S_A') \leq 2A\,\kappa(S_A') \tag{2.2.28}$$

For, $S_A' \subset \bigcup_{i=1}^{p} U_i' = \bigcup_{i=1}^{p} U_i \subseteq S$ by (2.2.14) and (2.2.22), so that by (2.2.22)

(2.2.27) and (2.2.15),

$$\begin{aligned}\mu(S_A') &\leq \sum_{i=1}^{p} \mu(U_i') \\ &\leq 2A\,\kappa\big(\bigcup_{i=1}^{p} U_i'\big) \\ &\leq 2A\,\kappa(S) \\ &< 2A\,\kappa(S_A') + \varepsilon\end{aligned}$$

Since ε was chosen arbitrarily, (2.2.28) now follows. This result applies equally to $S_A^p = S_A \cap [p-1,p]$ for each p in $\mathbb{Z}$, and so

$$\mu(S_A) \leq \sum_{p\in\mathbb{Z}} \mu(S_A^p) \leq 2A \sum_{p\in\mathbb{Z}} \kappa(S_A^p) = 2A\,\kappa(S_A)$$

as was to be proved.

(ii) Let S_A be a Borel subset of $\mathbb{R}$ and let $\varepsilon > 0$ be given.

Let $S_A' = S_A \cap [0,1]$

Since μ is a regular measure there exists an open set S such that

$[0,1] \setminus S_A' \subseteq S$

and $\mu(S) < \mu([0,1] \setminus S_A') + \varepsilon$

Define $S_1 = S \cap [0,1]$ so that $S_1 \subseteq S$ and

$$[0,1] \setminus S_1 \subseteq S_A' \tag{2.2.29}$$

Then
$$\begin{aligned}\mu(S_1) &\leq \mu(S) \\ &< \mu([0,1] \setminus S_A') + \varepsilon \\ &= \mu([0,1] \setminus S_1) + \mu(S_1) - \mu(S_A') + \varepsilon\end{aligned}$$

so that

$$\mu(S_A') < \mu([0,1] \setminus S_1) + \varepsilon$$

Since $[0,1] \setminus S_1$ is closed, we have by (2.2.29) and the result of (i)

$$\begin{aligned}\mu(S_A') &< 2A\,\kappa([0,1] \setminus S_1) + \varepsilon \\ &\leq 2A\,\kappa(S_A') + \varepsilon\end{aligned}$$

By the arbitrariness of ε

$$\mu(S_A') \leq 2A\,\kappa(S_A')$$

and hence, as in (i), $\mu(S_A) \leq 2A\,\kappa(S_A)$.

Clearly if E is any measurable subset of S_A which has κ-measure zero

$$\mu(E) \leq 2m\,\kappa(E) = 0$$

and so by (2.2.4)

$$\mu \ll \kappa \text{ on } S_A \tag{2.2.30}$$

(iii) We first show that

$$\mu(S_A) = \int_{S_A} \frac{d\mu}{d\kappa}\, d\kappa \tag{2.2.31}$$

For any subinterval $E = (a,b]$ of $\mathbb{R}$

$$\begin{aligned}\mu_{a.c.}(E) &= \rho_{a.c.}(b) - \rho_{a.c.}(a) \\ &= \int_a^b \frac{d\rho_{a.c.}(\lambda)}{d\lambda}\, d\lambda \\ &= \int_a^b \frac{d\rho(\lambda)}{d\lambda}\, d\lambda \\ &= \int_E \frac{d\mu}{d\kappa}\, d\kappa \end{aligned} \tag{2.2.32}$$

by (2.2.12) and the remarks following Lebesgue's Theorem (2.2.13). Using the Hahn Extension Theorem (2.2.2) we see that (2.2.32) also holds for arbitrary measurable subsets E of $\mathbb{R}$, in particular for $E = S_A$. Since $\mu_{s.}(S_A) = 0$ by (2.2.30), we have $\mu(S_A) = \mu_{a.c.}(S_A)$ by (2.2.8) and so (2.2.31) is proved.

From the hypothesis and Lebesgue's Theorem, $\frac{d\mu}{d\kappa}$ exists and is less than or equal to A κ-almost everywhere on S_A. Hence by (2.2.31), $\mu(S_A) \leq A\,\kappa(S_A)$ as was to be proved.

2.2. <u>Corollary</u>: If S_A is a measurable subset of $\mathbb{R}$ such that, for each x in S_A, $\frac{d\mu(x)}{d\kappa}$ exists and equals zero, then $\mu(S_A) = 0$

<u>Proof</u>:

The condition implies that if $\varepsilon > 0$ is given, then for each x in S_A there exists $\Delta_x > 0$ such that $\frac{\mu(I_x)}{\kappa(I_x)} < \varepsilon$ for all intervals I_x containing x with $\kappa(I_x) < \Delta_x$. If $\kappa(S_A) < \infty$, the corollary is immediate by Lemma 2.1 and the arbitrariness of ε. Applying this result to $S_A \cap (p-1,p]$ for each $p \in \mathbb{Z}$, the case $\kappa(S_A)=\infty$ follows by the countable additivity of μ.

2.3 <u>Lemma:</u> If S_A is a measurable subset of $\mathbb{R}$ with the property that for each x in S_A there exists $\Delta_x > 0$ such that $\frac{\mu(I_x)}{\kappa(I_x)} \geqslant A$ for all intervals I_x containing x with $\kappa(I_x) < \Delta_x$, then $\mu(S_A) \geqslant A\,\kappa(S_A)$.

<u>Proof</u>:

The hypothesis and Lebesgue's Theorem (2.2.13) imply that $\frac{d\mu}{d\kappa}$ exists and is greater than or equal to A κ-almost everywhere on S_A. Hence from (2.2.8) and (2.2.32) $\mu(S_A) \geqslant \mu_{a.c.}(S_A) \geqslant A\kappa(S_A)$ which proves the result.

In order to determine more precisely the sets on which the absolutely continuous, singular continuous and discrete spectra are concentrated we now investigate the set S which consists of all points of $\mathbb{R}$ at which $\frac{d\mu}{d\kappa}$ does not exist finitely or infinitely. It follows from Lebesgue's Theorem (2.2.13) that $\kappa(S)=0$; we shall now establish that $\mu(S)=0$ also. Our proof is adapted from Theorem 9.1 of [SA], and is geometric in character.

We require some notation and definitions, and a preliminary Lemma.

Let λ, y be rectangular Cartesian co-ordinates in the plane, and let $f(\lambda)$ be a function of bounded variation. Let the discontinuities of $f(\lambda)$ which are most countable, be denoted by $\{c_i\}$.

The <u>curve Γ generated by $f(\lambda)$</u> is the continuous curve whose graph is obtained from that of f by adding to the latter segments of each of the lines $\lambda = c_i$.

If the curve Γ is defined in the plane by parametric equations $\lambda = R(t), y = Y(t)$, then the <u>length of Γ</u> on the t-interval $[a,b]$ is defined to be

$$\sup \sum_{l=1}^{r} d(t_{l-1}, t_l) \qquad (2.2.33)$$

where $a = t_0 < t_1 < \ldots\ldots t_r = b$ is any partition of $[a,b]$ and

$$d(t_{l-1}, t_l) = ([R(t_l) - R(t_{l-1})]^2 + [Y(t_l) - Y(t_{l-1})]^2)^{\frac{1}{2}}$$

Now let Γ be the curve generated by the spectral function $\rho(\lambda)$ and let s be the length of Γ measured from an arbitrary fixed point of Γ in the direction of $\rho(\lambda)$ increasing. For convenience we shall also use s to refer to the point of Γ at which the length of Γ is s, whenever there is no ambiguity. Let R(s) and Y(s) denote the λ and y co-ordinates of the point s, let I denote any interval of Γ containing s, and let $R(I), \kappa(I)$ denote $\kappa(\{R(s) : s \in I\})$ and $\kappa(\{s : s \in I\})$ respectively.

We prove the following:

2.4 <u>Lemma</u>: For Lebesgue almost all s the derivatives

$$R'(s) = \lim_{\kappa(I) \to 0} \frac{R(I)}{\kappa(I)} \quad , \quad Y'(s) = \lim_{\kappa(I) \to 0} \frac{Y(I)}{\kappa(I)}$$

exist and

$$([R'(s)]^2 + [Y'(s)]^2)^{\frac{1}{2}} = 1$$

<u>Proof</u>

R(s) and Y(s) are monotonically increasing functions of s, so that $R'(s)$ and $Y'(s)$ simultaneously exist and are finite at Lebesgue almost all points of $\mathbb{R}^+$ by Lebesgue's Theorem (2.2.13).

Also, by Pythagorean geometry,

$$([R(I)]^2 + [Y(I)]^2)^{\frac{1}{2}} \leq \kappa(I)$$

for all intervals I, which implies

$$([R'(s)]^2 + [Y'(s)]^2)^{\frac{1}{2}} \leq 1$$

whenever $R'(s)$ and $Y'(s)$ both exist.

We require therefore to show that $\kappa(U) = 0$, where $U = \{ s \in \mathbb{R}^+ : R'(s), Y'(s)$ exist and $([R'(s)]^2 + [Y'(s)]^2)^{\frac{1}{2}} < 1 \}$.

Let U_k denote $U \cap [0,k]$ for each $k \in \mathbb{N}$. Since $\kappa(U) \leq \sum_{k=1}^{\infty} \kappa(U_k)$ we need only prove that $\kappa(U_k)=0$ for each $k \in \mathbb{N}$. Define

$$U_{n,k} = \{ s \in U_k : \left(\left[\frac{R(I)}{\kappa(I)}\right]^2 + \left[\frac{Y(I)}{\kappa(I)}\right]^2 \right)^{\frac{1}{2}} + \frac{1}{n} \leq 1$$

for all intervals I containing s with diameter $< \frac{1}{n}$ } (2.2.34)

To prove that $\kappa(U_k)=0$ for given $k \in \mathbb{N}$, we show that

$$U_k \subseteq \bigcup_{n=1}^{\infty} U_{n,k} \tag{2.2.35}$$

and $$\kappa(U_{n,k}) = 0 \tag{2.2.36}$$

for each $n \in \mathbb{N}$.

For each $s \in U_k$ there exist $l \in \mathbb{N}$ such that

$$([R'(s)]^2 + [Y'(s)]^2)^{\frac{1}{2}} + \frac{2}{l} \leq 1$$

and $m \in \mathbb{N}$ such that

$$\left| \left(\left[\frac{R(I)}{\kappa(I)}\right]^2 + \frac{Y(I)}{\kappa(I)}\Big]^2 \right)^{\frac{1}{2}} - ([R'(s)]^2 + [Y'(s)]^2)^{\frac{1}{2}} \right| < \frac{1}{l}$$

for all intervals I containing s such that $\kappa(I) < \frac{1}{m}$. Hence

$$\left(\left[\frac{R(I)}{\kappa(I)}\right]^2 + \left[\frac{Y(I)}{\kappa(I)}\right]^2 \right)^{\frac{1}{2}} + \frac{1}{l} \leq 1$$

for all intervals I containing s such that $\kappa(I) < \frac{1}{m}$ from which we see that $s \in U_{q,k}$ where $q = \min\{\frac{1}{l}, \frac{1}{m}\}$.

Thus (2.2.35) is proved; we now establish (2.2.36).

For n = p, let $\varepsilon > 0$ be given. By (2.2.33) there exists a sequence $\{s_0, \ldots s_m\}$ of points of Γ, with $s_0 = 0, s_m = k$ and $s_l < s_{l+1}$ for each $l \in \{0, \ldots m-1\}$, such that

$$s_{l+1} - s_l < \frac{1}{p} \tag{2.2.37}$$

for all $l \in \{0, \ldots\ldots m-1\}$ and

$$k = \sum_{l=0}^{m-1} (s_{l+1} - s_l) \leq \sum_{l=0}^{m-1} d(s_l, s_{l+1}) + \frac{\varepsilon}{p} \tag{2.2.38}$$

For each $l \in \{0, \ldots m-1\}$ for which $U_{p,k} \cap (s_l, s_{l+1}] \neq \phi$ we may, by (2.2.37), set $I = (s_l, s_{l+1}]$ in the defining inequality in (2.2.34) to give

$$d(s_l, s_{l+1}) + \frac{s_{l+1} - s_l}{p} \leq s_{l+1} - s_l \tag{2.2.39}$$

Hence, if $\sum_l^{(p)}$ denotes summation over all indices l for which $U_{p,k} \cap (s_l, s_{l+1}] \neq \phi$, we have by (2.2.38) and (2.2.39)

$$\begin{aligned} \kappa(U_{p,k}) &\leq \sum_l^{(p)} (s_{l+1} - s_l) \\ &\leq p \sum_l^{(p)} [(s_{l+1} - s_l) - d(s_l, s_{l+1})) \\ &< \epsilon \end{aligned}$$

Since $\epsilon > 0$ and $p \in \mathbb{N}$ were chosen arbitrarily, (2.2.36) is proved for all $n \in \mathbb{N}$; this completes the proof of the lemma.

In the following we shall refer to arbitrary points of the λ-axis as x. This is merely a convenience of notation bearing in mind the contexts in which Proposition 2.5 will be applied later.

2.5 <u>Proposition</u>: $\mu(\{x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x)$ does not exist finitely or infinitely $\}) = 0$.

<u>Proof</u>:

Let Γ, R(s), Y(s) be as in Lemma 2.4.

Define

$$U = \{x \in [R(0), \infty) : \frac{d\mu}{d\kappa}(x) \text{ does not exist finitely or infinitely}\} \tag{2.2.40}$$

We show that $\mu(U) = 0$.

From Lemma 2.4, $R'(s)$ and $Y'(s)$ both exist and are not simultaneously zero for Lebesgue almost all $s \geqslant 0$. Hence, noting the remarks following Lebesgue's Theorem (2.2.13), we have

$$\frac{d\mu}{d\kappa}(R(s)) = \left[\frac{d\rho(R(s))}{ds}\bigg/\frac{d\,R(s)}{ds}\right] = \frac{\Upsilon'(s)}{R'(s)}$$

for κ-almost all $s \geq 0$, so that $\kappa(\{ s \in \mathbb{R}^+ : \frac{d\mu}{d\kappa}(R(s))$ does not exist finitely or infinitely $\}) = 0$ (2.2.41)

Now the length of Γ generates a measure on $[R(0),\infty)$. For, defining $S(a,b] = R^{-1}(b+) - R^{-1}(a+)$ for all a,b in $[R(0),\infty)$, we see that S is a measure on the algebra $\mathcal{Q}'$ of half open subintervals of $[R(0),\infty)$ which by the Hahn Extension Theorem (2.2.2) may be extended to a unique measure on a σ-algebra containing $\mathcal{Q}'$. Moreover, by Pythagorean geometry,

$$S(E) \geq \mu(E)$$

for all subintervals E of $[R(0),\infty)$, and consequently the same is also true of arbitrary measurable subsets E of $[R(0),\infty)$.

Since (2.2.41) may now be expressed in the equivalent form:

$S(\{ x \in [R(0),\infty) : \frac{d\mu}{d\kappa}(R(s))$ does not exist finitely or infinitely$\})$ $= 0$, (2.2.40) and (2.2.41) imply $\mu(U) = 0$. The proposition now follows from the arbitrariness of the point s = 0.

We remark that the analogue of Proposition 2.5 for κ-measure is Lebesgue's Theorem, viz: $\kappa(\{ x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x)$ does not exist finitely$\})$ $= 0$. However, Proposition 2.5 leaves open the question of whether the set $\{ x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x) = \infty \}$, which has zero Lebesgue measure, can have positive μ-measure; by a process of elimination, we shall see that μ_s is concentrated on precisely this set. We first use the foregoing proposition to generalise Lemma 2.1:

2.6 <u>Proposition</u>: If S is a measurable subset of $\mathbb{R}$ with the property that for each x in S there exists $C_x < \infty$ such that $\lim_{n\to\infty} g_n(x) < C_x$

where

$$g_n(x) = \inf_{\kappa(I_x)<\frac{1}{n}} \left\{ \frac{\mu(I_x)}{\kappa(I_x)} : I_x \text{ is an interval containing } x \right\}$$

then $\mu \ll \kappa$ on S.

Proof:

Define $h_n(x) = \dfrac{\mu((x-\frac{1}{n}, x+\frac{1}{n}])}{\frac{2}{n}}$

Rewriting $h_n(x) = \frac{n}{2}(\rho(x+\frac{1}{n}) - \rho(x-\frac{1}{n}))$, we see

that $h_n(x)$ is a function of bounded variation, and hence is measurable for each $n \in \mathbb{N}$. It follows that $\liminf_{n\to\infty} h_n(x)$ is a measurable function

Define $S' = \{x \in S : \frac{d\mu}{dx}(x) \text{ exists}\}$; as we noted in the remarks following (2.2.13), S' is a Borel set. Hence, for each k in $\mathbb{N}$,

$$\begin{aligned} S_k &= S' \cap \{x \in S : k-1 \leq \liminf_{n\to\infty} h_n(x) < k\} \\ &= \{x \in S' : k-1 \leq \frac{d\mu}{d\kappa}(x) < k\} \end{aligned}$$

is a measurable set, and by Lemma 2.1, $\mu \ll \kappa$ on S_k. Consequently, $\mu \ll \kappa$ on $\bigcup_{k\in\mathbb{N}} S_k$; since, moreover, $\mu(S \setminus \bigcup_{k\in\mathbb{N}} S_k) = 0$ by Proposition 2.5 and the definition of S, the result is proved.

2.7 Corollary: (i) If $\mu \perp \kappa$, then $\mu(\{x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x) \neq \infty\}) = 0$.

(ii) $\mu \perp \kappa$ if and only if $\frac{d\mu}{d\kappa} = 0$ κ-almost everywhere on $\mathbb{R}$.

Proof:

Proof of (i):

If $\mu \perp \kappa$, then there exists a measurable set S such that $\kappa(S) = 0$ and $\mu(\mathbb{R} \setminus S) = 0$ by (2.2.5). Hence $\mu(\{x \in \mathbb{R} \setminus S : \frac{d\mu}{d\kappa}(x) \neq \infty\}) = 0$ and since $\mu \ll \kappa$ on $\{x \in S : \frac{d\mu}{d\kappa}(x) \neq \infty\}$ by Proposition 2.6,

$\mu(\{x \in S : \frac{d\mu}{d\kappa}(x) \neq \infty\}) = 0$. Hence result.

<u>Proof of (ii)</u>:

Let $E = \{x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x)$ exists and $0 < \frac{d\mu}{d\kappa}(x) < \infty\}$

If $\mu \perp \kappa$, then $\mu(E) = 0$, by (i); hence $\kappa(E) = 0$ by Lemma 2.3, so that, by Lebesgue's Theorem (2.2.13), $\frac{d\mu}{d\kappa} = 0$ κ-almost everywhere on $\mathbb{R}$.

Let $F = \{x \in \mathbb{R} : \frac{d\mu}{dx}(x)$ exists and equals $0\}$.

Then F is measurable, $\mu(F) = 0$ by Corollary 2.2, so if $\kappa(\mathbb{R} \setminus F) = 0$, $\mu \perp \kappa$ by (2.2.5).

This completes the proof of (ii), and hence,of the corollary.

We are now in a position to relate the decomposition of the spectrum to properties of the derivative $\frac{d\mu}{d\kappa}$. First we need to clarify the concept of a measure being "concentrated" on a subset of $\mathbb{R}$.

2.8 <u>Definition</u>: A subset of $\mathbb{R}$ is said to be a <u>minimal support</u> of a measure ι if the following conditions are satisfied:

(i) $\iota(\mathbb{R} \setminus S) = 0$

(ii) If S_0 is a subset of S such that $\iota(S_0) = 0$, then $\kappa(S_0) = 0$.

We remark that, in general, the spectrum of H need not be the same set as any of the minimal supports of the spectral measure μ. To see this, we note that according to the inverse method of Gel'fand and Levitan ([GL]), if $a, b \in \mathbb{R}$ with $-\infty < a < b < \infty$, there exists an operator H with spectral measure μ such that $\mu(\{x\}) > 0$ if $x \in [a, b] \cap \mathbb{Q}$ and $\mu(\{x\}) = 0$ if $x \in [a, b] \setminus \mathbb{Q}$. Clearly the spectrum of H, being a closed set, contains $[a, b]$ whereas, since $[a, b] \cap \mathbb{Q}$ is a minimal support of μ on $[a,b]$, $\kappa(\mathfrak{m} \cap [a,b])=0$ for every other minimal support $\mathfrak{m}$ of μ.

2.9 Theorem: Minimal supports $\mathfrak{m}, \mathfrak{m}_{a.c.}, \mathfrak{m}_{s.}, \mathfrak{m}_{s.c.}$ and $\mathfrak{m}_{d.}$ of $\mu, \mu_{a.c.}, \mu_{s.}, \mu_{s.c.}$ and $\mu_{d.}$ respectively are as follows, where

$E = \{x \in \mathbb{R} : \frac{d\mu(x)}{d\kappa}$ exists$\}$:

(i) $\mathfrak{m} = \{x \in E : 0 < \frac{d\mu}{d\kappa}(x) \leq \infty\}$

(ii) $\mathfrak{m}_{a.c.} = \{x \in E : 0 < \frac{d\mu}{d\kappa}(x) < \infty\}$

(iii) $\mathfrak{m}_{s.} = \{x \in E : \frac{d\mu}{d\kappa}(x) = \infty\}$

(iv) $\mathfrak{m}_{s.c.} = \{x \in E : \frac{d\mu}{d\kappa}(x) = \infty, \mu(\{x\}) = 0\}$

(v) $\mathfrak{m}_{d.} = \{x \in E : \frac{d\mu}{d\kappa}(x) = \infty, \mu(\{x\}) > 0\}$

Proof:

We need only prove (ii), (iv) and (v), since (iv) and (v) imply (iii), and (iii) and (ii) imply (i).

Proof of (ii):

Let $S = \{x \in E : 0 < \frac{d\mu}{d\kappa}(x) < \infty\}$

From Corollary 2.2, Proposition 2.5 and Lebesgue's Theorem (2.2.13), $\mu_{a.c.}(\mathbb{R} \setminus S) = 0$. To show that S is minimal we prove that if $S_0 \subseteq S$ is measurable and $\kappa(S_0) > 0$, then $\mu_{a.c.}(S_0) > 0$.

Define $S_m = \{x \in S_0 : \frac{d\mu}{d\kappa}(x) > \frac{1}{m}\}$ for each $m \in \mathbb{N}$. Since $S_0 = \bigcup_{m \in \mathbb{N}} S_m$ and $\kappa(S_0) > 0$, there exists $n \in \mathbb{N}$ such that $\kappa(S_n) > 0$. Hence $\mu(S_n) \geq \frac{1}{n}\kappa(S_n)$ (cf. proof of Lemma 2.3), so that $\mu(S_n) > 0$. Since $\mu_{s.}(S_n) = 0$ by Proposition 2.6, this implies that $\mu_{a.c.}(S_n) > 0$ by (2.2.8)

Hence $\mu_{a.c.}(S_0) > 0$, and the minimality of S is proved.

Proof of (iv):

Let

$$S = \{ x \in E : \frac{d\mu}{d\kappa}(x) = \infty \ , \ \mu(\{x\}) = 0 \}$$

Since

$$\mathbb{R} \setminus S = \{ x \in E : \frac{d\mu(x)}{d\kappa} = \infty , \mu(\{x\}) > 0 \}$$
$$\cup \{ x \in \mathbb{R} : \liminf_{\delta \to 0} \frac{\mu[x-\delta, x+\delta]}{2\delta} < \infty \}$$

$\mu_{s.c.}(\mathbb{R} \setminus S) = 0$ by (2.2.6) and Proposition 2.6.

Since $\kappa(S) = 0$ by Lebesgue's Theorem (2.2.13), S is minimal.

Proof of (v):

By definitions (2.2.7) and 2.8, m_d is the smallest support of μ_d.

We remark that according to Definition 2.8, m, as defined in the above theorem, is a minimal support of both μ and $\mu_{a.c.}$. While it is quite possible to recast the definition so as to ensure that minimal supports of orthogonal measures are always disjoint, we prefer to retain Definition 2.8 for a number of reasons. Firstly, restricting the definition would mean that a further condition needed to be checked each time a subset of was shown to be a minimal support, thus complicating proofs. Secondly, there is no difficulty, at least in principle, in obtaining disjoint supports from non-disjoint supports of mutually singular measures, on account of (2.2.5). Finally, as we shall show in §3, the set of all minimal supports of a measure, as defined in 2.8, is an equivalence class, and this property is frequently useful when establishing minimal supports.

We also note that, if one type of spectrum is absent on a subset S of $\mathbb{R}$, then, although there exists a minimal support of the corresponding measure which is empty on S, the appropriate minimal support of Theorem 2.9 need not be empty on S. To illustrate this point we give a couple of examples; we shall have occasion to refer to these again in Chapter III when we investigate the correlation between the parts of the spectrum and the existence or otherwise of subordinate solutions of the Schrödinger equation (2.1.1).

2.10 Example: It is well-known that an absolutely continuous function may have an infinite derivative on a non-empty set of points whose Lebesgue measure is zero. Indeed, given any bounded open interval (a,b) of $\mathbb{R}$ and any countable or uncountable subset S of (a,b) having Lebesgue measure zero, there exists an absolutely continuous function whose derivative is infinite at all points of S ([T1], §11.83, Lemma 1). Hence, by the inverse method of Gel'fand and Levitan ([GL]), an operator H exists which has no singular spectrum on (a,b), but for which $\mathcal{M}_S \cap [a,b]$ is an uncountable set.

2.11 Example: We show that an operator H exists which has no absolutely continuous spectrum on (-2,2) but for which $\frac{d\mu}{dK}(0) = 1$.

According to the inverse method of Gel'fand and Levitan ([GL]), an operator H exists whose spectral function is equal to $\rho(\lambda)$ on [-2,2], where $\rho(\lambda)$ is defined on [-2,2] as follows:

$$\rho(\lambda) = \begin{cases} -\frac{\pi}{2} & \lambda \in [-2, \sin\left(-\frac{\pi}{2}\right)) \\ \frac{\pi}{2} & \lambda \in [\sin\frac{\pi}{2}, 2] \end{cases}$$

$$\rho(0) = 0$$

and for each n in $\mathbb{N}$,

$$\rho(\lambda) = \begin{cases} (\sin)^{2n}\left(-\frac{\pi}{2}\right) & \lambda \in [(\sin)^{2n-1}\left(-\frac{\pi}{2}\right), (\sin)^{2n+1}\left(-\frac{\pi}{2}\right)) \\ (\sin)^{2n}\left(\frac{\pi}{2}\right) & \lambda \in [(\sin)^{2n+1}\left(\frac{\pi}{2}\right), (\sin)^{2n-1}\left(\frac{\pi}{2}\right)) \end{cases}$$

where $(\sin)^n k$ denotes $(\sin(\sin....(\sin k)...))$ n times

To show that $\rho(\lambda)$ is defined on all of $[-2,2]$, we prove that $(\sin)^n\left(-\frac{\pi}{2}\right)$ and $(\sin)^n\left(\frac{\pi}{2}\right)$ are increasing and decreasing sequences respectively which converge to 0.

Since $0<\sin x<x$ for x in $(0,\frac{\pi}{2}]$, $(\sin)^n \frac{\pi}{2}$ decreases with n and converges to some $l \geqslant 0$ as $n \to \infty$. If $l \gneqq 0$, $\sin l \lneqq l$ and so, by the continuity of sin, there exists $\varepsilon>0$ such that $\sin(l+\varepsilon) \lneqq l$. Since there exists $N_\varepsilon \in \mathbb{N}$ such that $(\sin)^{n+1}\left(\frac{\pi}{2}\right)<l+\varepsilon$ if $n \geqslant N_\varepsilon$, we have that $(\sin)^{n+1}\left(\frac{\pi}{2}\right)<\sin(l+\varepsilon) \lneqq l$ for $n \geqslant N_\varepsilon$. This is impossible since $(\sin)^n\left(\frac{\pi}{2}\right)$ converges to l from above, and hence $l = 0$.

The proof for $(\sin)^n\left(-\frac{\pi}{2}\right)$ is similar.

Now for each $n \in \mathbb{N}$, if

$$(\sin)^{2n+1}\left(\frac{\pi}{2}\right) \leqslant \lambda < (\sin)^{2n-1}\left(\frac{\pi}{2}\right)$$

then $(\sin)^{2n}\left(\frac{\pi}{2}\right) \leqslant \sin^{-1}\lambda$, $\sin\lambda < (\sin)^{2n}\left(\frac{\pi}{2}\right)$

and $\rho(\lambda) = (\sin)^{2n}\left(\frac{\pi}{2}\right)$

It follows that

$$\sin\lambda \leqslant \rho(\lambda) < \sin^{-1}\lambda \quad \text{on } (0,1]$$

and, similarly,

$$\sin^{-1}\lambda < \rho(\lambda) \leqslant \sin\lambda \quad \text{on } [-1,0)$$

Since $\rho(0)=0$ this implies

$$\frac{\sin\delta}{\delta} \leqslant \frac{\mu[0,\delta]}{\delta} \leqslant \frac{\sin^{-1}\delta}{\delta}$$

and, similarly,

$$\frac{\sin\delta}{\delta} \leqslant \frac{\mu[-\delta,0]}{\delta} \leqslant \frac{\sin^{-1}\delta}{\delta}$$

From $\lim_{\delta\to 0} \frac{\sin\delta}{\delta} = \lim_{\delta\to 0} \frac{\sin^{-1}\delta}{\delta} = 1$ we deduce that $\frac{d\mu}{d\kappa}(0) = 1$ as required. Since $\rho(\lambda)$ is a saltus function on $[-2,0) \cup (0,2]$

we have $\mathfrak{M}_{a.c.} \cap [-2,2] = \{0\} \neq \phi$ but no absolutely continuous spectrum on $(-2,2)$.

In the next section we shall use Theorem 2.9 to obtain a new set of minimal supports in terms of a function which is analytic in the upper half-plane where it has positive imaginary part.

§3 The function m(z)

We stated at the beginning of this chapter that if a self-adjoint operator H is defined by the Schrödinger equation together with a boundary condition at r = 0, then the corresponding differential expression is in the limit point case at ∞, and in the limit circle case at 0. To clarify this remark, we briefly indicate some of the theoretical background.

In 1909-10 Hermann Weyl produced three remarkable papers on second order differential equations, which developed and generalised the work of earlier mathematicians such as Fourier, Sturm and Liouville ([W1], [W2],[W3]). He obtained an eigenfunction expansion theorem of great generality and established the theory of the limit point and limit circle which is, in outline, as follows:

In considering a general equation lu = zu of the Sturm-Liouville type with a regular end-point at 0, it is found that only the following possibilities can occur:

1) Limit Point Case: For every z in $\mathbb{C} \setminus \mathbb{R}$, lu = zu has just one solution u which is in $L_2[0,\infty)$, and for every real z there is no more than one solution in $L_2[0,\infty)$.

2) Limit Circle Case: all solutions of lu = zu are in $L_2[0,\infty)$ for every z in $\mathbb{C}$.

The same distinction may be applied to each of the intervals $(0,a]$, $[a,\infty)$ where $0 < a < \infty$, in particular if 0 is a singular endpoint ([CL] Ch.9,§).

The geometric terminology arises because the locus $C_b(z)$ of the set $\{m_b(z,\beta):\beta\in[0,\pi)$, and $b\in(0,\infty)$, $z\in\mathbb{R}\setminus\mathbb{C}$ are fixed$\}$ is a circle in the complex plane, where $m_b(z,\beta)$ is defined by the condition that the solution $u(r)=u_2(r,z,\alpha)+m_b(z,b)u_1(r,z,\alpha)$ of lu = zu satisfies the real boundary condition

$$u(b)\cos\beta + u'(b)\sin\beta = 0$$

$u_1(r,z,\alpha)$ and $u_2(r,z,\alpha)$ being solutions of lu = zu satisfying

$$u_1(0,z,\alpha) = -u_2'(0,z,\alpha) = -\sin\alpha$$

$$u_2(0,z,\alpha) = u_1'(0,z,\alpha) = \cos\alpha \qquad (2.3.1)$$

If $b_2 > b_1$, then for each z in $\mathbb{C}\setminus\mathbb{R}$, $C_{b_2}(z)$ lies entirely inside $C_{b_1}(z)$ and the set of nesting circles $\{C_b(z)\}$ converges either to a point, the "limit point", or to a circle, the "limit circle" as $b\to\infty$. In the first case the problem is self-adjoint, whereas in the limit circle case an additional boundary condition is required at ∞ ([CL] Ch.9, §4). If m(z) is the limit point, or any point on the limit circle, $u_m(r,z,\alpha)=u_2(r,z,\alpha)+m(z)u_1(r,z,\alpha)$ is in $L_2[0,\infty)$ and, if $\|\cdot\|$ denotes the $L_2[0,\infty)$ norm,

$$\|u_m(r,z,\alpha)\|^2 = \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} \qquad (2.3.2)$$

where m(z) depends on α.

Possibly due to the influence of functional analysis, with its emphasis on abstract structure in a wider context, the work of Weyl was not significantly developed for another thirty years. It was E.C.Titchmarsh who,aware of the importance to mathematical physics, was primarily responsible for a revival of interest in second order differential equations of the Sturm-Liouville type in the 1940s. An important outcome of his work was the formula

$$\rho(\lambda)-\rho(\nu) = \lim_{y\downarrow 0}\frac{1}{\pi}\int_{\nu}^{\lambda}\operatorname{Im} m(x+iy)\,dx \qquad (2.3.3)$$

for points of continuity λ,ν of $\rho(\lambda)$. The derivation of the related formula (a slightly different formulation is required if $\alpha = 0$, see [EK] §2.3)

$$m(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z}\, d\rho(\lambda) + \cot\alpha \tag{2.3.4}$$

for z in $\mathbb{C}\setminus\mathbb{R}$ stems from the analyticity of m(z) in either half-plane, which was proved by Weyl in 1935 ([W4]). There are other equivalent representations of m(z) in current use (see[EK] §2.3); (2.3.3) and (2.3.4) are derived in ([LS] Ch.2, §5). It should be noted that the spectral function $\rho(\lambda)$, although often most conveniently analysed using (2.3.3), originated as the limit as $b\to\infty$ of step functions $\rho_b(\lambda)$ arising from the Sturm-Liouville problem on the finite interval $[0,b]$; the jumps of $\rho_b(\lambda)$ being at each eigenvalue, with the discontinuity equal to the inverse of the square of the $L_2[0,b]$ norm of the corresponding eigenvector ([CL] Ch.9,§3).

Analysis of the spectrum until the late 1950s was generally based on the idea of locating the points of increase of a spectral function and only discriminated between the discrete and continuous parts. Titchmarsh recognised that the isolated points of the spectrum occurred at the poles of m(z), and that if the set of points to which $\operatorname{Im} m(z)$ converged as $y\downarrow 0$ was bounded above and away from zero on an interval I, then there was continuous spectrum on I. ([T2] Ch.5). A subtler appreciation of the relationship between the spectrum and the boundary behaviour of $\operatorname{Im} m(z)$ was achieved in 1957 by N.Aronszajn ([A] §2]. His standard supports of the decomposed parts of the spectral measure are similar to the set of minimal supports we obtain in Theorem 2.17.

No doubt because the importance of boundary properties of analytic functions for spectral analysis of differential operators was only recognised comparatively recently, most available literature lacks even a rudimentary account of those aspects of analytic function theory that are relevant. In the proofs of our results, we shall have frequent occasion to use properties of m(z) as z approaches points on the real axis, so it seems appropriate to give an indication of some of the relevant theory. As this theory developed quite independently, there was no special consideration of functions analytic in the upper half-plane with positive imaginary part such as m(z). It was usual, in fact, to consider the behaviour of a function which was meromorphic on the interior of a unit disc in the complex plane as a point on the perimeter of the disc was approached radially, or "non-tangentially". Accordingly, we shall first cite some results for this case, and then show how properties of conformal mappings may be used to give analogous results for a function meromorphic in the upper half-plane, with particular reference to m(z).

We require some notation and definitions:

Let $f(\omega)$ be a function from $\mathbb{C}$ to $\mathbb{C}$ which is meromorphic on an open region R, bounded by a smooth boundary B.

A triangular neighbourhood $\Delta_{p,\alpha}(U)$ of a point p on B is defined to be the intersection of a neighbourhood U of p in $\mathbb{C}$ with an open region lying entirely in R and bounded by two straight lines intersecting at p; these straight lines are reflections of each other about the normal to the boundary at p, and subtend an angle 2α at p $(0<\alpha<\frac{\pi}{2})$.

Let $S_{p,\alpha}(U)$ be the set of limit points of $f(\omega)$ in $\Delta_{p,\alpha}(U)$ (this may include the "point" ∞); and define $C_{p,\alpha} = \bigcap_{U} S_{p,\alpha}(U)$. $\bigcup_{0<\alpha<\frac{\pi}{2}} C_{p,\alpha}$ is called the <u>cluster set</u> at p.

The function $f(\omega)$ is said to have a <u>non-tangential limit</u> at a point p on B if the cluster set at p consists of just one point.

The function $f(\omega)$ is said to <u>behave restrictedly</u> at a point p on B if there exists a triangular neighbourhood $\Delta_{p,\alpha}(U)$ such that $f(\Delta_{p,\alpha}(U)) = \{f(\omega) : \omega \in \Delta_{p,\alpha}(U)\}$ is not dense in the whole complex plane.

By the Lebesgue measure of a subset of the perimeter of the unit disc we refer to its length in the usual sense; thus the Lebesgue measure of an arc on the perimeter which subtends an angle θ at the centre is θ.

The following result was first proved for bounded analytic functions by F and M. Riesz in 1916 ([RI]) and later for general analytic functions by Lusin and Privalov ([LP]), who also showed that, in general, it is not possible to replace the condition of non-tangential convergence by radial convergence. The extension to meromorphic functions was accomplished by A. Plessner in 1927 ([PL] Satz II).

(A) If $f(\omega)$ and $g(\omega)$ are functions which are meromorphic on the interior of a unit disc, and which have the same non-tangential limit on a subset of its perimeter having positive Lebesgue measure, then $f(\omega) \equiv g(\omega)$.

If we take $g(\omega)$ to be a constant function then the following corollary is immediate:

(B) If $f(\omega)$ is non-constant and meromorphic on the interior of a unit disc, then the set of points on its perimeter for which $f(\omega)$ has a given fixed value as its non-tangential limit has Lebesgue measure zero.

The next result is due to A. Plessner ([PL] Sätze I, IV).

(C) If $f(\omega)$ is meromorphic on the interior of a unit disc, and if E is the subset of points of the perimeter at which $f(\omega)$ behaves restrictedly, then $f(\omega)$ has a finite non-tangential limit Lebesgue almost everywhere on E.

Points on the perimeter at which $f(\omega)$ does not behave restrictedly are sometimes referred to as Plessner points ([NO] Ch.III).

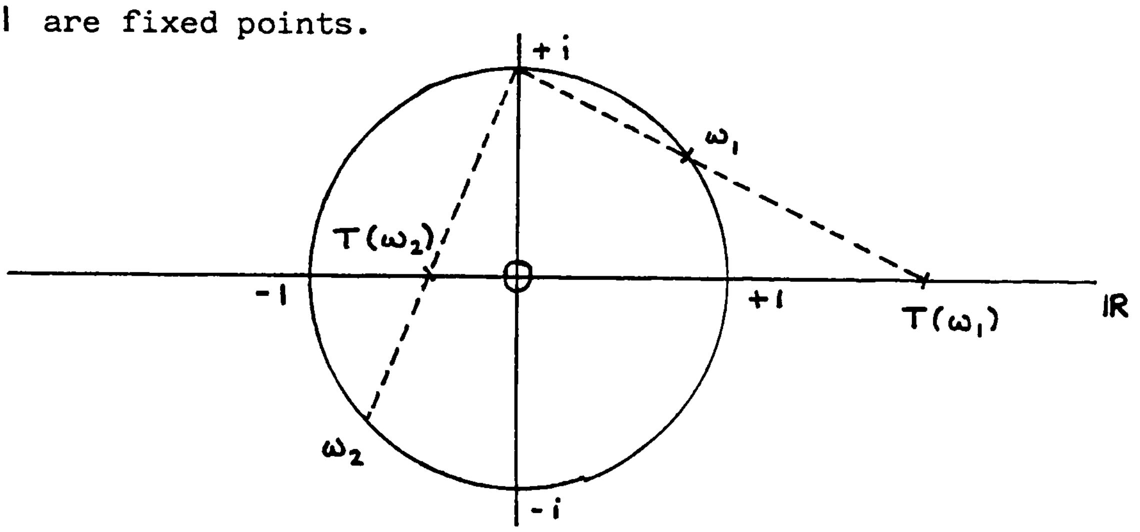
Further discussion and refinements to the above results may be found in [CC], [NO].

We now indicate how properties of conformal mappings are used to show that (A) - (C) also hold for a function which is meromorphic in the upper half plane, the "perimeter" in this case being the real line.

Consider the Möbuis transformation ([M] §33):

$$z = T(\omega) = \frac{i\omega - 1}{i - \omega} = \frac{2x - i(x^2 + y^2 - 1)}{x^2 + (1 - y)^2}$$

where $\omega = X + i\zeta$. It is clear that the circle $|\omega| = 1$ is mapped onto the real line $\mathbb{R}$, with the pole, $\omega = i$, being mapped to the "improper point" ∞. Moreover, the region outside the circle $|\omega| = 1$ is mapped conformally onto the lower half plane, and the region inside the circle onto the upper half plane. Indeed, it is easily ascertained using elementary geometry that $T(\omega)$ maps the circle $|\omega| = 1$ stereographically onto $\mathbb{R}$, with $\omega = i$ as pole; $\omega = \pm 1$ are fixed points.



Consider now a subset E of $B = \{\omega : |\omega| = 1\}$ which has Lebesgue measure zero and is such that the pole $\omega = i$ is not in the closure of E. We may cover E with a countable collection of sets $\{C_i\}$ which are such that each $C_i \subseteq B$ is open in B and the pole $\omega = i$ is not contained in the closure of $\bigcup_i C_i$. This last requirement will ensure that the "magnification" of the C_i is bounded under T; that is, that there exists $M \in \mathbb{R}^+$ such that

$$\kappa(T(C_i)) \leqslant M \kappa(C_i)$$

for all i. Hence if $\varepsilon > 0$ is given, we may choose our cover to be bounded away from $\omega = i$ as above, and such that

$$\sum_i \kappa(C_i) < \frac{\varepsilon}{M}$$

whence it follows that

$$\sum_i \kappa(T(C_i)) < \varepsilon$$

Thus any subset E of the circumference of the circle $|\omega| = 1$, which has Lebesgue measure zero and whose closure does not contain the pole $\omega = i$, is mapped by T onto a bounded subset T(E) of $\mathbb{R}$ which also has Lebesgue measure zero.

Now the unit circle $|\omega|=1$, less the point $\omega=i$, may be expressed as a countable union of closed sets $\{S_i\}$, each of which does not contain the point $\omega=i$. Therefore, if $E \subseteq B \setminus \{\omega=i\}$ has Lebesgue measure zero, $T(E\cap S_i)$ has Lebesgue measure zero for each i, by above. Hence

$$\kappa(T(E)) = \kappa(\bigcup_i T(E\cap S_i)) \leq \sum_i \kappa(T(E\cap S_i)) = 0$$

and so <u>T maps any subset of $B\setminus\{\omega=i\}$ with Lebesgue measure zero onto a subset of $\mathbb{R}$ with Lebesgue measure zero</u>. (i)

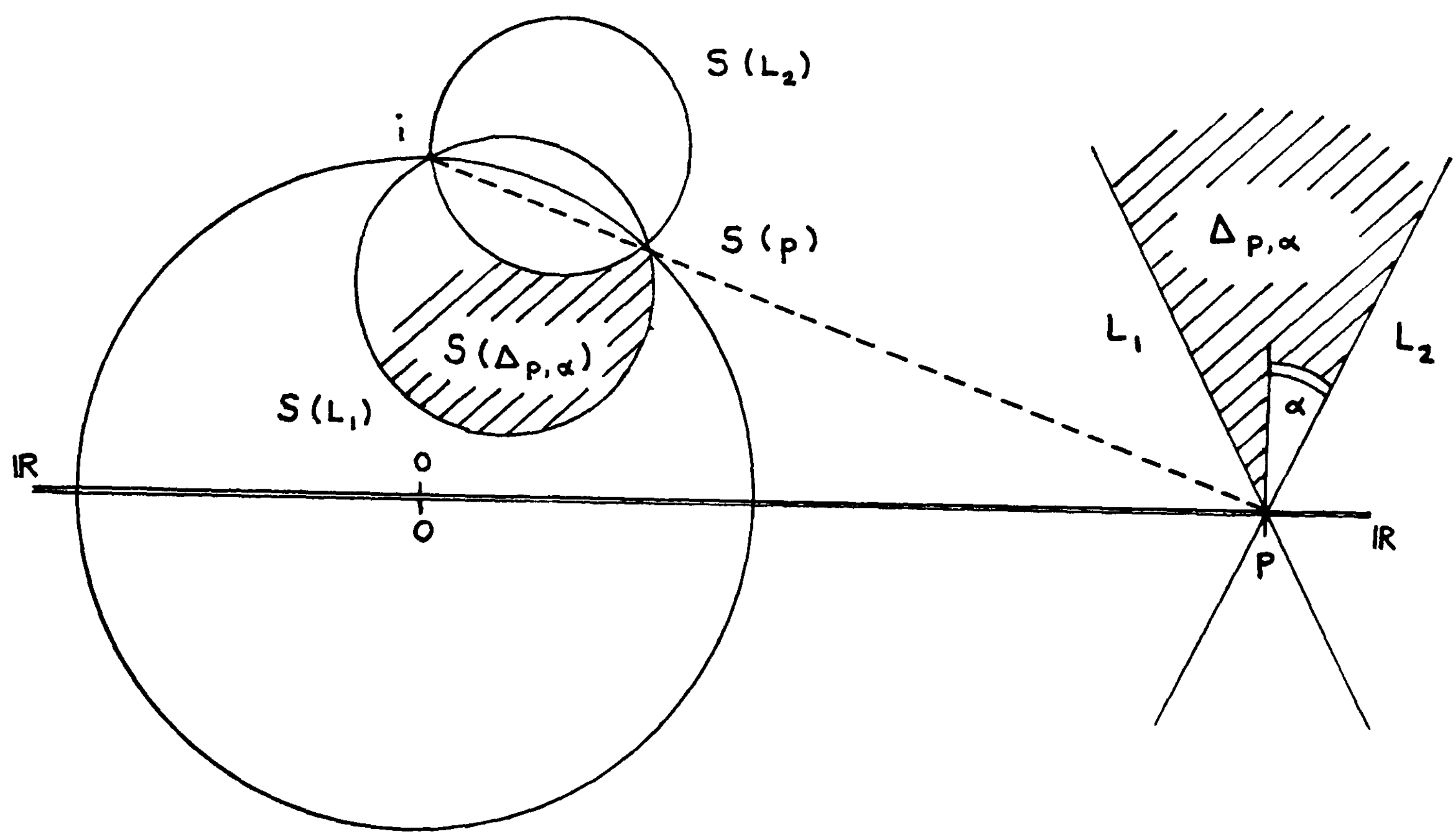
Let S(z) denote $T^{-1}(z)$. We remark that $S(z) = \dfrac{1+iz}{i+z}$ is also a Möbius transformation and a continuous one-to-one mapping of $\mathbb{C}\setminus\{-i\}$ onto $\mathbb{C}\setminus\{i\}$.

We now show that if a variable point z in the upper half-plane approaches a point $p\in\mathbb{R}$ in such a way that it eventually remains in a particular triangular neighbourhood of p, $\Delta_{p,\alpha}(V)$, then there exists a triangular neighbourhood of S(p), $\Delta_{S(p),\beta}(U)$, such that S(z) eventually remains in $\Delta_{S(p),\beta}(U)$.

Since every Möbius transformation is circle preserving ([M] §45) the straight lines L_1 and L_2 which bound $\Delta_{p,\alpha}(V)$ are mapped by S onto circles $S(L_1)$ and $S(L_2)$ in the ω-plane, each of which passes through S(p) and the pole $\omega=i$. (If one of L_1, L_2 passes through the point $\omega=-i$, then it will be mapped by S onto a straight line passing through S(p) and $\omega=i$). As a variable point z in $\Delta_{p,\alpha}(V)$ approaches p, its image S(z) approaches S(p) from the region within the circle $|\omega|=1$ which is bounded by $S(L_1)$ and $S(L_2)$. Since $S(p)\neq i$, neither $S(L_1)$ nor $S(L_2)$ is tangential to the circle $S(\mathbb{R})=\{\omega:|\omega|=1\}$ and hence S(z) is eventually contained in some triangular neighbourhood $\Delta_{S(p),\beta}(U)$ of S(p). (The diagram illustrates the case where $|p|>1$).

Now if f(z) is a function which is meromorphic in the upper half plane, but which does not have a non-tangential limit at some point $p\in\mathbb{R}$, then there exists a triangular neighbourhood $\Delta_{p,\alpha}(V)$ and a sequence of points

$\{z_i\}$ in $\Delta_{p,\alpha}(V)$ such that $z_i \to p$ but $f(z_i)$ does not converge to a limit as $i \to \infty$. We conclude from our previous remarks that $\{\omega_i : \omega_i = S(z_i)\}$ is contained in some triangular neighbourhood $\Delta_{S(p),\beta}(U)$ of S(p) in the interior of the disc $|\omega| \leq 1$; and,



using the properties of S, we see that the sequence of points $\{\omega_i : \omega_i = S(z_i)\}$ in the interior of the disc $|\omega| \lneq 1$ converges to the point $\omega = S(p)$ on its circumference as $\{z_i\}$ converges to p. However, the function $(fT)(\omega)$, which is meromorphic in the interior of the unit disc, does not have a non-tangential limit at $\omega = S(p)$ since $\{(fT)(\omega_i)\} = \{f(z_i)\}$ does not converge to a limit as $\{\omega_i\}$ converges to S(p). Thus if f(z) does not have a non-tangential limit at the real point z = p, then $(fT)(\omega)$ does not have a non-tangential limit at the point $\omega = S(p)$ of the circle $|\omega| = 1$. It may likewise be shown that the converse of this statement is also true and we have, equivalently:

<u>f(z) has a non-tangential limit at the real point z = p if and only if $(fT)(\omega)$ has a non-tangential limit at the point $\omega = S(p)$ on the unit circle $|\omega| = 1$</u> (ii)

Now if g(z) is also a complex function which is meromorphic in the upper half plane, and if f(z) and g(z) have the same non-tangential limit on a subset of $\mathbb{R}$ having positive Lebesgue measure, then $(fT)(\omega)$ and $(gT)(\omega)$ are meromorphic in the interior of the unit disc $|\omega| \leq 1$, and have the same non-tangential limit on a subset of $\{\omega : |\omega| = 1\}$ having positive Lebesgue measure, by (i) and (ii). Thus we may state:

(A)' If f(z) and g(z) are functions which are meromorphic in the upper half-plane, and have the same non-tangential limit on a subset of the real line having positive Lebesgue measure, then f(z) $\equiv$ g(z).

As before, the corollary is immediate:

(B)' If f(z) is non-constant and meromorphic on the upper half-plane, then the subset of points of $\mathbb{R}$ for which f(z) has a given fixed value as its non-tangential limit has Lebesgue measure zero.

Using the fact that conformal mappings preserve the angles at which curves intersect ([M] §23), we see that there exists a triangular neighbourhood of a real point p on which f(z) is not dense in the whole complex plane if and only if there exists a triangular neighbourhood of $S(p) \in \{\omega : |\omega| = 1\}$ on which $(fT)(\omega)$ is not dense in the whole complex plane. Hence if E is the subset of $\mathbb{R}$ on which f(z) behaves restrictedly then S(E) is the subset of the unit circle $|\omega| = 1$ at which $(fT)(\omega)$ behaves restrictedly. By (C), $(fT)(\omega)$ has a finite non-tangential limit Lebesgue almost everywhere on S(E), and hence by (i) and (ii), f(z) has a finite non-tangential limit Lebesgue almost everywhere on E. We have therefore:

(C)' If f(z) is meromorphic in the upper half-plane, and if E is the subset of $\mathbb{R}$ at which f(z) behaves restrictedly, then f(z) has a finite non-tangential limit Lebesgue almost everywhere on E.

We now return to the function m(z) which is analytic in the upper half-plane. Unless otherwise stated we shall assume that the differential expression L, defined as in (2.1.2), is in the limit point case at ∞, in which case m(z) may be defined by condition (2.1.3). (As necessary, the α dependence of m(z), $u_1(r,z)$, $u_2(r,z)$ will be indicated by $m(z,\alpha)$ $u_1(r,z,\alpha)$ and $u_2(r,z,\alpha)$ respectively). From (2.3.2) $\operatorname{Im} m(z) > 0$ if

$\operatorname{Im} z > 0$ so that m(z) behaves restrictedly at all points of $\mathbb{R}$.

We shall say that m(z) has a <u>normal limit</u> at the point $x \in \mathbb{R}$ if m(z) converges to a finite limit or to ∞ as z approaches x from above along the normal to the real axis at x.

The following is now easily deduced from (A)', (B)' and (C)':

2.12 <u>Theorem</u>: The function m(z), defined and analytic in the upper half-plane, has the following properties:

(i) m(z) has a finite non-tangential limit Lebesgue almost everywhere on $\mathbb{R}$; in particular, m(z) has a finite normal limit Lebesgue almost everywhere on $\mathbb{R}$.

(ii) The subset of points of $\mathbb{R}$ at which m(z) has a given fixed value as its normal limit has Lebesgue measure zero.

(iii) If g(z) is analytic in the upper half-plane, behaves restrictedly at all points of $\mathbb{R}$, and has the same normal limit as m(z) on a subset of $\mathbb{R}$ having positive Lebesgue measure, then $g(z) \equiv m(z)$.

It is presupposed in (ii) that m(z) is not a constant; this is certainly true in all the cases we consider. If m(z) has a normal limit at $x \in \mathbb{R}$ we denote this limit by $m_+(x)$; similarly, if $\operatorname{Im} m(z)$ has a normal limit at x, we denote this by $\operatorname{Im} m_+(x)$. Since z = x + iy, it is evident that $m_+(x) = \lim_{y \downarrow 0} m(x+iy)$, and from (2.3.4):

$$\operatorname{Im} m_+(x) = \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) \qquad (2.3.5)$$

whenever the limits exist.

We now prove a number of results relating $\operatorname{Im} m_+(x)$ to the derivative $\frac{d\mu}{d\kappa}(x)$; our first result is valid irrespective of whether the spectral measure μ is finite or infinite:

2.13 Lemma: If $\operatorname{Im} m_+(x)$ exists and equals zero, then $\frac{d\mu(x)}{d\kappa}$ also exists and equals zero.

Proof:

We first note that $\frac{d\mu(x)}{d\kappa}$ exists and equals zero if and only if

$\lim_{\delta\to 0} \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta}$ exists and equals zero.

Hence it is sufficient to prove that the hypothesis implies that $\lim_{\delta\to 0} \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta}$ exists and equals zero. Since $\frac{\delta}{(\lambda-x)^2+\delta^2} \geqslant \frac{1}{2\delta}$

on $(x-\delta, x+\delta]$, we have

$$0 \leqslant \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta} \leqslant \int_{x-\delta}^{x+\delta} \frac{\delta}{(\lambda-x)^2+\delta^2}\, d\rho(\lambda) \leqslant \int_{-\infty}^{\infty} \frac{\delta}{(\lambda-x)^2+\delta^2}\, d\rho(\lambda)$$

The result is now immediate from the hypothesis and (2.3.5).

The following proposition is also true irrespective of whether μ is a finite or an infinite measure. We use the fact that, for sufficiently small y, $\frac{y}{(\lambda-x)^2+y^2}$ decreases with y for every λ outside a certain neighbourhood of x.

2.14 Proposition: If $\frac{d\mu(x)}{d\kappa}$ exists finitely or infinitely, then $\operatorname{Im} m_+(x)$ also exists, and $\frac{d\mu(x)}{d\kappa} = \frac{1}{\pi} \operatorname{Im} m_+(x)$

Proof:

We note that if $\frac{d\mu(x)}{d\kappa}$ exists then $\lim_{\delta\to 0} \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta}$ also exists, and the two limits are equal. For the purposes of the proof it is convenient to use the function $\tilde{\rho}(\lambda)$ instead of $\rho(\lambda)$ where $\tilde{\rho}(\lambda)$ is defined as follows:

$$\tilde{\rho}(\lambda) = 0 \qquad \lambda < x$$

$$\tilde{\rho}(\lambda) = \rho(\lambda) - \rho(2x-\lambda) + \mu(\{2x-\lambda\}) \qquad \lambda \geqslant x$$

If $\lim_{\delta\to 0} \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta}$ exists, we see that $\lim_{\delta\to 0} \frac{\tilde{\rho}(x+\delta)}{2\delta}$

also exists and both are equal. Also because of the symmetry of the integrand about x, $\int_{-\infty}^{\infty} \frac{y}{(x-\lambda)^2+y^2}\, d\rho(\lambda) = \int_{x}^{\infty} \frac{y}{(x-\lambda)^2+y^2}\, d\tilde{\rho}(\lambda)$ for all $y > 0$, so that $\operatorname{Im} m_+(x) = \lim_{y \downarrow 0} \int_{x}^{\infty} \frac{y}{(x-\lambda)^2+y^2}\, d\tilde{\rho}(\lambda)$ whenever the limit exists.

Suppose now that $\frac{d\mu(x)}{d\kappa}$ exists finitely and equals l.

Then, if $\varepsilon > 0$ is given, there exists $\zeta > 0$ such that, whenever $\lambda \in (x, x+\zeta)$,

$$2(\lambda - x)(l - \tfrac{\varepsilon}{2}) < \tilde{\rho}(\lambda) < 2(\lambda - x)(l + \tfrac{\varepsilon}{2}) \tag{2.3.6}$$

We prove that (i) $\limsup_{y \downarrow 0} \frac{1}{\pi} \int_{x}^{\infty} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda) \leq l$

(ii) $\liminf_{y \downarrow 0} \frac{1}{\pi} \int_{x}^{\infty} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda) \geq l$

(i) We first show that we may choose an $M \in \mathbb{R}^+$ such that $\int_{(M,\infty)} \frac{y}{(\lambda-x)^2+y^2}\, d\tilde{\rho}(\lambda)$ is small for sufficiently small y.

For each $y > 0$

$$\int_{[x,\infty)} \frac{y}{(x-\lambda)^2+y^2}\, d\tilde{\rho}(\lambda) = y \| u_m(r,z) \|^2 < \infty$$

by (2.3.2). Hence if $y_k \in (0,1)$ is fixed and $\varepsilon > 0$ is given there exists $M > \max\{1, \zeta\}$ such that

$$\int_{(M,\infty)} \frac{y_k}{(\lambda - x)^2 + y_k^2}\, d\tilde{\rho}(\lambda) < \frac{\pi\varepsilon}{4} \tag{2.3.7}$$

We now prove that this inequality also holds for all $y < y_k$.

Since $0 < y < y_k < 1$ and $\lambda - x \geq M > 1$ on (M,∞) , $(y_k - y) y y_k < (y_k - y)(\lambda - x)^2$ for all λ in (M,∞) . Hence, if $\lambda \in (M,\infty)$

$$\frac{y}{(\lambda - x)^2 + y^2} < \frac{y_k}{(\lambda - x)^2 + y_k^2}$$

so that for all $y \leq y_k$

$$\int_{(M,\infty)} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda) < \frac{\pi\varepsilon}{4} \tag{2.3.8}$$

by (2.3.7). Now there exists $K \in \mathbb{R}^+$ such that $\tilde{\rho}(\lambda) \leqslant K$ on $[x, x + M]$, so we have, using integration by parts on $[x, x + M]$, (2.3.8) and the right-hand inequality in (2.3.6):

$$\int_{[x,\infty)} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda)$$

$$\leqslant \frac{yK}{M^2 + y^2} + \int_x^{x+\zeta} \frac{2y(\lambda - x)\rho(\lambda)}{((\lambda - x)^2 + y^2)}\, d\lambda$$

$$+ \int_{x+\zeta}^{x+M} \frac{2y(\lambda - x)\rho(\lambda)}{((\lambda - x)^2 + y^2)}\, d\lambda + \int_{(M,\infty)} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda)$$

$$\leqslant \frac{yK}{M^2 + y^2} + \int_x^{x+\zeta} \frac{4y(\lambda - x)^2(l + \frac{\varepsilon}{2})}{((\lambda - x)^2 + y^2)}\, d\lambda$$

$$+ \int_{x+\zeta}^{x+M} \frac{2y(\lambda - x)K}{((\lambda - x)^2 + y^2)^2}\, d\lambda + \frac{\pi\varepsilon}{4}$$

$$< \frac{yK}{M^2} + 4y\left(l + \frac{\varepsilon}{2}\right)\left[\frac{-(\lambda - x)}{2((\lambda - x)^2 + y^2)} + \frac{1}{2y}\tan^{-1}\frac{\lambda - x}{y}\right]_x^{x+\zeta}$$

$$+ \left[\frac{yK}{(\lambda - x)^2 + y^2}\right]_{x+\zeta}^{x+M} + \frac{\pi\varepsilon}{4}$$

$$< \frac{yK}{M^2} + 2\left(l + \frac{\varepsilon}{2}\right)\tan^{-1}\frac{\zeta}{y} + \frac{yK}{M^2} + \frac{\pi\varepsilon}{4}$$

$$< \left(l + \frac{\varepsilon}{2}\right)\pi + \frac{\pi\varepsilon}{2} \qquad \text{if } \; y < \frac{M^2\pi\varepsilon}{8K}$$

$$= \pi(l + \varepsilon)$$

Thus if $y < \min\{y_K, \frac{M^2\pi\varepsilon}{8K}\}$

$$\frac{1}{\pi}\int_{[x,\infty)} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda) < l + \varepsilon$$

where ε was chosen arbitrarily. Hence

$$\limsup_{y \downarrow 0} \frac{1}{\pi}\int_{[x,\infty)} \frac{y}{(\lambda - x)^2 + y^2}\, d\tilde{\rho}(\lambda) \leqslant l$$

as was to be proved.

(ii) Using integration by parts on $[x, x+\zeta)$ as in (i), we obtain:

$$\int_{[x,\infty)} \frac{y}{(\lambda-x)^2+y^2}\, d\tilde{\rho}(\lambda) > \int_x^{x+\zeta} \frac{4y(\lambda-x)^2(l-\frac{\varepsilon}{2})}{((\lambda-x)^2+y^2)^2}\, d\lambda$$

$$= 4y(l-\tfrac{\varepsilon}{2})\left[\frac{-(\lambda-x)}{2((\lambda-x)^2+y^2)} + \frac{1}{2y}\tan^{-1}\frac{\lambda-x}{y}\right]_x^{x+\zeta}$$

For sufficiently small y, $\tan^{-1}\frac{\zeta}{y}$ is close to $\frac{\pi}{2}$; that is, $y_1 > 0$ exists such that $\tan^{-1}\frac{\zeta}{y} > \frac{\pi}{2} - \frac{\pi\varepsilon}{8l}$ if $y < y_1$.

Hence, if $y < \min\{y_1, \frac{\pi\zeta\varepsilon}{4(l-\frac{\varepsilon}{2})}\}$

$$\int_{[x,\infty)} \frac{y}{(\lambda-x)^2+y^2}\, d\tilde{\rho}(\lambda)$$

$$> 4y(l-\tfrac{\varepsilon}{2})\left[\frac{-\zeta}{2(\zeta^2+y^2)} + \frac{1}{2y}\tan^{-1}\frac{\zeta}{y}\right]$$

$$> \frac{-2y(l-\frac{\varepsilon}{2})}{\zeta} + 2(l-\tfrac{\varepsilon}{2})(\tfrac{\pi}{2} - \tfrac{\pi\varepsilon}{8l})$$

$$> \pi(l-\varepsilon)$$

The arbitrariness of ε implies $\liminf_{y\downarrow 0} \frac{1}{\pi}\int_{[x,\infty)} \frac{y}{(\lambda-x)^2+y^2}\, d\tilde{\rho}(\lambda) \geqslant l$ as required.

(i) and (ii) together imply that if $\frac{d\mu}{dK}(x)$ exists finitely, then $\frac{1}{\pi}\operatorname{Im} m_+(x)$ also exists and both are equal.

If $\frac{d\mu}{dK}(x) = \infty$, then for each $P \in \mathbb{R}^+$ there exists $\zeta_P > 0$ such that $\tilde{\rho}(\lambda) > 2(\lambda-x)P$ if $\lambda \in (x, \zeta)$.

Proceeding as in (ii), we obtain

$$\int_{[x,\infty)} \frac{y}{(\lambda-x)^2+y^2}\, d\tilde{\rho}(\lambda) > \int_{[x,\infty)} \frac{4y(\lambda-x)^2 P}{((\lambda-x)^2+y^2)^2}\, d\lambda$$

$$> \pi(P-\varepsilon)$$

for sufficiently small y. The conclusion that $\operatorname{Im} m_+(x) = \infty$ follows from the arbitrariness of P.

The proof of the proposition is now complete.

Although it will not be important for our purposes, it is natural to enquire whether a converse of Proposition 2.14 is also true. It follows from the work of L.H. Loomis ([L]) and P. Fatou ([F]) that (i) $\operatorname{Im} m(z)$ has a finite non-tangential limit l at x if and only if $\frac{d\mu(x)}{d\kappa}$ exists and equals $\frac{l}{\pi}$.

(ii) $\operatorname{Im} m+(x)$ exists finitely and equals l if and only if the same is true of $\lim_{\delta \to 0} \frac{\rho(x+\delta) - \rho(x-\delta)}{2\delta}\pi$. It is not clear whether these results still hold if l is infinite; however, it is clear that the converse of Proposition 2.14 cannot be true in general. For, since we may choose $\rho(\lambda)$ as we please on a finite subinterval of $\mathbb{R}$ by the inverse method of Gel'fand and Levitan ([GL]), there exists a spectral function ρ which is continuous but not "smooth" at some point $x \in \mathbb{R}$, so that the generalised symmetric derivative $\lim_{\delta \to 0} \frac{\rho(x+\delta)-\rho(x-\delta)}{2\delta}$ exists finitely, but $\frac{d\mu}{d\kappa}(x)$ does not. In such a case $\operatorname{Im} m+(x)$ exists by (ii) (or, indeed, by our proof of Proposition 2.14), and so the converse of Proposition 2.14 is refuted by counterexample.

However, as we now show, a converse of Proposition 2.14 is true κ- and μ- almost everywhere on $\mathbb{R}$.

2.15 <u>Corollary</u>: (i) $\frac{\operatorname{Im} m+(x)}{\pi}$ and $\frac{d\mu(x)}{d\kappa}$ simultaneously exist and are finite κ-almost everywhere on $\mathbb{R}$, and are equal when they both exist.

(ii) $\frac{\operatorname{Im} m+(x)}{\pi}$ and $\frac{d\mu}{d\kappa}(x)$ simultaneously exist (finitely or infinitely) μ-almost everywhere on $\mathbb{R}$, and are equal when they both exist.

<u>Proof</u>:

<u>Proof of (i)</u>:

This is immediate by Lebesgue's Theorem (2.2.13) and Proposition 2.14.

<u>Proof of (ii)</u>:

This follows from Propositions 2.5 and 2.14.

Lemma 2.13 and Proposition 2.14 enable us to obtain a new set of minimal supports of the decomposed parts of μ from Theorem 2.9. We require the following preliminary result:

2.16 Lemma: If S is a minimal support of a measure ι and S' is a subset of $\mathbb{R}$ such that the symmetric difference $((S \setminus S') \cup (S' \setminus S))$ has κ- and μ-measure zero, then S' is also a minimal support of ι.

Proof:

We verify that S' satisfies the two conditions of Definition 2.8.

(i) $\mathbb{R} \setminus S' = [(\mathbb{R} \setminus S) \cup (S \setminus S')] \setminus (S' \setminus S)$

so
$$\begin{aligned} \iota(\mathbb{R} \setminus S') &\leq \iota(\mathbb{R} \setminus S) + \iota(S \setminus S') - \iota(S' \setminus S) \\ &\leq \iota(\mathbb{R} \setminus S) + 2\iota((S \setminus S') \cup (S' \setminus S)) \\ &= \iota(\mathbb{R} \setminus S) \end{aligned}$$

Since S is a minimal support of ι, $\iota(\mathbb{R} \setminus S) = 0$ and hence $\iota(\mathbb{R} \setminus S') = 0$.

(ii) Suppose S_0 is a subset of S' such that $\kappa(S_0) > 0$. Since S is a minimal support of ι, and $S_0 \cap S$ is a subset of S such that
$$\begin{aligned} \kappa(S_0 \cap S) &= \kappa(S_0 \cap S') - \kappa(S_0 \cap (S' \setminus S)) \\ &= \kappa(S_0) > 0 \end{aligned}$$
we have $\iota(S_0 \cap S) > 0$ by Definition 2.8(ii).

Hence $\iota(S_0) = \iota(S_0 \cap S) + \iota(S_0 \cap (S' \setminus S)) > 0$.

Thus S' satisfies the required conditions and the lemma is proved.

There is now no difficulty in deriving our new set of minimal supports in terms of $\operatorname{Im} m_+(x)$. For the set $U = \{x \in \mathbb{R} : \operatorname{Im} m_+(x)$ exists finitely or infinitely, but $\frac{d\mu}{d\kappa}(x)$ does not$\}$ is contained in the Borel set $\{x \in \mathbb{R} : \frac{d\mu}{d\kappa}(x)$ does not exist finitely or infinitely$\}$ which has κ- and μ-measure zero by Lebesgue's Theorem (2.2.13) and Proposition 2.5. Since the measures κ and μ are complete, U is κ- and μ-measurable and $\kappa(U) = \mu(U) = 0$. If $\mathfrak{m}$ is defined as in Theorem 2.9 and $\mathfrak{m}'$ denotes $\{x \in \mathbb{R} : 0 < \operatorname{Im} m_+(x) \leq \infty\}$, then $\mathfrak{m} \subseteq \mathfrak{m}'$ by Lemma 2.13 and Proposition 2.14, and $\mathfrak{m}' \setminus \mathfrak{m} \subseteq U$. Hence, by Lemma 2.16, $\mathcal{M}'$ is a minimal support of μ. Analogous results for $\mu_{a.c.}$, $\mu_{s.}$ etc. follow in the same way, and we have:

2.17 Theorem: Minimal supports $\mathcal{M}'$, $\mathcal{M}'_{a.c.}$, $\mathcal{M}'_{s.}$, $\mathcal{M}'_{s.c.}$ and $\mathcal{M}_{d.}$ of μ, $\mu_{a.c.}$, $\mu_{s.}$, $\mu_{s.c.}$ and $\mu_{d.}$ are as follows, where $E' = \{x \in \mathbb{R} : \operatorname{Im} m_+(x) \text{ exists}\}$:

(i) $\mathcal{M}' = \{x \in E' : 0 < \operatorname{Im} m_+(x) \leq \infty\}$

(ii) $\mathcal{M}'_{a.c.} = \{x \in E' : 0 < \operatorname{Im} m_+(x) < \infty\}$

(iii) $\mathcal{M}'_{s.} = \{x \in E' : \operatorname{Im} m_+(x) = \infty\}$

(iv) $\mathcal{M}'_{s.c.} = \{x \in E' : \operatorname{Im} m_+(x) = \infty, \mu(\{x\}) = 0\}$

(v) $\mathcal{M}'_{d.} = \{x \in E' : \operatorname{Im} m_+(x) = \infty, \mu(\{x\}) > 0\}$

Our interest in the support of Theorem 2.17 stems from our eventual aim to derive minimal supports in terms of the asymptotic behaviour of solutions of the Schrödinger equation. Since the set of solutions of the equation Lu = xu does not depend on the particular boundary condition which is imposed at r = 0, whereas the function m(z) does, we first need to investigate the effect on m(z) of a change of boundary condition.

Let $m(z,\alpha)$ denote the function which is defined for $\operatorname{Im} z > 0$ by the condition that $u_2(r,z,\alpha) + m(z,\alpha)u_1(r,z,\alpha)$ be in $L_2[0,\infty)$, where $u_1(r,z,\alpha)$ and $u_2(r,z,\alpha)$ are solutions of Lu = zu satisfying (2.3.1). We refer to the corresponding self-adjoint operator H as the operator arising from L with boundary condition α and denote it by $H(\alpha)$; every function f(r) in the domain of $H(\alpha)$ satisfies

$$\cos\alpha\, f(0) + \sin\alpha\, f'(0) = 0 \tag{2.3.9}$$

as we shall see in §4. We prove the following:

2.18 <u>Lemma</u>: If L is in the limit point case at ∞, then

$$m(z,\alpha_2) = \frac{1 + \cot\gamma\, m(z,\alpha_1)}{\cot\gamma - m(z,\alpha_1)}$$

where $\gamma = (\alpha_1 - \alpha_2)$ and $\alpha_2 \neq \alpha_1 \pmod{\pi}$.

<u>Proof</u>:

Since there is, up to a multiplicative constant, just one solution of Lu = zu which is in $L_2[0,\infty)$ if $\operatorname{Im} z \neq 0$, we have

$$u_2(r,z,\alpha_1) + m(z,\alpha_1)u_1(r,z,\alpha_1) = k\,[u_2(r,z,\alpha_2) + m(z,\alpha_2)\,u_1(r,z,\alpha_2)]$$

for some fixed $k \in \mathbb{C}$ and all r in $[0,\infty)$. Using the boundary conditions (2,3,1) this implies

$$\cos\alpha_1 - m(z,\alpha_1)\sin\alpha_1 = k(\cos\alpha_2 - m(z,\alpha_2)\sin\alpha_2)$$

$$\sin\alpha_1 + m(z,\alpha_1)\cos\alpha_1 = k(\sin\alpha_2 + m(z,\alpha_2)\cos\alpha_2)$$

Eliminating k we obtain

$$\sin(\alpha_1-\alpha_2)[1 + m(z,\alpha_1)m(z,\alpha_2)] + \cos(\alpha_1-\alpha_2)[m(z,\alpha_1) - m(z,\alpha_2)] = 0$$

from which the desired result follows.

Equating the real and imaginary parts of both sides in the expression for $m(z,\alpha_2)$ yields

$$\operatorname{Im} m(z,\alpha_2) = \frac{\operatorname{Im} m(z,\alpha_1)(1+\cot^2\gamma)}{(\cot\gamma - \operatorname{Re} m(z,\alpha_1))^2 + (\operatorname{Im} m(z,\alpha_1))^2} \qquad (2.3.10)$$

from which we shall now ascertain, at least up to sets of κ-measure zero, the behaviour as z approaches x normally of $\operatorname{Im} m(z,\alpha_2)$ relative to that of $\operatorname{Im} m(z,\alpha_1)$ in certain fundamental cases. This will enable us to determine where the spectrum of $H(\alpha_2)$ is concentrated relative to the spectrum of $H(\alpha_1)$, and also, up to a set of μ- and κ-measure zero, the subset of $\mathbb{R}$ on which $\operatorname{Im} m_+(x,\alpha)$ exists and equals zero for at least one boundary condition α.

2.19 <u>Lemma:</u> (i) For κ-almost all x in $\mathbb{R}$ for which $\operatorname{Im} m_+(x,\alpha_1)$ exists and equals zero, $\operatorname{Im} m_+(x,\alpha_2)$ also exists and equals zero for every boundary condition $\alpha_2 \neq \alpha_1$ (mod π) except, at most, one.

(ii) For all x in $\mathbb{R}$ such that $\operatorname{Im} m+(x,\alpha_1)$ exists infinitely, $\operatorname{Im} m+(x,\alpha_2)$ also exists and is zero whenever $\alpha_2 \neq \alpha_1$ (mod π).

(iii) For κ-almost all x in $\mathbb{R}$ for which $\operatorname{Im} m+(x,\alpha_1)$ exists and $0 < \operatorname{Im} m+(x,\alpha_1) < \infty$, $\operatorname{Im} m+(x,\alpha_2)$ also exists and $0 < \operatorname{Im} m+(x,\alpha_2) < \infty$ for every $\alpha_2 \neq \alpha_1$ (mod π).

Proof:

Proof of (i):

By (2.3.10), if $\operatorname{Re} m+(x,\alpha_1)$ exists, then, unless $\cot(\alpha_1-\alpha_2) = \operatorname{Re} m+(x,\alpha_1)$, $\operatorname{Im} m+(x,\alpha_2)$ exists and is zero. Since $\operatorname{Re} m+(x,\alpha_1)$ exists κ-almost everywhere on $\mathbb{R}$ by Theorem 2.12(i), the result is proved.

Proof of (ii):

From (2.3.10),

$$\operatorname{Im} m(z,\alpha_2) \leq \frac{1 + \cot^2\gamma}{\operatorname{Im} m(z,\alpha_1)}$$

The result is now immediate since $\cot^2\gamma < \infty$ if $\alpha_2 \neq \alpha_1$ (mod π).

Proof of (iii):

This follows from (2.3.10) since $\operatorname{Re} m+(x,\alpha_1)$ exists finitely for κ-almost all x in $\mathbb{R}$ by Theorem 2.12(i).

We may further refine part (i) of Lemma 2.19 for all real x which are in the resolvent set $R(\alpha_1)$ of $H(\alpha_1)$. Since $m(z,\alpha_1)$ may be continued analytically to include all points of the resolvent set ([CE] §5), $m+(x,\alpha_1)$ exists finitely and is real for all x in $R(\alpha_1)$. Hence by (2.3.10), $\operatorname{Im} m+(x,\alpha_2) = 0$ for all boundary conditions α_2 for which $\cot(\alpha_1-\alpha_2) \neq m+(x,\alpha_1)$. In the exceptional case we note from Lemma 2.18 and the invariance of the essential spectrum under finite dimensional perturbations, that x is an isolated pole of $m(z,\alpha_2)$ and hence is in the discrete spectrum of $H(\alpha_2)$. Using Proposition 2.14 we then have that $\operatorname{Im} m+(x,\alpha_2)$ exist infinitely.

In order to make precise the implications of Lemma 2.19 for the relationship between the spectra of $H(\alpha_1)$ and $H(\alpha_2)$ we first prove an elementary lemma. To lighten the notation, we denote the symmetric difference $(S \setminus S') \cup (S' \setminus S)$ of two sets S and S' by $S \triangle S'$.

2.20 Lemma: If S, S' are subsets of $\mathbb{R}$ which are ι- and κ- measurable, and if the relation $\sim$ is defined by : $S \sim S'$ if and only if $S \triangle S'$ has ι- and κ-measure zero, then $\sim$ is an equivalence relation. Moreover, the set of all minimal supports of the measure ι is an equivalence class under $\sim$.

Proof:

Evidently the relation $\sim$ is reflexive and symmetric, and transitivity follows from the inclusion

$$S \triangle S'' \subseteq (S \triangle S') \cup (S' \triangle S'')$$

Hence $\sim$ is an equivalence relation.

Let $\mathcal{M}_\iota$ be a minimal support of ι and let $E_\iota = \{ S \subseteq \mathbb{R} : S \sim \mathcal{M}_\iota \}$. We prove that E_ι is the set of all minimal supports of ι.

If $S \in E_\iota$, then S is a minimal support of ι by Lemma 2.16. If $\mathcal{M}_\iota' \neq \mathcal{M}_\iota$ is a minimal support of ι, we prove that $\mathcal{M}_\iota' \in E_\iota$. Since

$$\begin{aligned} \mathcal{M}_\iota \triangle \mathcal{M}_\iota' &= (\mathbb{R} \setminus (\mathcal{M}_\iota \cap \mathcal{M}_\iota')) \setminus (\mathbb{R} \setminus (\mathcal{M}_\iota \cup \mathcal{M}_\iota')) \\ &= ((\mathbb{R} \setminus \mathcal{M}_\iota) \cup (\mathbb{R} \setminus \mathcal{M}_\iota')) \setminus ((\mathbb{R} \setminus \mathcal{M}_\iota) \cap (\mathbb{R} \setminus \mathcal{M}_\iota')) \end{aligned}$$

and $\mathcal{M}_\iota, \mathcal{M}_\iota'$ are minimal supports of ι, we have $\iota(\mathcal{M}_\iota \triangle \mathcal{M}_\iota') = 0$ by condition (i) of Definition 2.8. Hence $\iota(\mathcal{M}_\iota \setminus \mathcal{M}_\iota') = 0$ and, since $\mathcal{M}_\iota \setminus \mathcal{M}_\iota' \subseteq \mathcal{M}_\iota$ we have by condition (ii) of Definition 2.8, $\kappa(\mathcal{M}_\iota \setminus \mathcal{M}_\iota') = 0$. Similarly $\kappa(\mathcal{M}_\iota' \setminus \mathcal{M}_\iota) = 0$ and so $\kappa(\mathcal{M}_\iota \triangle \mathcal{M}_\iota') = 0$. Hence $\mathcal{M}_\iota' \sim \mathcal{M}_\iota$, so that $\mathcal{M}_\iota' \in E_\iota$. The proof of the lemma is now complete since E_ι is an equivalence class under $\sim$.

Let $\mu^{(\alpha)}$ denote the spectral measure of the operator H arising from L with boundary condition α. Let $E_{a.c.}(\alpha)$, $E_s(\alpha)$ denote the equivalence classes of minimal supports of $\mu^{(\alpha)}_{a.c.}$ and $\mu^{(\alpha)}_{s.}$ respectively. The next theorem indicates the striking contrast between the behaviour of $E_{a.c.}(\alpha)$ and $E_{s.}(\alpha)$ as α varies.

2.21 Theorem: (i) $E_{a.c.}(\alpha_1) = E_{a.c.}(\alpha_2)$ for all boundary conditions α_1 and α_2.

(ii) If $\mu_{s.}^{(\alpha_1)}(\mathbb{R}) > 0$ then $E_{s.}(\alpha_2) \neq E_{s.}(\alpha_1)$ for any $\alpha_2 \neq \alpha_1$ (mod π); moreover, for each pair of distinct boundary conditions (α_1, α_2) there exist $\mathcal{M}(\alpha_1) \in E_{s.}(\alpha_1)$ and $\mathcal{M}(\alpha_2) \in E_{s.}(\alpha_2)$ such that $\mathcal{M}(\alpha_1) \cap \mathcal{M}(\alpha_2) = \emptyset$.

Proof:

Proof of (i):

Let $\overset{(1)}{\sim}$, $\overset{(2)}{\sim}$ denote the equivalence relation of Lemma 2.20 for $\iota = \mu_{a.c.}^{(\alpha_1)}, \mu_{a.c.}^{(\alpha_2)}$ respectively, and let the supports of Theorem 2.17(ii) for boundary condition α_1 and α_2 be denoted by $\mathcal{M}'_{a.c.}(\alpha_1)$ and $\mathcal{M}'_{a.c.}(\alpha_2)$ respectively.

If $\mathcal{M}(\alpha_1) \in E_{a.c.}(\alpha_1)$ then $\mathcal{M}(\alpha_1) \overset{(1)}{\sim} \mathcal{M}'_{a.c.}(\alpha_1)$ so that $\kappa((\mathcal{M}(\alpha_1) \Delta \mathcal{M}'(\alpha_1)) = 0$. Hence, by the absolute continuity of $\mu_{a.c.}^{(\alpha_2)}$, $\mu_{a.c.}^{(\alpha_2)}(\mathcal{M}(\alpha_1) \Delta \mathcal{M}'(\alpha_1)) = 0$ from which we have $\mathcal{M}(\alpha_1) \overset{(2)}{\sim} \mathcal{M}'_{a.c.}(\alpha_1)$. Now $\kappa((\mathcal{M}'_{a.c.}(\alpha_1) \Delta \mathcal{M}'_{a.c.}(\alpha_2)) = 0$ by Lemma 2.19(iii), which implies, by the absolute continuity of $\mu_{a.c.}^{(\alpha_2)}$ that $\mathcal{M}'_{a.c.}(\alpha_1) \overset{(2)}{\sim} \mathcal{M}'_{a.c.}(\alpha_2)$. Hence, by the transitivity of $\overset{(2)}{\sim}$, $\mathcal{M}(\alpha_1) \overset{(2)}{\sim} \mathcal{M}_{a.c.}(\alpha_2)$, so that $\mathcal{M}(\alpha_1) \in E_{a.c.}(\alpha_2)$.

We need only interchange the suffices 1 and 2 in the above argument to see that if $\mathcal{M}(\alpha_2) \in E_{a.c.}(\alpha_2)$ then $\mathcal{M}(\alpha_2) \in E_{a.c.}(\alpha_1)$. This completes the proof of (i).

Proof of (ii):

If $\mu_{s.}^{(\alpha_1)}(\mathbb{R}) > 0$, then by Theorem 2.17 (iii) the set $\mathcal{M}(\alpha_1) = \{x \in \mathbb{R} : \text{Im } m_+(x, \alpha_1) \text{ exists infinitely}\}$ is contained in $E_{s.}(\alpha_1)$ and is non-empty. Moreover, if $\mathcal{M}(\alpha_2) = \{x \in \mathbb{R} : \text{Im } m_+(x, \alpha_2) \text{ exists infinitely}\}$, $\mathcal{M}(\alpha_2)$ is contained in $E_{s.}(\alpha_2)$ and $\mathcal{M}(\alpha_1) \cap \mathcal{M}(\alpha_2) = \emptyset$ by Lemma 2.19 (ii). We have now proved (ii), and the theorem is complete.

We see from this result that whereas the absolutely continuous parts of the respective spectral measures are equivalent under a change of boundary condition, the singular parts are orthogonal. This is not altogether to be expected, given the invariance of the essential spectrum under finite dimensional perturbations ([WE1]§9.2), and has interesting implications in situations where there is dense singular spectrum ([A] §5).

In the next chapter we shall relate the boundary behaviour of $\operatorname{Im} m(z)$ as z approaches $x \in \mathbb{R}$ normally to the nature of the solutions of Lu = xu. The crucial distinction will be between those x for which $\operatorname{Im} m_+(x)$ exists finitely and is non-zero, and those x for which there is a boundary condition α such that $\operatorname{Im} m_+(z,\alpha)$ exists and is zero. We anticipate these results in the next proposition which follows easily at this stage from Lemma 2.19 and Theorem 2.21.

As $E_{a.c.}(\alpha)$ is independent of α, it will now be referred to simply as $E_{a.c.}$.

2.22 Theorem: The set $S = \{x \in \mathbb{R}$: there is no boundary condition α for which $\operatorname{Im} m_+(x,\alpha)$ exists and equals zero $\}$ is in $E_{a.c.}$.

Proof:

By Theorem 2.21 it is only necessary to show that S is a minimal support of $\mu_{a.c.}(\alpha_1)$ for some boundary condition α_1. Using the notation of Theorem 2.21 we shall therefore show that $S \overset{(1)}{\sim} M'_{a.c.}(\alpha_1)$. Because of the absolute continuity of $\mu^{(\alpha_1)}_{a.c.}$, this will be established if we prove that

$$\kappa(S \,\Delta\, M'_{a.c.}(\alpha_1)) = 0 \qquad (2.3.11)$$

Now for κ- almost all x in $M'_{a.c.}(\alpha_1)$, $\operatorname{Im} m_+(x,\alpha_2)$ exists and is strictly greater than zero for every $\alpha_2 \neq \alpha_1$ (mod π) by Lemma 2.19(iii). Hence $\kappa(S \,\Delta\, M'_{a.c.}(\alpha_1)) = 0$.

Also $\operatorname{Im} m_+(x,\alpha_1)$ exists and is finite κ- almost everywhere on $\mathbb{R}$, so that $\operatorname{Im} m_+(x,\alpha_1)$ exists and $0 \leq \operatorname{Im} m_+(x,\alpha_1) < \infty$ κ-almost everywhere on S. Since, however, $\{x \in \mathbb{R} : \operatorname{Im} m_+(x,\alpha_1)$ exists and is zero $\} \cap S = \phi$ by definition of S, $\operatorname{Im} m_+(x,\alpha_1)$ exists and $0 < \operatorname{Im} m_+(x,\alpha_1) < \infty$ κ- almost everywhere on S. This implies $\kappa(S \setminus M_{a.c.}(\alpha_1)) = 0$.

(2.3.11) is now immediate, and so the theorem is proved.

Before proceeding to consider the relationship between the boundary behaviour of m(z) and the nature of solutions of the Schrödinger equation we introduce some relevant ideas from Hilbert space theory.

§4 The Schrödinger Operator

The early work of Hermann Weyl was gradually absorbed into the wider framework of linear operators on Hilbert spaces during the twenties and thirties. Particularly noteworthy was the contribution of Marshall Stone whose "Linear Transformations in Hilbert Space" contains a very thorough treatment of second order differential operators ([S] Ch.X, §3). We briefly indicate some of the more important features of the theory from this point of view.

The relevant Hilbert spaces are of measurable functions which are Lebesgue square integrable with respect to a given measure. If the measure is not Lebesgue measure, this will be indicated by a superfix: for example, the space of μ-measurable functions $G(\lambda)$ for which $\int_{-\infty}^{\infty} |G(\lambda)|^2 d\rho(\lambda) < \infty$ will be denoted by $L_2^{\rho}(-\infty,\infty)$.

In considering the question of self-adjointness of operators arising from the differential expression $L = -\frac{d^2}{dr^2} + V(r)$ Green's formula is of fundamental importance:

$$\int_0^{\infty} ((Lf)\bar{g} - f(Lg))\,dr = W_{\infty}(f,g) - W_0(f,g)$$

where $W_0(f,g) = \lim_{b \downarrow 0} W_b(f,g)$ and $W_{\infty}(f,g) = \lim_{b \to \infty} W_b(f,g)$, where $W_b(f,g)$ is the Wronskian of f and g evaluated at r = b. Evidently, if an operator $\tilde{H}$ arising from L is to be symmetric it is necessary that $W_\infty(f,g) - W_0(f,g) = 0$ for all f and g in $\mathfrak{D}(\tilde{H})$, and hence if an operator H is to be self-adjoint, the same must be true for all f and g in $\mathfrak{D}(H)$.

Let $\tilde{\mathfrak{D}}^*$ denote the set of all measurable functions f(r) on $[0,\infty)$ for which

(i) f(r) and f′(r) are absolutely continuous functions on every closed subinterval of $[0,\infty)$,

(ii) f(r) and Lf(r) are in $L_2[0,\infty)$, and set

$\tilde{\mathfrak{D}} = \{f(r) \in \tilde{\mathfrak{D}}^* : W_\infty(f,g) - W_0(f,g) = 0$ for all g in $\tilde{\mathfrak{D}}^*\}$.

Then the operator $\tilde{H}$ mapping f(r) into Lf(r) with domain $\tilde{\mathfrak{D}}$ is symmetric and closed, and its deficiency indices (the co-dimensions of the ranges of $\tilde{H}+iI$ and $\tilde{H}-iI$ in $\mathcal{H} = L_2[0,\infty)$) are either (0,0), (1,1) or (2,2) ([S] Thm. 10.11). It is found that if L has a regular endpoint at r = 0, then the corresponding operator $\tilde{H}$ with domain $\tilde{\mathfrak{D}}$ has deficiency indices (1,1) if and only if L is in the limit point case at ∞, and deficiency indices (2,2) if and only if L is in the limit circle case at ∞ ([S] Thms. 10.13, 10.14). In either case $\tilde{H}$ has self-adjoint extensions since the deficiency indices are equal. ([RN] §123). L is limit point at both 0 and ∞ if and only if the deficiency indices are (0,0), and in this case $\tilde{H}$ with domain $\tilde{\mathfrak{D}}$ is self-adjoint.

If L is regular at 0 and in the limit point case at ∞, then $W_\infty(f,g) = 0$ for all f, g in $\tilde{\mathfrak{D}}^*$ and $\tilde{\mathfrak{D}} = \{f(r) \in \mathfrak{D}^* : f(0) = f'(0) = 0\}$. If $\tilde{\mathfrak{D}}$ is extended to $\mathfrak{D} = \{f(r) \in \tilde{\mathfrak{D}}^* : \cos\alpha\, f(0) + \sin\alpha\, f'(0) = 0\}$, then the operator $H(\alpha)$ mapping f(r) to Lf(r) with domain $\mathfrak{D}$ is a proper closed extension of $\tilde{H}$, and all proper closed extensions of $\tilde{H}$ are of this form. ([S] Thm. 10.16). So in this case $\{H(\alpha) : \alpha \in \mathbb{R}\}$ is the set of all self-adjoint operators arising from L.

If L is regular at 0 and in the limit circle case at ∞, then the restriction of $\tilde{\mathfrak{D}}^*$ obtained by imposing the requirements $\cos\alpha\, f(0) + \sin\alpha\, f'(0) = 0$

and $W_\infty(f(r), u_m(r, z_0, \alpha)) = 0$ where $u_m(r, z_0, \alpha) = u_2(r, z_0, \alpha) + m(z_0, \alpha) u_1(r, z_0, \alpha)$ (see (2.3.1)), $\operatorname{Im} z_0 \neq 0$ and $m(z_0, \alpha)$ is any point on the limit circle, yields a domain $\mathfrak{D}$ on which the operator H mapping f(r) to Lf(r) is self-adjoint ([CL] Ch.9, Thm. 4.1).

In the case where L is regular at 0 and limit point at ∞ there exists an isometric Hilbert space isomorphism from $L_2[0, \infty)$ onto $L_2^{\rho}(-\infty, \infty)$ ([CL] Ch.9, Thms. 3.1 and 3.2). Specifically, if f(r) is in $L_2[0, \infty)$ then

$$F(\lambda) = \underset{\omega \to \infty}{\mathrm{l.i.m.}} \int_0^\omega u_1(r, \lambda, \alpha) f(r) dr \tag{2.4.1}$$

exists and

$$\int_{-\infty}^{\infty} |F(\lambda)|^2 d\rho_\alpha(\lambda) = \int_0^\infty |f(r)|^2 dr \tag{2.4.2}$$

where $\rho_\alpha(\lambda)$ is the spectral measure of $H(\alpha)$.

Likewise, if $G(\lambda)$ is in $L_2^{\rho_\alpha}(-\infty, \infty)$, then

$$g(r) = \mathrm{l.i.m.} \int_{-\omega}^{\omega} u_1(r, \lambda, \alpha)\, G(\lambda)\, d\rho_\alpha(\lambda) \tag{2.4.3}$$

and $\int_0^\infty |g(r)|^2 dr = \int_{-\infty}^{\infty} |G(\lambda)|^2 d\rho_\alpha(\lambda)$

Moreover, the "eigenfunction expansion"

$$f(r) = \underset{\omega \to \infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} u_1(r, \lambda, \alpha)\, F(\lambda)\, d\rho_\alpha(\lambda) \tag{2.4.4}$$

where $F(\lambda)$ satisfies (2.4.1), is valid for arbitrary f(r) in $L_2[0, \infty)$. Analogous results hold if L is in the limit circle case at ∞; however, since the spectrum is discrete in this case ([CL] Ch.9, Thm. 4.1) it is usual to express the expansion (2.4.4) as a series.

The results of the previous paragraph were originally obtained for the specific case considered, but are also related to a far more general result. The spectral theory of ordinary differential operators arising from differential expressions of the Sturm-Liouville type has now been generalised to include suitable operators of the nth order, irrespective of whether the endpoints of the interval under consideration are regular or singular. ([DS] Ch.XIII, §5) We shall show in Chapter IV that the general theory may sometimes be simplified, and that in such cases, relationships formally analogous to (2.4.1), (2.4.2), (2.4.3) and (2.4.4) may be deduced. Moreover, many results from the general theory may, with suitable modifications, be applied to these cases, as we shall prove. We anticipate this development by applying a result which is well documented for the general case to the situation where L is regular at 0 and limit point at ∞.

Let T denote the transform which maps $f(r) \in L_2(0,\infty)$ to $F(\lambda)$ as in (2.4.1). The following is an application of the general Weyl-Kodaira theorem ([DS] Ch. XIII, §5, Thm. 13(ii)) to our simplified situation (see also Chapter IV).

If ϕ is a Borel measurable function on $\mathbb{R}$ with support in $(0,\infty)$ then

$$T((\phi(H)f)(r)) = \phi(\lambda)\,T(f(r)) = \phi(\lambda)\,F(\lambda) \tag{2.4.5}$$

for all f in the domain of $\phi(H)$. This implies, in particular, that

$$\int_0^\infty |(\phi(H)f)(r)|^2\,dr = \int_{-\infty}^\infty |\phi(\lambda)F(\lambda)|^2\,d\rho(\lambda) \tag{2.4.6}$$

and $$\langle (\phi(H)f)(r), f(r) \rangle = \int_{-\infty}^\infty \overline{\phi(\lambda)}\,|F(\lambda)|^2\,d\rho(\lambda) \tag{2.4.7}$$

Corresponding to every self-adjoint operator A is a unique "spectral family" or "resolution of the identity" $\{E_\lambda\}$ with the following properties:

(i) $E_{\lambda'} \leq E_\lambda$ if $\lambda' \leq \lambda$

(ii) $E_\lambda = \underset{\varepsilon \to 0}{\text{s. lim.}}\, E_{\lambda+\varepsilon}$

(iii) $\underset{\lambda \to -\infty}{\text{s. lim.}}\, E_\lambda = 0$, $\underset{\lambda \to +\infty}{\text{s. lim.}}\, E_\lambda = I$ (2.4.8)

By means of the spectral family A may be expressed as

$$A = \int_{-\infty}^\infty \lambda\, dE_\lambda$$

which is known as the Spectral Theorem. A related result is that if is measurable, finite and defined almost everywhere with respect to $\langle E_\lambda f, f \rangle$ for some $f \in \mathcal{H}$, then $\phi(A)$ commutes with E_λ for each λ and

$$(\phi(A)f)(r) = \int_{-\infty}^{\infty} \phi(\lambda)\, dE_\lambda f$$

$$\langle(\phi(A)f)(r), f(r)\rangle = \int_{-\infty}^{\infty} \overline{\phi(\lambda)}\, d\langle E_\lambda f, f\rangle$$

for all f in the domain of $\phi(A)$, this being defined to be the set of all f in $\mathcal{H}$ for which

$$\int_{-\infty}^{\infty} |\phi(\lambda)|^2\, d\langle E_\lambda f, f\rangle = \|\phi(A)f\|^2$$

converges. ([RN] Ch.IX). Clearly (2.4.5) - (2.4.7) are closely related to these results, and, indeed, the general theory of Weyl and Kodaira may be derived using the Spectral Theorem (see eg. [KO]).

In the context of Hilbert space theory, it is usual to characterise the spectrum of H in terms of the resolvent as $\Sigma(H) = \mathbb{R} \setminus \{x \in \mathbb{R} : (H - xI)^{-1}$ is a bounded linear operator on $\mathcal{H}\}$. It has been shown that $\{x \in \mathbb{R} : (H - xI)^{-1}$ is bounded on $\mathcal{H}\}$ $=$ $\{x \in \mathbb{R}$: $m(z)$ is regular at $x\}$, where m(z) is said to be regular at $x \in \mathbb{R}$ if there exists a neighbourhood of x into which m(z) may be continued analytically. ([CE] §5). Now from (2.3.2) it is evident that $m(\bar{z}) = \overline{m(z)}$, and hence, using Lemma 2.13, we have $\{x \in \mathbb{R} : (H - xI)^{-1}$ is bounded on $\mathcal{H}\}$ $=$ $\{x \in \mathbb{R}$: there exists a neighbourhood N_x of x such that $\operatorname{Im} m_+(x)$ exists and is zero for all λ in $N_x \cap \mathbb{R}\}$ $=$ $\{x \in \mathbb{R}$: there exists a neighbourhood N_x such that $\frac{d\mu}{d\kappa}(\lambda)$ exists and is zero for all λ in $N_x \cap \mathbb{R}\}$. Since $\frac{d\mu}{d\kappa}(x) = 0$ if and only if $\left.\frac{d\rho(\lambda)}{d\lambda}\right|_x = 0$, the characterisation of the spectrum in terms of the resolvent determines precisely the same set as do the points of increase of the spectral function.

From this discussion, it is evident that the behaviour of m(z) precisely reflects that of the resolvent on the resolvent set. It is not therefore surprising to find that, by means of m(z) and the supports of Theorem 2.17, we can find minimal supports of $\mu_{a.c.}$ and $\mu_{s.}$ in terms of the behaviour of the resolvent.

We shall use the following properties of the dense subset $\mathcal{D}$ of $\mathcal{H} = L_2(0,\infty)$ which consists of those elements of $\mathcal{H}$ which vanish outside a finite interval:

(i) If $f(r) \in \mathfrak{D}$, then $F(\lambda)$, as defined by (2.4.1), is an analytic function of λ in the entire complex plane.

(ii) For each finite interval Δ of $\mathbb{R}$ there exists f(r) in $\mathfrak{D}$ such that $F(\lambda)$ does not vanish on Δ.

These and further properties of $\mathfrak{D}$ were established by M.G.Krein ([AG] Appendix II, §7). We prove the following:

2.23 Proposition: If $\operatorname{Im} m_+(x)$ exists finitely or infinitely, then

$\|R_z f\| = O(y^{-1/2})$ as $y = \operatorname{Im} z \downarrow 0$ for all f in $\mathfrak{D}$ if and only if $\operatorname{Im} m_+(x) < \infty$. Moreover, if $\operatorname{Im} m_+(x) = 0$, then $\|R_z f\| = o(y^{-1/2})$ as $y = \operatorname{Im} z \downarrow 0$ for all f in $\mathfrak{D}$.

Proof:

We have (cf.(2.4.6), [LS] Ch.II, §3)

$$\|R_z f\| = \int_{-\infty}^{\infty} \frac{|F(\lambda)|^2}{(\lambda - x)^2 + y^2}\, d\rho(\lambda) \tag{2.4.10}$$

We show first that if $f(r) \in \mathfrak{D}$ and $\operatorname{Im} m_+(x) = l < \infty$, then $\|R_z f\| = O(y^{-1/2} l)$ as $y \downarrow 0$.

Let $f(r) \in \mathfrak{D}$; we may suppose without loss of generality that $\|f\| = 1$, so that by (2.4.2)

$$\int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\rho(\lambda) = 1 \tag{2.4.11}$$

From (2.4.10)

$$y\, \|R_z f\|^2 = \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 \, d\rho(\lambda) \tag{2.4.12}$$

and from (2.3.5)

$$\operatorname{Im} m_+(x) = \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2}\, d\rho(\lambda) \tag{2.4.13}$$

Now if ε such that $0 < \varepsilon < l$ is given, there exists K in $\mathbb{R}^+$ such that, on account of (2.4.11),

$$\int_{-K}^{K} |F(\lambda)|^2 \, d\rho(\lambda) > 1 - \varepsilon^{1/2}$$

and

$$\frac{y}{(\lambda - x)^2 + y^2} < \varepsilon^{1/2}$$

for all $y < 1$ if $|\lambda| \geqslant K$. Hence, if $S \subseteq (-\infty, K) \cup (K, \infty)$,

$$\int_S \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 d\rho(\lambda) < \varepsilon^{1/2} \int_S |F(\lambda)|^2 d\rho(\lambda) < \varepsilon \qquad (2.4.14)$$

Since $f(r) \in \mathbb{D}$, $F(\lambda)$ is bounded on the compact set $[-K,K]$ by property (i) above; hence there exists C_f in $\mathbb{R}^+$, which depends on f, such that for λ in $[-K,K]$

$$|F(\lambda)|^2 < \frac{1}{2}(C_f - 1) \qquad (2.4.15)$$

Since $\operatorname{Im} m_+(x) = l$ there exists Y_f, depending on f, such that $0 < Y_f < 1$ and

$$\int_{-K}^{K} \frac{y}{(\lambda - x)^2 + y^2} d\rho(\lambda) < 2l$$

for all $y < Y_f$ by (2.4.13).

Hence, if $0 < y < Y_f$, we have by (2.4.14) and (2.4.15),

$$\int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 d\rho(\lambda)$$

$$= \int_{-K}^{K} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 d\rho(\lambda) + \int_S \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 d\rho(\lambda)$$

$$< \frac{1}{2}(C_f - 1)\, 2l + \varepsilon$$

$$< C_f\, l$$

since ε was chosen to be less than l. It follows from (2.4.10) that $\| R_z f \| = O(y^{-1/2} l)$ as $y \downarrow 0$ and this is true for all f in $\mathbb{D}$ as was to be proved.

It is now immediate that if $0 < l < \infty$ then $\| R_z f \| = O(y^{-1/2})$ as $y \downarrow 0$ for all f in $\mathbb{D}$, and that if $l = 0$, then $\| R_z f \| = o(y^{-1/2})$ as $y \downarrow 0$; note that we do not assert the uniformity of this convergence.

It remains to show that if $\operatorname{Im} m_+(x) = \infty$, then there exists f(r) in $\mathbb{D}$ such that $\| R_z f \| \neq O(y^{-1/2})$ as $y \downarrow 0$.

If $\varepsilon > 0$ is given, we may, as in part (i) of the proof of Proposition 2.14, choose $M \in \mathbb{R}^+$ and $Y > 0$ such that

$$\int_S \frac{y}{(\lambda - x)^2 + y^2} d\rho(\lambda) < \varepsilon \qquad (2.4.16)$$

whenever $y < Y$, where $S = (-\infty, -M) \cup (M, \infty)$. Moreover, by property (ii) above, there exists f(r) in $\mathbb{D}$ such that $T(f(r)) = F(\lambda)$ does not vanish on $[-M,M]$. Using property (i) above we see that there exists

$k > 0$ such that

$$|F(\lambda)|^2 > k > 0 \tag{2.4.17}$$

for all λ in $[-M,M]$.

If $\text{Im}\, m+(x) = \infty$, then if C in $\mathbb{R}^+$ is given, there exists $Y_c < Y$ such that

$$\text{Im}\, m(z) > \frac{C}{k} + \varepsilon \tag{2.4.18}$$

for all $y < Y_c$. Therefore, by (2.4.16), (2.4.17) and (2.4.18), for each C in $\mathbb{R}^+$ there exists $Y_c > 0$ such that

$$\begin{aligned} y \, \| R_z f \|^2 &= \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 \, d\rho(\lambda) \\ &\geqslant \int_{-M}^{M} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 \, d\rho(\lambda) \\ &> k\,(\text{Im}\, m(z) - \varepsilon) \\ &> C \end{aligned}$$

if $y < Y_c$. Hence $\| R_z f \| \neq O(y^{-1/2})$ as $y \downarrow 0$. The proof of the proposition is now complete.

In order to show that this proposition is not generally true for all f in $L_2(0,\infty)$, we show that it fails for those f in $L_2(0,\infty)$ for which $T(f(r)) = F(\lambda) \to \infty$ as $\lambda \to x$.

Let $x \in \mathbb{R}$ be such that $\text{Im}\, m+(x) = l$ for some l such that $0 < l < \infty$. The isometric isomorphism T from $L_2(0,\infty)$ onto $L_2^{\rho}(-\infty,\infty)$ ensures that if $F(\lambda)$ is defined μ-almost everywhere on $\mathbb{R}$ in such a way that $\int_{-\infty}^{\infty} |F(\lambda)|^2 d\rho(\lambda) = 1$ and $\lim_{\varepsilon \to 0+} F(x-\varepsilon) = \lim_{\varepsilon \to 0+} F(x+\varepsilon) = \infty$, then there exists f(r) in $L_2(0,\infty)$ such that $T(f(r)) = F(\lambda)$.

From our description of $F(\lambda)$, it is evident that if $C > 0$ is given, there exists $\delta > 0$ such that for all λ in $[x-\delta, x+\delta]$

$$|F(\lambda)|^2 > \frac{4C}{l} \tag{2.4.19}$$

Moreover, if $y < \delta$, $\dfrac{y}{(\lambda - x)^2 + y^2}$ decreases with y, and hence it follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{y \downarrow 0} \int_{\mathbb{R} \setminus [x-\delta, x+\delta]} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) = 0$$

since by (2.3.2) and (2.3.4)

$$\int_{\mathbb{R}} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) < \infty$$

for all $y > 0$.

Therefore, if $\varepsilon > 0$ is given, $Y_1 > 0$ exists such that

$$\int_{\mathbb{R} \setminus [x-\delta, x+\delta]} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) < \varepsilon \tag{2.4.20}$$

for all $y < Y_1$. Also, since $\operatorname{Im} m_+(x) = l$, where $0 < l < \infty$, there exists $Y_2 > 0$ such that

$$\int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) > \frac{l}{2}$$

whenever $y < Y_2$. Combining this with (2.4.20) yields

$$\int_{[x-\delta, x+\delta]} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) > \frac{l}{2} - \varepsilon \tag{2.4.21}$$

for all $y < Y$, where $Y = \min\{Y_1, Y_2\}$. We may choose $\varepsilon = \frac{l}{4}$, and then, using (2.4.19) and (2.4.21), we have for all $y < Y$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 \, d\rho(\lambda) \\
&\geqslant \int_{[x-\delta, x+\delta]} \frac{y}{(\lambda - x)^2 + y^2} |F(\lambda)|^2 \, d\rho(\lambda) \\
&> \frac{4C}{l} \int_{[x-\delta, x+\delta]} \frac{y}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) \\
&> C
\end{aligned}$$

It follows from (2.4.10) and the arbitrariness of C that there exists $f(r)$ in $L_2(0, \infty)$ such that $\| R_z f \| \neq O(y^{-1/2})$ as $y \downarrow 0$ although $\operatorname{Im} m_+(x) = l$ where $0 < l < \infty$.

Thus Proposition 2.23 is, in a sense, optimal. Although we shall not use this proposition later, it is nonetheless relevant to spectral analysis in that it provides criteria for characterising minimal supports of $\mu_{a.c.}$ and $\mu_{s.}$ in terms of the resolvent operator, and incidentally highlights the close relationship between m(z) and $(H-z)^{-1}$. We have:

2.24 Theorem: Minimal supports $\mathcal{M}''_{a.c.}$, $\mathcal{M}''_{s.}$ of $\mu_{a.c.}$ and $\mu_{s.}$ respectively are as follows:

$$\mathcal{M}''_{a.c.} = \{ x \in \mathbb{R} : \|R_z f\| = O(y^{-1/2}) \text{ as } y \downarrow 0 \text{ for all f in } \mathcal{D} \}$$
$$\setminus \{ x \in \mathbb{R} : \|R_z f\| = o(y^{-1/2}) \text{ as } y \downarrow 0 \text{ for all f in } \mathcal{D} \}$$
$$\mathcal{M}''_{s.} = \{ x \in \mathbb{R} : \text{there exists f in } \mathcal{D} \text{ such that } \|R_z f\| \neq O(y^{-1/2}) \text{ as } y \downarrow 0 \}$$

Proof:

This follows immediately from Proposition 2.23 and Lemma 2.16, since by Propositions 2.5 and 2.14 and Theorem 2.12 the set of x in $\mathbb{R}$ for which $\operatorname{Im} m_+(x)$ does not exist has no K- or μ-measure.

We remark that Theorem 2.24 is in many respects similar to a result of K. Gustafson and G.Johnson which states that the absolutely continuous subspace $\mathcal{H}_{a.c.}(H)$ of H is the closure of the set of f in $L_2(0,\infty)$ for which $\|R_z f\| = O(y^{-1/2})$ as $y \downarrow 0$ uniformly over all x in $\mathbb{R}$ ([GJ]). Both results feature a dense subset of $\mathcal{H}$, and characterise absolutely continuous properties in terms of the growth rate of the resolvent.

Returning to the relationship between m(z) and $(H-z)^{-1}$ we note that if $F(\lambda)$ is the characteristic function of a bounded λ-interval (a,b], then $\int_{-\infty}^{\infty} |F(\lambda)|^2 d\rho(\lambda) < \infty$ since the μ-measure of bounded subsets of $\mathbb{R}$ is finite. Because T is an isometric isomorphism from $L_2(0,\infty)$ onto $L_2^{\rho}(-\infty,\infty)$, there exists f(r) in $L_2(0,\infty)$ such that

$T(f(r)) = F(\lambda)$, and, for this f,

$$\langle R_z f, f \rangle = \int_{(a,b]} \frac{1}{\lambda - z}\, d\rho(\lambda) \tag{2.4.22}$$

(cf.(2.4.7)). Now for every x in (a,b) there exists $\delta_x > 0$ such that $[x-\delta_x, x+\delta_x] \subseteq (a,b]$ so by (2.4.20)

$$\lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{y}{(\lambda-x)^2 + y^2}\, d\rho(\lambda) = \lim_{y \downarrow 0} \int_{(a,b]} \frac{y}{(\lambda-x)^2 + y^2}\, d\rho(\lambda)$$

for each x in (a,b) for which $\mathrm{Im}\, m_+(x)$ exists. For each such x, by (2.4.22),

$$\mathrm{Im}\, m_+(x) = \lim_{y \downarrow 0} \mathrm{Im} \langle R_z f, f \rangle \tag{2.4.23}$$

which expresses the close relationship between the boundary behaviour of m(z) and that of the resolvent operator.

Retaining the same f, we have (cf.(2.4.7)).

$$\begin{aligned} \rho(x) - \rho(a) &= \int_{(a,x]} d\rho(\lambda) \\ &= \int_{(a,x]} |F(\lambda)|^2\, d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \chi_{(a,x]} |F(\lambda)|^2\, d\rho(\lambda) \\ &= \langle (E_x - E_a) f, f \rangle \\ &= \langle E_x f, f \rangle \end{aligned} \tag{2.4.24}$$

for all x in (a,b], where χ denotes the characteristic function. It is interesting to note that in (2.4.23) and (2.4.24) the relationship between the spectral function $\rho(\lambda)$ and the spectral family $\{E_\lambda\}$ is similar to that between m(z) and the resolvent $(H-z)^{-1}$ at the boundary of their domains of definition.

To conclude, by relating the spectrum to the spectral function, and the spectral function to m(z) we have shown that each part of the spectrum is concentrated on subsets of the real line which can be unambiguously identified in terms of the boundary behaviour of $m(z)$. The link we have established between the boundary behaviour of $m(z)$ and the growth rate of the resolvent gives further insight into the structural relationships involved, and, in particular, highlights the close relationship between $m(z)$ and the resolvent operator.

In the next chapter we shall relate the boundary behaviour of $m(z)$ to the relative asymptotic behaviour of solutions of the Schrödinger equation at each point x, and thereby establish a fundamental correlation between the asymptotic behaviour of solutions and minimal supports of each part of the spectral measure μ.

CHAPTER III

SUBORDINACY AND THE SPECTRUM

§1. The Concept of a Subordinate Solution

There is little doubt that the study of the spectrum of the Schrödinger equation will continue to engage the attention of mathematical physicists for much time to come. Although a great diversity of sufficient conditions are known, each of which, if satisfied by the potential V(r), ensures a certain type of spectrum (eg. [G] §§31, 33), the results to date are far from comprehensive. One of the more systematic approaches was by E. C. Titchmarsh, who made use of the relation (2.3.3) between the spectral function and m(z) to obtain a complete analysis in many important cases ([T2]) In theory this approach provides the solution to the problem; m(z) is uniquely determined by the condition (2.1.3) and from m(z), as we have shown in Theorem 2.17, minimal supports of the spectral measure, and of its decomposed parts, may be obtained. In practise, however, the method is frequently inoperable because it is impossible to obtain sufficiently detailed information about the solutions to derive m(z) explicitly. What seems to be required, therefore, is an approach that is no less systematic but not dependent on such precise information.

In this chapter we shall use the minimal supports of Theorem 2.17 to derive a new set of minimal supports which are characterised in terms of the existence or otherwise of a certain type of solution of Lu = xu at each real point x. Thus we shall use the systematic correlation between m(z) and the spectrum to obtain an equally systematic correlation between the behaviour of solutions of the Schrödinger equation and the spectrum. As a result we shall obviate the need to determine m(z) explicitly, and so a less detailed knowledge of solutions will be required. Indeed, it will only be necessary to decide for each real x whether there is one solution of Lu = xu which is "smaller" than the others at infinity, and, if so, whether this solution satisfies the boundary condition (2.3.9) at 0; information about solutions of $Lu = zu$ for z in $\mathbb{C} \setminus \mathbb{R}$ will no longer be required.

Analysis of the spectrum of Schrödinger operator through the study of the behaviour of solutions of the Schrödinger equation is nothing new. For example, it has long been known that the discrete part of the spectrum consists of all real x such that $Lu = xu$ has a solution in $L_2(0,\infty)$ satisfying the boundary condition at 0. The physicists' rule of thumb that the spectrum is the set of all real x for which the solution of $Lu = xu$ satisfying the required boundary condition at 0 is bounded (eg. [BR] Ch.10, §16, [KR] pp.71,82), while not proved for the general case ([G],§58), nevertheless suggests that a close correlation between the spectrum and the behaviour of solutions exists. Some interesting results in this connection have been obtained by E.E.Shnol' ([G], Ch.V) and J. Weidmann ([WE2]).

In introducing the concept of subordinacy it is instructive to consider the case where $V(r) = 0$ for all r in $[0,\infty)$. For every x in $\mathbb{R}^-$, there is just one solution of $-\dfrac{d^2u}{dx^2} = xu$ in $L_2[0,\infty)$ and for every x in $[0,\infty)$ there are no solutions in $L_2[0,\infty)$; L is therefore in the limit point case at ∞ by Ch.II §3(1). According to the boundary condition (2.3.9) at 0, there may be just one negative eigenvalue or no negative eigenvalues at all, and for every boundary condition at 0, there is absolutely continuous spectrum on $[0,\infty)$ ([AG] Appendix II, §9). For each x in $\mathbb{R}^-$, we see that the solution $u(r) = e^{-\sqrt{(-x)}r}$ is much smaller at infinity than all other linearly independent solutions. Not only is its $L_2[0,\infty)$ norm finite whereas the norm of the others is infinite, it is also "pointwise subordinate" in the sense that the ratio $\dfrac{e^{-\sqrt{(-x)}r}}{\tilde{u}(r)}$ converges to zero as $r \to \infty$ for every solution $\tilde{u}(r)$ which is not a constant multiple of $u(r)$. However, for each x in $\mathbb{R}^+$, the concept of pointwise subordinacy of one solution relative to another is not applicable, since all solutions are oscillatory, and it is not possible to compare the $L_2[0,\infty)$ norms of linearly independent solutions since all are infinite. Nevertheless, it is clear that in this case there is a sense in which linearly independent solutions are of the same size; one way of making this idea precise, and which seems apt in the context of Hilbert space theory, arises from the observation that for x in $(0,\infty)$ the ratio

$$\frac{\left(\int_0^N |u(r,x)|^2 dr\right)^{\frac{1}{2}}}{\left(\int_0^N |\tilde{u}(r,x)|^2 dr\right)^{\frac{1}{2}}} \tag{3.1.1}$$

converges to a limit in $\mathbb{R}^+$ as $N \to \infty$ for every pair of linearly independent solutions $u(r,x)$ and $\tilde{u}(r,x)$ of $-\frac{d^2u}{dr^2} = xu$. In contrast, this situation does not hold for any x in $(-\infty,0]$; and because for each such x there is one solution which is "smaller" than the others at infinity, we may formulate the distinction between x in $(-\infty,0]$ and x in $\mathbb{R}^+$ as follows: For each x in $(-\infty,0]$ there is a solution $u(r,x)$ of $Lu = xu$ which is such that the ratio (3.1.1) converges to zero for every linearly independent solution $\tilde{u}(r,x)$, and for each x in $\mathbb{R}^+$, there is no such solution. We require some notation and a definition.

Let $\|f(r)\|_N$ denote $\left(\int_0^N |f(r)|^2 dr\right)^{\frac{1}{2}}$.

3.1 <u>Definition</u>: (i) If L is regular at 0 and limit point at infinity, then a solution $u_s(r,z)$ of the Schrödinger equation

$$-\frac{d^2u(r,z)}{dr^2} + V(r)u(r,z) = z\,u(r,z)$$

is said to be subordinate at infinity if, for every linearly independent solution $u(r,z)$,

$$\lim_{N\to\infty} \frac{\|u_s(r,z)\|_N}{\|u(r,z)\|_N} = 0$$

(ii) If L is not regular at 0, then a solution $u_s(r,z)$ of the Schrödinger equation is said to be subordinate at 0 (respectively infinity) if for every linearly independent solution $u(r,z)$

$$\lim_{a \downarrow 0} \frac{\left(\int_a^c |u_s(r,z)|^2\, dr\right)^{\frac{1}{2}}}{\left(\int_a^c |u(r,z)|^2\, dr\right)^{\frac{1}{2}}} = 0$$

(respectively $\lim_{b \to \infty} \frac{\left(\int_c^b |u_s(r,z)|^2\, dr\right)^{\frac{1}{2}}}{\left(\int_c^b |u(r,z)|^2\, dr\right)^{\frac{1}{2}}} = 0$)

where c is an arbitrary fixed number in $\mathbb{R}^+$ and $a < c < b$.

It is trivial to observe that for each fixed z in $\mathbb{C}$ there can be no more than one linearly independent solution of Lu = zu which is subordinate at infinity (respectively 0). Moreover, if for fixed z in $\mathbb{C}$ there exist solutions $u_s(r,z)$ and u(r,z) such tnat $u_s(r,z)$ is subordinate to u(r,z) at infinity (respectively 0), then $u_s(r,z)$ is subordinate at infinity (respectively 0) to every solution u(r,z) of Lu = zu which is not a constant multiple of $u_s(r,z)$.

In this chapter we shall only be concerned with the case where L is regular at 0; therefore, where we do not qualify the term "subordinate" it should be understood in the sense of Definition 3.1(i).

Returning to the case of zero potential, we note that 1 and r are linearly independent solutions at the point z = 0, so that here a subordinate solution is $u_s(r,0) = 1$, which is not in $L_2[0,\infty)$. This example shows that subordinate solutions can exist which are not square integrable; in due course it will become apparent that such solutions are of central importance where there is singular continuous spectrum.

It is not hard to show that if V(r) = 0, then

$$m(z,\alpha) = \frac{\sin\alpha\,\cos\alpha\,(z-1) + i\sqrt{z}}{\cos^2\alpha + z\sin^2\alpha} \tag{3.1.2}$$

For, if $\operatorname{Im} z > 0$, the solutions $u_1(r,z,\alpha)$ and $u_2(r,z,\alpha)$ as defined in (2.3.1) may be aetermined explicitly and expressed as linear combinations or $e^{i\sqrt{z}r}$, which is in $L_2[0,\infty)$, and $e^{-i\sqrt{z}r}$ which is not. Hence, on account of (2.1.3), $m(z,\alpha)$ may be evaluated using the condition that tne coefficient of $e^{-i\sqrt{z}r}$ in $u_2(r,z,\alpha) + m(z,\alpha)u_1(r,z,\alpha)$

be zero.

From (3.1.2), if x is in $\mathbb{R}^+$ then $0 < \operatorname{Im} m_+(x,\alpha) < \infty$ for all boundary conditions α, whereas if x is in $(-\infty, 0]$, $\operatorname{Im} m_+(x,\alpha) = 0$ unless $x = -\cot^2\alpha$ and $\cot\alpha > 0$, when $\operatorname{Im} m_+(x,\alpha) = \infty$. (Note that the numerator and denominator of (3.1.2) both contain the factor $\sqrt{z}\sin\alpha + i\cot\alpha$, so $x = -\cot^2\alpha$ is not a pole if $\cot\alpha < 0$). Therefore in this case there is a subordinate solution for precisely those x in $\mathbb{R}$ for which $\operatorname{Im} m_+(x,\alpha) = 0$ for at least one boundary condition α, and there is no subordinate solution for precisely those x for which $0 < \operatorname{Im} m_+(x,\alpha) < \infty$ for every boundary condition α. It turns out that, with the possible exception of subsets of $\mathbb{R}$ having κ- and μ- measure zero, this situation holds quite generally; one of the main purposes of this chapter is to prove this assertion, and to assess the implications for the location of the spectrum.

In the next section we establish some continuity properties of $\|u(r,z)\|_N$ as a function of y for sets of solutions $\{u(r,z) : z \in \mathbb{C}\}$ having certain common properties. This is an important prerequisite to the proofs of our main results in §3.

§2. Properties of the norm in a finite interval

For each fixed z in $\mathbb{C}$, define unique solutions $u_1(r,z)$, $u_2(r,z)$ and $u_{(k)}(r,z)$ of Lu = zu to satisfy

$$u_1(0,z) = -\sin\alpha \qquad u_1'(0,z) = \cos\alpha$$

$$u_2(0,z) = \cos\alpha \qquad u_2'(0,z) = \sin\alpha$$

$$u_{(k)}(r,z) = u_2(r,z) + k\,u_1(r,z) \qquad k \in \mathbb{C}$$

For each fixed z in $\mathbb{C} \setminus \mathbb{R}$, define

$$u_m(r,z) = u_2(r,z) + m(z)\,u_1(r,z)$$

where $m(z) = m(z,\alpha)$ (See remarks preceding Lemma 2.18). For those x for which $m_+(x)$ exists finitely and is real, define

$$m(x) = m+(x)$$

$$u_m(r,x) = u_2(r,x) + m(x)\,u_1(r,x)$$

We shall now regard $u_1(r,z)$, $u_{(k)}(r,z)$ and $u_2(r,z)$ as functions of both r and z, and examine the behaviour of $\|u_1(r,z)\|_N$, $\|u_{(k)}(r,z)\|_N$ and $\|u_m(r,z)\|_N$ when both x and $N<\infty$ are fixed. Since $z = x+iy$, these norms are functions of y, defined on $(-\infty,\infty)$ in the case of $\|u_1(r,z)\|_N$ and $\|u_{(k)}(r,z)\|_N$, and, in general, on $\mathbb{R}\setminus\{0\}$ in the case of $\|u_m(r,z)\|_N$. We shall derive some detailed estimates of $\big|\|u(r,z_2)\|_N - \|u(r,z_1)\|_N\big|$ for $u(r,z) = u_1(r,z)$, $u_{(k)}(r,z)$, $u_m(r,z)$, where $z_1 = x+iy_1$, $z_2 = x+iy_2$; and from these obtain continuity properties of $\|u_1(r,z)\|_N$, $\|u_{(k)}(r,z)\|_N$ and $\|u_m(r,z)\|_N$ as functions of y.

The proofs of the estimates are contained in the following four lemmas. Since the method is the same in each case, we shall omit some of the details in the later proofs. We shall assume throughout that V(r) is integrable on every finite interval $[0,N]$.

3.2 Lemma: Let $z_1 = x+iy_1$, with $y_1>0$ be fixed. Then if $z_2 = x+iy_2$, and $|y_2-y_1|$ is sufficiently small

$$\big|\|u_1(r,z_2)\|_N - \|u_1(r,z_1)\|_N\big| < \frac{\gamma_1}{1-\gamma_1}\|u_1(r,z_1)\|_N$$

where $\gamma_1 = 2\,|y_2-y_1|\;\|u_1(r,z_1)\|_N\,\|u_m(r,z_1)\|_N$. If m(x) is defined, then this inequality also holds for $z_1 = x$.

Proof:

The hypothesis ensures that $u_m(r,z_1)$ is defined and that $u_1(r,z_1)$ and $u_m(r,z_1)$ are linearly independent solutions of $Lu = z_1u$. Reformulating the equation $Lu = z_2u$ as $(L - z_1)u = i(y_2 - y_1)u$, and applying the "variation of constants" formula ([CL] Ch.3, Thm. 6.4) we have, since $W(u_m(r,z_1),\ u_1(r,z_1)) = 1$:

$$\begin{aligned} u_1(r,z_2) = {} & u_1(r,z_1) + u_m(r,z_1)\int_0^r u_1(t,z_1)\,i(y_2-y_1)\,u_1(t,z_2)\,dt \\ & - u_1(r,z_1)\int_0^r u_m(t,z_1)\,i(y_2-y_1)\,u_1(t,z_2)\,dt \qquad (3.2.1)\end{aligned}$$

We use an iterative scheme to obtain the required estimate from this equation. Let

$$u_1^{(n+1)}(r,z_2) = u_1(r,z_1) + u_m(r,z_1)\int_0^r u_1(t,z_1)\,i(y_2-y_1)\,u_1^{(n)}(t,z_2)\,dt$$
$$- u_1(r,z_1)\int_0^r u_m(t,z_1)\,i(y_2-y_1)\,u_1^{(n)}(t,z_2)\,dt$$

and set $u_1^{(1)}(r,z_2) = u_1(r,z_1)$ (3.2.2)

Then

$$u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2) = u_m(r,z_1)\int_0^r u_1(t,z_1)\,i(y_2-y_1)\,u_1(t,z_1)\,dt$$
$$- u_1(r,z_1)\int_0^r u_m(t,z_1)\,i(y_2-y_1)\,u_1(t,z_1)\,dt$$

so that if $r \leq N$,

$$|u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2)|$$
$$\leq |y_2-y_1|\,|u_m(r,z_1)|\int_0^N |u_1(t,z_1)|^2\,dt$$
$$+ |y_2-y_1|\,|u_1(r,z_1)|\int_0^N |u_m(t,z_1)u_1(t,z_1)|\,dt \qquad (3.2.3)$$

Since this inequality is preserved if we take the $L_2[0,N]$ norm of both sides, we have, using the Minkowski and Cauchy-Schwarz inequalities:

$$\| u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2)\|_N$$
$$\leq 2|y_2-y_1|\,\|u_m(r,z_1)\|_N\,\|u_1(r,z_1)\|_N^2$$

Similarly, if $r \leq N$,

$$|u_1^{(3)}(r,z_2) - u_1^{(2)}(r,z_2)|$$
$$\leq |y_2-y_1|\,|u_m(r,z_1)|\int_0^N |u_1(t,z_1)|\,|u_1^{(2)}(t,z_2) - u_1^{(1)}(t,z_2)|\,dt$$
$$+ |y_2-y_1|\,|u_1(r,z_1)|\int_0^N |u_m(t,z_1)|\,|u_1^{(2)}(t,z_2) - u_1^{(1)}(t,z_2)|\,dt \qquad (3.2.4)$$

so that

$$\| u_1^{(3)}(r,z_2) - u_1^{(2)}(r,z_2)\|$$
$$\leq 2|y_2-y_1|\,\|u_1(r,z_1)\|_N\,\|u_m(r,z_1)\|_N\,\|u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2)\|_N$$
$$\leq (2|y_2-y_1|\,\|u_1(r,z_1)\|_N\,\|u_m(r,z_1)\|_N)^2\,\|u_1(r,z_1)\|_N$$

and, in general,

$$\|u_1^{(n+1)}(r,z_2) - u_1^{(n)}(r,z_2)\|_N$$

$$\leq (2|y_2-y_1| \|u_1(r,z_1)\|_N \|u_m(r,z_1)\|_N)^n \|u_1(r,z_1)\|_N \qquad (3.2.5)$$

Since $\|u_1(r,z_1)\|_N$ and $\|u_m(r,z_1)\|_N$ are finite and do not depend on y_2, the iterations converge in $L_2[0,N]$ norm for all z_2 such that

$$|y_2-y_1| < \frac{1}{2\|u_1(r,z_1)\|_N \|u_m(r,z_1)\|_N}$$

In order to show that the iterations converge to the solution, we first prove uniform pointwise convergence of the iterations $\{u_1^{(n)}(r,z_2)\}$ on $[0,N]$.

Since L is regular at 0, there exists K in $\mathbb{R}^+$ such that $|u_1(r,z_1)|, |u_m(r,z_1)| \leq K$ for all r in $[0,N]$. Hence if $r \in [0,N]$, from (3.2.3), $|u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2)| \leq 2|y_2-y_1| K^3 N$

and from (3.2.4),

$$|u_1^{(3)}(r,z_2) - u_1^{(2)}(r,z_2)| \leq 2|y_2-y_1| K^2 N |u_1^{(2)}(r,z_2) - u_1^{(1)}(r,z_2)|$$
$$\leq K(2K^2N|y_2-y_1|)^2$$

so that, in general,

$$|u_1^{(n+1)}(r,z_2) - u_1^{(n)}(r,z_2)| \leq K(2K^2N|y_2-y_1|)^n$$

It follows that there is uniform pointwise convergence of the iterations $\{u_1^{(n)}(r,z_2)\}$ for all z_2 such that

$$|y_2-y_1| < \frac{1}{2K^2N}$$

Let $\phi(r)$ denote $\lim\limits_{n\to\infty} u_1^{(n)}(r,z_2)$.

From (3.2.2),

$$\phi(r) = u_1(r,z_1) + \lim_{n\to\infty} u_m(r,z_1)\int_0^r u_1(t,z_1)\, i(y_2-y_1)\, u_1^{(n)}(t,z_2)\,dt$$
$$-\lim_{n\to\infty} u_1(r,z_1)\int_0^r u_m(t,z_1)\, i(y_2-y_1)\, u_1^{(n)}(t,z_2)\,dt$$

The uniform convergence of the $\{u_1^{(n)}(r,z_2)\}$ ensures that $\phi(r)$ is continuous, and hence bounded on $[0,N]$. We may therefore use the Lebesgue Dominated Convergence Theorem to take the limits inside the integrals, and hence by (3.2.1) and the uniqueness of solutions,

$$\phi(r) = u_1(r, z)$$

Thus if y_2 is sufficiently close to y_1,

$$u_1(r, z_2) = u_1(r, z_1) + \sum_{n=1}^{\infty} \left[u_1^{(n+1)}(r, z_2) - u_1^{(n)}(r, z_2) \right]$$

for each r in $[0,N]$, and hence by (3.2.5)

$$\left| \|u_1(r,z_2)\|_N - \|u_1(r,z_1)\|_N \right| \leq \|u_1(r,z_2) - u_1(r,z_1)\|_N$$

$$\leq \sum_{n=1}^{\infty} \| u_1^{(n+1)}(r, z_2) - u_1^{(n)}(r, z_2) \|_N$$

$$\leq \frac{2|y_2 - y_1| \|u_1(r,z_1)\|_N^2 \|u_m(r,z_1)\|_N}{1 - 2|y_2 - y_1| \|u_1(r,z_1)\|_N \|u_m(r,z_1)\|_N}$$

$$= \frac{\gamma_1}{1 - \gamma_1} \|u_1(r, z_1)\|_N \tag{3.2.6}$$

if $|y_2 - y_1|$ is sufficiently small.

If m+(x) exists finitely and is real, there is no difficulty in extending this result to the case $y_1 = x$, for $u_m(r,x)$ is defined and has all the required properties.

The proof of the lemma is now complete.

3.3 <u>Lemma</u>: If m(x) is defined, and $k \in \mathbb{C}$ is such that $k \neq m(x)$ then if $y > 0$ is sufficiently small and z = x+iy,

$$\left| \|u_{(k)}(r,z)\|_N - \|u_{(k)}(r,x)\|_N \right| \leq \frac{\gamma_k}{1-\gamma_k} \|u_{(k)}(r,z)\|_N$$

where $\gamma_k = \dfrac{4y \|u_{(k)}(r,z)\|_N \|u_m(r,z)\|_N}{|k - m(x)|}$

<u>Proof</u>:

From the definitions of $u_{(k)}(r,z)$, $u_m(r,z)$, $W(u_{(k)}(r,z), u_m(r,z)) = m(z)-k$. Since $k \neq m(x)$, and m(z), regarded as a function of y, is continuous there exists $y_k > 0$ such that

$$|k - m(z)| > \frac{|k - m(x)|}{2} \tag{3.2.7}$$

for all y such that $|y| < y_k$. Hence, if $y > 0$ is sufficiently small, $u_m(r,z)$ and $u_{(k)}(r,z)$ are linearly independent solutions of $Lu = zu$.

Reformulating $Lu = xu$ as

$$Lu - zu = -iyu$$

and using the "variation of constants" formula as before we obtain for $0 < y < y_k$

$$u_{(k)}(r,x) = u_{(k)}(r,z) + \frac{u_{(k)}(r,z)\int_0^r u_m(t,z)(-iy)u_{(k)}(t,x)\,dt}{(m(z)-k)} - \frac{u_m(r,z)\int_0^r u_{(k)}(t,z)(-iy)u_{(k)}(t,x)\,dt}{(m(z)-k)}$$

If we form the iterative scheme

$$u_{(k)}^{(n+1)}(r,x) = u_{(k)}(r,z) + \frac{u_{(k)}(r,z)\int_0^r u_m(t,z)(-iy)u_{(k)}^{(n)}(t,x)\,dt}{(m(z)-k)} - \frac{u_m(r,z)\int_0^r u_{(k)}(t,z)(-iy)u_{(k)}^{(n)}(t,x)\,dt}{(m(z)-k)}$$

and set $u_{(k)}^{(1)}(r,x) = u_{(k)}(r,z)$, then using the method of Lemma 3.2 together with (3.2.7) yields the result.

Thus the lemma is proved.

3.4 Lemma: With the hypothesis and notation of Lemma 3.3, if $y > 0$ is sufficiently small

$$\left| \|u_m(r,z)\|_N - \|u_m(r,x)\|_N \right|$$

$$\leq \frac{2}{1-\gamma_k}\left|\frac{m(z)-m(x)}{k-m(x)}\right| \|u_{(k)}(r,z)\|_N$$

$$+ 2\left[\frac{\gamma_k}{1-\gamma_k} + \left|\frac{m(z)-m(x)}{k-m(x)}\right|\right] \|u_m(r,z)\|_N$$

Proof:

Define $u_{m_x}(r,z) = u_2(r,z)+m(x)u_1(r,z)$. We have

$$\big|\, \|u_m(r,z)\|_N - \|u_m(r,x)\|_N \,\big|$$

$$\leq \|u_m(r,z) - u_m(r,x)\|_N$$

$$\leq \|u_m(r,z) - u_{m_x}(r,z)\|_N + \|u_{m_x}(r,z) - u_m(r,x)\|_N$$

$$= |m(z)-m(x)|\,\|u_1(r,z)\|_N + \|u_{m_x}(r,z) - u_m(r,x)\|_N \quad (3.2.8)$$

Since

$$u_1(r,z) = \frac{u_m(r,z) - u_k(r,z)}{(m(z)-k)}$$

if $0 < y < y_k$, we deduce from (3.2.7) and Minkowski's inequality

$$\|u_1(r,z)\|_N \leq \frac{2}{|k-m(x)|}\left(\|u_m(r,z)\|_N + \|u_{(k)}(r,z)\|_N\right) \quad (3.2.9)$$

for sufficiently small y.

Now $u_{m_x}(r,z)$ and $u_m(r,x)$ satisfy the same boundary conditions at $r = 0$, and if $y < y_k$, $u_m(r,z)$ and $u_{(k)}(r,z)$ are linearly independent solutions of $Lu = zu$. Since $Lu = xu$ may be reformulated as $Lu-zu = -iyu$ we have therefore by the "variation of constants" formula:

$$u_m(r,x) = u_{m_x}(r,z) + \frac{u_{(k)}(r,z)\int_0^r u_m(t,z)(-iy)u_m(t,x)\,dt}{(m(z)-k)}$$
$$- \frac{u_m(r,z)\int_0^r u_{(k)}(t,z)(-iy)u_m(t,x)\,dt}{(m(z)-k)}$$

Iterating this equation as before with $u_m^{(1)}(r,x) = u_{m_x}(r,z)$, we find that for sufficiently small y

$$\|u_m(r,x) - u_{m_x}(r,z)\|_N \leq \frac{\gamma_k}{1-\gamma_k}\|u_{m_x}(r,z)\|_N \quad (3.2.10)$$

Now

$$u_{m_x}(r,z) = \frac{(m(z)-m(x))}{(m(z)-k)}\, u_{(k)}(r,z) + \frac{(m(x)-k)}{(m(z)-k)}\, u_m(r,z)$$

Hence for sufficiently small y, by Minkowski's inequality and (3.2.7)

$$\|u_{m_x}(r,z)\|_N \leq 2\left|\frac{m(z)-m(x)}{k-m(x)}\right| \|u_{(k)}(r,z)\|_N + 2\,\|u_m(r,z)\|_N$$

This, together with (3.2.8), (3.2.9) and (3.2.10) gives the result, and the lemma is proved.

3.5 Lemma: Let $z_1 = x+iy_1$ be fixed. Then if $z_2 = x+iy_2$ and $y_1, y_2 > 0$

$$\big|\,\|u_m(r,z_2)\|_N - \|u_m(r,z_1)\|_N\big|$$

$$\leq \frac{\gamma_1}{1-\gamma_1}\,\|u_m(r,z_1)\|_N + \frac{1}{1-\gamma_1}\,|m(z_2)-m(z_1)|\,\|u_1(r,z_1)\|_N$$

whenever $|y_2-y_1|$ is sufficiently small, where γ_1 is as in Lemma 3.2. If m(x) is defined, then this inequality also holds for z_1 or $z_2 = x$.

Proof:

Define $u_{m_2}(r,z_1) = u_2(r,z_1) + m(z_2)u_1(r,z_1)$. By Minkowski's inequality

$$\big|\,\|u_m(r,z_2)\|_N - \|u_m(r,z_1)\|_N\big| \leq \|u_m(r,z_2) - u_m(r,z_1)\|_N$$

$$\leq \|u_m(r,z_2) - u_{m_2}(r,z_1)\|_N + \|u_{m_2}(r,z_1) - u_m(r,z_1)\|_N$$

$$\leq \|u_m(r,z_2) - u_{m_2}(r,z_1)\|_N + |m(z_2)-m(z_1)|\,\|u_1(r,z_1)\|_N \qquad (3.2.11)$$

Now if $y_1 > 0$, $u_1(r,z_1)$ and $u_m(r,z_1)$ are linearly independent solutions of $Lu = z_1u$, and $W(u_m(r,z_1),u_1(r,z_1)) = 1$; and if m(x) is defined, $u_1(r,x)$ and $u_m(r,x)$ are linearly independent solutions of $Lu = xu$, with $W(u_m(r,x),u_1(r,x)) = 1$. Moreover, $u_m(r,z_2)$ and $u_{m_2}(r,z_2)$ satisfy the same boundary conditions at r = 0. Hence, reformulating the equation $Lu = z_2u$ as $Lu - z_1u = i(y_2-y_1)u$, we have by the "variation of constants" formula:

$$u_m(r,z_2) = u_{m_2}(r,z_1) + u_m(r,z_1)\int_0^r u_1(t,z_1)\, i(y_2-y_1)\, u_m(t,z_2)\,dt$$
$$- u_1(r,z_1)\int_0^r u_m(t,z_1)\, i(y_2-y_1)\, u_m(t,z_2)\,dt$$

Forming the iterative scheme

$$u_m^{(n+1)}(r,z_2) = u_{m_2}(r,z_1) + u_m(r,z_1)\int_0^r u_1(t,z_1)\, i(y_2-y_1)\, u_m^{(n)}(t,z_2)\,dt$$
$$- u_1(r,z_1)\int_0^r u_m(t,z_1)\, i(y_2-y_1)\, u_m^{(n)}(t,z_2)\,dt$$

and setting $u_m^{(1)}(r,z_2) = u_{m_2}(r,z_1)$ we obtain, as in Lemma 3.2,

$$\left|\|u_m(r,z_2)\|_N - \|u_{m_2}(r,z_1)\|_N\right| \leqslant \frac{\gamma_1}{1-\gamma_1}\|u_{m_2}(r,z_1)\|_N \qquad (3.2.12)$$

Since $u_{m_2}(r,z_1) = u_m(r,z_1)+(m(z_2)-m(z_1))u_1(r,z_1)$,

$$\|u_{m_2}(r,z_1)\|_N \leqslant \|u_m(r,z_1)\|_N + |m(z_2)-m(z_1)|\,\|u_1(r,z_1)\|_N$$

by Minkowski's inequality. Hence the result follows from (3.2.11) and (3.2.12), so the lemma is proved.

3.6 Corollary: Let $x\in\mathbb{R}$ and $N<\infty$ be fixed. Then if $m(x)$ is defined, $\|u_1(r,z)\|_N$ and $\|u_m(r,z)\|_N$ are continuous functions of y on $[0,\infty)$. If in addition, $k\in\mathbb{C}$ is such that $k\neq m(x)$ then $\|u_{(k)}(r,z)\|_N$ is also a continuous function of y on $[0,\infty)$.

Proof:

It is immediate from Lemma 3.2 that for each y_1 in $[0,\infty)$

$\frac{\gamma_1}{1-\gamma_1}\|u_1(r,z_1)\|_N$ is arbitrarily small for y_2 sufficiently close to y_1, since $\|u_1(r,z_1)\|_N$, $\|u_m(r,z_1)\|_N$ are defined and finite. Hence $\|u_1(r,z)\|_N$ is a continuous function of y on $[0,\infty)$.

Since $m(x)$ is defined the function $m(z)$ is, for fixed x, a continuous function of y on $\mathbb{R}$. The continuity of $\|u_m(r,z)\|_N$ on $[0,\infty)$ therefore follows from Lemma 3.5.

Now $u_{(k)}(r,z) = u_m(r,z)+(k-m(z))u_1(r,z)$ for all y in $\mathbb{R}$. Hence if $y_1, y_2 \geqslant 0$

$$\left| \|u_{(k)}(r,z_2)\|_N - \|u_{(k)}(r,z_1)\|_N \right|$$
$$\leqslant \|u_{(k)}(r,z_2) - u_{(k)}(r,z_1)\|_N$$
$$\leqslant \|u_m(r,z_2) - u_m(r,z_1)\|_N + |k-m(z_2)| \, \|u_1(r,z_2) - u_1(r,z_1)\|_N$$
$$+ |m(z_2) - m(z_1)| \, \|u_1(r,z_1)\|_N$$

Now the bounds of Lemmas 3.2 and 3.5 for $\left|\|u_1(r,z_2)\|_N - \|u_1(r,z_1)\|_N\right|$ and $\left|\|u_m(r,z_2)\|_N - \|u_m(r,z_1)\|_N\right|$ also apply to $\|u_1(r,z_2) - u_1(r,z_1)\|_N$ and $\|u_m(r,z_2) - u_m(r,z_1)\|_N$ respectively by (3.2.6) and (3.2.11), so the final right hand side of this inequality may be made arbitrarily small by choosing y_2 sufficiently close to y_1. Since this is true for all y in $[0,\infty)$, $\|u_{(k)}(r,z)\|_N$ is a continuous function of y on $[0,\infty)$.

The proof is now complete.

3.7 Remarks:

(1) It was necessary to stipulate that m(x) be defined in Corollary 3.6 only because Lemmas 3.2 and 3.5 were used in the proof. However, it is possible to show that the continuity on $[0,\infty)$ of $\|u_1(r,z)\|_N$ and $\|u_{(k)}(r,z)\|_N$ for any k in $\mathbb{C}$ is not dependent on the existence of m+(x) as a finite real limit. To see this, it is only necessary to use the iterative method of Lemma 3.2 on the formulae:

$$u_1(r,z_2) = u_1(r,z_1) + u_{(k)}(r,z_1)\int_0^r u_1(t,z_1)\, i(y_2-y_1)\, u_1(t,z_2)\,dt$$
$$- u_1(r,z_1)\int_0^r u_{(k)}(t,z_1)\, i(y_2-y_1)\, u_1(t,z_2)\,dt$$
$$u_{(k)}(r,z_2) = u_{(k)}(r,z_1) + u_{(k)}(r,z_1)\int_0^r u_1(t,z_1)\, i(y_2-y_1)\, u_{(k)}(t,z_2)\,dt$$
$$- u_1(r,z_1)\int_0^r u_{(k)}(t,z_1)\, i(y_2-y_1)\, u_{(k)}(t,z_2)\,dt$$

(2) If m+(x) exists and $0 < \operatorname{Im} m_+(x) < \infty$, then if $u_m(r,x)$ is taken to be $u_2(r,x) + m_+(x)u_1(r,x)$, Lemmas 3.2 and 3.5 still hold

for $y_1 = 0$, $y_2 > 0$ (or vice versa), and Lemmas 3.3 and 3.4 still hold for $y > 0$.

(3) If m+(x) does not exist but there exists a sequence $\{\Upsilon_n\}$ in $\mathbb{R}^+$ such that $\Upsilon_n \to 0$ and $m(x+i\Upsilon_n) \to l$, where $|l| < \infty$, as $n \to \infty$ then, if $u_m(r,x)$ is taken to be $u_2(r,x)+lu_1(r,x)$,

(i) Lemmas 3.2 and 3.5 hold in the sense of Remark (2) if y_2 (respectively y_1) $\in \{\Upsilon_n\}$ and "$|y_2-y_1|$ sufficiently small" is replaced by "n sufficiently large".

(ii) Lemmas 3.3 and 3.4 hold in the sense of Remark (2) if $y \in \{\Upsilon_n\}$ and "$y > 0$ sufficiently small" is replaced by "n sufficiently large".

(4) It is not hard to see that Lemmas 3.2 - 3.5, Corollary 3.6 and Remarks (1) - (3) are also valid for z_1, z_2, z in the lower half-plane provided the necessary obvious adjustments are made. Hence $\|u_1(r,z)\|_N$, $\|u_{(k)}(r,z)\|_N$ are continuous functions of y on $\mathbb{R}$, and, if m(x) is defined, $\|u_m(r,z)\|_N$ is also a continuous function of y on $\mathbb{R}$.

We are now in a position to prove our main theorems on subordinacy. These will be used to derive a new set of minimal supports of the spectral measure, and of its decomposed parts, in terms of subordinate solutions.

§3. On the existence of subordinate solutions.

Our first objective is to show that if x is fixed and m(z) = m(x+iy) converges to a finite real limit m(x) as $y \downarrow 0$, then there exists a subordinate solution of the equation Lu = xu. We need a few preliminary results.

3.8 Lemma: Let $x \in \mathbb{R}$ be fixed and suppose that m(z) converges to a finite real limit as $y \downarrow 0$. Then if $\varepsilon > 0$ is given, there exists N in $\mathbb{R}^+$ such that

$$\frac{\|u_m(r,x)\|_N}{\|u_1(r,x)\|_N} < \varepsilon$$

Proof:

Let ε be chosen to be less than ½.

We may choose $Y > 0$ to satisfy

$$\frac{\varepsilon^2}{32} = \sup_{0<y\leq Y} |m(z) - m(x)| + Y \tag{3.3.1}$$

To see this, note that since the right hand side of (3.3.1) is continuous, strictly increasing with Y and convergent to 0 as $Y \downarrow 0$, it assumes every value in $(0, \infty)$. If Y is not sufficiently small to ensure the convergence of the estimates of Lemmas 3.2 and 3.5 for $z_2 = x$ and $z_1 = z$ when $y < Y$, choose $Y_1 > 0$ such that $Y_1 < Y$ and the convergence is assured for all $y \leq Y_1$. We then determine $\varepsilon_1 > 0$ by the condition

$$\frac{\varepsilon_1^2}{32} = \sup_{0<y\leq Y_1} |m(z) - m(x)| + Y_1 \tag{3.3.2}$$

Let $\tilde{\varepsilon}, \tilde{y}$ denote ε, Y respectively if convergence of the estimates of Lemmas 2.2 and 2.5 is assured for all $y \leq Y$, and. if not, let $\tilde{\varepsilon}, \tilde{y}$ denote ε_1, Y_1 respectively. In either case, (3.3.1) or (3.3.2), as appropriate, ensures that

$$|m(\tilde{z}) - m(x)| < \frac{\varepsilon}{4} \tag{3.3.3}$$

and $$|\mathrm{Im}\, m(\tilde{z})| < \frac{\tilde{\varepsilon}^2}{32} \tag{3.3.4}$$

where $\tilde{z} = x + i\tilde{y}$.

Since $u_1(r, \tilde{z})$ is not in $L_2[0, \infty)$, there exists $N(\tilde{y}) \in \mathbb{R}^+$ such that

$$|\tilde{y}|^{\frac{1}{2}} \|u_1(r, \tilde{z})\|_{N(\tilde{y})} = \sqrt{2} \tag{3.3.5}$$

For $\tilde{y}$ and $N(\tilde{y})$ chosen in this way, we have by (2.3.2)

$$\frac{\|u_m(r, \tilde{z})\|_{N(\tilde{y})}}{\|u_1(r, \tilde{z})\|_{N(\tilde{y})}} \leq \frac{|\mathrm{Im}\, m(\tilde{z})|^{\frac{1}{2}}}{|\tilde{y}|^{\frac{1}{2}} \|u_1(r, \tilde{z})\|_{N(\tilde{y})}}$$

$$< \frac{\tilde{\varepsilon}}{8} \tag{3.3.6}$$

and $\gamma_1 = 2|\tilde{y}| \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \|u_m(r,\tilde{z})\|_{N(\tilde{y})} < \frac{\tilde{\varepsilon}}{2} < \frac{1}{4}$ (3.3.7)

The estimates of Lemmas 3.2 and 3.5 now enable us to relate $\dfrac{\|u_m(r,x)\|_{N(\tilde{y})}}{\|u_1(r,x)\|_{N(\tilde{y})}}$ to the ratio of norms in (3.3.6) .

From Lemma 3.2

$$\left| \|u_1(r,x)\|_{N(\tilde{y})} - \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \right| \leq \frac{\gamma_1}{1-\gamma_1} \|u_1(r,\tilde{z})\|_{N(\tilde{y})}$$

so that, by (3.3.7),

$$\begin{aligned} \|u_1(r,x)\|_{N(\tilde{y})} &\geq \frac{1-2\gamma_1}{1-\gamma_1} \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \\ &> \frac{1}{2} \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \end{aligned}$$

From Lemma 3.5

$$\begin{aligned} &\left| \|u_m(r,x)\|_{N(\tilde{y})} - \|u_m(r,\tilde{z})\|_{N(\tilde{y})} \right| \\ &\qquad \leq \frac{\gamma_1}{1-\gamma_1} \|u_m(r,\tilde{z})\|_{N(\tilde{y})} + \frac{|m(\tilde{z})-m(x)|}{1-\gamma_1} \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \end{aligned}$$

so that, by (3.3.3) and (3.3.7),

$$\|u_m(r,x)\|_{N(\tilde{y})} \leq \frac{4}{3} \left[\|u_m(r,\tilde{z})\|_{N(\tilde{y})} + \frac{\tilde{\varepsilon}}{4} \|u_1(r,\tilde{z})\|_{N(\tilde{y})} \right]$$

Hence, by (3.3.6) and the definition of $\tilde{\varepsilon}$,

$$\begin{aligned} \frac{\|u_m(r,x)\|_{N(\tilde{y})}}{\|u_1(r,x)\|_{N(\tilde{y})}} &\leq \frac{8}{3} \frac{\|u_m(r,\tilde{z})\|_{N(\tilde{y})}}{\|u_1(r,\tilde{z})\|_{N(\tilde{y})}} + \frac{2}{3}\tilde{\varepsilon} \\ &< \tilde{\varepsilon} \\ &\leq \varepsilon \end{aligned}$$

Thus there exists $N = N(\tilde{y})$ with the required property, so the lemma is proved.

Lemma 3.8 shows that if $m+(x)$ exists finitely and is real, then the relative smallness of the solution $u_m(r,z)$ for $\operatorname{Im} z > 0$ is reflected in a similar relative smallness of the solution $u_m(r,x)$ in the sense of the ratio of norms on carefully chosen finite intervals. To deduce subordinacy, however, we must first prove the existence of a continuous dependence of $N(\tilde{y})$ on $\tilde{\varepsilon}$, such that $N(\tilde{y})$ becomes arbitrarily large as $\tilde{\varepsilon} \to 0$.

3.9 Lemma: With the hypothesis and notation of Lemma 3.8, $N(\tilde{y}) \to \infty$ as $\tilde{\varepsilon} \to 0$.

Proof:

we first prove by contradiction that $N(\tilde{y}) \to \infty$ as $\tilde{y} \to 0$.

Suppose it is not true that given any $M > 0$, there exists $\tilde{y}_M > 0$ such that $N(\tilde{y}) > M$ for all $\tilde{y} < \tilde{y}_M$. Then there exists an $M > 0$ such that for any $\eta > 0$, there exists $\tilde{y}_\eta < \eta$ with $N(\tilde{y}_\eta) \leq M$. Hence there exists a sequence $\{\tilde{y}_k\}$ such that $\tilde{y}_k \to 0$ as $k \to \infty$ and $N(\tilde{y}_k) \leq M$ for each k.

Using the definition (3.3.5) of $N(\tilde{y})$, we see that for each $\tilde{y}_k$

$$\|u_1(r, x+i\tilde{y}_k)\|_M \geq \|u_1(r, x+i\tilde{y}_k)\|_{N(\tilde{y}_k)} = \sqrt{\frac{2}{\tilde{y}_k}}$$

which implies that, as $\tilde{y}_k \downarrow 0$,

$$\|u_1(r, x+i\tilde{y}_k)\|_M \to \infty$$

This is impossible since, by Corollary 3.6, $\|u_1(r,z)\|_M$ is a continuous function of y on $\mathbb{R}$. This contradiction proves that $N(\tilde{y}) \to \infty$ as $\tilde{y} \to 0$.

Since $\tilde{\varepsilon}$ and $\tilde{y}$ are related by the formula

$$\frac{\tilde{\varepsilon}^2}{32} = \sup_{0 < \operatorname{Im} z \leq \tilde{y}} |m(z) - m(x)| + \tilde{y} \tag{3.3.8}$$

it is clear that $\tilde{y} \to 0$ as $\tilde{\varepsilon} \to 0$. Hence $N(\tilde{y}) \to \infty$ as $\tilde{\varepsilon} \to 0$ and the proof is now complete.

As in (3.3.5), if $y > 0$, define $N(y)$ to satisfy

$$|y|^{\frac{1}{2}} \|u_1(r, x+iy)\|_{N(y)} = \sqrt{2} \tag{3.3.9}$$

The following result will be needed when we show that $N(\tilde{y})$ is a continuous function of $\tilde{y}$.

3.10 Lemma: With the hypothesis and notation of Lemma 3.8, $N(\tilde{y})$ is locally bounded.

Proof:

We prove by contradiction that for each $\tilde{y} > 0$ there exists $\beta_{\tilde{y}} > 0$ and $M \in \mathbb{R}^+$ such that $N(\tilde{y}) \leq M$ on $[\tilde{y} - \beta_{\tilde{y}}, \tilde{y} + \beta_{\tilde{y}}]$.

Suppose this statement is not true, ie. that $N(\tilde{y})$ is not locally bounded. Then if $\eta > 0$ and $K \in \mathbb{R}^+$ are given, there exists $y_\eta > 0$ such that $|y_\eta - \tilde{y}| < \eta$ and $N(y_\eta) > K$. Since $u_1(r, z) \notin L_2[0, \infty)$ if $Im\, z \neq 0$, $N \in \mathbb{R}^+$ exists such that

$$\|u_1(r, \tilde{z})\|_N = 3\, \|u_1(r, \tilde{z})\|_{N(\tilde{y})} \qquad (3.3.10)$$

In particular, for this N and given $\eta > 0$, there exists $y_\eta > 0$ such that $|y_\eta - \tilde{y}| < \eta$ and $N(y_\eta) > N$. That is, using (3.3.5), there exists $y_\eta > 0$ such that $|y_\eta - \tilde{y}| < \eta$ and, if z_η denotes $x + iy_\eta$,

$$\begin{aligned} |y_\eta|^{\frac{1}{2}} \|u_1(r, z_\eta)\|_N &< |y_\eta|^{\frac{1}{2}} \|u_1(r, z_\eta)\|_{N(y_\eta)} \\ &= |\tilde{y}|^{\frac{1}{2}} \|u_1(r, \tilde{z})\|_{N(\tilde{y})} \end{aligned} \qquad (3.3.11)$$

Now by Corollary 3.6, $\|u_1(r,z)\|_N$ is a continuous function of y, and hence there exists $\nu > 0$ such that

$$\big|\, \|u_1(r, z)\|_N - \|u_1(r, \tilde{z})\|_N \,\big| < \|u_1(r, \tilde{z})\|_{N(\tilde{y})}$$

whenever $|y - \tilde{y}| < \nu$. Using (3.3.10), this implies that

$$\|u_1(r, z)\|_N > 2\, \|u_1(r, \tilde{z})\|_{N(\tilde{y})} \qquad (3.3.12)$$

whenever $|y - \tilde{y}| < \nu$.

We may choose $\eta = \min\{\nu, \frac{3\tilde{y}}{4}\}$. Then there exists y_η with $|y_\eta - \tilde{y}| < \frac{3\tilde{y}}{4}$ for which (3.3.10) is satisfied; and since this y_η

also satisfies $|y_\eta - \tilde{y}| < \nu$ we have by (3.3.12)

$$\|u_1(r, z_\eta)\|_N > 2\|u_1(r, \tilde{z})\|_{N(\tilde{y})} \qquad (3.3.13)$$

However $|y_\eta - \tilde{y}| < \dfrac{3\tilde{y}}{4}$ implies $\left|\dfrac{\tilde{y}}{y_\eta}\right|^{\frac{1}{2}} < 2$ so since this y_η satisfies (3.3.11) we have

$$\|u_1(r, z_\eta)\|_N < 2\|u_1(r, \tilde{z})\|_{N(\tilde{y})}$$

which contradicts (3.3.13).

Thus the lemma has been proved by contradiction.

We are now in a position to establish the continuity of $N(\tilde{y})$ on $\mathbb{R}^+$.

3.11 Lemma: With the hypothesis and notation of Lemma 3.8, $N(\tilde{y})$ is a continuous function of $\tilde{y}$ on $\mathbb{R}^+$.

Proof:

It is sufficient to prove that for each $\tilde{y}_1 > 0$ if $\delta > 0$ is given there exists $\zeta_\delta > 0$ such that

$$\left|\|u_1(r, \tilde{z}_1)\|_{N(\tilde{y}_1)} - \|u_1(r, \tilde{z}_1)\|_{N(\tilde{y})}\right| < \delta \qquad (3.3.14)$$

whenever $|\tilde{y} - \tilde{y}_1| < \zeta_\delta$, where $N(\tilde{y}_1)$, is as in (3.3.5). To see this, we show that if $N(\tilde{y})$ is discontinuous at some point $\tilde{y}_1 > 0$, then the condition (3.3.14) fails to hold.

If $N(\tilde{y})$ is discontinuous at $\tilde{y}_1 > 0$, there exists $K \in \mathbb{R}^+$ such that for any given $\eta > 0$ there exists $\tilde{y} > 0$ with $|\tilde{y} - \tilde{y}_1| < \eta$ and $|N(\tilde{y}) - N(\tilde{y}_1)| > K$. Hence there exists $M \in \mathbb{R}^+$ such that for any given $\eta > 0$ there exists $\tilde{y} > 0$ with $|\tilde{y} - \tilde{y}_1| < \eta$ and

$$\left|\|u_1(r, \tilde{z}_1)\|_{N(\tilde{y}_1)} - \|u_1(r, \tilde{z}_1)\|_{N(\tilde{y})}\right| > M$$

For, were this not so, we should have $u_1(r, \tilde{z}) = 0$ on a non-trivial interval, which is impossible.

Hence to prove the lemma we verify condition (3.3.14).

By (3.3.5), if $\tilde{y} > 0$,

$$\|u_1(r,\tilde{z})\|_{N(\tilde{y})} = \sqrt{\frac{2}{\tilde{y}}}$$

so that $\|u_1(r,\tilde{z})\|_{N(\tilde{y})}$ is a continuous function of $\tilde{y}$ on $\mathbb{R}^+$. Hence if $\delta > 0$ is given, there exists $\xi_\delta > 0$ such that

$$\left| \|u_1(r,\tilde{z})\|_{N(\tilde{y})} - \|u_1(r,\tilde{z}_1)\|_{N(\tilde{y}_1)} \right| < \frac{\delta}{2} \tag{3.3.15}$$

if $\tilde{y}, \tilde{y}_1 > 0$ and $|\tilde{y} - \tilde{y}_1| < \xi_\delta$

By Lemma 3.10, there exists $M \in \mathbb{R}^+$ and $\beta_{\tilde{y}} > 0$ such that $N(\tilde{y}) \leq M$ whenever $|\tilde{y} - \tilde{y}_1| < \beta_{\tilde{y}}$. Moreover, using (3.2.6) we see that ν_δ exists such that

$$\|u_1(r,\tilde{z}) - u_1(r,\tilde{z}_1)\|_M < \frac{\delta}{2}$$

whenever $|\tilde{y} - \tilde{y}_1| < \nu_\delta$. Hence, if $|\tilde{y} - \tilde{y}_1| < \min\{\beta_{\tilde{y}}, \nu_\delta\}$

$$\begin{aligned} &\left| \|u_1(r,\tilde{z})\|_{N(\tilde{y})} - \|u_1(r,\tilde{z}_1)\|_{N(\tilde{y})} \right| \\ &\leq \|u_1(r,\tilde{z}) - u_1(r,\tilde{z}_1)\|_{N(\tilde{y})} \\ &\leq \|u_1(r,\tilde{z}) - u_1(r,\tilde{z}_1)\|_M \\ &< \frac{\delta}{2} \end{aligned} \tag{3.3.16}$$

Let $\zeta_\delta = \min\{\xi_\delta, \beta_{\tilde{y}}, \nu_\delta\}$. Then from (3.3.15) and (3.3.16)

$$\left| \|u_1(r,\tilde{z}_1)\|_{N(\tilde{y}_1)} - \|u_1(r,\tilde{z}_1)\|_{N(\tilde{y})} \right| < \delta$$

whenever $\tilde{y}, \tilde{y}_1 > 0$ and $|\tilde{y} - \tilde{y}_1| < \zeta_\delta$. This is equivalent to the condition (3.3.14), so the lemma is proved.

3.12 <u>Corollary</u>: If $\tilde{\varepsilon}$ is sufficiently small, $N(\tilde{y})$ is a continuous function of $\tilde{\varepsilon}$ satisfying the inequality

$$\frac{\|u_m(r,x)\|_{N(\tilde{y})}}{\|u_1(r,x)\|_{N(\tilde{y})}} < \tilde{\varepsilon}$$

Proof:

As we noted in the proof of Lemma 3.8, the convergence of the estimates of Lemmas 3.2 and 3.5 for $z_2 = x$ and $z_1 = \tilde{z}$ is assured if $\tilde{y}$ is sufficiently small. For such $\tilde{y}$,

$\dfrac{\|u_m(r,x)\|_{N(\tilde{y})}}{\|u_1(r,x)\|_{N(\tilde{y})}} < \tilde{\varepsilon}$ (see proof of Lemma 3.8), Lemmas 3.9 - 3.11 are applicable, and $\tilde{\varepsilon}$ and $\tilde{y}$ are related by (3.3.8).

From (3.3.8) it is clear that if there exists $\beta > 0$ such that some property holds for all $\tilde{y} < \beta$, then the property also holds for all $\tilde{\varepsilon} < 2^{5/2}\big(\sup\limits_{0<y\leq\beta} |m(z) - m(x)| + \beta\big)^{1/2}$. Hence $\dfrac{\|u_m(r,x)\|_{N(\tilde{y})}}{\|u_1(r,x)\|_{N(\tilde{y})}} < \tilde{\varepsilon}$ for sufficiently small $\tilde{\varepsilon}$.

It is also evident from (3.3.8) that for all $\tilde{y}$ which are sufficiently small in the sense indicated above, $\tilde{y}$ is a continuous function of $\tilde{\varepsilon}$. It follows from Lemma 3.11 that $N(\tilde{y})$ is a continuous function of $\tilde{\varepsilon}$ for sufficiently small $\tilde{y}$, and hence, also, for sufficiently small $\tilde{\varepsilon}$.

The proof of the corollary is now complete.

We are now able to deduce the subordinacy of $u_m(r,x)$ from Lemma 3.8.

3.13 Theorem: Let $x \in \mathbb{R}$ be fixed and suppose that $m(z)$ converges to a finite real limit as $y \downarrow 0$. Then $u_m(r,x)$ is a subordinate solution of $Lu = xu$.

Proof:

Since $N(\tilde{y})$ is a continuous function of $\tilde{\varepsilon}$ for sufficiently small $\tilde{\varepsilon}$, we relabel $N(\tilde{y})$ as $N(\tilde{\varepsilon})$.

By Corollary 3.12 there exists an interval $(0,a]$ such that $N(\tilde{\varepsilon})$ is a continuous function of $\tilde{\varepsilon}$ on $(0,a]$ satisfying

$$\frac{\|u_m(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}} < \tilde{\varepsilon} \tag{3.3.17}$$

The continuity of $N(\tilde{\varepsilon})$ implies that if η such that $0<\eta<a$ is given, then there exists $K \in \mathbb{R}^+$ such that K is the least upper bound of $N(\tilde{\varepsilon})$ on $[\eta, a]$.

By Lemma 3.9, $N(\tilde{\varepsilon}) \to \infty$ as $\tilde{\varepsilon} \to 0$. Hence, since $N(\tilde{\varepsilon})$ is continuous on (0,a], $N(\tilde{\varepsilon})$ takes every value in $[K,\infty)$ as $\tilde{\varepsilon}$ ranges over (0,a], by the Intermediate Value Theorem. Moreover, if $\tilde{\varepsilon}$ is in (0,a], then whenever $N(\tilde{\varepsilon}) > K$, $\tilde{\varepsilon}$ is in $(0,\eta)$ by the definition of K.

We may reformulate this last statement as follows: If η such that $0<\eta<a$ is given, then there exists N_η(any number greater than K will do) such that $\tilde{\varepsilon}<\eta$ whenever $N(\tilde{\varepsilon})>N_\eta$, $\tilde{\varepsilon} \in (0,a]$.

Hence, from (3.3.17), if η such that $0<\eta<a$ is given then there exists N_η in $\mathbb{R}^+$ such that

$$\frac{\|u_m(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}} < \eta$$

whenever $N(\tilde{\varepsilon}) > N_\eta$, $\tilde{\varepsilon} \in (0,a]$.

Since $N(\tilde{\varepsilon})$ takes every value in $[N_\eta, \infty)$, it follows that

$$\lim_{N\to\infty} \frac{\|u_m(r,x)\|_N}{\|u_1(r,x)\|_N} = 0$$

so that, by Definition 3.1, $u_m(r,x)$ is a subordinate solution of Lu = xu. This completes the proof.

In addition to proving that a subordinate solution of Lu = xu exists whenever m(z) converges to a finite real limit as $y \downarrow 0$, Theorem 3.13 identifies the set of subordinate solutions in this case as scalar multiples of $u_m(r,x)$. Given that $u_m(r,z)$ is in $L_2[0,\infty)$ for each z in $\mathbb{C} \setminus \mathbb{R}$, it is not altogether surprising to find that $u_m(r,x)$ is subordinate where it is defined.

Of course, m(z) depends on the boundary condition α_1 at r = 0 (see Chapter II, §3); we may indicate this dependence by the notation $m(z,\alpha_1)$. Clearly Theorem 3.13 may also be applied in those situations where, although the hypothesis is not satisfied for $m(z,\alpha_1)$, another boundary condition α_2 exists for which $m(z,\alpha_2)$ does converge to a finite real limit as $y \downarrow 0$. This is the idea behind the following complementary result:

3.14 Theorem: Let $x \in \mathbb{R}$ be fixed and suppose that $m(z) \to \infty$ as $y \downarrow 0$. Then $u_1(r,x)$ is a subordinate solution of Lu = xu.

Proof:

The hypothesis implies that $m_+(x,\alpha_1) = \infty$ for some given boundary condition α_1. Hence, by Lemma 2.18, $m_+(x,\alpha_2) = -\cot(\alpha_1-\alpha_2)$ for any distinct boundary condition α_2. For each such α_2, $m_+(x,\alpha_2)$ is finite and real, so by Theorem 3.13

$$u_m(r,x,\alpha_2) = u_2(r,x,\alpha_2) + m_+(x,\alpha_2)\, u_1(r,x,\alpha_2)$$

is a subordinate solution of Lu = xu. Now

$$u_m(0,x,\alpha_2) = \cos\alpha_2 + \cot(\alpha_1-\alpha_2)\sin\alpha_2$$
$$= \frac{\sin\alpha_1}{\sin(\alpha_1-\alpha_2)}$$

and

$$u_m'(0,x,\alpha_2) = \sin\alpha_2 - \cot(\alpha_1-\alpha_2)\cos\alpha_2$$
$$= \frac{-\cos\alpha_1}{\sin(\alpha_1-\alpha_2)}$$

which implies, by uniqueness,

$$u_1(r,x,\alpha_1) = \sin(\alpha_2-\alpha_1)\, u_m(r,x,\alpha_2)$$

That is, $u_1(r,x,\alpha_1)$ is a scalar multiple of a subordinate solution of Lu = xu, and so is itself subordinate.

Reverting to the notation of the hypothesis, we have shown that if $m(z) \to \infty$ as $y \downarrow 0$ then $u_1(r,x)$ is subordinate; thus the theorem is proved.

We now prove conversely that, whenever a subordinate solution of

$Lu = xu$ exists then, as $y \downarrow 0$, either $m(z)$ converges to a finite real limit or $m(z) \to \infty$. We first need some further estimates of solutions.

3.15 Lemma: If $m+(x)$ exists and is finite, and $\operatorname{Im} m+(x) = l_x$ then there exists $K \in \mathbb{R}^+$, which is independent of x, such that

$$\frac{\| u_{m+}(r,x) \|_N}{\| u_1(r,x) \|_N} \leq K l_x$$

as $N \to \infty$, where $u_{m+}(r,x) = u_2(r,x) + m+(x)\, u_1(r,x)$.

Proof:

The method of proof follows the same pattern as that of Theorem 3.13 and requires preliminary arguments similar to those of Lemmas 3.8 - 3.11 and Corollary 3.12.

We first show that if $\varepsilon > 0$ is given then there exist $\tilde{\varepsilon}$ such that $0 < \tilde{\varepsilon} \leq \varepsilon$ and $N(\tilde{\varepsilon})$ which is a continuous function of $\tilde{\varepsilon}$ satisfying

$$\frac{\| u_{m+}(r,x) \|_{N(\tilde{\varepsilon})}}{\| u_1(r,x) \|_{N(\tilde{\varepsilon})}} < 30\, l_x + \tilde{\varepsilon} \tag{3.3.18}$$

Let $\varepsilon > 0$ be given. Subject to the condition that $\tilde{y} > 0$ be sufficiently small to ensure that for all $y < \tilde{y}$ the estimates of Lemmas 3.2 and 3.5, with $z_2 = x$ and $z_1 = z$, converge, we choose $\tilde{y} > 0$ and $\tilde{\varepsilon} \leq \varepsilon$, as in Lemma 3.8, to satisfy

$$\frac{\tilde{\varepsilon}^2}{40^2 l_x} = \sup_{0 < \operatorname{Im} z \leq \tilde{y}} |m(z) - m+(x)| + \tilde{y} \tag{3.3.19}$$

together with the requirement

$$\tilde{\varepsilon} < 8\, l_x \tag{3.3.20}$$

For this $\tilde{\varepsilon}$ and $\tilde{y}$,

$$|m(\tilde{z}) - m+(x)| < \frac{\tilde{\varepsilon}^2}{40^2 l_x} < \frac{\tilde{\varepsilon}}{20} \tag{3.3.21}$$

and $|\operatorname{Im} m(\tilde{z}) - \operatorname{Im} m+(x)|^{\frac{1}{2}} < \dfrac{\tilde{\varepsilon}}{40\, l_x^{\frac{1}{2}}}$

so that, using $||a|^{\frac{1}{2}} - |b|^{\frac{1}{2}}| \leq |a-b|^{\frac{1}{2}}$,

$$|\operatorname{Im} m(\tilde{z})|^{\frac{1}{2}} < l_x^{\frac{1}{2}} + \frac{\tilde{\varepsilon}}{40\, l_x^{1/2}} \tag{3.3.22}$$

Define $N(\tilde{\varepsilon})$ to satisfy

$$\tilde{y}^{\frac{1}{2}} \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} = \frac{1}{8\, l_x^{1/2}} \tag{3.3.23}$$

Then, by (2.3.2) and (3.3.22)

$$\begin{aligned} \frac{\|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})}}{\|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})}} &\leq \frac{|\operatorname{Im} m(\tilde{z})|^{\frac{1}{2}}}{\tilde{y}^{\frac{1}{2}} \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})}} \\ &< \left[l_x^{1/2} + \frac{\tilde{\varepsilon}}{40\, l_x^{1/2}} \right] 8\, l_x^{1/2} \\ &< 8\, l_x + \frac{\tilde{\varepsilon}}{5} \end{aligned} \tag{3.3.24}$$

and, also using (3.3.20),

$$\begin{aligned} \gamma_1 &= 2\tilde{y}\, \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \\ &< \frac{1}{4\, l_x^{1/2}} \left[l_x^{1/2} + \frac{1}{40\, l_x^{1/2}} \right] \\ &< \frac{3}{10} \end{aligned} \tag{3.3.25}$$

We now use the estimates of Lemmas 3.2 and 3.5 to relate

$\dfrac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}}$ to the ratio of norms in (3.3.24).

From Lemma 3.2,

$$\left| \|u_1(r,x)\|_{N(\tilde{\varepsilon})} - \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \right| \leq \frac{\gamma_1}{1-\gamma_1} \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})}$$

so that, by (3.3.25),

$$\begin{aligned} \|u_1(r,x)\|_{N(\tilde{\varepsilon})} &\geq \frac{1-2\gamma_1}{1-\gamma_1} \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \\ &> \frac{4}{10} \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \end{aligned}$$

From Lemma 3.5, (3.3.21) and (3.3.25),

$$\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})} \leq \frac{1}{1-\gamma_1} \left[\|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})} + |m(\tilde{z}) - m_+(x)| \|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})} \right]$$

$$< \frac{10}{7}\left[\|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})} + \frac{\tilde{\varepsilon}}{20}\|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})}\right]$$

Hence, by (3.3.24),

$$\frac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}} < \frac{25}{7}\frac{\|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})}}{\|u_1(r,\tilde{z})\|_{N(\tilde{\varepsilon})}} + \frac{5\tilde{\varepsilon}}{28}$$

$$< \frac{25.8}{7} l_x + \frac{5}{7}\tilde{\varepsilon} + \frac{5}{28}\tilde{\varepsilon}$$

$$< 30 l_x + \tilde{\varepsilon}$$

so that we have proved (3.3.18).

Just as in Lemmas 3.9 - 3.11 it may be proved that for sufficiently small $\tilde{\varepsilon}$, $N(\tilde{\varepsilon})$ is a continuous function of $\tilde{\varepsilon}$ and that as $\tilde{\varepsilon} \to 0$, $N(\tilde{\varepsilon}) \longrightarrow \infty$. Moreover, for sufficiently small $\tilde{\varepsilon}$, the inequality (3.3.18) holds, so that arguing as in the proof of Theorem 3.13, we find that if a sufficiently small η is given then there exists $N_\eta \in \mathbb{R}^+$ such that $N(\tilde{\varepsilon})$ takes every value in $[N_\eta, \infty)$ if $\tilde{\varepsilon} < \eta$, and

$$\frac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}} < 30 l_x + \eta$$

for all $N(\tilde{\varepsilon}) > N_\eta$. It follows that

$$\frac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_1(r,x)\|_{N(\tilde{\varepsilon})}} = O(l_x)$$

as $N \longrightarrow \infty$, so the proof of the lemma is now complete.

3.16 <u>Lemma</u>: With the hypothesis and notation of Lemma 3.15, suppose also that $k \in \mathbb{C}$ is such that $k \neq m+(x)$. Then there exists $K_{k,x} \in \mathbb{R}^+$ which depends on k and x such that

$$\frac{\|u_{m+}(r,x)\|_N}{\|u_{(k)}(r,x)\|_N} \leq K_{k,x}\, l_x$$

for sufficiently large N.

<u>Proof</u>:

The method of proof is similar to that of Lemma 3.15.

We first show that if $\varepsilon > 0$ is given then there exist $\tilde{\varepsilon}$ such that $0 < \tilde{\varepsilon} \leq \varepsilon$ and $N(\tilde{\varepsilon})$ which is a continuous function of $\tilde{\varepsilon}$ satisfying

$$\frac{\| u_{m+}(r,x)\|_{N(\tilde\varepsilon)}}{\| u_1(r,x)\|_{N(\tilde\varepsilon)}} < \frac{80\, l_x}{|m_+(x)-k|} + \tilde\varepsilon \tag{3.3.26}$$

Let $\varepsilon > 0$ be given. Subject to the condition that $\tilde y > 0$ be sufficiently small to ensure that for all $y < \tilde y$ the estimates of Lemmas 3.3 and 3.4, with $z_2 = x$ and $z_1 = z$, converge, we may, as in Lemma 3.8, choose $\tilde y > 0$ and $\tilde\varepsilon \leq \varepsilon$ such that

$$\frac{\tilde\varepsilon^4 |m_+(x)-k|}{32^2 . 25^2 l_x} = \sup_{0 < \operatorname{Im} z \leq \tilde y} |m(z) - m_+(x)| + \tilde y \tag{3.3.27}$$

and, also, so that $\tilde\varepsilon$ satisfies $\tilde\varepsilon < 1$ and

$$\frac{\tilde\varepsilon}{25\,|m_+(x)-k|^{1/2}} < \frac{1}{10}\,,\quad \frac{\tilde\varepsilon\,|m_+(x)-k|^{1/2}}{16^2 . 25\; l_x} < \frac{3}{40}$$
$$\max\left\{\frac{5\tilde\varepsilon}{48 . 25^2 |k - m_+(x)|},\ \frac{\tilde\varepsilon}{150 . 32^2 l_x},\ \frac{\tilde\varepsilon^2}{3 . 32^2 25^2 l_x}\right\} < \frac{1}{4} \tag{3.3.28}$$

For this $\tilde\varepsilon$ and $\tilde y$

$$|m(\tilde z) - m_+(x)| < \frac{\tilde\varepsilon^4 |m_+(x)-k|}{32^2 . 25^2\, l_x} < \frac{\tilde\varepsilon^2 |m_+(x)-k|}{32^2 . 25^2 l_x} \tag{3.3.29}$$

and, as in Lemma 3.15,

$$|\operatorname{Im} m(\tilde z)|^{1/2} < l_x^{1/2} + \frac{\tilde\varepsilon^2 |m_+(x)-k|^{1/2}}{32 . 25\;\; l_x^{1/2}} \tag{3.3.30}$$

Define $N(\tilde\varepsilon)$ to satisfy

$$\tilde y^{1/2} \| u_{(k)}(r,\tilde z)\|_{N(\tilde\varepsilon)} = \frac{|m_+(x) - k|}{32\;\; l_x^{1/2}} \tag{3.3.31}$$

Then by (2.3.2), (3.3.30) and (3.3.28)

$$\begin{aligned}\frac{\| u_m(r,\tilde z)\|_{N(\tilde\varepsilon)}}{\| u_{(k)}(r,\tilde z)\|_{N(\tilde\varepsilon)}} &\leq \frac{|\operatorname{Im} m(\tilde z)|^{1/2}}{\tilde y^{1/2} \| u_{(k)}(r,\tilde z)\|_{N(\tilde\varepsilon)}} \\ &< \frac{32\;\; l_x^{1/2}}{|m_+(x)-k|}\left(l_x^{1/2} + \frac{\tilde\varepsilon^2 |m_+(x)-k|^{1/2}}{32 . 25\;\; l_x^{1/2}}\right) \\ &= \frac{32\;\; l_x}{|m_+(x)-k|} + \frac{\tilde\varepsilon}{10}\end{aligned} \tag{3.3.32}$$

and, by (3.3.28),

$$\gamma_k = \frac{4\tilde{y}\ \|u_{(k)}(r,\tilde{z})\|_{N(\tilde{\varepsilon})}\ \|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})}}{|m+(x) - k|}$$

$$< \frac{1}{8\, l_x^{1/2}}\left(l_x^{1/2} + \frac{\tilde{\varepsilon}^2 |m+(x) - k|^{1/2}}{32.25\ l_x^{1/2}}\right)$$

$$< \frac{1}{5} \tag{3.3.33}$$

The estimates of Lemmas 3.3 and 3.4 now enable us to relate

$\dfrac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_{(k)}(r,x)\|_{N(\tilde{\varepsilon})}}$ to the ratio of norms in (3.3.32).

From Lemma 3.3 and (3.3.33)

$$\|u_{(k)}(r,x)\|_{N(\tilde{\varepsilon})} \geqslant \frac{1-2\gamma_k}{1-\gamma_k}\ \|u_{(k)}(r,\tilde{z})\|_{N(\tilde{\varepsilon})}$$

$$> \frac{3}{5}\ \|u_{(k)}(r,\tilde{z})\|_{N(\tilde{\varepsilon})}$$

so, using Lemma 3.4 (see Remarks 3.7(2)), we have

$$\frac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_{(k)}(r,x)\|_{N(\tilde{\varepsilon})}} \leqslant \frac{5}{3}\left[\frac{(1+\gamma_k)}{(1-\gamma_k)} + 2\,\frac{|m(\tilde{z}) - m+(x)|}{|k - m+(x)|}\right]\frac{\|u_m(r,\tilde{z})\|_{N(\tilde{\varepsilon})}}{\|u_{(k)}(r,\tilde{z})\|_{N(\tilde{\varepsilon})}}$$

$$+ \frac{5}{3}\,\frac{2}{(1-\gamma_k)}\ \frac{|m(\tilde{z}) - m+(x)|}{|k - m+(x)|}$$

Hence, by (3.3.28), (3.3.29), (3.3.32) and (3.3.33)

$$\frac{\|u_{m+}(r,x)\|_{N(\tilde{\varepsilon})}}{\|u_{(k)}(r,x)\|_{N(\tilde{\varepsilon})}} \leqslant \frac{5}{3}\left(\frac{3}{2} + \frac{2\tilde{\varepsilon}^2}{32^2.\,25^2 l_x}\right)\left(\frac{32\, l_x}{|k - m+(x)|} + \frac{\tilde{\varepsilon}}{10}\right) + \frac{\tilde{\varepsilon}^2}{150.\,32^2 l_x}$$

$$= \frac{80\, l_x}{|m+(x) - k|} + \frac{\tilde{\varepsilon}}{4} + \frac{5\tilde{\varepsilon}^2}{48.\,25^2|k - m+(x)|} + \frac{\tilde{\varepsilon}^2}{150.\,32^2 l_x} + \frac{\tilde{\varepsilon}^3}{3.\,32^2 25^2 l_x}$$

$$< \frac{80\, l_x}{|m+(x) - k|} + \tilde{\varepsilon}$$

so we have proved (3.3.26). It now follows, just as in Lemma 3.15, that, in the sense indicated in the statement of the lemma.

$$\frac{\|u_{m+}(r,x)\|_N}{\|u_{(k)}(r,x)\|_N} = O(l_x)$$

as $N \to \infty$, and the proof of the lemma is complete.

We remark that, by and large, no particular significance is attached to the precise numbers involved in the proofs of Lemmas 3.8, 3.15 and 3.16. However, in view of the complexity of the relationships involved, it is necessary to exercise considerable care in these proofs, and working with particular numbers gives a precision which cannot be achieved by such notions as "sufficiently small" alone.

Just as in the case of our earlier estimates, Lemmas 3.15 and 3.16 remain valid in a modified sense if there exists a sequence $\{Y_n\}$ in $\mathbb{R}^+$ such that $Y_n \to 0$ and $m(x+iY_n)$ converges to a finite limit as $n \to \infty$. As this is easily verified by making suitable minor adjustments to the arguments of Lemmas 3.15 and 3.16, we omit detailed proofs:

3.17 <u>Remarks</u> If $m+(x)$ does not exist, but there is a sequence $\{Y_n\}$ in $\mathbb{R}^+$ such that $Y_n \to 0$ and $m(x+iY_n)$ converges to a finite limit l as $n \to \infty$, then there exist sequences $\{M_p\}$ and $\{N_q\}$ in $\mathbb{R}^+$ such that $M_p, N_q \to \infty$ as $p, q \to \infty$ and, if $k \neq l$,

$$\frac{\| u_{(l)}(r,x) \|_{M_p}}{\| u_1(r,x) \|_{M_p}} = O(\mathrm{Im}\, l), \quad \frac{\| u_{(l)}(r,x) \|_{N_q}}{\| u_{(k)}(r,x) \|_{N_q}} = O(\mathrm{Im}\, l) \qquad (3.3.34)$$

as $p, q \to \infty$, where $u_l(r,x) = u_2(r,x) + l u_1(r,x)$. To see this, let $\{\tilde{y}_p\}, \{\tilde{y}_q\} \subseteq \{Y_n\}$ and corresponding sequences $\{\tilde{\varepsilon}_p\}, \{\tilde{\varepsilon}_q\}$ be related according to equations (3.3.19) and (3.3.27) respectively in such a way as to ensure the convergence of the iterations of Lemmas 3.2, 3.5 and Lemmas 3.3, 3.4 respectively in the sense of Remarks 3.7(3). Then the sequences of numbers $\{M_p\}$ and $\{N_q\}$ satisfying

$$\tilde{y}_p^{\frac{1}{2}} \| u_1(r, \tilde{z}_p) \|_{M_p} = \frac{1}{8 (\mathrm{Im}\, l)^{1/2}}$$

and
$$\tilde{y}_q^{\frac{1}{2}} \| u_{(k)}(r, \tilde{z}_q) \|_{N_q} = \frac{|l-k|}{32 (\mathrm{Im}\, l)^{1/2}}$$

(cf. (3.3.23) and (3.3.31) respectively) also satisfy

$$\frac{\|u_{(l)}(r,x)\|_{M_p}}{\|u_1(r,x)\|_{M_p}} < 30\,\mathrm{Im}\,l + \tilde{\varepsilon}_p$$

and

$$\frac{\|u_{(l)}(r,x)\|_{N_q}}{\|u_{(k)}(r,x)\|_{N_q}} < \frac{80\,\mathrm{Im}\,l}{|l-k|} + \tilde{\varepsilon}_q$$

respectively, which imply (3.3.34)

We are now able to prove the following converse to Theorems 3.13 and 3.14:

3.18 Theorem: If a subordinate solution of Lu = xu exists then, as $y \downarrow 0$; either m(z) converges to a finite real limit or $m(z) \to \infty$.

Proof:

We may regard the subordinate solutions as the set of scalar multiples of $u(r,x) = au_1(r,x)+bu_2(r,x)$ for some $a \in \mathbb{R}$ and $b \in \mathbb{C}$. Since $\|u(r,x)\|_N = \|\overline{u(r,x)}\|_N$ for all N in $\mathbb{R}^+$, $\overline{u(r,x)}$ is also a subordinate solution of Lu = xu, and must therefore be a scalar multiple of u(r,x). It follows that if $a \neq 0$, then b is real, so that a subordinate solution is always a scalar multiple of a real solution.

To prove the theorem, it is sufficient to show that

(i) if a subordinate solution of Lu = xu exists at the real point x, and m(z) converges to a finite limit m+(x) as $y \downarrow 0$, then m+(x) must be real

(ii) if m(z) does not converge to a limit as $y \downarrow 0$ then no subordinate solution can exist.

Proof of (i):

Suppose there is a subordinate solution of Lu = xu at the real point x, and that m+(x) exists finitely; and let $l_x = \mathrm{Im}\, m_+(x)$,

$u_{m+}(x) = u_2(r,x) + m_+(x)\, u_1(r,x)$.

Then by Lemmas 3.15 and 3.16 respectively

$$\frac{\| u_{m+}(r,x) \|_N}{\| u_1(r,x) \|_N} = O(l_x) \tag{3.3.35}$$

and

$$\frac{\| u_{m+}(r,x) \|_N}{\| u_{(k)}(r,x) \|_N} = O(l_x) \tag{3.3.36}$$

as $N \to \infty$, where $k \in \mathbb{C}$ is such that $k \neq m+(x)$. Now suppose that $u_{m+}(r,x)$ is not subordinate; then there exists a solution $u(r,x)$ which is not a scalar multiple of $u_{m+}(r,x)$ such that

$$\frac{\| u(r,x) \|_N}{\| u_{m+}(r,x) \|_N} \longrightarrow 0$$

as $N \to \infty$ by Definition 3.1. That is there exists a solution $u(r,x)$, such that if $K > 0$ is given, $N_K \in \mathbb{R}^+$ exists with

$$\frac{\| u_{m+}(r,x) \|_N}{\| u(r,x) \|_N} > K \quad \text{for all } N > N_K \tag{3.3.37}$$

However, $u(r,x)$ is a scalar multiple of some element of the solution set $\{u_1(r,x)\} \cup \{u_{(k)}(r,x) : k \in \mathbb{C},\ k \neq m+(x)\}$ since there are no other solutions of $Lu = xu$ which are linearly independent of $u_{m+}(r,x)$. Hence (3.3.37) is not compatible with (3.3.35) and (3.3.36), so the supposition that $u_{m+}(r,x)$ is not subordinate is false.

Therefore $u_{m+}(r,x)$ is subordinate, and a scalar multiple of a real solution. This implies that $m+(x)$ is real, as was to be proved.

Proof of (ii):

If $m(z)$ does not converge as $y \downarrow 0$, let us first suppose that there exist sequences $\{y_m\}$ and $\{Y_n\}$ in $\mathbb{R}^+$ and l, L in $\mathbb{C}$ with $l \neq L$, $|l|, |L| < \infty$ such that $y_m, Y_n \to 0$, $m(x+iy_m) \to l$ and $m(x+iY_n) \to L$ as $m, n \to \infty$.

Let $u_{(l)}(r,x) = u_2(r,x) + l u_1(r,x)$

By Remark 3.17, there exist sequences $\{M_p\}$ and $\{N_q\}$ in $\mathbb{R}^+$ such that $M_p, N_q \to \infty$ as $p, q \to \infty$, and

$$\frac{\| u_{(l)}(r,x) \|_{M_p}}{\| u_1(r,x) \|_{M_p}} = O(\operatorname{Im} l) \tag{3.3.38}$$

as $p \to \infty$ and

$$\frac{\| u_{(l)}(r,x) \|_{N_q}}{\| u_{(k)}(r,x) \|_{N_q}} = O(\operatorname{Im} l) \tag{3.3.39}$$

as $q \to \infty$ for all k in $\mathbb{C}$ such that $k \neq l$. If there were a subordinate solution u(r,x) of Lu = xu which was not a scalar multiple of $u_{(l)}(r,x)$, then, as above, given any $K > 0$, there would exist P and Q in $\mathbb{N}$ such that

$$\frac{\| u_{(l)}(r,x) \|_{M_p}}{\| u(r,x) \|_{M_p}} > K \qquad \text{for all } p > P$$

and

$$\frac{\| u_{(l)}(r,x) \|_{N_q}}{\| u(r,x) \|_{N_q}} > K \qquad \text{for all } q > Q$$

As before, this is not possible in view of (3.3.38) and (3.3.39), so if there were a subordinate solution of Lu = xu, $u_{(l)}(r,x)$ would be subordinate.

Similarly $u_{(L)}(r,x) = u_2(r,x)+Lu_1(r,x)$ would be subordinate; however, this cannot be the case since $u_{(l)}(r,x)$ and $u_{(L)}(r,x)$ are linearly independent. Hence in the case we have considered with $|l|, |L| < \infty$, $l \neq L$ no subordinate solution of Lu = xu can exist.

Now suppose that $l = \infty$, $|L| < \infty$.

Using the method of Theorem 3.14 together with Remark 3.17 we see that there exists a boundary condition α_2 and sequences $\{M_p\}$ and $\{N_q\}$ in $\mathbb{R}^+$ such that $M_p, N_q \to \infty$ as $p, q \to \infty$.

$$\frac{\| u_{(l)}(r,x,\alpha_2) \|_{M_p}}{\| u_1(r,x,\alpha_2) \|_{M_p}} \to 0$$

as $p \to \infty$ and

$$\frac{\| u_{(l)}(r,x,\alpha_2) \|_{N_q}}{\| u_{(k)}(r,x,\alpha_2) \|_{N_q}} \to 0$$

as $q \to \infty$ for all $k \neq -\cot(\alpha_1 - \alpha_2)$ where α_1 is the original boundary condition, and

$$u_{(L)}(r,x,\alpha_2) = u_2(r,x,\alpha_2) - \cot(\alpha_1 - \alpha_2)\, u_1(r,x,\alpha_2)$$

Hence, if a subordinate solution were to exist, $u_L(r,x,\alpha_2)$ would be subordinate by the arguments above; and, as in the proof of Theorem 3.14, this implies that $u_1(r,x) = u_1(r,x,\alpha_1)$ would be subordinate.

However, this is not possible, since $u_{(L)}(r,x)$ would also be subordinate, and $u_1(r,x)$ and $u_{(L)}(r,x)$ are linearly independent. Hence in this case also, no subordinate solution of Lu = xu can exist.

It follows from the two cases we have considered that if m(z) does not converge to a limit as $y \downarrow 0$, then no subordinate solution of Lu = xu exists at the real point x.

The proofs of (i) and (ii) are now complete, and so the theorem is proved.

Theorems 3.13, 3.14 and 3.18 together form a complete set of necessary and sufficient conditions for the existence of a subordinate solution of Lu = xu at the real point x in terms of the behaviour of the function m(z) as z approaches x along the normal to the real axis at x. For convenience we also express these three existence tneorems as a single result:

3.19 <u>Theorem</u>: A subordinate solution of Lu = xu exists at the real point x if and only if as $y \downarrow 0$ either m(z) converges to a finite real limit, in which case $u_m(r,x)$ is subordinate, or $m(z) \to \infty$, in which case $u_1(r,x)$ is subordinate.

A discussion of some consequences of this theorem is contained in [P5].

We recall that $u_1(r,x)$ satisfies the boundary condition at $r = 0$.

As we saw in the proof of Theorem 3.14, if $m+(x,\alpha_1) = \infty$ for some boundary condition α_1, then for any distinct boundary condition α_2, $m+(x,\alpha_2) = -\cot(\alpha_1 - \alpha_2)$ which is finite and real. This means that the existence theorem may also be expressed in the following alternative form:

3.20 <u>Theorem</u>: A subordinate solution of $Lu = xu$ exists at the real point x if and only if there exists a boundary condition α such that $m(z,\alpha)$ converges to a finite real limit as $y \downarrow 0$, in which case

$$u_m(r,x,\alpha) = u_2(r,x,\alpha) + m+(x,\alpha)u_1(r,x,\alpha)$$

is subordinate.

It is now straightforward to derive our ultimate set of minimal supports of the decomposed parts of the spectral measure μ:

3.21 <u>Theorem</u>: Minimal supports $\mathcal{M}'''$, $\mathcal{M}'''_{a.c.}$, $\mathcal{M}'''_{s.}$, $\mathcal{M}'''_{s.c.}$, $\mathcal{M}'''_{d.}$ of μ, $\mu_{a.c.}$, $\mu_{s.}$, $\mu_{s.c.}$ and $\mu_{d.}$ are as follows:

(i) $\mathcal{M}''' = \mathbb{R} \setminus \{x \in \mathbb{R}$: a subordinate solution of $Lu = xu$ exists but does not satisfy the boundary condition at $0\}$

(ii) $\mathcal{M}'''_{a.c.} = \{x \in \mathbb{R}$: no subordinate solution of $Lu = xu$ exists$\}$

(iii) $\mathcal{M}'''_{s.} = \{x \in \mathbb{R}$: a subordinate solution of $Lu = xu$ exists which satisfies the boundary condition at $0\}$

(iv) $\mathcal{M}'''_{s.c.} = \{x \in \mathbb{R}$: a subordinate solution of $Lu = xu$ exists which satisfies the boundary condition at 0 but is not in $L_2[0,\infty)\}$

(v) $\mathcal{M}'''_{d.} = \{x \in \mathbb{R}$: a subordinate solution of $Lu = xu$ exists which satisfies the boundary condition at 0 and is in $L_2[0,\infty)\}$

<u>Proof</u>:

We need only prove (ii) and (iii) since (v) is well-known, (iii) and (v) imply (iv), and (iii) and (ii) imply (i).

Proof of (ii):

This is immediate by Theorems 2.22 and 3.20.

Proof of (iii):

To prove that $\mathcal{M}_{s.}'''$ is a minimal support of $\mu_{s.}$ we prove that $\mathcal{M}_{s.}''' \sim \mathcal{M}_{s.}'$ where $\sim$ is the equivalence relation of Lemma 2.20, and $\mathcal{M}_{s.}'$ is as in Theorem 2.17.

Since scalar multiplies of $u_1(r,x)$ are the only solutions of $Lu = xu$ which satisfy the boundary condition (2.3.9) at $r = 0$, we see by Theorem 3.19 that a subordinate solution of $Lu = xu$ exists which satisfies the boundary condition at 0 if and only if $m(z) \to \infty$ as $y \downarrow 0$.

Let S denote $\{x \in \mathbb{R} : m(z) \to \infty,\ \operatorname{Im} m(z) \not\to \infty \text{ as } y \downarrow 0\}$.

Clearly $\mathcal{M}_{s.}''' = \mathcal{M}_{s.}' \cup S$, so to prove $\mathcal{M}_{s.}''' \sim \mathcal{M}_{s.}'$ it is sufficient to show that $\mu(S) = \kappa(S) = 0$.

Now $\kappa(S) = 0$ by Theorem 2.12(i), and μ is absolutely continuous with respect to κ on S by Lemma 2.1 and Proposition 2.14. Hence $\mu(S) = 0$, so that $\mathcal{M}_{s.}''' \sim \mathcal{M}_{s.}'$ and (iii) is proved.

The proof of the theorem is now complete.

We survey the implications of these results in the following section.

§4. Ramifications

In the context of ordinary differential equations of the Sturm-Liouville type, subordinate solutions may be regarded as a generalisation of square integrable solutions. Indeed, where L is regular at 0 and in the limit point case at infinity, subordinate solutions bear precisely the same relation to the minimal supports of the singular spectral measure as do the square integrable solutions to the pure point measure, as has been shown in Theorem 3.21.

The invariance of minimal supports of the absolutely continuous measure and the contrasting orthogonality of minimal supports of the singular measure under a change of boundary condition at r = 0 which was proved in Theorem 2.21 was not an immediately obvious corollary to Theorem 2.17. However, the necessity of this behaviour is apparent at once from Theorem 3.21; for clearly the presence or absence of a subordinate solution of Lu = xu at a given point x is independent of the boundary condition at r =0, whereas it is impossible for more than one distinct boundary condition of the type (2.3.9) to be satisfied by a subordinate solution at x.

Since it is customary when considering possible energy levels of a system to use the spectrum rather than minimal supports of the spectral measure, we comment briefly on the relationship between the minimal supports of Theorem 3.21 and the relevant spectra. As we noted in Chapter II §2, the correlation between minimal supports and the relevant parts of the spectrum is not exact; it may even happen, as in the case of dense pure point spectrum, that every minimal support of the spectral measure differs from the spectrum by a set having positive Lebesgue measure. All our earlier observations in respect of Theorem 2.17 may be applied, with modifications, to Theorem 3.21; thus it follows from Example 2.10, using Proposition 2.14 and Theorem 3.14, that an operator H exists which has no singular spectrum on a subinterval (a,b) of $\mathbb{R}$, although an uncountable subset X of (a,b) exists such that for every x in X, Lu = xu has a subordinate solution satisfying the boundary condition at 0. Likewise, using Proosition 2.14 and Theorem 3.19, we deduce from Example 2.11 that a real point x may exist at which there is no absolutely continuous spectrum of the Hamiltonian in a neighbourhood of x.

The most striking finding of Theorem 3.21 is undoubtedly that relating to the support of the singular continuous measure μ_{s}. We have already commented on the importance of including singular continuous spectrum in any complete treatment of spectral theory of Schrödinger operators, and note that it is especially in those cases where singular continuous or dense point spectrum occurs in conjunction with other types of spectrum that the discrimination afforded by Theorem 3.21 may be most useful. It is worth noting that Theorem 3.21 also gives a new criterion for locating the absolutely continuous spectrum, viz. that $\mu_{a.c.}$ is concentrated on those real x for which no subordinate solution of Lu = xu exists. Where the spectrum is known to consist solely of absolutely continuous and isolated pure point parts the absolutely continuous spectrum may be identified with the closure of the set of all real x for which no $L_2(0,\infty)$ solution of Lu = xu exists. In such cases Theorem 3.21(ii) is unlikely to be of further assistance. However, where there is a possibility of other types of spectrum, the characterisation of Theorem 3.21(ii) enables the absolutely continuous spectrum to be distinguished from the other constituents of the essential spectrum, at least in theory.

Theorem 3.21 applies to all self-adjoint operators of the Sturm-Liouville type which are regular at 0 and limit point at infinity; however the decomposition of the spectrum considered here is of particular relevance to Schrödinger operators.

To conclude, we have introduced the concept of subordinate solutions, and shown that supports of each part of the spectral measure may be characterised in terms of this concept where the differential operator $L = \frac{-d^2}{dr^2} + V(r)$ is regular at 0 and limit point at infinity. We shall see in the next chapter that the description of the spectrum in terms of subordinate solutions is possible under more general conditions, and that subordinate solutions are of fundamental importance in certain eigenfunction expansions when the spectrum of H is simple and L is limit point at 0.

CHAPTER IV

SIMPLIFICATION OF THE WEYL-KODAIRA THEOREM

§1. The Weyl-Kodaira Theorem

In this chapter we no longer suppose that $L = \frac{-d^2}{dr^2} + V(r)$ is regular at $r = 0$ and in the limit point case at infinity, but instead suppose that $V(r)$ is in $L_1[a,\infty)$ for each $a > 0$ and that the behaviour of $V(r)$ in a neighbourhood of 0 is such that for some $a > 0$, the spectrum of the Schrödinger operator H_a arising from L acting on $(0,a]$ is singular. Of course, these new conditions on $V(r)$ imply that L is in the limit point case at ininity ([N] §23, Satz 3), and if L is regular at 0 the spectrum of H_a consists of isolated eigenvalues ([N] §24, Satz 5), so there is a wide class of potentials satisfying both our former and our present assumptions. However we can no longer assume that L is regular at 0; indeed, L may be in the limit point case at 0, or, even if L is limit circle at 0, 0 may be a singular endpoint ([HP] Lemma 10.4.15). In fact, unless the spectrum of H_a consists entirely of isolated eigenvalues and their accumulation points, L must be in the limit point case at 0 ([N] §19, Bemerkung 2).

In general, therefore, the theory we have described and used in Chapters II and III no longer applies; however, if the interval $(0,\infty)$ is decomposed into two parts $(0,a]$ and $[a,\infty)$, the earlier theory may be applied to each of the intervals $(0,a]$ and $[a,\infty)$ (the precise location of the point a in $(0,\infty)$ is immaterial), and from this a general theory has been constructed which applies to the entire interval $(0,\infty)$. The principal architect of this generalisation was K.Kodaira who in 1949, at the invitation of H.Weyl, undertook the task of unifying and generalising previous related work by Weyl, Stone and Titchmarsh ([KO],[KO2]). We shall now state the Weyl-Kodaira Theorem in the particular form that we require, while noting that in its most general form it applies to arbitrary subintervals of $\mathbb{R}$ and to suitable differential operators of any order ([DS] Ch.XIII, Thm.13).

We choose a = 1 for simplicity, and first describe some notation. Let $y_1(r,z)$ and $y_2(r,z)$ be solutions of $Lu = zu$ satisfying $y_1(1,z) = y_2'(1,z) = 1$ and $y'(1,z) = y_2(1,z) = 0$. If L is in the limit point case at both 0 and infinity let $m_0(z)$ and $m_\infty(z)$ be defined by the requirements that, if $\operatorname{Im} z \neq 0$, $y_1(r,z) + m_0(z)\, y_2(r,z)$ be in $L_2(0,1]$ and $y_1(r,z) + m_\infty(z)\, y_2(r,z)$ be in $L_2[1,\infty)$ respectively. If L is in the limit circle case at 0, and 0 is a regular endpoint, then $m_0(z)$ is defined by the condition that the solution $y(r,z) = y_1(r,z) + m_0(z)\, y_2(r,z)$ satisfies the boundary condition

$$\cos\alpha\, y(0,z) + \sin\alpha\, y'(0,z) = 0 \tag{4.1.1}$$

for some α in $[0, 2\pi)$. If L is limit circle at 0, but 0 is a singular endpoint, then if z_0 in $\mathbb{C} \setminus \mathbb{R}$ is fixed, $m_0(z)$ is defined by the condition that the solution $y(r,z)$ above satisfy

$$\lim_{r \downarrow 0} W(y(r,z), y_1(r,z_0) + \hat{m}_0(z_0)\, y_2(r,z_0)) = 0 \tag{4.1.2}$$

where $\hat{m}_0(z_0)$ is some point on the limit circle ([CL], Ch.9, Thm.4.1).

Let $\{M_{ij}(z) : i,j = 1,2\}$ be defined for $\operatorname{Im} z \neq 0$ by the relations

$$M_{11}(z) = (m_0(z) - m_\infty(z))^{-1}$$

$$M_{12}(z) = M_{21}(z) = \tfrac{1}{2}(m_0(z) + m_\infty(z))(m_0(z) - m_\infty(z))^{-1}$$

$$M_{22}(z) = m_0(z)\, m_\infty(z)\, (m_0(z) - m_\infty(z))^{-1} \tag{4.1.3}$$

4.1 <u>Weyl-Kodaira Theorem</u>:

Let $V(r)$ be integrable on every compact subinterval of $(0,\infty)$, and let $L = -\frac{d^2}{dr^2} + V(r)$ be limit point at both 0 and ∞. Let H be the self-adjoint operator arising from L. Then there exists a positive 2 x 2 Hermitian matrix with elements ρ_{ij} which satisfy

$$\rho_{ij}(\mu) - \rho_{ij}(\nu) = \lim_{\delta \downarrow 0} \lim_{y \downarrow 0} \frac{1}{\pi} \int_{\nu+\delta}^{\mu-\delta} \operatorname{Im} M_{ij}(x + iy)\, dx \tag{4.1.4}$$

for i,j = 1,2, such that if f(r) is in $L_2(0,\infty)$, the limit

$$(Tf)_i(\lambda) = \underset{\substack{\sigma\to 0\\ \omega\to\infty}}{\text{l.i.m.}} \int_\sigma^\omega f(r)\, y_i(r,\lambda)\, dr$$

converges in $L_2^{\rho_{ij}}(-\infty,\infty)$, where $L_2^{\rho_{ij}}(-\infty,\infty)$ is the Hilbert space of vectors $G(\lambda) = (g_1(\lambda), g_2(\lambda))$ with inner product

$$\langle G(\lambda), H(\lambda)\rangle = \sum_{i,j=1,2} \int_{-\infty}^{\infty} \overline{g_i(\lambda)}\, h_j(\lambda)\, d\rho_{ij}(\lambda) \qquad (4.1.5)$$

Moreover, the mapping T defines an isometric Hilbert space isomorphism from $L_2(0,\infty)$ onto $L_2^{\rho_{ij}}(-\infty,\infty)$, so that if f(r) is in $L_2(0,\infty)$,

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} \sum_{i,j=1,2} u_1(r,\lambda)(Tf)_j(\lambda)\, d\rho_{ij}(\lambda)$$

and if θ is a Borel measurable function on $\mathbb{R}$ such that $\theta(H)f(r)$ is in $L_2(0,\infty)$ then

$$(T\,\theta(H)f)_i(\lambda) = \theta(\lambda)(Tf)_i(\lambda)$$

for i = 1,2.

For ease of reference, we present the results of this theorem in a slightly modified form. Let $\phi_i(\lambda)$ denote $(Tf)_i(\lambda)$ for i = 1,2. Then if f(r) is in $L_2(0,\infty)$,

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} \sum_{i,j=1,2} y_i(r,\lambda)\, \phi_j(\lambda)\, d\rho_{ij}(\lambda) \qquad (4.1.6)$$

and

$$\int_0^\infty |f(r)|^2 dr = \sum_{i,j=1,2} \int_{-\infty}^{\infty} \overline{\phi_i(\lambda)}\, \phi_j(\lambda)\, d\rho_{ij}(\lambda)$$

where

$$\phi_i(\lambda) = \underset{\substack{\sigma\to 0\\ \omega\to\infty}}{\text{l.i.m.}} \int_\sigma^\omega y_i(r,\lambda)\, f(r)\, dr \qquad (4.1.7)$$

for i = 1,2. The convergence of the integral (4.1.6) is in $L_2(0,\infty)$, whereas the convergence in (4.1.7) is in $L_2^{\rho_{ij}}(-\infty,\infty)$; that is

$$\lim_{\substack{\sigma\to 0\\ \omega\to\infty}} \sum_{i,j=1,2} \int_{-\infty}^{\infty} \overline{\left(\phi_i(\lambda) - \int_\sigma^\omega y_i(r,\lambda) f(r)\, dr\right)} \left(\phi_j(\lambda) - \int_\sigma^\omega y_j(r,\lambda) f(r)\, dr\right) d\rho_{ij}(\lambda)$$

$$= 0 \qquad (4.1.8)$$

If θ is a Borel measurable function on $\mathbb{R}$ such that $(\theta(H)f)(r)$ is in $L_2(0,\infty)$, then

$$(\theta(H)f)(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} \sum_{i,j=1,2} \theta(\lambda)\, y_i(r,\lambda)\, \phi_j(\lambda)\, d\rho_{ij}(\lambda) \qquad (4.1.9)$$

where the integral converges in $L_2(0,\infty)$. In particular, if $-\infty < \nu < \mu < \infty$,

$$((E_\mu - E_\nu)f)(r) = \int_\nu^\mu \sum_{i,j=1,2} y_i(r,\lambda)\, \phi_j(\lambda)\, d\rho_{ij}(\lambda) \qquad (4.1.10)$$

For further details, see [KO], [DS] Ch.XIII, §5. We remark that the Weyl-Kodaira Theorem is also valid where L is in the limit circle case at 0 or ∞ or both; in these cases the spectral matrix is also unique up to an additive constant once suitable boundary conditions are applied at one or both endpoints, as appropriate.

It seems not unlikely that, where L is regular at 0, the relationships of Theorem 4.1 are but an alternative expression of (2.4.1) to (2.4.7). Before considering more general cases, we tested this conjecture for the case $V(r) = 0$; this trivial potential satisfies the requirements that $V(r)$ be in $L_1[1,\infty)$ and that the spectrum of the operator arising from $L = \frac{-d^2}{dr^2}$ on $(0,1]$ be singular (it is, of course, discrete). Moreover, $m(z)$ and $\rho(\lambda)$ as defined in Chapter II, and $m_0(z)$, $m_\infty(z)$, $\{\rho_{ij}(\lambda) : i,j = 1,2\}$ may, without undue difficulty, be calculated exactly. It was found that the Weyl-Kodaira theory described above did indeed reduce to the simpler theory of Chapter II, §4, for all boundary conditions α at $r = 0$ (see (2.3.9)).

In this chapter we shall show that there are many other situations where the Weyl-Kodaira expansion reduces to a simpler form; and that, where L is regular at 0, this simplification reduces to the expansion (2.4.4), at least for those $f(r)$ in the absolutely continuous subspace of H. Thus the process of simplifying Theorem 4.1 may also be regarded as one of extending the theory we described in Chapter II.

Let us first establish some properties of the spectral matrix (ρ_{ij}).

§2. Properties of the spectral matrix

The elements $\{\rho_{ij}(\lambda) : i,j = 1,2\}$ of the spectral matrix are functions of bounded variation on every finite λ-interval, continuous on the right, and unique up to an additive constant ([CL] Ch.9, Thm.5.1). For convenience we shall suppose that $\rho_{ij}(0) = 0$ for each i,j = 1,2. Moreover, the matrix itself is positive semi-definite ([KO] Thm.1.13), and the elements $\rho_{11}(\lambda)$, $\rho_{22}(\lambda)$ are non-decreasing functions of λ as may be ascertained by inspection of (4.1.4) in conjunction with the formulae

$$\operatorname{Im} M_{11}(z) = \frac{m_{\infty I} - m_{0I}}{(m_{0R} - m_{\infty R})^2 + (m_{0I} - m_{\infty I})^2} \tag{4.2.1}$$

$$\operatorname{Im} M_{12}(z) = \operatorname{Im} M_{21}(z) = \frac{m_{0R}\, m_{\infty I} - m_{0I}\, m_{\infty R}}{(m_{0R} - m_{\infty R})^2 + (m_{0I} - m_{\infty I})^2} \tag{4.2.2}$$

$$\operatorname{Im} M_{22}(z) = \frac{m_{\infty I}(m_{0I}^2 + m_{0R}^2) - m_{0I}(m_{\infty I}^2 + m_{\infty R}^2)}{(m_{0R} - m_{\infty R})^2 + (m_{0I} - m_{\infty I})^2} \tag{4.2.3}$$

which are derived from (4.1.3). Note that, for conciseness, we have denoted $\operatorname{Im} m_0(z), \operatorname{Im} m_\infty(z), \operatorname{Re} m_0(z), \operatorname{Re} m_\infty(z)$ by m_{0I}, $m_{\infty I}$, m_{0R} and $m_{\infty R}$ respectively, and that, for $\operatorname{Im} z > 0$, $m_{0I} \leqslant 0$, $m_{\infty I} \geqslant 0$ ([CL], Ch.9, §5).

The spectrum of an operator H whose spectral matrix is (ρ_{ij}) is the set of points of increase of (ρ_{ij}); that is, it is the complement of the set of points x for which $\rho_{ij}(\lambda)$ is constant in some neighbourhood of x for each i,j = 1,2 ([DS], Ch.XIII §5, Cor.15).

The spectral matrix (ρ_{ij}) generates a positive matrix measure (see [DS] Ch.XIII, §5, Def.6), whose elements $\{\mu_{ij} : i,j = 1,2\}$ are obtained by extending the measures $\{\mu_{ij}^* : i,j = 1,2\}$, which are defined on the algebra of half open subintervals (a,b] of $\mathbb{R}$ by

$$\mu_{ij}^{*}(a,b] = \rho_{ij}(b) - \rho_{ij}(a)$$

(see (2.2.2)). It should be noted that μ_{12} and μ_{21} are signed measures (see [H] Ch.6, §28); that this is so follows from the fact that $\rho_{12}(\lambda)$ and $\rho_{21}(\lambda)$ are functions of bounded variation, and hence each is the sum of an increasing and a decreasing function which are unique up to additive constants.

The inverse of (4.1.4) is

$$\operatorname{Im} M_{ij}(z) = \int_{-\infty}^{\infty} \frac{y}{(\lambda-x)^2 + y^2}\, d\rho_{ij}(\lambda) \qquad (4.2.4)$$

for each i,j = 1,2 ([KO],Thm. 1.13).

Now $m_0(z)$ and $m_\infty(z)$ are analytic in $\mathbb{C} \setminus \mathbb{R}$ and so each of $\{M_{ij}(z) : i,j = 1,2\}$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ by (4.1.3). Moreover, by (4.2.1) and (4.2.3), $M_{11}(z)$ and $M_{22}(z)$ have positive imaginary part in the upper half-plane, and hence behave restrictedly at all points of $\mathbb{R}$. Therefore by (C'), Chapter II §3, $M_{11}(z)$ and $M_{22}(z)$ have a finite non-tangential limit Lebesgue almost everywhere on $\mathbb{R}$, and, in particular, a finite normal limit Lebesgue almost everywhere on $\mathbb{R}$.

$\rho_{11}(\lambda)$ and $\rho_{22}(\lambda)$ are therefore related to $M_{11}(z)$ and $M_{22}(z)$ respectively and to μ_{11} and μ_{22} respectively in precisely the same way that the spectral function $\rho(\lambda)$ was related to the function m(z) and to the spectral measure μ respectively in Chapter II. Hence all the theory developed in Chapter II from the basic relationships and properties of $\rho(\lambda)$, m(z) and μ applies equally to $\rho_{jj}(\lambda)$, $M_{jj}(z)$ and μ_{jj} for j = 1,2.

We now derive a few simple properties which reflect the positive semi-definite character of the spectral matrix.

4.2 <u>Lemma</u>: For all z in $\mathbb{C} \setminus \mathbb{R}$,

$$(\operatorname{Im} M_{12}(z))^2 = (\operatorname{Im} M_{21}(z))^2 \leq \operatorname{Im} M_{11}(z) \operatorname{Im} M_{22}(z)$$

and, if $\mu, \nu \in \mathbb{R}$,

$$\int_\nu^\mu |\operatorname{Im} M_{12}(z)|\,dx = \int_\nu^\mu |\operatorname{Im} M_{21}(z)|\,dx$$
$$\leq \left(\int_\nu^\mu \operatorname{Im} M_{11}(z)\,dx \int_\nu^\mu \operatorname{Im} M_{22}(z)\,dx\right)^{\frac{1}{2}}$$

Proof:

Using (4.2.1), (4.2.2) and (4.2.3), we obtain

$$(\operatorname{Im} M_{12}(z))^2 - \operatorname{Im} M_{11}(z) \operatorname{Im} M_{22}(z) = m_{0I}\, m_{\infty I}$$

Since m_{0I} and $m_{\infty I}$ have opposite signs in $\mathbb{C} \setminus \mathbb{R}$ ([CL] Ch.9, §5) and $M_{12}(z) = M_{21}(z)$, the first part of the lemma is proved. The second part of the lemma follows from the first by the Cauchy-Schwarz inequality; the proof is now complete.

4.3 Lemma: Every measurable subset of $\mathbb{R}$ which has μ_{11} - or μ_{22}- measure zero has μ_{12}- and μ_{21}- measure zero.

Proof:

Since $\mu_{12} = \mu_{21}$ we need only prove the result for μ_{12}.

Let S be a measurable subset of $\mathbb{R}$.

Now μ_{11} and μ_{22} are positive measures in the usual sense whereas μ_{12} is a signed measure or charge. Thus

$$\mu_{12}(S) = \mu_{12}^+(S) - \mu_{12}^-(S)$$

where μ_{12}^+ and μ_{12}^- are the upper and lower variations of μ_{12} respectively ([H] §28). Moreover, each of $\{\rho_{ij}(\lambda) : i,j = 1,2\}$ is a function of bounded variation on each finite λ-interval, so the μ_{11}-, μ_{22}-, μ_{12}^+-, μ_{12}^-- measures of bounded subsets of $\mathbb{R}$ are finite. It follows that $\mu_{11}, \mu_{22}, \mu_{12}^+$ and μ_{12}^- are regular measures ([R], Thm.2.18). Hence

$$\mu_{12}^{\pm}(S) = \inf\{\mu_{12}^{\pm}(U) : U \text{ is open and } S \subseteq U \subseteq \mathbb{R}\} \qquad (4.2.5)$$

and for j = 1,2,

$$\mu_{jj}(S) = \inf\{\mu_{jj}(U) : U \text{ is open and } S \subseteq U \subseteq \mathbb{R}\} \qquad (4.2.6)$$

Let us now suppose that S is also a bounded subset of $\mathbb{R}$. Then there exists a bounded open cover U of S and an M in $\mathbb{R}^+$ such that $\mu_{11}(U), \mu_{22}(U) \leq M$.

Let $\varepsilon > 0$ be given. Then, from (4.2.5) and (4.2.6), there exists a bounded open cover U_s of S which is contained in U, and for which

$$0 \leq \mu_{12}^{\pm}(U_s) \leq \mu_{12}^{\pm}(S) + \frac{\varepsilon}{3} \tag{4.2.7}$$

and, for j = 1,2,

$$\mu_{jj}(U_s) \leq \mu_{jj}(S) + \frac{\varepsilon^2}{9M} \tag{4.2.8}$$

Evidently, also,

$$\mu_{jj}(U_s) \leq \mu_{jj}(U) \leq M \tag{4.2.9}$$

for j = 1,2.

Since U_s is open, we may write $U_s = \bigcup_i U_{s,i}$ where each $U_{s,i}$ is an open subinterval of $\mathbb{R}$ such that $U_{s,i} \cap U_{s,j} = \emptyset$ if $i \neq j$. There is no loss of generality if we suppose the endpoints of each $U_{s,i}$ to be points of continuity of $\rho_{ij}(\lambda)$ for each i,j = 1,2, since the points of discontinuity are, at most, countably infinite. We have from (4.1.4) and Lemma 4.2

$$\begin{aligned}
|\mu_{12}(U_s)| &= \left|\sum_i \mu_{12}(U_{s,i})\right| \leq \sum_i |\mu_{12}(U_{s,i})| \\
&= \sum_i \left| \lim_{y \downarrow 0} \frac{1}{\pi} \int_{U_{s,i}} \operatorname{Im} M_{12}(z)\, dx \right| \\
&\leq \frac{1}{\pi} \sum_i \lim_{y \downarrow 0} \int_{U_{s,i}} |\operatorname{Im} M_{12}(z)|\, dx \\
&\leq \sum_i \left(\lim_{y \downarrow 0} \frac{1}{\pi} \int_{U_{s,i}} \operatorname{Im} M_{11}(z)\, dx \right)^{\frac{1}{2}} \left(\lim_{y \downarrow 0} \frac{1}{\pi} \int_{U_{s,i}} \operatorname{Im} M_{22}(z)\, dx \right)^{\frac{1}{2}} \\
&= \sum_i \left(\mu_{11}(U_{s,i})\, \mu_{22}(U_{s,i}) \right)^{\frac{1}{2}} \\
&\leq \left(\sum_i \mu_{11}(U_{s,i}) \right)^{\frac{1}{2}} \left(\sum_i \mu_{22}(U_{s,i}) \right)^{\frac{1}{2}} \\
&= (\mu_{11}(U_s))^{\frac{1}{2}} (\mu_{22}(U_s))^{\frac{1}{2}}
\end{aligned} \tag{4.2.10}$$

If $\mu_{11}(S) = 0$, then $\mu_{11}(U_s) \leq \frac{\varepsilon^2}{9M}$ by (4.2.8); it therefore follows from (4.2.10), since $\mu_{22}(U_s) < M$ by (4.2.9), that $|\mu_{12}(U_s)| < \frac{\varepsilon}{3}$. Consequently by (4.2.7),

$$\begin{aligned}|\mu_{12}(S)| &\leq |\mu_{12}(S) - \mu_{12}(U_s)| + |\mu_{12}(U_s)| \\ &\leq |\mu_{12}^+(S) - \mu_{12}^+(U_s)| + |\mu_{12}^-(S) - \mu_{12}^-(U_s)| + \frac{\varepsilon}{3} \\ &\leq \varepsilon\end{aligned}$$

The arbitrariness of ε implies that $\mu_{12}(S) = 0$.

Similarly we may show that if $\mu_{22}(S) = 0$ then $\mu_{12}(S) = 0$.

The extension to the case where S is an unbounded subset of $\mathbb{R}$ is immediate, since S may be decomposed into a countable union of disjoint bounded sets.

The proof of the lemma is now complete.

The application of Lemmas 4.2 and 4.3 is quite general; we shall now prove some results which are dependent on the potential V(r) satisfying specific conditions at 0 and ∞.

§3. The nature of the spectrum

We now suppose that V(r) is in $L_1[1,\infty)$ and is integrable on compact subsets of $(0,\infty)$, and that the spectrum of H_1 is singular. Note that these conditions on V(r) are equivalent to the condition that V(r) be in $L_1[a,\infty)$ for each a > 0. Also, the spectrum of H_1 is singular if and only if the same is true for each H_a, where H_a is a self adjoint operator arising from L in $L_2(0,a]$ with boundary conditions $y(a,\lambda) = 0$, $y'(a,\lambda) = 1$. To see this in the case where L is limit point at 0, note that, adapting Theorem 3.21 to the interval (0,a], H_a has singular spectrum if and only if $\kappa(\{\lambda$: no solution of $Lu = \lambda u$ on (0,a] exists which is subordinate at 0$\}) = 0$.

The truth of our assertion is now immediate since the existence or otherwise of a solution of $Lu = \lambda u$ on (0,a] which is subordinate at 0 is independent of the precise location of the point a. If L is limit circle at 0, every self-adjoint extension on (0,a] has discrete spectrum, irrespective of the position of a ([CL] Ch.9, Thm.4.1).

Let us denote the self-adjoint operators arising from L in $(0,\infty)$ and $[1,\infty)$ by H and H_∞ respectively. Note that L is regular at the decomposition point 1, and all functions f(r) in the domains of H_1 and H_∞ satisfy the condition f(1) = 0. If L is limit circle at 0, a boundary condition is also needed at r = 0 to render H_1 and H self-adjoint (as (4.1.1) or (4.1.2)), whereas if L is limit point at 0, this is not required.

To investigate the spectrum of H, we first need some information about $m_\infty(z)$. The given conditions ensure that the spectrum H_∞ is absolutely continuous on $(0,\infty)$, and consists of isolated eigenvalues on $(-\infty,0)$ ([HI] Thm. 10.3.7). The point 0 can only be an eigenvalue of H if there is an $L_2[1,\infty)$ solution of Lu = 0. There are many potentials V(r) in $L_1[1,\infty)$ for which there is no such solution (see eg[WE1],Thm.10.30, [LS] Ch.IV, proof of Lemma 3.2). However, some quite simple potentials in $L_1[1,\infty)$ do have a solution of Lu = 0 in $L_2[1,\infty)$; for example, if $V(r) = \frac{2}{r^2}$, $\frac{1}{r}$ is a solution of $\frac{-d^2u}{dr^2} + V(r)u = 0$. We shall therefore take account of this possibility in what follows.

Now $m_\infty(z)$ may be analytically continued across the axis at all points of the resolvent set ([CE], §5, Thm.(i)). Hence, defining $m_\infty(x)$ to be $m_\infty^+(x)$ whenever the latter exists finitely and is real, $m_\infty(z)$ is bounded on $[\nu,\mu]\times[0,Y]$ for each compact interval $[\nu,\mu]$ of $\mathbb{R}$ which is contained in the resolvent set and each Y > 0; this property will be used in a number of the following proofs.

On $(0,\infty)$, an explicit expression for $m_\infty^+(x)$ may be obtained; for full details of the method, consult [T2], Chapter V, §4.2. We shall summarise the relevant results.

Applying the "variation of constants" formula ([CL], Ch.3, Thm.6.1) to the Schrödinger equation for r > 1,

$$y_i(r,z) = \tau_i(z)\sin(\sqrt{z}(r-1)) + \sigma_i(z)\cos(\sqrt{z}(r-1)) + o(1) \quad (4.3.1)$$

for each i = 1,2, where

$$\tau_1(z) = \frac{1}{\sqrt{z}}\int_1^\infty \cos(\sqrt{z}(s-1))\,V(s)\,y_1(s,z)\,ds$$

$$\tau_2(z) = \frac{1}{\sqrt{z}} + \frac{1}{\sqrt{z}}\int_1^\infty \cos(\sqrt{z}(s-1))V(s)\,y_2(s,z)\,ds$$

$$\sigma_1(z) = 1 - \frac{1}{\sqrt{z}}\int_1^\infty \sin(\sqrt{z}(s-1))V(s)\,y_1(s,z)\,ds$$

$$\sigma_2(z) = -\frac{1}{\sqrt{z}}\int_1^\infty \sin(\sqrt{z}(s-1))\,V(s)\,y_2(s,z)\,ds \quad (4.3.2)$$

After further refinements to the estimates (4.3.1) for $y_1(r,z)$ and $y_2(r,z)$ in the case x,y > 0, the formula $m_\infty(z) = -\lim_{R\to\infty} \frac{y_1(R,z)}{y_2(R,z)}$ (cf.([CL] Ch.9, §2, (2.13)) yields

$$m_\infty^+(x) = -\frac{\tau_1(x) - i\,\sigma_1(x)}{\tau_2(x) - i\,\sigma_2(x)} \quad (4.3.3)$$

for each x in $(0,\infty)$. For these x, $\operatorname{Im} m_\infty^+(x) = \dfrac{\tau_2(x)\,\sigma_1(x) - \tau_1(x)\,\sigma_2(x)}{\tau_2^2(x) + \sigma_2^2(x)}$.

Since $W(y_1(r,z),y_2(r,z)) = 1$ for all r > 1, it follows from (4.3.1) that

$$\sqrt{z}\,(\sigma_1(z)\,\tau_2(z) - \sigma_2(z)\,\tau_1(z)) = 1$$

for each z in $\mathbb{C}$. This implies that $\sigma_2(x)$ and $\tau_2(x)$ cannot vanish simultaneously, and that

$$\operatorname{Im} m_\infty^+(x) = \frac{1}{\sqrt{x}\,(\tau_2^2(x) + \sigma_2^2(x))} \quad (4.3.4)$$

for each x in $(0,\infty)$. Since $\sigma_2(x)$ and $\tau_2(x)$ are continuous, it follows that $\operatorname{Im} m_\infty^+(x)$ is bounded above and away from zero on each compact subinterval of $(0,\infty)$; we shall use this property in several of the following proofs.

It should be noted that the differences in sign between (4.3.3),(4.3.4) and the analogous results in [T2], loc.cit., are due to the difference in the boundary conditions.

We now prove a proposition that gives some insight into the nature and location of the negative spectrum of H. The set E of eigenvalues of H which occurs in this and later results cannot be ignored (although in particular cases it may be empty), because the elements of E can be eigenvalues of H and hence may have positive spectral measure. However, E will not occur explicitly in the simplified expansion which is derived in Theorem 4.9.

4.4. Proposition: If $V(r)$ is in $L_1[1,\infty)$ and the spectrum of H_1 is singular then

(i) the spectrum of H is singular on $(-\infty, 0]$.

(ii) if E is the set of eigenvalues of H_∞, and if $x \in E$, then x is an eigenvalue of H if and only if x is also an eigenvalue of H_1.

(iii) if E is as in (ii), then for $i,j = 1,2$, $\mu_{ij}(\{x \in (-\infty, 0] \setminus E$: it is not the case that $m_0(x)$, $m_\infty(x)$ exist and are equal $\}) = 0$, where $m_0(x)$, $m_\infty(x)$ are defined to be $m_0^+(x), m_\infty^+(x)$ respectively whenever the limits exist finitely and are real.

Proof:

Proof of (i):

Since the spectra of H_1 and H_∞ are singular on $(-\infty, 0]$, it follows from Corollary 2.7 and Lemma 2.13, applied to $m_0(z)$ and μ_0, and to $m_\infty(z)$ and μ_∞, that $\operatorname{Im} m_0^+(x)$ and $\operatorname{Im} m_\infty^+(x)$ are zero Lebesgue almost everywhere on $(-\infty, 0]$. Moreover, $m_0(z) \not\equiv m_\infty(z)$ on $\mathbb{C} \setminus \mathbb{R}$, so the set $\{x \in \mathbb{R}: m_0+(x) = m_\infty+(x)\}$ has Lebesgue measure zero by Theorem 2.12(iii). Hence the denominators of (4.2.1)-(4.2.3) converge to non-zero limits as $y \downarrow 0$ Lebesgue almost everywhere on $\mathbb{R}$ so applying Theorem 2.12(i) to $m_0(z)$ and $m_\infty(z)$ we conclude that $\operatorname{Im} M_{ij}(z) \to 0$ as $y \downarrow 0$ for Lebesgue almost all x on $(-\infty, 0]$, for each $i,j = 1,2$. It now follows from Lemma 2.13 and Corollary 2.7 applied to $M_{jj}(z)$ and μ_{jj} for $j = 1,2$, and from Lemma 4.3, that the spectrum of H is singular on $(-\infty, 0]$.

Proof of (ii):

We shall use the relationship

$$\mu(\{x\}) = \rho(x+0) - \rho(x-0) = -\lim_{y\downarrow 0} iy\, m(x+iy) \qquad (4.3.5)$$

which holds quite generally for a function m(z) which is analytic with positive imaginary part in the upper half-plane and the measure μ related to m(z) by (2.2.1) and (2.3.3). ([EK] Ch.2, §3).

x is therefore an eigenvalue of H_∞ if and only if

$$\lim_{y\downarrow 0} -iy\, m_\infty(x+iy) > 0 \qquad (4.3.6)$$

and x is an eigenvalue of H_1 if and only if

$$\lim_{y\downarrow 0} iy\, m_0(x+iy) > 0 \qquad (4.3.7)$$

since $m_0(z)$ has negative imaginary part in the upper half plane. Since x is an eigenvalue of H if and only if at least one of $\mu_{11}(x)$, $\mu_{22}(x)$ is non-zero ([DS] Ch.XIII §5, p.1360), x is an eigenvalue of H if and only if at least one of the limits

$$-\lim_{y\downarrow 0} iy\, M_{11}(x+iy) \quad , \quad -\lim_{y\downarrow 0} iy\, M_{22}(x+iy)$$

is strictly positive. From (4.1.3)

$$-\lim_{y\downarrow 0} iy\, M_{11}(x+iy) = -\lim_{y\downarrow 0} \frac{iy}{m_0 - m_\infty}$$

It is clear that this limit cannot be strictly positive unless $m_0 - m_\infty$ converges to zero as $y \downarrow 0$; that is, since m_{0I} and $m_{\infty I}$ have opposite signs in the upper half plane, unless m_{0I} and $m_{\infty I}$ both converge to zero as $y \downarrow 0$. However, if x is an eigenvalue of H_∞ then $\mathrm{Im}\, m_\infty^+(x) = \infty$ by Propositions 2.6 and 2.14 and hence $\mu_{11}(\{x\}) = 0$ for all x in E.

Also from (4.1.3)

$$-\lim_{y\downarrow 0} iy\, M_{22}(x+iy) = \lim_{y\downarrow 0} \frac{-iy\, m_\infty \cdot iy\, m_0}{iy\, m_0 - iy\, m_\infty}$$

It is evident from (4.3.6) and (4.3.7) that this limit can only be strictly positive if x is both an eigenvalue of H_0 and an eigenvalue of H_∞.

We conclude that if x is an eigenvalue of H_∞, then $\mu_{ij}(\{x\}) = 0$ for $i,j = 1,2$ unless x is also an eigenvalue of H_1. In this case

$\mu_{11}(\{x\}) = \mu_{12}(\{x\}) = \mu_{21}(\{x\}) = 0$, $\mu_{22}(\{x\}) > 0$.

Proof of (iii):

Since the spectrum of H is singular on $(-\infty, 0]$ by (i),

$\mu_{jj}(\{x \in (-\infty, 0] : \text{Im } M_{jj}+(x)$ does not exist infinitely $\}) = 0$

for j = 1,2, by Propositions 2.6 and 2.14 applied to μ_{11} and μ_{22}.

Let I be a compact subinterval of $\mathbb{R}$ which is contained in the open interval between two consecutive eigenvalues of H_∞. As we noted earlier, m_∞ is bounded in any rectangular region of the form I x $[0,Y]$, and $m_\infty+(x)$ exists finitely and is real at all x in I. It follows therefore from (4.2.1) and (4.2.3) that for j = 1,2, $\text{Im } M_{jj}+(x)$ cannot exist infinitely on I unless $m_0+(x)$ exists finitely and equals $m_\infty+(x)$. Since this is true for all such intervals I, and $\text{Im } m_\infty+(x) \geqslant 0$, $\text{Im } m_0+(x) \leqslant 0$, the assertion is proved.

The proof of the proposition is now complete.

We shall now show that the spectrum of H is absolutely continuous on $(0, \infty)$.

4.5 Proposition: With the hypothesis of Proposition 4.4, H has purely absolutely continuous spectrum on $(0, \infty)$.

Proof:

Let I be a compact subinterval of $(0, \infty)$.

From (4.3.4), $\text{Im } m_\infty+(x)$ exists finitely at all points of I, and by our earlier remarks, there exist k, K in $\mathbb{R}^+$ such that $0 < k < K < \infty$ and $2k < \text{Im } m_\infty+(x) < \frac{K}{2}$ for all x in I. If we identify $\text{Im } m_\infty(x)$ with the limit $\text{Im } m_\infty+(x)$, the continuity of $m_\infty(z)$ in the upper half plane implies that $Y > 0$ exists such that

$$k < \operatorname{Im} m_\infty(z) < K \tag{4.3.8}$$

for all z in I x [0,Y].

Since $m_{0I} \leq 0$, $m_{\infty I} \geq 0$ for $\operatorname{Im} z > 0$ we have from (4.2.1)

$$\operatorname{Im} M_{11}(z) \leq \frac{1}{m_{\infty I} - m_{0I}} \leq \frac{1}{m_{\infty I}}$$

for all z in the upper half plane. Hence by (4.3.8),

$$\operatorname{Im} M_{11}(z) < \frac{1}{k}$$

for all z in I x [0,Y], and

$$\limsup_{y \downarrow 0} \operatorname{Im} M_{11}(z) < \frac{1}{k}$$

for all x in I. Hence, applying the Lebesgue Dominated Convergence Theorem to (4.1.4) with i = j = 1, and using the fact that $\operatorname{Im} m_0+(x) = 0$ for Lebesgue almost all x in I, we have

$$\begin{aligned} \rho_{11}(\mu) - \rho_{11}(\nu) &= \frac{1}{\pi}\int_\nu^\mu \lim_{y\downarrow 0} \operatorname{Im} M_{11}(x+iy)\, dx \\ &= \frac{1}{\pi}\int_\nu^\mu \frac{\operatorname{Im} m_\infty+(x)}{|m_0+(x) - m_\infty+(x)|^2}\, dx \end{aligned} \tag{4.3.9}$$

for all points μ, ν of I for which $\mu < \nu$. Since this holds for all such intervals I, we conclude that $\rho_{11}(\lambda)$ is an absolutely continuous function on compact subintervals of $(0,\infty)$ and that

$$\frac{d\rho_{11}(\lambda)}{d\lambda} = \frac{\operatorname{Im} m_\infty+(\lambda)}{\pi\,|m_0+(\lambda) - m_\infty+(\lambda)|^2} \tag{4.3.10}$$

for Lebesgue almost all λ in $(0,\infty)$ ([HS] Thm.18.17).

Likewise, using $m_{0I} \leq 0$, $m_{\infty I} \geq 0$ for $\operatorname{Im} z > 0$ we have from (4.2.3)

$$\operatorname{Im} M_{22}(z) \leq \frac{m_{\infty I}\, m_{0I}^2}{(m_{\infty I} - m_{0I})^2} - \frac{m_{0I}\, m_{\infty I}^2}{(m_{\infty I} - m_{0I})^2} + \frac{m_{\infty I}\, m_{0R}^2}{(m_{0R} - m_{\infty R})^2 + m_{\infty I}^2} - \frac{m_{0I}\, m_{\infty R}^2}{(m_{\infty I} - m_{0I})^2}$$

If $\operatorname{Im} z > 0$, the first two terms on the right hand side above are bounded above by $m_{\infty I}$, and the last term by $\dfrac{m_{\infty R}^2}{m_{\infty I}}$. Using the inequality

$$\frac{y^2}{(y+b)^2 + c^2} \leq 1 + \left(\frac{b}{c}\right)^2 \tag{4.3.11}$$

with $y = m_{0R}$, $b = -m_{\infty R}$, $c = m_{\infty I}$, we see that the third term is bounded above by $\dfrac{|m_\infty|^2}{m_{\infty I}}$. From these bounds and (4.3.3), (4.3.4) it is evident that $M \in \mathbb{R}^+$ exists such that $\operatorname{Im} M_{22}(z) < M$ for all z in I x (0,Y] and $\limsup_{y \downarrow 0} \operatorname{Im} M_{22}(z) < M$ for all x in I. As in the case of $\rho_{11}(\lambda)$ we conclude that $\rho_{22}(\lambda)$ is an absolutely continuous function on $(0,\infty)$ and that

$$\frac{d\rho_{22}(\lambda)}{d\lambda} = \frac{(m_0^+(\lambda))^2 \operatorname{Im} m_\infty^+(\lambda)}{\pi\,|m_0^+(\lambda) - m_\infty^\pm(\lambda)|^2} \tag{4.3.12}$$

for Lebesgue almost all λ in $(0,\infty)$. Note that we have used the fact that $m_0^+(\lambda)$ exists Lebesgue almost everywhere on $\mathbb{R}$ (see Theorem 2.12(i)).

It follows from Lemma 4.3 that $\rho_{12}(\lambda)$ and $\rho_{21}(\lambda)$ are also absolutely continuous functions of λ on compact subintervals of $(0,\infty)$; since the same is true of $\rho_{11}(\lambda)$ and $\rho_{22}(\lambda)$ the spectrum of H is absolutely continuous on $(0,\infty)$.

The proof of the proposition is now complete.

We are now in a position to establish the main results of this chapter.

§4. The simplified expansion

We shall prove that if V(r) is in $L_1[1,\infty)$ and the spectrum of H_1 is singular, then for each f in $L_2(0,\infty)$ an eigenfunction expansion exists which is formally similar to (2.4.4). In the case where L is regular at 0, we shall relate our results to the theory we described in Chapter II, §4.

We need some preliminary lemmas:

4.6 <u>Lemma</u>: Let V(r) be in $L_1[1,\infty)$ and the spectrum of H_1 be singular. Then if $[\nu,\mu]$ is a compact subinterval of $\mathbb{R}$ which is in the resolvent set of H_∞, and if ν,μ are points of continuity of $\rho_{ij}(\lambda)$ for each i,j = 1,2

$$\rho_{12}(\mu) - \rho_{12}(\nu) = \lim_{y\downarrow 0} \frac{1}{\pi}\int_\nu^\mu \mathrm{Re}\, m_0(z)\, \mathrm{Im}\, M_{11}(z)\, dx$$

$$\rho_{22}(\mu) - \rho_{22}(\nu) = \lim_{y\downarrow 0} \frac{1}{\pi}\int_\nu^\mu \mathrm{Re}\, m_0(z)\, \mathrm{Im}\, M_{12}(z)\, dx$$

Proof:

From (4.1.4) and (4.2.2),

$$\begin{aligned}\rho_{12}(\mu) - \rho_{12}(\nu) &= \lim_{y\downarrow 0} \frac{1}{\pi}\int_\nu^\mu \frac{m_{OR}\, m_{\infty I} - m_{OI}\, m_{\infty R}}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}\, dx \\ &= \lim_{y\downarrow 0} \frac{1}{\pi}\int_\nu^\mu \frac{m_{OR}(m_{\infty I} - m_{OI})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}\, dx \\ &+ \lim_{y\downarrow 0} \frac{1}{\pi}\int_\nu^\mu \frac{m_{OI}(m_{OR} - m_{\infty R})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}\, dx\end{aligned} \tag{4.4.1}$$

Now the integrand in the last term of (4.4.1) converges to zero Lebesgue almost everywhere on $[\nu,\mu]$ (cf. proof of Proposition 4.4(i)), and, using $m_{OI} \leq 0$, $m_{\infty I} \geq 0$ for $\mathrm{Im}\, z > 0$ and the inequality $a^2 + b^2 \geq |2ab|$,

$$\left|\frac{m_{OI}(m_{OR} - m_{\infty R})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}\right| \leq \frac{1}{2} \tag{4.4.2}$$

if $y > 0$. Hence by the Lebesgue Dominated Convergence Theorem the final term in (4.4.1) is zero. The first part of the lemma now follows from (4.2.1).

From (4.2.3),

$$\mathrm{Im}\, M_{22}(z) = -\frac{m_{\infty I}\, m_{OI}(m_{\infty I} - m_{OI})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}$$
$$+ \frac{m_{OI}\, m_{\infty R}(m_{OR} - m_{\infty R})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2} + \frac{m_{OR}(m_{OR}\, m_{\infty I} - m_{OI}\, m_{\infty R})}{(m_{OR} - m_{\infty R})^2 + (m_{OI} - m_{\infty I})^2}$$

The first term on the right hand side is positive and bounded above by $m_{\infty I}$ if $y > 0$, and the second term is absolutely bounded by $\frac{1}{2}|m_{\infty R}|$ by (4.4.2); moreover, as $y \downarrow 0$ each of these terms converges pointwise to zero Lebesgue almost everywhere on $[\nu,\mu]$ (cf. proof of Proposition 4.4(i)). Since $m_\infty(z)$ may be analytically continued across the real axis at all points of $[\nu,\mu]$, the second part of the lemma now follows by the Lebesgue Dominated Convergence Theorem and (4.2.2).

The proof of the lemma is now complete.

4.7 Lemma: With the hypothesis and notation of Lemma 4.6,

$$\lim_{y \downarrow 0} \frac{1}{\pi}\int_{\nu}^{\mu} \operatorname{Re} m_0(z)\, \operatorname{Im} M_{1j}(z)\, dx = \lim_{y \downarrow 0} \frac{1}{\pi}\int_{\nu}^{\mu} \operatorname{Re} m_\infty(z)\, \operatorname{Im} M_{1j}(z)\, dx$$

for j = 1,2.

Proof:

By (4.2.1), $(\operatorname{Re} m_0(z) - \operatorname{Re} m_\infty(z))\operatorname{Im} M_{11}(z)$ is absolutely bounded by ½ and converges to zero Lebesgue almost everywhere on $[\nu,\mu]$ (cf. proof of Proposition 4.4(i)). Hence, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{y \downarrow 0} \frac{1}{\pi}\int_{\nu}^{\mu} (\operatorname{Re} m_0(z) - \operatorname{Re} m_\infty(z))\operatorname{Im} M_{11}(z)\, dx = 0$$

The result now follows for j = 1, since

$$\lim_{y \downarrow 0} \frac{1}{\pi}\int_{\nu}^{\mu} \operatorname{Re} m_0(z)\, \operatorname{Im} M_{11}(z)\, dx$$

exists by Lemma 4.6. By (4.2.2),

$$|(\operatorname{Re} m_0(z) - \operatorname{Re} m_\infty(z))\operatorname{Im} M_{12}(z)|$$
$$\leq \frac{(m_{0R} - m_{\infty R})^2\, m_{\infty I}}{(m_{0R} - m_{\infty R})^2 + (m_{0I} - m_{\infty I})^2} + \left|\frac{(m_{0R} - m_{\infty R})m_{\infty R}(m_{0I} - m_{\infty I})}{(m_{0R} - m_{\infty R})^2 + (m_{0I} - m_{\infty I})^2}\right|$$
$$\leq m_{\infty I} + \frac{1}{2}|m_{\infty R}|$$

Since $m_\infty(z)$ may be analytically continued across the axis at all points of $[\nu,\mu]$, the result follows for j = 2 by the Lebesgue Dominated Convergence Theorem and Lemma 4.6.

The lemma is now proved.

4.8 Lemma: Let V(r) be in $L_1[1,\infty)$ and suppose that the spectrum of H_1 is singular. Then if E is the set of eigenvalues of H ,

(i) $d\rho_{12}(\lambda) = d\rho_{21}(\lambda) = m_0(\lambda)\, d\rho_{11}(\lambda)$ and $d\rho_{22}(\lambda) = (m_0(\lambda))^2\, d\rho_{11}(\lambda)$

μ_{11} -almost everywhere on $\mathbb{R} \setminus E$, where $m_0(\lambda) = m_0+(\lambda)$ whenever the limit exists finitely and is real.

(ii) $\mu_{11}(\{\lambda\}) = \mu_{12}(\{\lambda\}) = \mu_{21}(\{\lambda\}) = 0$ for all λ in E.

Proof:

Proof of (i):

Let us first consider λ in $\mathbb{R}^+$.

From (4.3.10) and (4.3.12), $d\rho_{22}(\lambda) = (m_0(\lambda))^2 d\rho_{11}(\lambda)$ for Lebesgue almost all λ in $(0,\infty)$; the absolute continuity of each of the measures μ_{ij} ensures that this is also true for almost all λ with respect to μ_{ij} , for i,j = 1,2.

Using $m_{0I} \leqslant 0$, $m_{\infty I} \geqslant 0$ for $\operatorname{Im} z > 0$, we have from (4.2.2)

$$|\operatorname{Im} M_{12}(z)| \leqslant \frac{m_{\infty I}|m_{0R}|}{(m_{0R}-m_{\infty R})^2 + (m_{0I}-m_{\infty I})^2} + \frac{m_{0I}|m_{\infty R}|}{(m_{0I}-m_{\infty I})^2}$$

$$\leqslant \frac{|m_{0R}|}{((m_{0R}-m_{\infty R})^2 + m_{\infty I}^2)^{1/2}} + \frac{|m_{\infty R}|}{m_{\infty I}}$$

By (4.3.11) the first term on the right hand side is bounded by $\dfrac{|m_\infty|}{(m_{\infty I})^{1/2}}$;

hence, as in Proposition 4.5 we may use the Lebesgue Dominated Convergence Theorem on compact subinterals of $(0,\infty)$ to obtain

$$\frac{d\rho_{12}(\lambda)}{d\lambda} = \frac{m_0+(\lambda)\operatorname{Im} m_\infty+(\lambda)}{\pi\,|m_0+(\lambda) - m_\infty+(\lambda)|^2} \tag{4.4.3}$$

for Lebesgue almost all λ in $(0,\infty)$. This, together with (4.3.10) yields

$$d\rho_{12}(\lambda) = d\rho_{21}(\lambda) = m_0(\lambda)\, d\rho_{11}(\lambda)$$

for almost all λ with respect to each of the measures μ_{ij} , i,j = 1,2.

Let us now consider λ in $\mathbb{R}^-$.

Let ν,μ be as in the hypothesis of Lemma 4.6. Since $m_\infty^+(x)$ exists and is real at all points x in $[\nu,\mu]$, we shall denote $m_\infty+(x)$ by $m_\infty(x)$ for all x in this interval. We first show that

$$\int_{\nu}^{\mu} d\rho_{12}(\lambda) = \int_{\nu}^{\mu} m_{\infty}(\lambda)\, d\rho_{11}(\lambda) \tag{4.4.4}$$

For conciseness, let g(z) denote $\operatorname{Im} M_{11}(z)$ (see (4.2.1)). From Lemmas 4.6 and 4.7,

$$\int_{\nu}^{\mu} d\rho_{12}(\lambda) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_{\nu}^{\mu} \operatorname{Re} m_{\infty}(z)\, g(z)\, dx \tag{4.4.5}$$

We prove that (4.4.4) implies (4.4.3).

Let $\varepsilon > 0$ be given, and $M = \mu_{11}([\nu,\mu])$.

Our choice of ν,μ implies that $m_{\infty}(z)$ is uniformly continuous on the compact set $[\nu,\mu] \times [0,K]$ for each $K > 0$. Hence there exists $Y_K > 0$ such that if $y < Y_K$

$$| m_{\infty R}(z) - m_{\infty}(x) | < \frac{\varepsilon}{7M} \tag{4.4.6}$$

for all x in $[\nu,\mu]$. Moreover, there exists a bounded step function $F(x) = \sum_{i=1}^{P} \alpha_i \chi_i$ on $[\nu,\mu]$, where χ_i is the characteristic function of an interval S_i, such that

$$| F(x) - m_{\infty}(x) | < \frac{\varepsilon}{7M} \tag{4.4.7}$$

for all x in $[\nu,\mu]$. There is no loss of generality if we suppose that the endpoints of each S_i are points of continuity of $\rho_{ij}(\lambda)$ for each i,j = 1,2. (4.4.6) and (4.4.7) together imply that if $y < Y_K$

$$| F(x) - m_{\infty R}(z) | < \frac{2\varepsilon}{7M} \tag{4.4.8}$$

for all x in $[\nu,\mu]$.

Let $Q = \max_{i \in \{1,\ldots,P\}} \alpha_i$

By (4.1.4), there exists $Y_L > 0$ such that if $y < Y_L$

$$\left| \frac{1}{\pi} \int_{\nu}^{\mu} g(z)\, dx - \int_{\nu}^{\mu} d\rho_{11}(\lambda) \right| < M \tag{4.4.9}$$

and for each i = 1,...P, there exists Y_i such that

$$\left| \frac{1}{\pi} \int_{S_i} g(z)\,dx - \int_{S_i} d\rho_{11}(\lambda) \right| < \frac{\varepsilon}{7PQ} \tag{4.4.10}$$

if $y < Y_i$. Moreover, by (4.4.5), there exists $Y_N > 0$ such that

$$\left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \int_\nu^\mu d\rho_{12}(\lambda) \right| < \frac{\varepsilon}{7} \tag{4.4.11}$$

Hence if $y < \min \{ Y_K, Y_L, Y_N \} \cup \{ Y_i : i = 1, \dots p \}$ we have by (4.4.7)-(4.4.11)

$$\begin{aligned}
&\left| \int_\nu^\mu d\rho_{12}(\lambda) - \int_\nu^\mu m_\infty(\lambda)\, d\rho_{11}(\lambda) \right| \\
&\quad \leqslant \left| \int_\nu^\mu d\rho_{12}(\lambda) - \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx \right| \\
&\qquad + \left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \int_\nu^\mu m_\infty(\lambda)\, d\rho_{11}(\lambda) \right| \\
&\quad \leqslant \frac{\varepsilon}{7} + \left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \int_\nu^\mu F(\lambda)\, d\rho_{11}(\lambda) \right| \\
&\qquad + \int_\nu^\mu \left| F(\lambda) - m_\infty(\lambda) \right| d\rho_{11}(\lambda) \\
&\quad \leqslant \frac{2\varepsilon}{7} + \left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \sum_{i=1}^{p} \alpha_i \int_{S_i} d\rho_{11}(\lambda) \right| \\
&\quad \leqslant \frac{2\varepsilon}{7} + \left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \frac{1}{\pi} \sum_{i=1}^{p} \alpha_i \int_{S_i} g(z)\,dx \right| \\
&\qquad + \sum_{i=1}^{p} |\alpha_i| \left| \frac{1}{\pi} \int_{S_i} g(z)\,dx - \int_{S_i} d\rho_{11}(\lambda) \right| \\
&\quad \leqslant \frac{3\varepsilon}{7} + \left| \frac{1}{\pi} \int_\nu^\mu m_{\infty R}(z)\, g(z)\,dx - \frac{1}{\pi} \int_\nu^\mu F(x)\, g(z)\,dx \right| \\
&\quad \leqslant \frac{3\varepsilon}{7} + \frac{1}{\pi} \int_\nu^\mu \left| m_{\infty R}(z) - F(x) \right| g(z)\,dx \\
&\quad \leqslant \frac{3\varepsilon}{7} + \frac{2\varepsilon}{7M} \frac{1}{\pi} \int_\nu^\mu g(z)\,dx \\
&\quad \leqslant \frac{3\varepsilon}{7} + \frac{2\varepsilon}{7M} \left| \frac{1}{\pi} \int_\nu^\mu g(z)\,dx - \int_\nu^\mu d\rho_{11}(x) \right| + \frac{2\varepsilon}{7M} \int_\nu^\mu d\rho_{11}(x) \\
&\quad < \varepsilon
\end{aligned}$$

The arbitrariness of ε implies

$$\int_\nu^\mu d\rho_{12}(\lambda) = \int_\nu^\mu m_\infty(\lambda)\, d\rho_{11}(\lambda)$$

and hence, from Proposition 4.4(iii),

$$\int_{\nu}^{\mu} d\rho_{12}(\lambda) = \int_{\nu}^{\mu} m_0(\lambda)\, d\rho_{11}(\lambda) \tag{4.4.12}$$

where the real limit $m_0(\lambda) = m_0^+(\lambda)$ exists μ_{11}-almost everywhere.

It may be proved in a similar way that

$$\int_{\nu}^{\mu} d\rho_{22}(\lambda) = \int_{\nu}^{\mu} m_0(\lambda)\, d\rho_{12}(\lambda) \tag{4.4.13}$$

However, in this case we may not assume that the analogue of g(z), viz. $\operatorname{Im} M_{12}(z)$, is positive nor that $\rho_{12}(\lambda)$ is increasing. To overcome the first difficulty we use the second inequality in Lemma 4.2, and to overcome the second difficulty we note that $\rho_{12}(\lambda)$, being a function of bounded variation on finite λ-intervals, may be expressed as the difference of two increasing functions.

Since each of the measures μ_{ij}, i,j = 1,2 is regular, (4.4.12) and (4.4.13) also hold for all half-open intervals $(\nu,\mu]$ which consist entirely of points of the resolvent set, irrespective of whether ν and μ are points of continuity of the measures μ_{ij}. It therefore follows from the Hahn Extension Theorem (2.2.2) that

$$\mu_{12}(I) = \int_I m_0(\lambda)\, d\rho_{11}(\lambda)$$

for arbitrary measurable real subsets I of the resolvent set. Hence

$$d\rho_{12}(\lambda) = d\rho_{21}(\lambda) = m_0(\lambda)\, d\rho_{11}(\lambda)$$

μ_{11}-almost everywhere on $\mathbb{R}^- \setminus E$. It follows, similarly, that

$$d\rho_{22}(\lambda) = (m_0(\lambda))^2\, d\rho_{11}(\lambda)$$

μ_{11}-almost everywhere on $\mathbb{R}^- \setminus E$.

It remains to consider the point 0 if $0 \notin E$. In this case either

a) 0 is an eigenvalue of H, or

b) 0 is not an eigenvalue of H.

In case a) there exists an $L_2[1,\infty)$ solution of Lu = 0 on $[1,\infty)$ which does not satisfy the boundary condition u(1,0) = 0. Hence the result of Theorem 3.19, applied to $m_\infty(z)$, implies that $m_\infty(y)$ converges to a finite real limit $m_\infty(0)$ as $y \downarrow 0$. Moreover, since 0 can only be an eigenvalue of H if at least one of $\mu_{ij}(\{0\}) > 0$, i,j = 1,2 ([DS] Ch.XIII §5, p.1360), $m_0^+(0)$ exists finitely and is real, and $m_0(0) = m_\infty(0)$ by Proposition 4.4(iii).

Therefore, applying the relationship (4.3.5) to $\mu_{11}(\{0\})$, $\mu_{22}(\{0\})$, $\mu_{12}^{\pm}(\{0\})$, $\mu_{21}^{\pm}(\{0\})$, we obtain from (4.1.3)

$$\mu_{11}(\{0\}) = \lim_{y\downarrow 0} \frac{-iy}{(m_0(y) - m_\infty(y))}$$

$$\mu_{12}(\{0\}) = \mu_{21}(\{0\}) = \lim_{y\downarrow 0} \frac{-iy(m_0(y) + m_\infty(y))}{2(m_0(y) - m_\infty(y))}$$

$$= m_0(0)\,\mu_{11}(\{0\})$$

$$\mu_{22}(\{0\}) = \lim_{y\downarrow 0} \frac{-iy\, m_0(y)\, m_\infty(y)}{(m_0(y) - m_\infty(y))}$$

$$= (m_0(0))^2\,\mu_{11}(\{0\})$$

In case b), $\mu_{ij}(\{0\}) = 0$ for each i,j = 1,2 ([DS] loc.cit.).

In either case these results are sufficient to complete the proof of (i).

Proof of (ii):

This has already been established in the proof of Proposition 2.4(ii).

The proof of the lemma is now complete.

We now show that under the hypothesis of Theorem 4.8, the Weyl Kodaira theory may be dramatically simplified.

We first describe some notation. If $S \subseteq \mathbb{R}$ is measurable, define

$$\tilde{\mu}(S) = \mu_{11}(S \setminus E) + \mu_{22}(S \cap E) \tag{4.4.14}$$

and for $\lambda \in \mathbb{R}$,

$$y_S(r,\lambda) = \begin{cases} y_2(r,\lambda) & \text{on } E \\ y_1(r,\lambda) + m_0(\lambda)\, y_2(r,\lambda) & \text{whenever} \end{cases} \tag{4.4.15}$$

$m_0(\lambda) = m_0^+(\lambda)$ exists finitely and is real on $\mathbb{R} \setminus E$ (this is, as we have shown in Lemma 4.8, μ_{11}-almost everywhere on $\mathbb{R} \setminus E$. Let $\tilde{\rho}(\lambda)$ be the right continuous increasing function arising from $\tilde{\mu}$ (cf. (2.2.1)), for which $\tilde{\rho}(0) = 0$.

4.9 Theorem: Let V(r) be in $L_1[1,\infty)$, and suppose the spectrum of H_1 is singular. Then each f(r) in $L_2(0,\infty)$ has the eigenfunction expansion

$$f(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} y_s(r,\lambda)\,\tilde{F}(\lambda)\,d\tilde{\rho}(\lambda)$$

where $\tilde{F}(\lambda) = \underset{\substack{\omega\to\infty \\ \sigma\to 0}}{\mathrm{l.i.m.}} \int_{\sigma}^{\omega} y_s(r,\lambda)\,f(r)\,dr$

the integrals being convergent in $L_2(0,\infty)$ and $L_2^{\tilde{\rho}}(-\infty,\infty)$ respectively. The transformation $\tilde{S}$ which maps f(r) to $\tilde{F}(\lambda)$ is an isometric Hibert space isomorphism from $L_2(0,\infty)$ onto $L_2^{\tilde{\rho}}(-\infty,\infty)$. Moreover, if $\theta : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function, and $\theta(H)f(r)$ is in $L_2(0,\infty)$, then

$$\theta(H)f(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} \theta(\lambda)\,y_s(r,\lambda)\,\tilde{F}(\lambda)\,d\tilde{\rho}(\lambda) \qquad (4.4.16)$$

where the integral converges in $L_2(0,\infty)$.

Proof:

Let f(r) be in $L_2(0,\infty)$.

Since E is a bounded subset of $\mathbb{R}$ ([LS] Ch.IV Thm.3.1), we have from (4.1.6) and Lemma 4.8,

$$\begin{aligned}
0 &= \lim_{\omega\to\infty} \int_0^{\infty} \Big| f(r) - \sum_{i,j=1,2} \int_{-\omega}^{\omega} y_i(r,\lambda)\,\phi_j(\lambda)\,d\rho_{ij}(\lambda)\Big|^2 dr \\
&= \lim_{\omega\to\infty} \int_0^{\infty} \Big| f(r) - \int_E y_2(r,\lambda)\,\phi_2(\lambda)\,d\rho_{22}(\lambda) - \int_{(-\omega,\omega)\setminus E} y_1(r,\lambda)\phi_1(\lambda)\,d\rho_{11}(\lambda) \\
&\qquad - \int_{(-\omega,\omega)\setminus E} (y_1(r,\lambda)\,\phi_2(\lambda) + y_2(r,\lambda)\,\phi_1(\lambda))\,m_0(\lambda)\,d\rho_{11}(\lambda) \\
&\qquad - \int_{(-\omega,\omega)\setminus E} y_2(r,\lambda)\,\phi_2(\lambda)\,(m_0(\lambda))^2\,d\rho_{11}(\lambda)\Big|^2 dr \\
&= \lim_{\omega\to\infty} \int_0^{\infty} \Big| f(r) - \int_E y_2(r,\lambda)\,\phi_2(\lambda)\,d\rho_{22}(\lambda) \\
&\qquad - \int_{(-\omega,\omega)\setminus E} (y_1(r,\lambda) + m_0(\lambda)\,y_2(r,\lambda))(\phi_1(\lambda) + m_0(\lambda)\,\phi_2(\lambda))\,d\rho_{11}(\lambda)\Big|^2 dr
\end{aligned}$$

$$= \lim_{\omega\to\infty} \int_0^\omega | f(r) - \int_{-\omega}^{\omega} y_s(r,\lambda)\, G(\lambda)\, d\tilde{\rho}(\lambda)|^2 dr \qquad (4.4.17)$$

where
$$G(\lambda) = \begin{cases} \phi_2(\lambda) \text{ on } E \\ \phi_1(\lambda) + m_0(\lambda)\phi_2(\lambda) \ \tilde{\mu}\text{-almost everywhere on} \end{cases}$$

$\mathbb{R}\setminus E$, where $\tilde{\mu}$ is the measure generated by $\tilde{\rho}(\lambda)$. From (4.1.8),

$$\begin{aligned} 0 = \lim_{\substack{\omega\to\infty\\ \sigma\to 0}} \Big[&\int_{\mathbb{R}\setminus E} |\phi_1(\lambda) - \int_\sigma^\omega y_1(r,\lambda) f(r)\, dr|^2 d\rho_{11}(\lambda) \\ &+ \int_{\mathbb{R}\setminus E} \overline{(\phi_1(\lambda) - \int_\sigma^\omega y_1(r,\lambda) f(r)\, dr)}(\phi_2(\lambda) - \int_\sigma^\omega y_2(r,\lambda) f(r)\, dr)\, m_0(\lambda)\, d\rho_{11}(\lambda) \\ &+ \int_{\mathbb{R}\setminus E} \overline{(\phi_2(\lambda) - \int_\sigma^\omega y_2(r,\lambda) f(r)\, dr)}(\phi_1(\lambda) - \int_\sigma^\omega y_1(r,\lambda) f(r)\, dr)\, m_0(\lambda)\, d\rho_{11}(\lambda) \\ &+ \int_{\mathbb{R}\setminus E} |\phi_2(\lambda) - \int_\sigma^\omega y_2(r,\lambda) f(r)\, dr|^2 (m_0(\lambda))^2 d\rho_{11}(\lambda) + \int_E |\phi_2(\lambda) - \int_\sigma^\omega y_2(r,\lambda) f(r)\, dr|^2 d\rho_{22}(\lambda) \Big] \\ = \lim_{\substack{\omega\to\infty\\ \sigma\to 0}} &\int_{-\infty}^{\infty} (G(\lambda) - \int_\sigma^\omega y_s(r,\lambda) f(r)\, dr)^2\, d\tilde{\rho}(\lambda) \end{aligned}$$

Hence $G(\lambda) = F(\lambda)$ $\tilde{\mu}$-almost everywhere on $\mathbb{R}$, so from (4.4.17)

$$f(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} y_s(r,\lambda)\, F(\lambda)\, d\tilde{\rho}(\lambda)$$

as required.

Let $f_1(r)$, $f_2(r)$ be in $L_2(0,\infty)$, and for $i,j = 1,2$ define

$$\phi_i^{(j)}(\lambda) = \underset{\substack{\omega\to\infty\\ \sigma\to 0}}{\mathrm{l.i.m.}} \int_\sigma^\omega y_i(r,\lambda) f_j(r)\, dr$$

By Theorem 4.1, Lemma 4.8, and (4.1.5)

$$\begin{aligned} \int_0^\infty \overline{f_1(r)} f_2(r)\, dr &= \sum_{i,j=1,2} \int_{-\infty}^{\infty} \overline{\phi_i^{(1)}(\lambda)}\, \phi_j^{(2)}(\lambda)\, d\rho_{ij}(\lambda) \\ &= \int_E \overline{\phi_2^{(1)}(\lambda)}\, \phi_2^{(2)}(\lambda)\, d\rho_{22}(\lambda) + \int_{\mathbb{R}\setminus E} \overline{\phi_1^{(1)}(\lambda)}\, \phi_1^{(2)}(\lambda)\, d\rho_{11}(\lambda) \\ &\quad + \int_{\mathbb{R}\setminus E} (\overline{\phi_1^{(1)}(\lambda)}\phi_2^{(2)}(\lambda) + \overline{\phi_2^{(1)}(\lambda)}\, \phi_1^{(2)}(\lambda))\, m_0(\lambda)\, d\rho_{11}(\lambda) + \int_{\mathbb{R}\setminus E} \overline{\phi_2^{(1)}(\lambda)}\, \phi_2^{(2)}(\lambda) (m_0(\lambda))^2 d\rho_{11}(\lambda) \\ &= \int_E \overline{\phi_2^{(1)}(\lambda)}\, \phi_2^{(2)}(\lambda)\, d\rho_{22}(\lambda) + \int_{\mathbb{R}\setminus E} \overline{(\phi_1^{(1)}(\lambda) + m_0(\lambda)\, \phi_2^{(1)}(\lambda))}(\phi_1^{(2)}(\lambda) + m_0(\lambda)\, \phi_2^{(2)}(\lambda))\, d\rho_{11}(\lambda) \\ &= \int_{-\infty}^{\infty} \overline{G_1(\lambda)}\, G_2(\lambda)\, d\tilde{\rho}(\lambda) \end{aligned}$$

where
$$G_j(\lambda) = \begin{cases} \phi_2^{(j)}(\lambda) & \text{if } \lambda \in E \\ \phi_1^{(j)}(\lambda) + m_0(\lambda)\, \phi_2^{(j)}(\lambda) & \mu_{11}\text{-almost everywhere} \end{cases}$$

on $\mathbb{R}\setminus E$ $(j = 1,2)$. As above, it is clear that $G_j(\lambda) = (\tilde{S} f_j)(\lambda)$ $\tilde{\mu}$-almost everywhere for $j = 1,2$.

Hence the transformation $\tilde{S}$ preserves inner products; and, in particular, $\tilde{S}$ is isometric. To complete the proof that $\tilde{S}$ is an isometric Hilbert space isomorphism from $L_2(0,\infty)$ onto $L_2^{\tilde{\rho}}(-\infty,\infty)$ it remains therefore to show that $\tilde{S}$ is linear, one-to-one and surjective. The proof of the surjective property is somewhat cumbersome and is contained in the Appendix. To prove linearity we show that if $f_1(r)$, $f_2(r)$ are in $L_2(0,\infty)$ and $\tilde{F}_1(\lambda), \tilde{F}_2(\lambda), \tilde{F}(\lambda)$ denote $(\tilde{S}f_1)(\lambda), (\tilde{S}f_2)(\lambda), (\tilde{S}(f_1+cf_2))(\lambda)$ respectively, where c is constant, then

$$\tilde{F}(\lambda) = \tilde{F}_1(\lambda) + c\,\tilde{F}_2(\lambda) \tag{4.4.18}$$

almost everywhere with respect to the measure $\tilde{\mu}$ generated by $\tilde{\rho}(\lambda)$.

Now if $\|\cdot\|$ denotes the $L_2^{\tilde{\rho}}(-\infty,\infty)$ norm,

$$\begin{aligned}
&\|\tilde{F}(\lambda) - (\tilde{F}_1(\lambda) + c\tilde{F}_2(\lambda))\| \\
&\quad\leq \|\tilde{F}(\lambda) - \int_{1/N}^{N} y_s(r,\lambda)(f_1(r) + c f_2(r))dr\| \\
&\qquad + \|\tilde{F}_1(\lambda) - \int_{1/N}^{N} y_s(r,\lambda) f_1(r) dr\| + |c|\,\|F_2(\lambda) - \int_{1/N}^{N} y_s(r,\lambda) f_2(r)\,dr\|
\end{aligned}$$

for all N in $\mathbb{N}$ by Minkowski's inequality. (4.4.18) now follows since the right hand side converges to zero as $N \to \infty$.

Now suppose $(\tilde{S}f_1)(\lambda) = (\tilde{S}f_2)(\lambda)$. Then, by the linearity of $\tilde{S}$, $(\tilde{S}(f_1 - f_2))(\lambda) = 0$, and hence, since $\tilde{S}$ is isometric,

$$\left(\int_0^\infty |f_1(r) - f_2(r)|^2\,dr\right)^{1/2} = 0$$

so that $f_1(r) = f_2(r)$ Lebesgue almost everywhere. It follows that $\tilde{S}$ is one-to-one.

Let $\theta : \mathbb{R} \to \mathbb{C}$ be a Borel measurable function. Then if $f(r)$, $\theta(H)f(r)$ are in $L_2(0,\infty)$, (4.4.16) may be proved from (4.1.9) in the same way that (4.4.17) was proved from (4.1.6).

The proof of the theorem is now complete.

§5. Discussion

Propositions 4.4 and 4.5 not only indicate the nature and location of the spectrum of H, they also give some insight into the behaviour of solutions of the Schrödinger equation associated with the different parts of the spectrum.

Let us suppose that L is limit point at 0, so that there is at most one linearly independent $L_2(0,1]$ solution of $Lu = xu$ for real x. We note from Proposition 4.4 that the singular spectrum is concentrated on two sets, viz. the set E of eigenvalues of H_∞, and the set A of all x for which $m_0^+(x)$ and $m_\infty^+(x)$ exist finitely and are equal. Since $\operatorname{Im} m_0(z)$ and $\operatorname{Im} m_\infty(z)$ have opposite signs in the upper half plane, $m_0^+(x)$ and $m_\infty^+(x)$ must also be real for all x in A. Hence, applying Theorem 3.19 to each of the intervals $(0,1]$ and $[1,\infty)$, we see that for each x in A there exists a solution of $Lu = xu$, viz.

$y_1(r,x)+m_0(x)y_2(r,x) = y_1(r,x)+m_\infty(x)y_2(r,x)$, which is subordinate both at 0 and at ∞. Moreover, the only other points x in $\mathbb{R}$ at which such solutions can exist are the eigenvalues of H_∞ and the point $x = 0$. To see this, note from Theorem 3.19 that if there is a solution of $Lu = xu$ which is subordinate at both 0 and ∞, but $m_0^+(x)$ and $m_\infty^+(x)$ do not exist as finite real limits, then this solution must be $u_2(r,x)$ and $m_0^+(x) = m_\infty^+(x) = \infty$. As has already been noted in §3, $m_\infty^+(x)$ exists as a finite limit for all x in $\mathbb{R} \setminus (E \cup \{0\})$, so $m_0^+(x)$ and $m_\infty^+(x)$ can only exist infinitely on $E \cup \{0\}$.

Now A has Lebesgue measure zero by Theorem 2.12(iii); hence so also has $A \cup E \cup \{0\}$. It follows that the set of all x for which $Lu = xu$ has a solution which is subordinate at both 0 and ∞ is a minimal support of the singular part of the simplified spectral measure $\tilde{\mu}$. This approach may be extended to the other parts of the spectrum to give the result below.

We recall that if L is limit point at 0, the spectrum of H_1 is singular if and only if the same is true of the spectrum of the unique self-adjoint operator H_a arising from L in $L_2(0,a]$ with boundary condition

$u(a,z) = 0$, for each a in $\mathbb{R}^+$. Also, since we have assumed throughout that $V(r)$ is integrable on compact subsets of $\mathbb{R}^+$ not containing the origin, $V(r)$ is in $L_1[1,\infty)$ if and only if $V(r)$ is in $L_1[a,\infty)$ for each $a > 0$.

Since the existence of solutions which are subordinate at 0, or at ∞, does not depend on the decomposition point a, we may state our result in its most general form, as follows:

4.10 <u>Theorem</u>: Let L be in the limit point case at 0, and $V(r)$ be in $L_1[a,\infty)$ for each $a > 0$. Suppose, moreover, that there exists an a in $\mathbb{R}^+$ such that the spectrum of H_a is singular. Then minimal supports $\tilde{M}$, $\tilde{M}_{a.c.}$, $\tilde{M}_{s.}$, $\tilde{M}_{s.c.}$ and $\tilde{M}_{d.}$ of the simplified spectral measure $\tilde{\mu}$, and of $\tilde{\mu}_{a.c.}$, $\tilde{\mu}_{s.}$, $\tilde{\mu}_{s.c.}$, $\mu_{d.}$ are as follows:

$\tilde{M} = \mathbb{R} \setminus \{x \in \mathbb{R}$: a solution of $Lu = xu$ exists which is not subordinate at 0 but is subordinate at $\infty\}$

$\tilde{M}_{a.c.} = \{x \in \mathbb{R}$: no solution of $Lu = xu$ exists which is subordinate at $\infty\}$

$\tilde{M}_{s.} = \{x \in \mathbb{R}$: a solution of $Lu = xu$ which is subordinate both at 0 and at $\infty\}$

$\tilde{M}_{s.c.} = \{x \in \mathbb{R}$: a solution of $Lu = xu$ exists which is subordinate both at 0 and at ∞, but is not in $L_2(0,\infty)\}$

$\tilde{M}_{d.} = \{x \in \mathbb{R}$: a solution of $Lu = xu$ exists which is subordinate both at 0 and at ∞, and is in $L_2(0,\infty)\}$

We note that if "subordinate at 0", "subordinate at infinity" and $\tilde{\mu}$ are replaced by "satisfies the boundary condition at 0", "subordinate" and μ respectively, then Theorem 4.10 reduces to a particular case of Theorem 3.21, which applies to the regular limit circle case at 0. Indeed, noting that for L in the limit circle case at 0, the solution $y_1(r,x)+m_0(x)y_2(r,x)$ satisfies the boundary condition at 0, the arguments above for the limit point case at 0 may be simply adapted to show that if $V(r) \in L_1[a,\infty)$ for each $a > 0$, then the conclusions of Theorem 3.21 hold in respect of $\tilde{\mu}$ for the singular limit circle case at 0.

Likewise, adaptations may be made to accommodate the case where L is limit circle at infinity; in this case $m_\infty(z)$ is a meromorphic function ([CL] Ch.9, §4), so that if the spectrum of H_a is singular, much of the theory of §§ 3,4, suitably modified, still holds (of course, here V(r) is no longer integrable at infinity ([N] §23, Satz 3).

The condition on V(r) at infinity in Theorem 4.10 is such that for each x in $\mathbb{R} \setminus \{0\}$ every solution of Lu = xu which is subordinate at infinity is in $L_2[a,\infty)$ for each a > 0. However we prefer to retain the characterisations of Theorem 4.10 as they stand, bearing in mind that further generalisations may be possible.

If L is in the limit point case at 0, it may happen that there is some absolutely continuous spectrum of H_a (for an example see [P2]). In this case Theorem 4.10 remains true if $\tilde{m}$, $\tilde{m}_{a.c.}$ etc., are now taken to be the minimal supports of $\tilde{\mu}$, $\tilde{\mu}_{a.c.}$ etc., on $\mathbb{R} \setminus \sigma_{a.c.}(H_a)$ where $\sigma_{a.c.}(H_a)$ is the absolutely continuous spectrum of H_a.

It seems not unlikely that when L is regular at 0, some quite straightforward relationship exists between the simplified spectral function $\tilde{\rho}(\lambda)$ and the spectral function $\rho(\lambda)$ described in Chapter II. The following result which we prove for $\lambda > 0$, suggests that such a relationship may hold quite generally.

4.11 Proposition: Let V(r) be in $L_1[0,\infty)$ and suppose that L is regular at 0. Then, if $u_1(r,z)$ is that solution of Lu = zu which satisfies $u_1(0,z) = -\sin\alpha$, $u_1'(0,z) = \cos\alpha$,

$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = (u_1(1,\lambda))^2 \frac{d\rho(\lambda)}{d\lambda}$$

for Lebesgue almost all λ in $(0,\infty)$.

Proof:

The hypothesis satisfies the conditions of Theorem 4.10, so there exists a simplified spectral measure $\tilde{\mu}$ of H satisfying (4.4.14). Moreover, since L is regular at 0 there exists a function m(z) which is analytic in the upper half plane, and a spectral function $\rho(\lambda)$ satisfying (2.3.3) and

(2.3.4). In addition the hypothesis implies that $V(r)$ is in $L_1[1,\infty)$ and so

$$\frac{d\rho(\lambda)}{d\lambda} = \frac{1}{\pi}\, \mathrm{Im}\, m+(\lambda) \qquad (4.5.1)$$

and
$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = \frac{d\rho_{11}(\lambda)}{d\lambda} = \frac{1}{\pi}\, \mathrm{Im}\, M_{11}+(x) \qquad (4.5.2)$$

for $\lambda > 0$ (cf. proof of Proposition 4.5).

Let $u_2(r,z)$ be the solution of $Lu = zu$ which satisfies $u_2(0,z) = \cos\alpha$, $u_1(0,z) = \sin\alpha$. By the definition of $m_\infty(z)$, $y_1(r,z) + m_\infty(z)y_2(r,z)$ is in $L_2[1,\infty)$ for $\mathrm{Im}\, z \neq 0$ and so, since L is regular at 0, $y_1(r,z) + m_\infty(z)y_2(r,z)$ is in $L_2(0,\infty)$ for $\mathrm{Im}\, z \neq 0$. Moreover, using $W(u_2(r,z),u_1(r,z)) = 1$, we have

$$y_1(r,z) = u_1'(1,z)u_2(r,z) - u_2'(1,z)u_1(r,z) \qquad (4.5.3)$$

and

$$y_2(r,z) = u_2(1,z)u_1(r,z) - u_1(1,z)u_2(r,z) \qquad (4.5.4)$$

by the uniqueness of solutions. Hence

$$(u_1'(1,z) - m_\infty(z)u_1(1,z))u_2(r,z) - (u_2'(1,z) - m_\infty(z)u_2(1,z))u_1(r,z)$$

is in $L_2(0,\infty)$ for $\mathrm{Im}\, z \neq 0$ which implies by (2.1.3) that

$$m(z) = -\frac{(u_2'(1,z) - m_\infty(z)\, u_2(1,z))}{(u_1'(1,z) - m_\infty(z)\, u_1(1,z))}$$

Using $W(u_2(1,z),u_1(1,z)) = 1$, this yields

$$\mathrm{Im}\, m(z) = \frac{\mathrm{Im}\, m_\infty(z)}{(u_1'(1,z) - \mathrm{Re}\, m_\infty(z)u_1(1,z))^2 + (\mathrm{Im}\, m_\infty(z)\, u_1(1,z))^2}$$

Hence, by (4.5.1),

$$\frac{d\rho(\lambda)}{d\lambda} = \frac{\mathrm{Im}\, m_\infty+(\lambda)}{\pi[(u_1'(1,\lambda) - \mathrm{Re}\, m_\infty+(\lambda)u_1(1,\lambda))^2 + (\mathrm{Im}\, m_\infty^+(\lambda)\, u_1(1,\lambda))^2]} \qquad (4.5.5)$$

for $\lambda > 0$.

Since L is regular at 0, $m_0(z)$ is defined by the boundary condition (4.1.1). Hence

$$\frac{y_1'(r,z) + m_0(z)\, y_2'(r,z)}{y_1(r,z) + m_0(z)\, y_2(r,z)} = \frac{u_1'(r,z)}{u_1(r,z)}$$

which yields

$$\begin{aligned} m_0(z) &= -\frac{W(y_1(r,z), u_1(r,z))}{W(y_2(r,z), u_1(r,z))} \\ &= -\frac{W(y_1(1,z), u_1(1,z))}{W(y_2(1,z), u_1(1,z))} \\ &= \frac{u_1'(1,z)}{u_1(1,z)} \end{aligned} \tag{4.5.6}$$

Hence by (4.3.10) and (4.5.2)

$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = \frac{\operatorname{Im} m_\infty^+(\lambda)\,(u_1(1,z))^2}{\pi[(u_1'(1,\lambda) - \operatorname{Re} m_\infty^{\pm}(\lambda)\, u_1(1,\lambda))^2 + (\operatorname{Im} m_\infty^+(\lambda)\, u_1(1,\lambda))^2]}$$

for $\lambda > 0$.

This, together, with (4.5.5) gives the result, so the proposition is proved.

Denoting the subspace of absolute continuity with respect to H by $\mathcal{H}_{a.c.}(H)$ ([KA] Ch.X, §2), we have the following:

4.12 Corollary: With the hypothesis of Proposition 4.11, the eigenfunction expansions of Theorem 4.9 and (2.4.4) are equivalent for all f in $\mathcal{H}_{a.c.}(H)$.

Proof:

Let f(r) be in $\mathcal{H}_{a.c.}(H)$. Then

$$f(r) = \underset{\omega\to\infty}{\text{s. lim.}}\ (E_\omega - E_0) f(r)$$

so by (2.4.4) and (4.4.16)

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^\omega u_1(r,\lambda)\, F(\lambda)\, d\rho(\lambda) \tag{4.5.7}$$

$$= \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^\omega y_s(r,\lambda)\,\tilde{F}(\lambda)\,d\tilde{\rho}(\lambda) \qquad (4.5.8)$$

where

$$F(\lambda) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^\omega u_1(r,\lambda)\,f(r)\,dr$$

and $$\tilde{F}(\lambda) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^\omega y_s(r,\lambda)\,f(r)\,dr$$

The last two integrals converge in $L_2^{\rho}(0,\infty)$ and $L_2^{\tilde{\rho}}(0,\infty)$ respectively.

Also by (4.4.15), (4.5.3), (4.5.4) and (4.5.6),

$$u_1(1,z)\,y_s(r,z) = \left[y_1(r,z) + \frac{u_1'(1,z)}{u_1(1,z)}\,y_2(r,z)\right]u_1(1,z)$$

$$= [u_1'(1,z)u_2(r,z) - u_2'(1,z)u_1(r,z)]\,u_1(1,z) + [u_2(1,z)u_1(r,z) - u_1(1,z)u_2(r,z)]\,u_1'(1,z)$$

$$= W(u_2(1,z),\,u_1(1,z))\,u_1(r,z)$$

$$= u_1(r,z)$$

Hence, by Proposition 4.11, (4.5.8) is but an alternative expression of (4.5.7), and so the corollary is proved.

Thus, where L is regular at 0 and V(r) is in $L_1[0,\infty)$, the Weyl-Kodaira expansion (4.1.6) simplifies to the expansion (2.4.4) described in Chapter II for all f in $\mathcal{H}_{a.c.}(H)$; it seems probable that this is also true for f(r) in the singular subspace $\mathcal{H}_s(H)$. If this is so, our simplified expansion is a natural extension of the expansion (2.4.4) for all f in $\mathcal{H}$.

We observe that when L is limit point at 0, the solutions of $Lu = \lambda u$ which feature in the transformation $\tilde{S}$ of Theorem 4.9 are, for $\tilde{\mu}$-almost all λ in $\mathbb{R}$, subordinate at 0 for $\lambda > 0$ and subordinate at both 0 and infinity for $\lambda < 0$. Comparing Theorem 4.9 with the analogous results for the regular limit circle case at 0 (see Ch.II, §4), we note that, as in the decomposition of the spectrum, subordinate solutions in the limit point case at 0 correspond to solutions satisfying the boundary condition in the limit circle case at 0.

According to the brief summary which is available in translation ([K2] §2), it appears to have been shown by Kac that a simplified expansion exists provided the intersection of the sets $\{ \lambda \in \mathbb{R} : \frac{d\mu_0(\lambda)}{dK}$ exists and $0 < \frac{d\mu_0(\lambda)}{dK} < \infty \}$ and $\{ \lambda \in \mathbb{R} : \frac{d\mu_\infty(\lambda)}{dK}$ exists and $0 < \frac{d\mu_\infty(\lambda)}{dK} < \infty \}$ has Lebesgue measure zero ([K1]); here μ_0 and μ_∞ are the spectral measures of H_1 and H_∞ respectively. This would imply by Theorems 2.9 and 3.21 that such an expansion exists provided a solution of Lu = xu exists which is subordinate at 0 or at ∞ (or both) for Lebesgue almost all λ in $\mathbb{R}$. It may well be the case, therefore, that the simplified expansion of Theorem 4.9 and the conclusions of Theorem 4.10 hold under weaker conditions than we have assumed.

However, the question of whether the simplified isometric transformation is surjective, and the relationship between the simplified expansion and expansions such as (2.4.4) which are obtained directly, do not appear to have been considered by Kac, nor is the role of subordinate solutions recognised. From the point of view of the applications to scattering theory which we condition in the following chapter, the conditions of Theorem 4.9 are sufficient, and the surjective property of the simplified transformation $\tilde{S}$, which is proved in the Appendix, is essential.

In conclusion, we note that under the conditions we have imposed on the potential in this chapter,

$$\frac{1}{\pi^2} \operatorname{Im} M_{11}+(\lambda) \operatorname{Im} M_{22}+(\lambda) = \frac{d\rho_{11}(\lambda)}{d\lambda} \frac{d\rho_{22}(\lambda)}{d\lambda}$$
$$= \frac{d\rho_{12}(\lambda)}{d\lambda} \frac{d\rho_{21}(\lambda)}{d\lambda} = \frac{1}{\pi^2} \operatorname{Im} M_{12}+(\lambda) \operatorname{Im} M_{21}+(\lambda) \quad (4.5.9)$$

for Lebesgue almost all λ in $(0,\infty)$, by (4.2.1), (4.2.2), (4.2.3), (4.3.10), (4.3.12) and (4.4.3). If we suppose that there is some absolutely continuous spectrum of H_1 in $(0,\infty)$, so that by Corollary 2.7 and Proposition 2.14 there exists a subset S of $(0,\infty)$ with positive Lebesgue measure such that $0 < \operatorname{Im} m_0+(\lambda) < \infty$ for all λ in S, then the relationships (4.5.9) cannot hold. To see this, note from the proof of Lemma 4.2 that $\operatorname{Im} M_{11}+(\lambda) \operatorname{Im} M_{22}+(\lambda) - \operatorname{Im} M_{12}+(\lambda) \operatorname{Im} M_{21}+(\lambda) = 0$ Lebesgue almost everywhere on $(0,\infty)$ only if $\operatorname{Im} m_0+(\lambda) \operatorname{Im} m_\infty+(\lambda) = 0$ Lebesgue almost everywhere on $(0,\infty)$; that is, since $\operatorname{Im} m_\infty^+(\lambda) > 0$ for all λ, only if $\operatorname{Im} m_0+(\lambda) = 0$ Lebesgue almost everywhere on $(0,\infty)$. Thus by Lemma 2.13 and Corollary 2.7, the relationships (4.5.9) can only hold if the spectrum of H_1 is singular. Since these relationships are crucial to the simplification of the Weyl-Kodaira theorem, (without them the results of Lemma 4.8 fail), it follows that, if V(r) is in $L_1[1,\infty)$, the conclusions of Theorem 4.9 only hold when the spectrum of H_1 is singular on $(0,\infty)$.

CHAPTER V

APPLICATIONS TO SCATTERING THEORY

§1 Wave and Scattering Operators

We now apply some of the results of the previous chapter to the scattering of a single non-relativistic particle in a spherically symmetric potential. As in spectral analysis, the three dimensional situation is most conveniently analysed in this case by considering each partial wave subspace separately (see [AJS], Ch.11).

We briefly indicate some of the relevant ideas and terminology. With fixed quantum numbers l and m, representing a fixed partial wave subspace, the one dimensional free Hamiltonian $H_{o,l}$ is the self-adjoint operator arising from the differential expression

$$L_{o,l} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \qquad (5.1.1)$$

In the case $l = 0$, $L_{0,l}$ is in the limit circle case at 0, so that $H_{o,o}$ is not unique; it is necessary, therefore to fix a boundary condition of the form (2.3.9). Now, defining the free Hamiltonian H_o in the customary way to be the unique self-adjoint extension of $-\Delta$ acting on $C_0^\infty(\mathbb{R}^3)$, it is found that $H_{o,o}$ is unitarily equivalent to the restriction of H_o to the angular momentum subspace $l = 0$, which is unique; it follows that the appropriate boundary condition is obtained by setting $\alpha = 0$ in (2.3.9) (see [AJS], §11.3).

The corresponding total Hamiltonian H_l arises from the differential expression

$$L_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \qquad (5.1.2)$$

A boundary condition may again be required at r = 0 for some or all $l \geqslant 0$, depending on the nature of V(r); since we shall assume throughout this chapter that V(r) is integrable at infinity, no boundary condition will be required at ∞ for any l ([HI] Thm.10.1.4).

Particles encountering the potential V(r) characteristically exhibit one of two modes of behaviour; either at large positive and negative times the particle is arbitrarily far from the scattering centre, or at all times it is located within a finite radius of the scatterer; the particle is said to be in a <u>scattering state</u>, or a <u>bound state</u>, respectively. In most of the cases usually considered, the scattering states span the absolutely continuous subspace $\mathcal{H}_{a.c.}(H)$ of H, and the bound states are identified with linear combinations of eigenvectors of H. This situation may break down, for example, if some states are asymptotically absorbed ([P2]); however, as we shall see, absorption cannot occur for the class of potentials considered in this chapter.

If for large positive and negative times, all particles in scattering states behave like free particles the system is said to be asymptotically complete. This idea may be formulated in a way that is mathematically more precise, using wave operators. Noting that e^{-iH_0t}, e^{-iHt} describe the free and perturbed time evolution of a state vector f, the <u>wave operators</u> are defined to be

$$\Omega_{\pm} = \underset{t \to \pm\infty}{\text{s. lim.}}\ e^{iHt} e^{-iH_0t} E_{a.c.}(H_0) \tag{5.1.3}$$

whenever these limits exist, where $E_{a.c.}(H_0)$ is the projection operator onto the absolutely continuous subspace $\mathcal{H}_{a.c.}(H_0)$ of H_0. The wave operators are partial isometries with initial set $\mathcal{H}_{a.c.}(H_0)$ and ranges subspaces of the absolutely continuous subspace $\mathcal{H}_{a.c.}(H)$ of H. If the ranges of $\Omega_{\pm}$ are equal to $\mathcal{H}_{a.c.}(H)$ the wave operators are said to be <u>asymptotically complete</u>; if, in addition, the singular continuous subspace $\mathcal{H}_{s.c.}(H)$ of H is empty, we refer to <u>strong asymptotic completeness</u> (cf. [AJS] §9.1).

The wave operators satisfy the following <u>intertwining relations</u>

$$\begin{aligned} e^{-iHt}\Omega_{\pm} &= \Omega_{\pm} e^{-iH_0t} \\ e^{-iH_0t}\Omega_{\pm}^{*} &= \Omega_{\pm}^{*} e^{-iHt} \end{aligned} \tag{5.1.4}$$

where $\Omega_\pm^*$ denote the adjoints of $\Omega_\pm$, viz.

$$\underset{t \to \pm\infty}{\text{s. lim.}} \; e^{iH_0 t} e^{-iHt} M^\pm \tag{5.1.5}$$

$M^\pm$ being the projection operators onto the ranges of $\Omega_\pm$.

Where the wave operators exist, the <u>scattering operator</u> S is defined by

$$S = \Omega_+^* \Omega_- \tag{5.1.6}$$

The scattering operator is a partial isometry with initial set $\mathcal{H}_{a.c.}(H_0)$ and range a subspace of $\mathcal{H}_{a.c.}(H_0)$. The range of S is equal to $\mathcal{H}_{a.c.}(H_0)$ if and only if the ranges of Ω_+ and Ω_- are equal, so that, in particular, S is unitary if the wave operators are asymptotically complete ([AJS], Prop .4.8). Moreover, S commutes with H_0; that is

$$S H_0 E_{a.c.}(H_0) = H_0 S$$

which implies that the unperturbed energy is conserved during the scattering process.

We may apply Theorem 4.9 to see that for each l, the operator $H_{0,l}$ in $L_2(0,\infty)$ is unitarily equivalent to multiplication by λ in $L_2^{\tilde{\rho}_{0,l}}(0,\infty)$, where $\tilde{\rho}_{0,l}(\lambda)$ is the simplified spectral function of $H_{0,l}$. Note that, if we take $\alpha = 0$ in (2.3.9) for the case $l = 0$, then for each l, $H_{0,l}$ has purely absolutely continuous spectrum which is concentrated on $(0,\infty)$, so that $L_2^{\tilde{\rho}_{0,l}}(0,\infty) = L_2^{\tilde{\rho}_{0,l}}(-\infty,\infty)$, $\mathcal{H}_{a.c.}(H_0) = \mathcal{H}$ and $E_{a.c.}(H_0) = I$.

Similarly, it has been shown (and we shall derive this result independently during our proofs) that when the wave operators are asymptotically complete, the restriction S_l of S to a partial wave subspace is unitarily equivalent to multiplication by a function of λ, $S_l(\lambda)$ for each l, where

$$S_l(\lambda) = \exp(2i\delta_l(\lambda)) \tag{5.1.7}$$

The function $S(\lambda)$ defined by

$$S(\lambda) = \sum_{l,m} S_l(\lambda) E_{lm} \tag{5.1.8}$$

where E_{lm} is the projection operator onto the one dimensional subspace of unit sphere $L_2(S^2)$ generated by the spherical harmonic Y_{lm}, is known as the S-matrix , and $\delta_l(\lambda)$ is known as the partial wave phase shift ([AJS] Prop.11.6).

The following definitions relate to the scattering of particles with energy λ.

The scattering X-section for a cone C with apex at the scattering centre is the number of particles scattered into C per unit time divided by the number of particles in the incoming beam per unit time and per unit surface area of the hyperplane orthogonal to the direction of motion of the incoming particles.

Now suppose that the incoming particles are approximately collimated in the direction $\underline{\omega}_1$, and that the axis of C lies in the direction $\underline{\omega}_2$. If the scattering X-section for C is divided by the magnitude of the solid angle $\Delta\omega$ subtended by C at its apex, then the square root of the limit of this quantity as $\Delta\omega \to 0$ is known as the scattering amplitude at energy λ , and is written $f(\lambda : \underline{\omega}_1 \to \underline{\omega}_2)$.

The square of the scattering amplitude, integrated over all final directions $\underline{\omega}_2$ gives the total scattering X-section $\Omega(\lambda)$.

Where the potential is spherically symmetric, and the wave operators are asymptotically complete the scattering amplitude and total cross section have the following representations:

$$f(\lambda : \underline{\omega}_1 \to \underline{\omega}_2) = \frac{1}{2i\sqrt{\lambda}} \sum_l (2l+1)(S_l(\lambda)-1) P_l(\underline{\omega}_1 \cdot \underline{\omega}_2) \qquad (5.1.9)$$

where P_l is the Legendre polynomial of degree l,

$$\Omega(\lambda) = \frac{1}{4\pi\lambda} \sum_l (2l+1) \sin^2 \delta_l(\lambda) \qquad (5.1.10)$$

(see [AJS] Prop. 11.7). Note that $\Omega(\lambda)$ is independent of the initial direction $\underline{\omega}_1$, and that, by (5.1.7), both $f(\lambda : \underline{\omega}_1 \to \underline{\omega}_2)$ and $\Omega(\lambda)$ are sums of continuous functions of the partial wave phase shifts.

In this chapter we shall derive explicit formulae for the phase shifts in cases where the potential may be sufficiently pathological at the origin to produce dense point or singular continuous spectrum of $H_{1,l}$ in at least one partial wave subspace. The only conditions we shall require of our spherically symmetric potential are:

(i) $V(r) = O(r^{-(1+\varepsilon)})$ as $r \to \infty$.

(ii) the spectrum of $H_{1,l}$ is singular in each partial wave subspace (for description of H_1, see Chapter IV).

The condition at 0 is considerably more general than that considered by Green and Lanford ([GR]). These authors required that V(r) be $O(r^{-(2-\varepsilon)})$ as $r \to 0$ which ensures that there is at least one solution of $L_1 u = xu$ in $L_2(0,1]$ for each x in $\mathbb{R}$, and all l (see [KO] §5). This implies that there is no singular continuous spectrum of $H_{1,l}$ for any l, by Theorem 3.21, and that the spectrum of $H_{1,l}$ is nowhere dense ([WE2] Satz 3.3).

Using the simplified expansion of Chapter IV, we shall adopt a method similar to that of Green and Lanford, and, as in their derivation, the existence and completeness of the wave operators will be demonstrated in the course of the proof. The existence of the wave operators and asymptotic completeness under conditions (i) and (ii) may be proved independently from other results which are already known. Kupsh and Sandhas have shown that the wave operators exist whenever the potential dies away at infinity more rapidly than the Coulomb potential $\frac{1}{r}$, irrespective of the behaviour of the potential at 0. ([KS]). Moreover, it has been proved by Kuroda that provided the wave operators exist, the absolutely continuous spectrum of H is contained in that of H_o, and the spectrum of H is simple in each partial wave subspace, then the theory is asymptotically complete ([KU1] Thm. 3.3; see also [DE] for amendment). Kuroda's second condition is satisfied on account of Proposition 4.5, and that the third condition is satisfied follows from a theorem of Kac, which proves simplicity of the spectrum whenever the Lebesgue measure of the set

$$\{x \in \mathbb{R} : 0 < \frac{d\mu_1(x)}{dK} < \infty\} \cap \{x \in \mathbb{R} : 0 < \frac{d\mu_\infty(x)}{dK} < \infty\}$$

is zero, where μ_1, μ_∞ are the spectral measures of H_1 and H_∞ in some partial wave subspace ([K2]). Under condition (i), therefore, asymptotic completeness can only fail if there is absolutely continuous spectrum of $H_{1,l}$ for some l. This, while unusual, can occur; an example is due to Pearson ([P2]).

The explicit formulae we shall obtain for the phase shifts will enable us to refute the accepted wisdom that the scattering amplitude and total cross section are continuous functions of energy. First, however, we shall reformulate the simplified expansion of Theorem 4.9 for elements of $\mathcal{H}_{a.c.}(H)$ in such a way that the simplified spectral function $\tilde{\rho}(\lambda)$ no longer occurs explicitly.

§2. Reformulation of the simplified expansion theorem

As we shall be solely concerned with a single partial wave subspace in both this and the following section, we shall as a matter of convenience regard the term $\frac{l(l+1)}{r^2}$ as included in the potential V(r), and denote the operators $H_{o,l}$, H_l by H_o and H respectively.

We begin by showing that if, with the hypothesis and notation of Theorem 4.9, the domain of $\tilde{S}$ is restricted to $\mathcal{H}_{a.c.}(H)$ then its range is $L_2^{\tilde{\rho}}(0,\infty)$, and this restriction of $\tilde{S}$ is an isometric Hilbert space isomorphism from $\mathcal{H}_{a.c.}(H)$ onto $L_2^{\tilde{\rho}}(0,\infty)$.

5.1 Proposition: Let V(r) be in $L_1[1,\infty)$ and suppose the spectrum of H_1 is singular. Then each f(r) in $\mathcal{H}_{a.c.}(H)$ has the eigenfunction expansion

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{(0,\omega]} y_s(r,\lambda)\,\tilde{F}(\lambda)\,d\tilde{\rho}(\lambda) \qquad (5.2.1)$$

where

$$\tilde{F}(\lambda) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{\sigma}^{\omega} y_s(r,\lambda)\,f(r)\,dr \qquad (5.2.2)$$

the integrals being convergent in $L_2(0,\infty)$ and $L_2^{\tilde{\rho}}(0,\infty)$ respectively. The transformation $\tilde{S}_{a.c.}$ which maps f(r) in $\mathcal{H}_{a.c.}(H)$ to $\tilde{F}(\lambda)$ is an isometric Hilbert space isomorphism from $L_2(0,\infty)$ onto $L_2^{\tilde{\rho}}(0,\infty)$. Moreover, if $\theta : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function, and $\theta(H)f(r)$ is in $L_2(0,\infty)$, then

$$\theta(H)f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{(0,\omega]} \theta(\lambda)\, y_s(r,\lambda)\, \tilde{F}(\lambda)\, d\tilde{\rho}(\lambda) \qquad (5.2.3)$$

where the integral converges in $L_2(0,\infty)$.

Proof:

By Proposition 4.4 the spectrum of H is singular on $(-\infty,0]$ and by Proposition 4.5, H has no singular spectrum on $(0,\infty)$. Hence f(r) is in $\mathcal{H}_{a.c.}(H)$ if and only if the probability $\int_{(a,b]} d\langle f, E_\lambda f\rangle$ that a measurement of the total energy of a system in the state f will yield a value in (a,b] is zero for all $(a,b] \subseteq (-\infty,0]$ (see [AJS] Ch.3, §2) So if f(r) is in $\mathcal{H}_{a.c.}(H)$ we have by (2.4.8)

$$f(r) = \underset{\omega\to\infty}{\text{s. lim.}} (E_\omega - E_0) f(r)$$

where $\{E_\lambda\}$ is the spectral family of H. The eigenfunction expansion above for f(r) now follows from Theorem 4.9.

To prove that $\tilde{S}_{a.c.}$ is an isometric Hilbert space isomorphism from $\mathcal{H}_{a.c.}(H)$ onto $L_2^{\tilde{\rho}}(0,\infty)$ it is only necessary in view of Theorem 4.9 and the fact that $\mathcal{H}_{a.c.}(H)$ and $L_2^{\tilde{\rho}}(0,\infty)$ are subspaces of $\mathcal{H}$ and $L_2^{\tilde{\rho}}(-\infty,\infty)$ respectively, to show that

(i) for each f(r) in $\mathcal{H}_{a.c.}(H)$, $(\tilde{S}_{a.c.}f)(\lambda)$ is zero $\tilde{\mu}$-almost everywhere on $(-\infty,0)$.

(ii) for each $\tilde{F}(\lambda)$ in $L_2^{\tilde{\rho}}(0,\infty)$ which is zero $\tilde{\mu}$-almost everywhere on $(-\infty,0)$, $(\tilde{S}_{a.c.}^{-1}\tilde{F})(r)$ is in $\mathcal{H}_{a.c.}(H)$.

If f(r) is in $\mathcal{H}_{a.c.}(H)$ and $\tilde{F}(\lambda)$ denotes $(\tilde{S}f)(\lambda)$ then by (4.4.16)

$$(E_b - E_a) f(r) = \int_{(a,b]} y_s(r,\lambda)\, \tilde{F}(\lambda)\, d\tilde{\rho}(\lambda) = 0$$

for all $(a,b] \subseteq (-\infty,0]$. Hence, using the surjective property of $\tilde{S}$ proved in the Appendix, $\tilde{F}(\lambda)=0$ $\tilde{\mu}$-almost everywhere on $(-\infty,0]$; since $(\tilde{S}_{a.c.}f)(\lambda)=(\tilde{S}f)(\lambda)$, (i) is established.

If $\tilde{F}(\lambda)$ is in $L_2^{\tilde{\rho}}(0,\infty)$ and is zero $\tilde{\mu}$-almost everywhere on $(-\infty,0]$, then $((E_b-E_a)(S_{a.c.}^{-1}\tilde{F}))(\lambda)=0$ for every $(a,b]\subseteq(-\infty,0]$. Hence the probability that a measurement of the total energy of a system in the state $(\tilde{S}_{a.c.}^{-1}\tilde{F})(r)$ will yield a value in $(-\infty,0]$ is zero, and so $(\tilde{S}_{a.c.}^{-1}\tilde{F})(r)$ is in $\mathcal{H}_{a.c.}(H)$. We have now proved (ii).

(5.2.1) follows from (4.4.16), by (i).

The proof of the proposition is now complete.

We remark that if we take $V(r) = \dfrac{l(l+1)}{r^2}$ for each l, Proposition 5.1 applies also to H_0 in each partial wave subspace. In general, $\tilde{\rho}(\lambda)$ depends both on the potential and on l.

We now state a result which will enable us to show that $y_s(r,\lambda)$ is bounded as $r\to\infty$. The proof is elementary and may be found in [T2], Chapter V, (Lemma 5.2); see also [W], Chapter I §1, III for a fuller account.

5.2 Gronwall's Inequality: Suppose $g(r)$, $h(r) \geq 0$, $g(r)$ is continuous, and $h(r)$ is integrable on $[R,\infty)$. If also there exists C in $\mathbb{R}^+$ such that

$$g(r) \leq C + \int_R^r g(s)h(s)\,ds$$

for all r in $[R,\infty)$, then

$$g(r) \leq C\exp\left(\int_R^r h(s)\,ds\right)$$

for all r in $[R,\infty)$.

This enables us to prove that if the spectrum of H_1 is singular, the solution $y_s(r,\lambda)$ of $Lu=\lambda u$ asymptotically approaches a solution of $-\dfrac{d^2u}{dr^2}=\lambda u$ as $r\to\infty$ for Lebesgue almost all λ in $(0,\infty)$.

5.3 Lemma: Let $V(r)$ be in $L_1[1,\infty)$ and suppose the spectrum of H_1 is singular. Then $y_s(r,\lambda)$ is defined, bounded on the r-interval $[1,\infty)$, and converges pointwise to a function of the form $g(\lambda)\sin(\sqrt{\lambda}r+\delta(\lambda))$ for Lebesgue almost all $\lambda > 0$.

Proof:

By (4.4.15), $y_s(r,\lambda)$ is defined on $(0,\infty)$ whenever $m_0^+(\lambda)$ exists as a finite real limit; since the spectrum of H_1 is singular this is Lebesgue almost everywhere on $(0,\infty)$ by Corollary 2.7(ii), Lemma 2.13, and Theorem 2.12(i).

Using the variation of constants formula ([CL] Ch.3, Thm. 6.4),

$$\begin{aligned} y_s(r,\lambda) &= \cos(\sqrt{\lambda}(r-1)) + m_0(\lambda)\frac{\sin(\sqrt{\lambda}(r-1))}{\sqrt{\lambda}} \\ &\quad + \frac{\sin(\sqrt{\lambda}(r-1))}{\sqrt{\lambda}}\int_1^r \cos(\sqrt{\lambda}(p-1))V(p)\,y_s(p,\lambda)\,dp \\ &\quad - \frac{\cos(\sqrt{\lambda}(r-1))}{\sqrt{\lambda}}\int_1^r \sin(\sqrt{\lambda}(p-1))V(p)\,y_s(p,\lambda)\,dp \end{aligned} \tag{5.2.4}$$

for all $r \geq 1$, and all $\lambda > 0$ for which $y_s(r,\lambda)$ is defined. For such r and λ, we have by Minkowski's inequality

$$|y_s(r,\lambda)| \leq 1 + \frac{|m_0(\lambda)|}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_1^r |V(p)|\,|y_s(p,\lambda)|\,dp$$

Identifying $1 + \dfrac{|m_0(\lambda)|}{\sqrt{\lambda}}$ with C, $\dfrac{|V(r)|}{\sqrt{\lambda}}$ with h(r)

and $|y_s(r,\lambda)|$ with g(r), it follows from Gronwall's Inequality that $y_s(r,\lambda)$ is a bounded function of r on $[1,\infty)$ for each fixed $\lambda > 0$ for which $y_s(r,\lambda)$ is defined.

If we set

$$\beta(\lambda) = 1 - \int_1^\infty \frac{\sin(\sqrt{\lambda}(p-1))}{\sqrt{\lambda}}V(p)\,y_s(p,\lambda)\,dp \tag{5.2.5}$$

and

$$\gamma(\lambda) = \frac{m_0(\lambda)}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_1^\infty \cos(\sqrt{\lambda}(p-1))V(p)\,y_s(p,\lambda)\,dp \tag{5.2.6}$$

we have from (5.2.4)

$$y_s(r,\lambda) = \beta(\lambda)\cos(\sqrt{\lambda}(r-1)) + \gamma(\lambda)\sin(\sqrt{\lambda}(r-1))$$

$$- \int_r^\infty \frac{\sin(\sqrt{\lambda}(r-p))}{\sqrt{\lambda}} V(p)\, y_s(p,\lambda)\, dp \qquad (5.2.7)$$

Since $y_s(r,\lambda)$ is bounded for $r \geqslant 1$, and V(r) is integrable on $[1,\infty)$ it is clear that the final term converges to zero as $r \to \infty$. Thus as $r \to \infty$ we have for Lebesgue almost all $\lambda > 0$

$$y_s(r,\lambda) \to [\beta^2(\lambda) + \gamma^2(\lambda)]^{\frac{1}{2}} \sin(\sqrt{\lambda}r + \delta(\lambda))$$

where

$$\tan(\delta(\lambda) + \sqrt{\lambda}) = \frac{\beta(\lambda)}{\gamma(\lambda)} \qquad (5.2.8)$$

Setting $g^2(\lambda) = \beta^2(\lambda) + \gamma^2(\lambda)$, the lemma is proved.

We now show that the factor $g(\lambda)$ which occurs in Lemma 5.3 also occurs in the derivative of the simplified spectral function $\tilde{\rho}(\lambda)$ Lebesgue almost everywhere on $(0,\infty)$.

5.4 Lemma: If V(r) is in $L_1[1,\infty)$ and the spectrum of H_1 is singular then

$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = \frac{1}{\pi\sqrt{\lambda}\, g^2(\lambda)}$$

for Lebesgue almost all $\lambda > 0$, where $g(\lambda)$ is as in Lemma 5.3.

Proof:

With the given conditions,

$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = \frac{\operatorname{Im} m_{\infty^+}(\lambda)}{\pi\,|m_0(\lambda) - m_{\infty^+}(\lambda)|^2}$$

Lebesgue almost everywhere on $(0,\infty)$ by (4.3.10) and (4.4.14). Hence for these λ, from (4.3.3) and (4.3.4)

$$\frac{d\tilde{\rho}(\lambda)}{d\lambda} = \frac{1}{\pi\sqrt{\lambda}\,([\sigma_1(\lambda) + m_0(\lambda)\sigma_2(\lambda)]^2 + [\tau_1(\lambda) + m_0(\lambda)\tau_2(\lambda)]^2)}$$

where $\sigma_1(\lambda), \sigma_2(\lambda), \tau_1(\lambda), \tau_2(\lambda)$ are as defined in (4.3.2). However, by inspection of (4.3.2), (5.2.5) and (5.2.6) we see that

$$\sigma_1(\lambda) + m_0(\lambda)\,\sigma_2(\lambda) = \beta(\lambda)$$

$$\tau_1(\lambda) + m_0(\lambda)\,\tau_2(\lambda) = \gamma(\lambda)$$

whenever $m_0(\lambda)$ is defined; that is, Lebesgue almost everywhere on $(0,\infty)$ Since $g^2(\lambda) = \beta^2(\lambda) + \gamma^2(\lambda)$ the desired relation follows, and the lemma is proved.

If we define

$$v_s(r,\lambda) = \frac{y_s(r,\lambda)}{g(\lambda)} \tag{5.2.9}$$

and

$$\phi(\lambda) = \frac{\tilde{F}(\lambda)}{g(\lambda)} \tag{5.2.10}$$

for those λ in $(0,\infty)$ for which $y_s(r,\lambda)$ is defined, we see from Proposition 5.1 and Lemma 5.4 that each f(r) in $\mathcal{H}_{a.c.}(H)$ has the eigenfunction expansion

$$f(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}} \int_0^{\omega} v_s(r,\lambda)\,\phi(\lambda)\,\frac{1}{\pi\sqrt{\lambda}}\,d\lambda \tag{5.2.11}$$

where

$$\phi(\lambda) = \underset{\substack{\omega\to\infty\\ \sigma\to 0}}{\mathrm{l.i.m.}} \int_{\sigma}^{\omega} v_s(r,\lambda) f(r)\,dr \tag{5.2.12}$$

the limits being convergent in $L_2(0,\infty)$ and $L_2^{\lambda}(0,\infty)$ respectively, where $L_2^{\lambda}(0,\infty)$ is the Hilbert space of functions $h(\lambda)$ for which

$$\int_0^{\infty} |h(\lambda)|^2 \frac{1}{\pi\sqrt{\lambda}}\,d\lambda < \infty$$

We observe that, from the isometric property of $\tilde{S}_{a.c.}$ (see Proposition 5.1), Lemma 5.4 and (5.2.10),

$$\left(\int_0^{\infty} |f(r)|^2\,dr\right)^{\frac{1}{2}} = \left(\int_0^{\infty} |\phi(\lambda)|^2 \frac{1}{\pi\sqrt{\lambda}}\,d\lambda\right)^{\frac{1}{2}} \tag{5.2.13}$$

Similarly from (5.2.3), if f(r) is in $\mathcal{H}_{a.c.}(H)$, $\theta : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function, and $\theta(H)f(r)$ is in $L_2(0,\infty)$, then by

Lemma 5.4, (5.2.9) and (5.2.10)

$$\theta(H)f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^{\omega} \theta(\lambda)\, v_s(r,\lambda)\, \phi(\lambda) \frac{1}{\pi\sqrt{\lambda}}\, d\lambda \qquad (5.2.14)$$

where the integral converges in $L_2(0,\infty)$.

As convenient, we shall use the formulation of Proposition 5.1 or (5.2.11)-(5.2.14) above when deriving an explicit expression for the phase shift. We shall sometimes also use a modified version of (5.2.11)-(5.2.14) above, obtained by substituting $\lambda = k^2$.

§3. An explicit formula for the phase shift

The strategy we shall use in deriving an explicit formula for the phase shift in a given partial wave subspace follows closely that of Green and Lanford ([GR]). However, as we noted in §1, our class of potentials contains elements whose behaviour at 0 is more singular than any considered by these authors; consequently, we may not assume certain properties of the solutions $\{v_s(r,\lambda)\}$ which were conveniently utilised in their proof. For example, we may not assume that $v_s(r,\lambda)$ is bounded or even integrable on any r-interval containing the origin, nor may we suppose that for fixed r, $v_s(r,\lambda)$ is a continuous function of λ. As it is frequently necessary to depart from the methods of Green and Lanford, we consider it best to present our results in full.

In this section, we shall prove the existence and completeness of the wave operators under conditions (i) and (ii) of §1, and an explicit formula for the phase shift will emerge incidentally. We note that the proof of the existence of the wave operators is formally the same as that of asymptotic completeness, the roles of H and H_0, and of free and scattering states, being reversed. We shall not therefore give separate proofs for existence and completeness, but merely indicate, when appropriate, the necessary adjustments required in either case.

To give an indication of the method, we outline the stages of the proof for the case of asymptotic completeness. The idea is to "find" for each scattering state, the free states to which it converges at large positive and negative times. The first step is to show that, if f(r) is in $\mathcal{H}_{a.c.}(H)$, and if we replace the solution $v_s(r,\lambda)$ in the right hand side of

$$f_t(r) = e^{-iHt}f(r) = \underset{\omega\to\infty}{\text{l.i.m.}}\ \frac{1}{\pi}\int_0^{\omega} e^{-i\lambda t} v_s(r,\lambda)\,\phi(\lambda)\,\frac{1}{\sqrt{\lambda}}\,d\lambda \tag{5.3.1}$$

by its pointwise asymptotic limit $\sin(\sqrt{\lambda}\,r + \delta(\lambda))$, then the resulting expression, which we shall denote by $\mathring{f}_t(r)$, is well-defined, and $f_t(r)$ converges strongly to $\mathring{f}_t(r)$ at large positive and negative times. Now $\mathring{f}_t(r)$ does not represent the time evolution of a free state; however it may be expressed as the sum of two time dependent functions, one of which converges strongly to a free state at large positive times and to zero at large negative times, and the other to zero at large positive times and to a free state at large negative times. Therefore $f_t(r)$ converges strongly to these free states at large positive and negative times, and completeness follows from the arbitrary choice of f(r). Note that these remarks refer to a fixed partial wave subspace; the general results will follow quite simply once the particular results for each subspace are established.

We follow the procedure outlined above, and note that (5.3.1) follows from (5.2.14), and that $v_s(r,\lambda) \to \sin(\sqrt{\lambda}\,r + \delta(\lambda))$ as $r \to \infty$ by (5.2.9) and Lemma 5.3. For f(r) in $\mathcal{H}_{a.c.}(H)$ define $f_{tN}(r) = (E_N - E_0)f_t(r)$, where $f_t(r) = e^{-iHt} f(r)$; by (5.2.14)

$$f_{tN}(r) = \frac{1}{\pi}\int_0^{N} e^{-i\lambda t}\, v_s(r,\lambda)\,\phi(\lambda)\,\frac{1}{\sqrt{\lambda}}\,d\lambda \tag{5.3.2}$$

Replacing $v_s(r,\lambda)$ by $\sin(\sqrt{\lambda}\,r + \delta(\lambda))$, we obtain:

5.5 <u>Lemma</u> : For each fixed N and t,

$$\mathring{f}_{tN}(r) = \frac{1}{\pi}\int_0^{N} e^{-i\lambda t}\sin(\sqrt{\lambda}\,r + \delta(\lambda))\,\phi(\lambda)\,\frac{1}{\sqrt{\lambda}}\,d\lambda$$

is in $L_2(0,\infty)$ and $\{\mathring{f}_{tN}(r)\}$ converges uniformly over t as $N \to \infty$ in the topology of $L_2(0,\infty)$.

Proof:

$$\overset{\circ}{f}_{tN}(r) = \frac{1}{\pi}\int_0^N e^{-i\lambda t}\sin\sqrt{\lambda}\,r\cos\delta(\lambda)\,\phi(\lambda)\frac{1}{\sqrt{\lambda}}\,d\lambda$$

$$+\frac{1}{\pi}\int_0^N e^{-i\lambda t}\cos\sqrt{\lambda}\,r\sin\delta(\lambda)\,\phi(\lambda)\frac{1}{\sqrt{\lambda}}\,d\lambda \qquad (5.3.3)$$

Using the theory of Chapter II, §4, especially (2.4.1) - (2.4.4), we see that if $H_o^{\,s}$ and $H_o^{\,c}$ are the self-adjoint operators arising from $-\dfrac{d^2u(r,\lambda)}{dr^2} = \lambda\,u(r,\lambda)$ with boundary conditions $\alpha = 0$ and $\alpha = \dfrac{3\pi}{2}$ respectively (see (2.3.9)), then the associated spectral functions satisfy $\dfrac{d\rho_o^{\,s}(\lambda)}{d\lambda} = \sqrt{\lambda}$ and $\dfrac{d\rho_o^{\,c}(\lambda)}{d\lambda} = \dfrac{1}{\sqrt{\lambda}}$ respectively for $\lambda > 0$, and $u_1(r,\lambda,0) = \dfrac{\sin\sqrt{\lambda}\,r}{\sqrt{\lambda}}$, $u_1(r,\lambda,\frac{3\pi}{2}) = \cos\sqrt{\lambda}\,r$. Since, by (5.2.13),

$$\int_0^N |\phi(\lambda)|^2\frac{1}{\pi\sqrt{\lambda}}\,d\lambda \leq \int_0^\infty |f(r)|^2\,dr < \infty$$

$\dfrac{\cos\delta(\lambda)\,\phi(\lambda)}{\sqrt{\lambda}}$ and $\sin\delta(\lambda)\,\phi(\lambda)$ are in $L_2^{\rho_o^s}(0,\infty)$ and $L_2^{\rho_o^c}(0,\infty)$ respectively. Hence by (2.4.3) and (2.4.5) there exist functions $h_s(r)$ and $h_c(r)$ in $\mathcal{H}_{a.c.}(H_o^s) = L_2(0,\infty)$ and $\mathcal{H}_{a.c.}(H_o^c)$ respectively such that

$$(E_N^s - E_o^s)\,e^{-iH_o^s t}h_s(r) = \frac{1}{\pi}\int_0^N e^{-i\lambda t}\sin\sqrt{\lambda}\,r\cos\delta(\lambda)\,\phi(\lambda)\frac{1}{\sqrt{\lambda}}\,d\lambda \qquad (5.3.4)$$

$$(E_N^c - E_o^c)\,e^{-iH_o^c t}h_c(r) = \frac{1}{\pi}\int_0^N e^{-i\lambda t}\cos\sqrt{\lambda}\,r\sin\delta(\lambda)\,\phi(\lambda)\frac{1}{\sqrt{\lambda}}\,d\lambda \qquad (5.3.5)$$

where $\{E_\lambda^s\}$, $\{E_\lambda^c\}$ are the spectral families of $H_o^{\,s}$ and $H_o^{\,c}$ respectively. It follows from (5.3.3) that

$$\|\overset{\circ}{f}_{tN}(r)\| \leq \|h_s(r)\| + \|h_c(r)\|$$

where $\|\cdot\|$ denotes the $L_2(0,\infty)$ norm; hence $\overset{\circ}{f}_{tN}(r)$ is in $L_2(0,\infty)$ for each fixed N and t. Moreover, by (5.3.3),(5.3.4) and (5.3.5),

$$\|\overset{\circ}{f}_{tN}(r) - \overset{\circ}{f}_{tM}(r)\| \leq \|(E_N^s - E_M^s)\,e^{-iH_o^s t}h_s(r)\| + \|(E_N^c - E_M^c)\,e^{-iH_o^c t}h_c(r)\|$$

$$= \|e^{-iH_o^s t}(E_N^s - E_M^s)h_s(r)\| + \|e^{-iH_o^c t}(E_N^c - E_M^c)h_c(r)\|$$

$$= \|(E_N^s - E_M^s)h_s(r)\| + \|(E_N^c - E_M^c)h_c(r)\|$$

for all t. Since s.lim. $E_N^{\,s}$, $E_N^{\,c} = I$, we conclude that $\{\mathring{f}_{tN}(r)\}$ converges uniformly over t as $N \to \infty$ in the topology of $L_2(0,\infty)$.

This completes the proof of the lemma.

It follows from Lemma 5.5 that $\mathring{f}_t(r) = \underset{N\to\infty}{\text{l.i.m.}}\ \mathring{f}_{tN}(r)$ is well-defined and in $L_2(0,\infty)$ for each t. We shall now show that $f_t(r)$ converges strongly to $\mathring{f}_t(r)$ at large positive and negative times.

5.6 <u>Lemma</u>:

With the notation of Lemma 5.5,

$$\lim_{|t|\to\infty} \| f_t(r) - \mathring{f}_t(r) \| = 0$$

where $\mathring{f}_t(r) = \underset{N\to\infty}{\text{l.i.m.}}\ \mathring{f}_{tN}(r)$ for each t.

<u>Proof</u>:

It is sufficient to show that

(i) $\displaystyle\lim_{|t|\to\infty} \int_0^P |f_t(r)|^2\, dr = 0$

for all P such that $0 < P < \infty$.

(ii) $\displaystyle\lim_{|t|\to\infty} \int_0^P |\mathring{f}_t(r)|^2\, dr = 0$

for all P such that $0 < P < \infty$

(iii) $\displaystyle\lim_{P\to\infty} \int_P^\infty |f_t(r) - \mathring{f}_t(r)|^2\, dr = 0$

uniformly with respect to t.

<u>Proof of (i)</u>:

Let P in $\mathbb{R}^+$ be fixed.

We may define the Dirichlet operator H_d to be the direct sum of H_p and H_∞, where H_p and H_∞ are self-adjoint operators arising from L in $L_2(0,P]$ and $L_2[P,\infty)$ respectively ([N] §24). We write

$$H_d = H_p \oplus H_\infty$$

Note that the domains of H_p and H_∞ are restricted by boundary conditions at r = P, and that if L is limit circle at r = 0 a further condition at 0 is required in the case of H_p. Irrespective of the boundary conditions which are chosen to fix H_p uniquely, the spectrum of H_p is singular owing to the conditions imposed on $V(r)$.

We shall use the Trace Theorem ([RSIII] Thm. XI. 9) to show that the wave operators

$$\Omega_{\pm} = \operatorname*{s.lim.}_{t \to \pm\infty} e^{iH_d t} e^{-iHt} M_{a.c.}(H) \tag{5.3.6}$$

$$\Omega^{*}_{\pm} = \operatorname*{s.lim.}_{t \to \pm\infty} e^{iHt} e^{-iH_d t} M_{a.c.}(H_d) \tag{5.3.7}$$

exist and deduce that $f_t(r)$ converges strongly to a state whose support is outside of $[0,P]$.

We first show that $((H+i)^{-1} - (H_d+i)^{-1})$ is a trace class operator.

Let $h(r)$ be in $L_2(0,\infty)$. Then there exist $g_1(r)$, $g_2(r)$ in $L_2(0,\infty)$ such that $g_1(r) = (H+i)^{-1} h(r)$ and $g_2(r) = (H_d+i)^{-1} h(r)$. Hence

$$((H+i)^{-1} - (H_d+i)^{-1})\,h(r) = g_1(r) - g_2(r)$$

and, since $g_1(r)$ is in $\mathcal{D}(H)$, $g_2(r)$ is in $\mathcal{D}(H_d)$,

$$\left(-\frac{d^2}{dr^2} + V(r) + i\right) g_1(r) = (H+i)\, g_1(r)$$

$$\left(-\frac{d^2}{dr^2} + V(r) + i\right) g_2(r) = (H_d+i)\, g_2(r)$$

It follows that

$$\left(-\frac{d^2}{dr^2} + V(r) + i\right)\left(g_1(r) - g_2(r)\right) = h(r) - h(r) = 0$$

so that

$$\{ g_1(r) - g_2(r) : ((H+i)^{-1} - (H_d+i)^{-1})h(r) = g_1(r) - g_2(r),\ h(r) \in L_2(0,\infty) \}$$

is a subspace of $\mathcal{H} = L_2(0,\infty)$ whose dimension cannot be greater than 4. That is, $((H+i)^{-1} - (H_d+i)^{-1})$ is an operator with rank less than or equal to 4, so is of trace class.

It is now immediate by the Trace Theorem that the wave operators (5.3.6) and (5.3.7) exist.

Let $\{E_\lambda(H)\}$, $\{E_\lambda(H_d)\}$ denote the spectral families of H and H_d respectively.

Since f(r) is in $\mathcal{H}_{a.c.}(H)$, $< f(r), E_\lambda(H) f(r) >$ is an absolutely continuous function of λ ; moreover, since the wave operators above exist,

$$E_\lambda(H_d)\Omega_\pm = \Omega_\pm E_\lambda(H)$$

(see [KA] Ch.X, proof of Thm. 3.2). It follows that

$< f(r), (\Omega_\pm^* E_\lambda(H_d)\Omega_\pm)(f(r)) >$ and, equivalently,

$< (\Omega_\pm f(r)), E_\lambda(H_d)(\Omega_\pm f(r)) >$ are absolutely continuous

functions of λ , so that $\Omega_\pm f(r)$ is in $\mathcal{H}_{a.c.}(H_d)$ ([KA] Ch X, §2).

This implies as we now show, that f(r), evolving under H, is evanescent in [0,P] (for terminology, see [APW]).

Let g(r) be in $\mathcal{H}_{a.c.}(H_d)$.

Now $g(r) = \{g_P(r), g_\infty(r)\}$, where $g_p(r)$ is in $L_2(0,P]$ and $g_\infty(r)$ is in $L_2[P,\infty)$; moreover,

$$E_\lambda(H_d) = E_\lambda(H_P) \oplus E_\lambda(H_\infty)$$

Hence

$$< \{g_P(r), g_\infty(r)\}, (E_\lambda(H_P) \oplus E_\lambda(H_\infty))\{g_P(r), g_\infty(r)\} >$$

$$= < g_P(r), E_\lambda(H_P) g_P(r) > + < g_\infty(r), E_\lambda(H_\infty) g_\infty(r) >$$

$$= \| E_\lambda(H_P) g_P(r) \| + \| E_\lambda(H_\infty) g_\infty(r) \|$$

is an absolutely continuous function of λ. Since the sum of two positive functions can only be absolutely continuous if each is absolutely continuous, we deduce that $< g_P(r), E_\lambda(H_P) g_P(r) >$ is an absolutely continuous function of λ , so that $g_p(r)$ is in $\mathcal{H}_{a.c.}(H_P)$.

However, the spectrum of H_p is singular, so that $\mathcal{H}_{a.c.}(H_P) = \phi$ and consequently $g_p(r) = 0$ Lebesque almost everywhere on [0,P].

Thus if g(r) is in $\mathcal{H}_{a.c.}(H_d)$, g(r) has no support in [0,P].

From our earlier remarks, since f(r) is in $\mathcal{H}_{a.c.}(H)$, $\Omega_\pm f(r)$ exist and are in $\mathcal{H}_{a.c.}(H_d)$. Hence there exist $g^\pm(r)$ with no support in [0,P] such that

$$\lim_{t \to \pm\infty} \| e^{iH_d t} e^{-iHt} f(r) - g^\pm(r) \| = 0$$

or, equivalently, such that

$$\lim_{t \to \pm\infty} \| f_t(r) - e^{-iH_d t} g^{\pm}(r) \| = 0$$

Hence

$$\lim_{t \to \pm\infty} \int_0^P | f_t(r) - e^{-iH_d t} g^{\pm}(r)|^2 \, dr = 0$$

which implies, since $g^{\pm}(r)$ have no support in $[0,P]$,

$$\lim_{t \to \pm\infty} \int_0^P | f_t(r)|^2 \, dr = 0$$

as was to be proved.

Proof of (ii):

We first use the Riemann-Lebesgue Lemma to show that

$$\lim_{|t| \to \infty} \int_0^P | \mathring{f}_{tN}(r)|^2 \, dr = 0$$

for fixed P in $\mathbb{R}^+$.

Now $\sin(\sqrt{\lambda}\, r + \delta(\lambda))\, \phi(\lambda) \frac{1}{\sqrt{\lambda}}$ is integrable with respect to λ on $[0,N]$ for each $N \in \mathbb{R}^+$ and each r in $(0,\infty)$; to see this note that by (5.2.10) and Lemma 5.4

$$\begin{aligned} \frac{1}{\pi}\int_0^N \frac{|\phi(\lambda)|}{\sqrt{\lambda}}\, d\lambda &= \int_0^N |g(\lambda)\tilde{F}(\lambda)| \frac{1}{\pi\sqrt{\lambda}\, g^2(\lambda)}\, d\lambda \\ &= \int_0^N |g(\lambda)\tilde{F}(\lambda)|\, d\tilde{\rho}(\lambda) \\ &\leq \left(\int_0^N g^2(\lambda)\, d\tilde{\rho}(\lambda)\right)^{\frac{1}{2}} \left(\int_0^N |\tilde{F}(\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\pi}\int_0^N \frac{1}{\sqrt{\lambda}}\, d\lambda\right)^{\frac{1}{2}} \| f(r) \| \qquad (5.3.8) \end{aligned}$$

Hence $\mathring{f}_{tN}(r)$ converges pointwise to zero as $|t| \to \infty$ for each fixed N in $\mathbb{R}^+$, by the Riemann-Lebesgue Lemma ([HS] 16.36, 16.37). Since for all r in $(0,\infty)$

$$\begin{aligned} |\mathring{f}_{tN}(r)|^2 &\leq \frac{1}{\pi}\int_0^N \frac{|\phi(\lambda)|}{\sqrt{\lambda}}\, d\lambda \\ &\leq \frac{2\sqrt{N}}{\pi} \| f(r) \|^2 \qquad (5.3.9) \end{aligned}$$

by (5.3.8), we may use the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{|t|\to\infty} \int_0^P |\mathring{f}_{tN}(r)|^2\, dr = 0 \tag{5.3.10}$$

for all N in $\mathbb{R}^+$ and all P such that $0 < P < \infty$.

Now, if $\varepsilon > 0$ is given, there exists N_ε such that

$$\begin{aligned}\left(\int_0^P |\mathring{f}_t(r)|^2 dr - \int_0^P |\mathring{f}_{tN}(r)|^2 dr\right)^{\frac{1}{2}} &\leq \left(\int_0^P |\mathring{f}_t(r) - \mathring{f}_{tN}(r)|^2 dr\right)^{\frac{1}{2}} \\ &\leq \|\mathring{f}_t(r) - \mathring{f}_{tN}(r)\| \\ &< \varepsilon\end{aligned}$$

for $N > N_\varepsilon$, since the $L_2(0,\infty)$ convergence of $\{\mathring{f}_{tN}(r)\}$ to $\mathring{f}_t(r)$ is uniform over t. It follows from (5.3.10) and the arbitrariness of ε that

$$\lim_{|t|\to\infty} \int_0^P |\mathring{f}_t(r)|^2\, dr = 0$$

as required.

<u>Proof of (iii)</u>:

We first show that for each fixed n in $\mathbb{R}^+$

$$\int_P^\infty |f_{tn}(r) - \mathring{f}_{tn}(r)|^2\, dr$$

converges uniformly over t to 0 as $P \to \infty$, where

$$\begin{aligned} f_{tn}(r) &= \int_{1/n}^{n} e^{-i\lambda t}\, v_s(r,\lambda)\, \phi(\lambda) \frac{1}{\sqrt{\lambda}}\, d\lambda \\ \mathring{f}_{tn}(r) &= \int_{1/n}^{n} e^{-i\lambda t} \sin(\sqrt{\lambda}\, r + \delta(\lambda))\, \phi(\lambda) \frac{1}{\sqrt{\lambda}}\, d\lambda \end{aligned}$$

We use the fact that $V(r) = O(r^{-(1+\varepsilon)})$ as $r \to \infty$ to show that

$$|f_{tn}(r) - \mathring{f}_{tn}(r)|^2 = O(r^{-(1+\varepsilon)}) \text{ as } r \to \infty .$$

Now by (5.2.9), (5.2.10) and Lemma 5.4

$$\begin{aligned}|f_{tn}(r) - \mathring{f}_{tn}(r)|^2 &= \left(\int_{1/n}^{n} e^{-i\lambda t}(y_s(r,\lambda) - g(\lambda) \sin(\sqrt{\lambda}\, r + \delta(\lambda))\, \tilde{F}(\lambda)\, d\tilde{\rho}(\lambda)\right)^2 \\ &\leq \left(\int_{1/n}^{n} |y_s(r,\lambda) - g(\lambda)\sin(\sqrt{\lambda}\, r + \delta(\lambda))\, \tilde{F}(\lambda)|\, d\tilde{\rho}(\lambda)\right)^2\end{aligned}$$

$$\leq \left(\int_{1/n}^{n} |\tilde{F}(\lambda)|^2 d\tilde{\rho}(\lambda)\right)\left(\int_{1/n}^{n} |y_s(r,\lambda) - g(\lambda)\sin(\sqrt{\lambda}r + \delta(\lambda))|^2 d\tilde{\rho}(\lambda)\right)$$

$$\leq \|f(r)\| \left(\int_{1/n}^{n} |y_s(r,\lambda) - g(\lambda)\sin(\sqrt{\lambda}r + \delta(\lambda))|^2 d\tilde{\rho}(\lambda)\right) \qquad (5.3.11)$$

From the proof of Lemma 5.3, in particular (5.2.7),

$$y_s(r,\lambda) = g(\lambda)\sin(\sqrt{\lambda}r + \delta(\lambda)) - \int_r^\infty \frac{\sin\sqrt{\lambda}(r-p)}{\sqrt{\lambda}} V(p)\, y_s(r,p)\,dp$$

Hence

$$\int_{1/n}^{n} |y_s(r,\lambda) - g(\lambda)\sin(\sqrt{\lambda}r + \delta(\lambda))|^2 d\tilde{\rho}(\lambda)$$

$$\leq n\int_{1/n}^{n} \left|\int_r^\infty \sin(\sqrt{\lambda}(r-p))V(p)\,y_s(r,p)\,dp\right|^2 d\tilde{\rho}(\lambda)$$

Since $V(p) = O(p^{-(1+\varepsilon)})$ as $p \to \infty$, $\sin(\sqrt{\lambda}(r-p))V(p)$ is in $L_2(0,\infty)$ for each fixed $r > 1$. Hence if we define

$$h_r(p) = \begin{cases} \sin(\sqrt{\lambda}(r-p))V(p) & p \geq r \\ 0 & p < r \end{cases}$$

then

$$\int_r^\infty \sin(\sqrt{\lambda}(r-p))V(p)\,y_s(r,p)\,dp = (\tilde{S}h_r)(\lambda)$$

for each $r > 1$, where $\tilde{S}$ is the transform of Theorem 4.9. Consequently, by Theorem 4.9,

$$\int_{1/n}^{n} \left|\int_r^\infty \sin(\sqrt{\lambda}(r-p))V(p)\,y_s(r,p)\,dp\right|^2 d\tilde{\rho}(\lambda)$$

$$\leq \int_{-\infty}^{\infty} \left|\int_r^\infty \sin(\sqrt{\lambda}(r-p))V(p)\,y_s(r,p)dp\right|^2 d\tilde{\rho}(\lambda)$$

$$= \int_0^\infty |h_r(p)|^2\,dp$$

$$\leq \int_r^\infty (V(p))^2\,dp$$

$$= O(r^{-(1+\varepsilon)})$$

as $r \to \infty$. Using this result in (5.3.11), we see that, as $r \to \infty$

$$|f_{tn}(r) - \mathring{f}_{tn}(r)|^2 = O(r^{-(1+\varepsilon)})$$

uniformly over t. Consequently for each fixed n in $\mathbb{R}^+$,

$$\int_p^\infty |f_{tn}(r) - \mathring{f}_{tn}(r)|^2\,dr \qquad (5.3.12)$$

converges uniformly over t to 0 as $P \to \infty$.

We now deduce (iii).

Let H_o^s and H_o^c be the self-adjoint operators described in Lemma 5.5, and let $\{E_\lambda^s\}$ and $\{E_\lambda^c\}$ denote their respective spectral families. Let $\{E_\lambda\}$ denote the spectral family of H, and $E(n)$, $E^s(n)$, $E^c(n)$ denote $(E_n - E_{\frac{1}{n}})$, $(E_n^s - E_{\frac{1}{n}}^s)$ and $(E_n^c - E_{\frac{1}{n}}^c)$ respectively.

Proceeding as in Lemma 5.5, we have for all t and n

$$\left(\int_P^\infty |f_t(r) - \mathring{f}_t(r)|^2 dr\right)^{\frac{1}{2}} - \left(\int_P^\infty |f_{tn}(r) - \mathring{f}_{tn}(r)|^2 dr\right)^{\frac{1}{2}}$$

$$\leq \left(\int_P^\infty |(f_t(r) - f_{tn}(r)) - (\mathring{f}_t(r) - \mathring{f}_{tn}(r))|^2 dr\right)^{\frac{1}{2}}$$

$$\leq \| f_t(r) - f_{tn}(r) \| + \| \mathring{f}_t(r) - \mathring{f}_{tn}(r) \|$$

$$\leq \|(I - E(n)) e^{-iHt} f(r)\| + \|(I - E^s(n)) e^{-iH_o^s t} h_s(r)\| + \|(I - E^c(n)) e^{-iH_o^c t} h_c(r)\|$$

Since $f(r)$, $h_s(r)$, $h_c(r)$ are in $\mathcal{H}_{a.c.}(H)$, $\mathcal{H}_{a.c.}(H_o^s)$ and $\mathcal{H}_{a.c.}(H_o^c)$ respectively, each of the terms in the final right hand side converges to zero as $n \to \infty$. From this result and (5.3.12) it follows that

$$\lim_{P \to \infty} \int_P^\infty |f_t(r) - \mathring{f}_t(r)|^2 dr = 0$$

uniformly with respect to t, as required.

This completes the proof of (iii), and hence the lemma.

With the notation of Lemma 5.5,

$$\mathring{f}_{tN}(r) = \mathring{f}_{tN}^{+}(r) + \mathring{f}_{tN}^{-}(r) \tag{5.3.13}$$

where

$$\mathring{f}_{tN}^{\pm}(r) = \frac{1}{2\pi} \int_0^N e^{-i\lambda t} e^{\pm i(\sqrt{\lambda}\, r - \frac{\pi}{2} + \delta(\lambda))} \phi(\lambda) \frac{1}{\sqrt{\lambda}} d\lambda \tag{5.3.14}$$

Moreover, reasoning as in Lemma 5.5, we see that

$$\mathring{f}_t^{\pm}(r) = \underset{N\to\infty}{\text{l.i.m.}} \; \mathring{f}_{tN}^{\pm}(r)$$

exist and are in $L_2(0,\infty)$, and that the convergence of $\{\mathring{f}_{tN}^{\pm}(r)\}$ to $\mathring{f}_t^{\pm}(r)$ is uniform over t. Hence, for sufficiently large N.

$$\| \mathring{f}_t^{+}(r) - \mathring{f}_{tN}^{+}(r) \| \,, \; \| \mathring{f}_t^{-}(r) - \mathring{f}_{tN}^{-}(r) \| \quad \text{and} \quad \| \mathring{f}_t(r) - \mathring{f}_{tN}(r) \|$$

are arbitrarily small for all t. Therefore, from

$$\| \mathring{f}_t(r) - \mathring{f}_t^{+}(r) - \mathring{f}_t^{-}(r) \| - \| \mathring{f}_{tN}(r) - \mathring{f}_{tN}^{+}(r) - \mathring{f}_{tN}^{-}(r) \|$$
$$\leq \| \mathring{f}_t(r) - \mathring{f}_{tN}(r) \| + \| \mathring{f}_t^{+}(r) - \mathring{f}_{tN}^{+}(r) \| + \| \mathring{f}_t^{-}(r) - \mathring{f}_{tN}^{-}(r) \|$$

and (5.3.13), it follows that for each t

$$\mathring{f}_t(r) = \mathring{f}_t^{+}(r) + \mathring{f}_t^{-}(r) \tag{5.3.15}$$

Lebesgue almost everywhere on $(0,\infty)$.

We now show that $\mathring{f}_t(r)$ converges strongly to $\mathring{f}_t^{+}(r)$ at large positive times and to $\mathring{f}_t^{-}(r)$ at large negative times.

5.7 <u>Lemma</u>: $\lim_{t\to\pm\infty} \| f_t(r) - \mathring{f}_t^{\pm}(r) \| = 0$ where

$$\mathring{f}_t^{\pm}(r) = \underset{N\to\infty}{\text{l.i.m.}} \; \frac{1}{2\pi} \int_0^N e^{-i\lambda t} \, e^{\pm i(\sqrt{\lambda}r - \frac{\pi}{2} + \delta(\lambda))} \phi(\lambda) \frac{1}{\sqrt{\lambda}} \, d\lambda$$

<u>Proof</u>:

It is sufficient to prove that

$$\lim_{t\to\pm\infty} \| \mathring{f}_t^{\mp}(r) \| = 0$$

on account of (5.3.15); we first show that

$$\lim_{t\to+\infty} \| \mathring{f}_t^{-}(r) \| = 0$$

It is convenient to substitute $\lambda = k^2$ in the expression for $\mathring{f}_t^{-}(r)$; this gives

$$\mathring{f}_t^{-}(r) = \underset{N\to\infty}{\text{l.i.m.}} \; \frac{1}{\pi} \int_0^{N^{1/2}} e^{-ikr} e^{-ik^2 t} \phi(k^2) \, e^{-i(\delta(k^2) - \frac{\pi}{2})} \, dk$$

Using the theory of Fourier transforms, we shall prove that for each $\varepsilon > 0$ there exists a step function $\sum_{i=1}^{n} \alpha_i \chi_i$ with compact support in $(0,\infty)$ for which

$$\left(\int_0^\infty \left| \frac{1}{\pi} \int_0^{N^{1/2}} e^{-ikr} e^{-ik^2t} \left(\phi(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} - \sum_{i=1}^{n} \alpha_i \chi_i \right) dk \right|^2 dr \right)^{\frac{1}{2}}$$

$$< \frac{\varepsilon}{3} \qquad (5.3.16)$$

$\| \overset{\circ}{f}{}_{tN}^{-}(r) \|$ may then be approximated with arbitrary precision by a finite sum of time dependent functions; we complete the proof using the Riemann-Lebesgue Lemma to show that each function in this sum converges to zero as $t \to \infty$.

Now by (5.2.13),

$$\int_0^\infty | \phi(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} |^2 \, dk = \frac{1}{2} \int_0^\infty | \phi(\lambda) |^2 \frac{1}{\sqrt{\lambda}} \, d\lambda = \frac{1}{2} \| f(r) \|^2 < \infty$$

Hence, since the step functions are dense in $L_2(0,\infty)$ ([HS],Thm.13.23), we may deduce that if $\varepsilon > 0$ is given, there exists a > 0 and a step function $\sum_{i=1}^{n} \alpha_i \chi_i$, which vanishes outside [a,b] for some b > a, such that

$$\left(\int_0^\infty | \phi(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} - \sum_{i=1}^{n} \alpha_i \chi_i |^2 dk \right)^{\frac{1}{2}} < \frac{\varepsilon}{3} \qquad (5.3.17)$$

We now derive (5.3.16).

Let N in $\mathbb{R}^+$ be such that $b < N^{1/2} < \infty$. From (5.3.14),

$$\overset{\circ}{f}{}_{tN}^{-}(r) = \frac{1}{\pi} \int_0^{N^{1/2}} e^{-ikr} e^{-ik^2t} \phi(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} dk \qquad (5.3.18)$$

Now $e^{-ik^2t} \phi(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})}$ is both integrable and square integrable on $(0,N^{1/2})$ (cf.(5.3.8) and (5.3.9)). Hence, defining

$$\phi_N(k^2) = \begin{cases} \phi(k^2) & \text{on } (0,N^{1/2}) \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$e^{-ik^2t} \left[\phi_N(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} - \sum_{i=1}^{n} \alpha_i \chi_i \right] \qquad (5.3.19)$$

is in $L_1(0,\infty) \cap L_2(0,\infty)$ so that

$$\frac{1}{\pi} \int_0^{N^{1/2}} e^{-ikr} e^{-ik^2t} \left[\phi_N(k^2) e^{-i(\delta(k^2) - \frac{\pi}{2})} - \sum_{i=1}^{n} \alpha_i \chi_i \right] dk$$

is the Fourier transform $\hat{f}_{tN}(r)$ of (5.3.19). Hence, by the isometric property of Fourier transforms ([HS] 21.52), and (5.3.17),

$$\left(\int_0^\infty |\hat{f}_{tN}(r)|^2 dr\right)^{\frac{1}{2}} \leqslant \left(\int_{-\infty}^\infty |\hat{f}_{tN}(r)|^2 dr\right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^\infty | e^{-ik^2t}[\phi_N(k^2)\, e^{-i(\delta(k^2)-\frac{\pi}{2})} - \sum_{i=1}^n \alpha_i \chi_i]|^2 dk\right)^{\frac{1}{2}}$$

$$= \left(\int_0^{N^{1/2}} |\phi(k^2)\, e^{-i(\delta(k^2)-\frac{\pi}{2})} - \sum_{i=1}^n \alpha_i \chi_i|^2 dk\right)^{\frac{1}{2}}$$

$$\leqslant \left(\int_0^\infty |\phi(k^2)\, e^{-i(\delta(k^2)-\frac{\pi}{2})} - \sum_{i=1}^n \alpha_i \chi_i|^2 dk\right)^{\frac{1}{2}}$$

$$< \frac{\varepsilon}{3}$$

which proves (5.3.16). It follows from (5.3.16) and (5.3.18) that

$$\|\mathring{f}^-_{tN}(r)\| \leqslant \left(\int_0^\infty \left|\int_0^{N^{1/2}} e^{-ikr}\, e^{-ik^2t} \sum_{i=1}^n \alpha_i \chi_i\, dk\right|^2 dr\right)^{\frac{1}{2}} + \frac{\varepsilon}{3}$$

$$= \left(\int_0^\infty \left|\sum_{i=1}^n \alpha_i \int_{I_i} e^{-ikr}\, e^{-ik^2t}\, dk\right|^2 dr\right)^{\frac{1}{2}} + \frac{\varepsilon}{3}$$

$$\leqslant \sum_{i=1}^n |\alpha_i| \left(\int_0^\infty \left|\int_{I_i} e^{-ikr}\, e^{-ik^2t}\, dk\right|^2 dr\right)^{\frac{1}{2}} + \frac{\varepsilon}{3} \qquad (5.3.20)$$

where we note that χ_i is the characteristic function of the interval I_i.

We now show that each of the finite collection of terms of the form

$$|\alpha_i| \left(\int_0^\infty \left|\int_{I_i} e^{-ikr}\, e^{-ik^2t}\, dk\right|^2 dr\right)^{\frac{1}{2}} \qquad (5.3.21)$$

converges to zero as $t \to +\infty$.

Consider the ith term and suppose $I_i = (a_i, b_i)$. From our construction of $\sum_{i=1}^n \alpha_i \chi_i$, $a_i \geqslant a > 0$.

Now, since $\lambda = k^2$,

$$\int_{I_i} e^{-ikr}\, e^{-ik^2t}\, dk = \int_{a_i^2}^{b_i^2} e^{-i\lambda t}\, \frac{e^{-i\sqrt{\lambda}\, r}}{2\sqrt{\lambda}}\, d\lambda$$

and $\dfrac{e^{-i\sqrt{\lambda}\, r}}{2\sqrt{\lambda}}$ is integrable with respect to λ on (a_i^2, b_i^2) for each r in $(0,\infty)$. Hence, by the Riemann-Lebesgue Lemma, $\left|\int_{I_i} e^{-ikr}\, e^{-ik^2t}\, dk\right|^2$ converges pointwise to zero.

Moreover, using integration by parts,

$$\left|\int_{I_i} e^{-ikr} e^{-ik^2t} dk\right| = \left|\int_{a_i}^{b_i} \frac{1}{-i(r+2kt)} \frac{d}{dk} \exp(-i(kr+k^2t))\, dk\right|$$

$$= \left|\left[\frac{-1}{i(r+2kt)} \exp(-i(kr+k^2t))\right]_{a_i}^{b_i} - \int_{a_i}^{b_i} \frac{2t}{i(r+2kt)^2} \exp(-i(kr+k^2t))\, dk\right|$$

$$\leq \frac{1}{r+2b_it} + \frac{1}{r+2a_it} + \int_{a_i}^{b_i} \frac{2t}{(r+2kt)^2}\, dk$$

$$\leq \frac{3}{r+2at}$$

Hence for $t \geq 1$, $\left|\int_{I_i} e^{-ikr} e^{-ik^2t} dk\right|^2$ is dominated by $\left(\frac{3}{r+2a}\right)^2$

which is in $L_1(0,\infty)$ and so by the Lebesgue Dominated Convergence Theorem (5.3.21) converges to zero as $t \to \infty$ for each $i = 1,\dots,n$.

It follows from (5.3.20) that there exists $T > 0$ such that for all $N > b^2$

$$\|\mathring{f}_{tN}^-(r)\| < \frac{2\varepsilon}{3} \tag{5.3.22}$$

whenever $t > T$.

Since $\mathring{f}_t^-(r) = \underset{N\to\infty}{\text{l.i.m.}}\ \mathring{f}_{tN}^-(r)$ for each t, there exists N_t, depending on t, such that, whenever $N > N_t$,

$$\|\mathring{f}_t^-(r) - \mathring{f}_{tN}^-(r)\| < \frac{\varepsilon}{3} \tag{5.3.23}$$

Hence for each $t > T$, we may choose $N > \max\{b^2, N_t\}$, so that the inequalities in (5.3.22) and (5.3.23) both hold, giving

$$\|\mathring{f}_t^-(r)\| < \varepsilon$$

The arbitrariness of ε implies $\lim_{t\to+\infty} \|\mathring{f}_t^-(r)\| = 0$; similarly, $\lim_{t\to-\infty} \|\mathring{f}_t^+(r)\| = 0$ and the lemma is proved.

It is not hard to see that Lemmas 5.3 - 5.7 also apply in the case of the free Hamiltonian in each partial wave subspace. For a fixed partial wave subspace we take $V(r) = \frac{l(l+1)}{r^2}$, and, to distinguish the results for H_0 from those of H, denote by $g_0(\lambda)$, $\delta_0(\lambda)$, $\tilde{\rho}_0(\lambda)$, $y_{s,0}(r,\lambda)$, $v_{s,0}(r,\lambda)$, $\tilde{S}^0_{a.c.}$ the analogues of $g(\lambda)$, $\delta(\lambda)$, $\tilde{\rho}(\lambda)$, $y_s(r,\lambda)$, $v_s(r,\lambda)$ and $\tilde{S}_{a.c.}$ respectively. Note that, with the proviso that a suitable boundary condition at 0 be chosen in the case $l = 0$, $\mathcal{H}_{a.c.}(H_0) = L_2(0,\infty)$ in each partial wave subspace. Hence, if g(r) is in $L_2(0,\infty)$,

$$g(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_0^{\omega} y_{s,0}(r,\lambda)\, \tilde{G}_0(\lambda)\, d\tilde{\rho}_0(\lambda)$$

where

$$\tilde{G}_0(\lambda) = \underset{\substack{\omega\to\infty \\ \sigma\to 0}}{\text{l.i.m.}} \int_{\sigma}^{\omega} y_{s,0}(r,\lambda)\, g(r)\, dr$$

the integrals being convergent in $L_2(0,\infty)$ and $L_2^{\tilde{\rho}_0}(0,\infty)$ respectively.

Bearing in mind the comments preceding Theorem 4.10, and the fact that absolutely continuous spectrum is preserved under a change of boundary condition (see Theorem 2.21), we note that conditions (i) and (ii) of §1, are equivalent to the hypothesis of the following theorem:

5.8 Theorem: Let V(r) be in $L_1[a,\infty)$ for each $a > 0$, and suppose there exists a finite interval $(0,b]$ and a self-adjoint operator H_b arising from L in $L_2(0,b]$ whose spectrum is purely singular. Then the wave operators exist and are complete.

Proof:

It is sufficient to prove:

(i) If g(r) in $L_2(0,\infty)$ is given, then $f^{\pm}(r)$ exist in $\mathcal{H}_{a.c.}(H)$ such that

$$\lim_{t\to\pm\infty} \| e^{-iH_0 t} g(r) - e^{-iHt} f^{\pm}(r) \| = 0$$

(ii) If f(r) in $\mathcal{H}_{a.c.}(H)$ is given, then $g^{\pm}(r)$ exist in $L_2(0,\infty)$ such that

$$\lim_{t\to\pm\infty} \| e^{-iH_0 t} g^{\pm}(r) - e^{-iHt} f(r) \| = 0$$

Proof of (i):

Let g(r) in $L_2(0,\infty)$ be given.

Then by (5.2.14)

$$g_t(r) = e^{-iH_0t}g(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{\pi}\int_0^\omega e^{-i\lambda t}\, v_{s,0}(r,\lambda)\,\phi_0(\lambda)\frac{1}{\sqrt{\lambda}}\,d\lambda \tag{5.3.24}$$

where $\phi_0(\lambda) = \dfrac{\tilde{G}_0(\lambda)}{g_0(\lambda)}$. Moreover

$$\overset{\circ}{g}{}_t^{\pm}(r) = \underset{\omega\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\int_0^\omega e^{-i\lambda t}\, e^{\pm(\sqrt{\lambda}r - \frac{\pi}{2} + \delta_0(\lambda))}\,\phi_0(\lambda)\,\frac{1}{\sqrt{\lambda}}\,d\lambda$$

is well-defined (cf. Lemma 5.5), and by Lemma 5.6

$$\lim_{t\to\pm\infty} \| g_t(r) - \overset{\circ}{g}{}_t^{\pm}(r)\| = 0 \tag{5.3.25}$$

We now show that there exist states f(r) and h(r) in $\mathcal{H}_{a.c.}(H)$ such that with the notation of Lemma 5.6 $\overset{\circ}{f}{}_t^-(r) = \overset{\circ}{g}{}_t^-(r)$, $\overset{\circ}{h}{}_t^+(r) = \overset{\circ}{g}{}_t^+(r)$.

Applying Lemma 5.4 in respect of H and H_0,

$$\begin{aligned}\int_0^\infty &\left|\frac{g(\lambda)\,\tilde{G}_0(\lambda)\,e^{\pm i(\delta(\lambda)-\delta_0(\lambda))}}{g_0(\lambda)}\right|^2 d\tilde{\rho}(\lambda)\\
&= \int_0^\infty \left|\frac{g(\lambda)\,\tilde{G}_0(\lambda)}{g_0(\lambda)}\right|^2 \frac{1}{\pi\sqrt{\lambda}\,g^2(\lambda)}\,d\lambda\\
&= \int_0^\infty |\tilde{G}_0(\lambda)|^2\,\frac{1}{\pi\sqrt{\lambda}\,g_0^2(\lambda)}\,d\lambda\\
&= \int_0^\infty |\tilde{G}_0(\lambda)|^2\,d\tilde{\rho}_0(\lambda)\\
&= \|g(r)\|^2\end{aligned} \tag{5.3.26}$$

so the functions

$$\frac{g(\lambda)\,\tilde{G}_0(\lambda)\,e^{\pm i(\delta(\lambda)-\delta_0(\lambda))}}{g_0(\lambda)}$$

are in $L_2^{\tilde{\rho}}(0,\infty)$. Hence by Proposition 5.1, (in particular, by the surjective property of $\tilde{S}_{a.c.}$) there exist f(r), h(r) in $\mathcal{H}_{a.c.}(H)$ such that

$$(\tilde{S}_{a.c.}f)(\lambda) = \frac{g(\lambda)\,\tilde{G}_0(\lambda)\,e^{+i(\delta(\lambda)-\delta_0(\lambda))}}{g_0(\lambda)} \tag{5.3.27}$$

$$(\tilde{S}_{a.c.}h)(\lambda) = \frac{g(\lambda)\,\tilde{G}_0(\lambda)\,e^{-i(\delta(\lambda)-\delta_0(\lambda))}}{g_0(\lambda)} \tag{5.3.28}$$

Evidently by (5.2.14) and (5.3.27)

$$f_t(r) = e^{-iHt} f(r)$$

$$= \underset{\omega\to\infty}{\text{l.i.m.}} \frac{1}{\pi} \int_0^{\omega} e^{-i\lambda t} v_s(r,\lambda)\, \phi_o(\lambda)\, e^{+i(\delta(\lambda)-\delta_o(\lambda))} \frac{1}{\sqrt{\lambda}}\, d\lambda$$

so that with the notation of Lemma 5.6,

$$\overset{\circ}{f}{}_t^-(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \frac{1}{\pi} \int_0^{\omega} e^{-i\lambda t} e^{-i(\sqrt{\lambda}r - \frac{\pi}{2} + \delta(\lambda))} \phi_o(\lambda)\, e^{+i(\delta(\lambda)-\delta_o(\lambda))} \frac{1}{\sqrt{\lambda}}\, d\lambda$$

$$= \overset{\circ}{g}{}_t^-(r)$$

Hence by Lemma 5.6

$$\lim_{t\to-\infty} \| e^{-iHt} f(r) - \overset{\circ}{g}{}_t^-(r)\| = 0$$

and, similarly,

$$\lim_{t\to+\infty} \| e^{-iHt} h(r) - \overset{\circ}{g}{}_t^+(r)\| = 0$$

Setting $f^-(r) = f(r)$, $f^+(r) = h(r)$, the result (i) follows from (5.3.25).

Proof of (ii):

The method of proof is identical to that of (i); we note that in this case the surjective nature of the transformation $\tilde{S}^{\circ}_{a.c.}$ from $\mathcal{H}_{a.c.}(H_o) = L_2(0,\infty)$ onto $L_2^{\tilde{\rho}_o}(0,\infty)$ ensures that the elements $g^{\pm}(r)$ exist.

The proof of the theorem is now complete.

Using the proof of Theorem 5.8 we may deduce explicit formulae for the wave and scattering operators in each partial wave subspace. We remark that, in general, $g(\lambda)$, $g_o(\lambda)$, $\tilde{S}_{a.c.}$, $\tilde{S}^{\circ}_{a.c.}$ etc. are dependent on the decomposition point used in the simplification of the Weyl-Kodaira Theorem; we chose this point, arbitrarily, to be r = 1. However, for almost all λ in $(0,\infty)$, $y_s(r,\lambda)$ is, as a function of r, uniquely defined up to a multiplicative constant; therefore, since $y_s(r,\lambda) \to g(\lambda)\sin(\sqrt{\lambda}r + \delta(\lambda))$ as $r\to\infty$ by Lemma 5.2, $\delta(\lambda)$ (and similarly $\delta_o(\lambda)$) is independent of the decomposition point. With this in mind, we have

5.9 Theorem: With the hypothesis of Theorem 5.8

$$\Omega_{\pm} = (\tilde{S}_{a.c.})^{-1} \frac{e^{\pm i\delta_0(\lambda)}\, g(\lambda)}{e^{\pm i\delta(\lambda)}\, g_0(\lambda)} \tilde{S}_{a.c.}$$

$$S = (\tilde{S}^{\circ}_{a.c.})^{-1} e^{2i(\delta(\lambda)-\delta_0(\lambda))} \tilde{S}^{\circ}_{a.c.}$$

Proof:

If g(r) is in $L_2(0,\infty) = \mathcal{H}_{a.c.}(H_0)$ we have from Theorem 5.8(i),(5.3.27) and (5.3.28),

$$\begin{aligned}
\Omega_{\pm}\, g(r) &= \operatorname*{s.lim.}_{t\to\pm\infty} e^{iHt} e^{-iH_0t} g(r) \\
&= f^{\pm}(r) \\
&= (\tilde{S}_{a.c.})^{-1} \left(\frac{e^{\pm i\delta_0(\lambda)}\, g(\lambda)}{e^{\pm i\delta(\lambda)}\, g_0(\lambda)} \tilde{G}_0(\lambda) \right) \\
&= (\tilde{S}_{a.c.})^{-1} \left(\frac{e^{\pm \delta_0(\lambda)}\, g(\lambda)}{e^{\pm i\delta(\lambda)}\, g_0(\lambda)} (\tilde{S}^{\circ}_{a.c.}\, g)(\lambda) \right)
\end{aligned}$$

Similarly, if f(r) is in $\mathcal{H}_{a.c.}(H)$,

$$\begin{aligned}
\Omega^*_{\pm} f(r) &= \operatorname*{s.lim.}_{t\to\pm\infty} e^{iH_0t} e^{-iHt} f(r) \\
&= (\tilde{S}^{\circ}_{a.c.})^{-1} \left(\frac{e^{\pm i\delta(\lambda)}\, g_0(\lambda)}{e^{\pm i\delta_0(\lambda)}\, g(\lambda)} (\tilde{S}_{a.c.} f)(\lambda) \right)
\end{aligned}$$

and so

$$S = \Omega^*_+ \Omega_- = (\tilde{S}^{\circ}_{a.c.})^{-1} e^{2i(\delta(\lambda)-\delta_0(\lambda))} \tilde{S}^{\circ}_{a.c.}$$

as required.

This completes the proof of the theorem.

It is straightforward to check that the wave operators are isometric (see, for example, (5.3.26)), and that the scattering operator is unitary.

From the definition in §1, we see from Theorem 5.9 that the partial wave phase shift is

$$(\delta(\lambda) - \delta_0(\lambda))$$

where $\delta(\lambda)$ is defined by

$$\tan(\delta(\lambda) + \sqrt{\lambda}) = \frac{\beta(\lambda)}{\gamma(\lambda)} \qquad (5.3.29)$$

$\beta(\lambda)$ and $\gamma(\lambda)$ being as in (5.2.5) and (5.2.6) respectively. Defining $\beta_0(\lambda)$, $\gamma_0(\lambda)$ in a similar way to $\beta(\lambda)$, $\gamma(\lambda)$ with $m_{0,0}(\lambda)$, $y_{s,0}(r,\lambda)$, $\frac{l(l+1)}{r^2}$ in place of $m_0(\lambda)$, $y_s(r,\lambda)$ and $V(r)$ respectively,

$$\tan(\delta_0(\lambda)+\sqrt{\lambda}) = \frac{\beta_0(\lambda)}{\gamma_0(\lambda)} \qquad (5.3.30)$$

Hence the partial wave phase shift is given by

$$(\delta(\lambda)-\delta_0(\lambda)) = \tan^{-1}\left(\frac{\beta(\lambda)\gamma_0(\lambda)-\beta_0(\lambda)\gamma(\lambda)}{\beta_0(\lambda)\beta(\lambda)+\gamma_0(\lambda)\gamma(\lambda)}\right)$$

Provided conditions (i) and (ii) of §1 are satisfied in each partial wave subspace, the existence and completeness of the wave operators for the full three dimensional problem is now immediate from Theorem 5.9. Therefore, indicating the l-dependence of $\delta(\lambda)$, $\delta_0(\lambda)$ by $\delta(\lambda,l)$ and $\delta_0(\lambda,l)$ respectively, we have the following formulations of the S-matrix and of the scattering amplitude from (5.1.8) and (5.1.9)

$$S(\lambda) = \sum_{l,m} \exp(2i(\delta(\lambda,l)-\delta_0(\lambda,l))$$

$$f(\lambda:\underline{\omega}_1\to\underline{\omega}_2) = \frac{1}{2i\sqrt{\lambda}}\sum_l (2l+1)\left(\exp[2i(\delta(\lambda,l)-\delta_0(\lambda,l))]-1\right)P_l\,\underline{\omega}_1\cdot\underline{\omega}_2$$

Our result includes that of Green and Lanford and significantly extends the class of potentials considered by them. We note that it may be possible to relax the condition on the potential at infinity so as to include all potentials which are in $L_1[a,\infty)$ for each $a > 0$. This has been achieved by Kuroda for the class of potentials satisfying Green and Lanford's conditions at 0. ([KU2]). It is certainly possible to weaken the condition at infinity so as to include all potentials for which

$$\int_r^\infty p\,[V(p)]^2\,dp < \infty \qquad (5.3.31)$$

for $r > 0$. To see this, note that (5.3.31) implies that

$$\int_k^\infty\int_r^\infty [V(p)]^2 dp\,dr = \left[r\int_r^\infty [V(p)]^2\,dp\right]_k^\infty + \int_k^\infty r\,[V(r)]^2 dr$$

$$\leq 2 \int_k^\infty p[V(p)]^2 dp < \infty$$

where we have used integration by parts. This is sufficient to ensure the validity of (5.3.12), and hence of Lemma 5.6.

We observe that our proof of the sufficiency of the condition $V(r) = O(r^{-(1+\varepsilon)})$ is considerably simpler than that of Green and Lanford (see [GR] §IV).

§4. An example of discontinuous scattering amplitude where the theory is asymptotically complete

It is known that in many cases where the wave operators exist and are complete, the scattering amplitude is a continuous function of the energy (see, for example, [AJS] Prop.11.16, [D] [LE]). The question arises whether this is true whenever the wave operators exist and are complete.

We must first consider what we mean by continuity in this context. From our proofs in §§ 2 and 3 it will be seen that for each l, $S_l(\lambda) = \exp(2i(\delta(\lambda) - \delta_0(\lambda))$ is defined for those $\lambda > 0$ for which $m_0{+}(\lambda, l)$ and $m_{0,0}^{+}(\lambda, l)$ exist as finite real limits. However, since each such $S_l(\lambda)$ is unitarily equivalent to the scattering operator in a given partial wave subspace, the $S_l(\lambda)$ we have considered is, strictly speaking, a particular representative of an equivalence class of functions under the norm $\left(\int_0^\infty |\cdot|^2 \frac{\lambda^{-1/2}}{\pi} d\lambda\right)^{1/2}$. To enquire whether, for a given l, $S_l(\lambda)$ is a continuous function of λ is more precisely, therefore, to enquire whether the equivalence class containing $S_l(\lambda)$ contains a continuous function.

Now the scattering amplitude (5.1.9) can only be a continuous function of energy if each term

$$(2l+1)(S_l(\lambda) - 1)\, P_l\,(\underline{\omega}_1 \cdot \underline{\omega}_2)$$

is a continuous function of λ, so in order to establish discontinuity of the scattering amplitude, it is sufficient to prove that just one of the terms $S_l(\lambda)$ is not continuous.

In this section we use the inverse method of Gel'fand and Levitan to show that a potential exists for which the equivalence class containing $S_0(\lambda)$ does not contain a continuous function although the wave operators exist and are complete.

The following definitions will be of assistance when describing and assessing our findings.

5.10 Definitions:

(i) A function $p(\lambda)$, defined Lebesgue almost everywhere on a subset D of $\mathbb{R}$ is said to be extendably continuous on D if there exists a function $q(\lambda)$ which is everywhere continuous on D such that $p(\lambda) = q(\lambda)$ whenever $p(\lambda)$ is defined.

(ii) A function $p(\lambda)$, defined Lebesgue almost everywhere on a subset D of $\mathbb{R}$ is said to be essentially continuous on D if there exists a subset E of D having Lebesgue measure zero such that the restriction of $p(\lambda)$ to $D \setminus E$ is extendably continuous on D.

(iii) A set is said to be nowhere connected if it contains no connected subsets.

Clearly extendable continuity implies essential continuity, and a subset of $\mathbb{R}$ is nowhere connected if and only if it contains no intervals.

In Example 5.12, the behaviour of the potential in a neighbourhood of 0 is such that $S_0(\lambda)$ is defined on a domain which is nowhere connected in [0,1], but which nevertheless contains almost all the points of [0,1], and, as we shall show, $S_0(\lambda)$ is not essentially continuous on [0,1]. First, however we establish that a class of potentials exists for which the spectrum of H_1 is bounded and is dense singular on [0,1].

5.11 Lemma: Let $\rho(\lambda)$ be a real monotonically increasing function on $\mathbb{R}$ with the following properties:

(i) $\rho(0) = 0$

(ii) $\rho(\lambda)$ is discontinuous at each point of a countable dense subset of [0,1], at each of the points $\lambda_n = (\pi n)^2$ and at $\lambda_0 = -1$; these are the only discontinuities of $\rho(\lambda)$.

(iii) $\rho(\lambda_n +) - \rho(\lambda_n -) = 2$

(iv) $\rho_{a.c.}(\infty) = \rho_{a.c.}(-\infty)$, $\rho_{s.c.}(\infty) = \rho_{s.c.}(-\infty)$

Then there exists a potential $\mathring{V}(r)$ on (0,1] such that $\rho(\lambda)$ is the spectral function of one of the self-adjoint operators arising from $\mathring{L} = -\dfrac{d^2}{dr^2} + \mathring{V}(r)$ in $L_2(0,1]$, r = 1 being a regular endpoint.

Proof:

We adapt the inverse method of Gel'fand and Levitan who consider the inverse problem on a finite interval $[0,l)$ with a boundary condition at 0 ([GL] §10). We wish to consider the inverse problem on the interval (0,1] with a boundary condition at 1, so shall make use of the transformation s = 1-r which maps the r-interval (0,1] onto the s-interval [0,1). Since $\dfrac{d^2}{dr^2} = \dfrac{d^2}{ds^2}$, for given $\mathring{V}(r)$ the equation

$$-\frac{d^2}{dr^2} u(r,\lambda) + \mathring{V}(r)\, u(r,\lambda) = \lambda\, u(r,\lambda) \tag{5.4.1}$$

with boundary conditions $u(1,\lambda) = 1$, $u'(1,\lambda) = h$ transforms to

$$-\frac{d^2}{ds^2} \tilde{u}(s,\lambda) + \tilde{V}(s)\, \tilde{u}(s,\lambda) = \lambda\, \tilde{u}(s,\lambda) \tag{5.4.2}$$

with boundary condition $\tilde{u}(0,\lambda) = 1$, $\tilde{u}'(0,\lambda) = -h$ where $\tilde{V}(s) = \mathring{V}(1-s)$, $\tilde{u}(s,\lambda) = u(1-s,\lambda)$; similarly (5.4.2) transforms to (5.4.1) using the substitution r = 1-s.

It is not hard to see, using Theorem 3.21, that the self-adjoint operators $\mathring{H}_1$ and $\tilde{H}_1$ associated with (5.4.1) and (5.4.2) and their respective boundary conditions have the same spectra; for the existence or otherwise of a certain type of solution of (5.4.1) at each point λ is not affected by our transformation. (Note that, as it stands, Theorem 3.21 applies to the r-interval $[0,\infty)$, 0 being a regular endpoint; however, it may be modified in the obvious way to apply to each of the intervals [0,1) and (0,1], with 0 and 1 respectively being regular endpoints).

Thus if λ is an eigenvalue of $\overset{\circ}{H}_1$, with corresponding eigenvector $u(r,\lambda)$, then λ is also an eigenvalue of $\tilde{H}_1$ with corresponding eigenvector $\tilde{u}(s,\lambda)$ and conversely. Moreover,

$$\int_0^1 |u(r,\lambda)|^2\, dr = \int_0^1 |u(1-s,\lambda)|^2\, ds = \int_0^1 |\tilde{u}(s,\lambda)|^2\, ds$$

so that the norms of these eigenvectors are equal for the same eigenvalue λ.

Now suppose that $\tau(\lambda)$ is a monotonically increasing saltus function which is known to be the spectral function of $\tilde{H}_1$ for some potential $\tilde{V}(s)$ and some $\tilde{h}$ in $\mathbb{R}$; then $\tau(\lambda)$ is also the spectral function of $\overset{\circ}{H}_1$ for $\overset{\circ}{V}(r) = \tilde{V}(1-r)$ with boundary condition $h = -\tilde{h}$. To see this, note that since $\tau(\lambda)$ is a saltus function, the spectrum of $\tilde{H}_1$ consists solely of eigenvalues and their accumulation points. By our remarks above, the spectrum of $\overset{\circ}{H}_1$ consists of the same eigenvalues and accumulation points; moreover, the "jump" in the spectral functions $\overset{\circ}{H}_1$ and $\tilde{H}_1$ will be the same at each eigenvalue, since the spectral measure at an eigenvalue is the square of the inverse of the corresponding eigenvector ([GL] p.253). The relationship between the boundary conditions of $\overset{\circ}{H}_1$ and $\tilde{H}_1$ is a consequence of the relationship between $\overset{\circ}{V}(r)$ and $\tilde{V}(r)$, as indicated above.

Therefore to show that the function $\rho(\lambda)$ in the hypothesis is the spectral function of some $\overset{\circ}{H}_1$, we need only show that there exists a potential $\tilde{V}(s)$ and an $\tilde{h}$ in $\mathbb{R}$ such that $\rho(\lambda)$ is the spectral function of the corresponding operator $\tilde{H}_1$. Sufficient conditions for this to be the case are as follows (see [GL] § 10.2):

(1) For each $s < 2$, the integral

$$\int_{-\infty}^{0} \cosh \sqrt{|\lambda|}\, s \; d\rho(\lambda)$$

exists. (Note that the upper limit of integration differs from that given in [GL]§10, which appears to us to be in error (cf. [GL] §4)).

(2) If $\sigma(\lambda) = \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}$ for $\lambda \geqslant 0$, the function

$$a(s) = \int_1^{\infty} \frac{\cos\sqrt{\lambda}\, s}{\lambda}\, d\sigma(\lambda)$$

has a continuous fourth derivative if $0 \leqslant s \leqslant 2$.

We note, firstly, that the existence or otherwise of the integral in (1) is independent of the behaviour of $\rho(\lambda)$ in any finite interval, and, secondly, that the existence or otherwise of a continuous fourth derivative of a(s) on [0,2] is independent of the behaviour of $\rho(\lambda)$ on $(-\infty,1]$. Therefore, provided $\rho(\lambda)$ is chosen suitably for $\lambda > 1$, and has an infinite set of points of increase on some finite interval ([GL] §§4,10), it may be otherwise arbitrarily chosen on any finite interval whose right hand endpoint is 1, and be constant on $(-\infty,c]$ for some $c < 1$. That conditions (ii)-(iv) of the hypothesis are sufficient to ensure suitable behaviour of $\rho(\lambda)$ for $\lambda > 1$, so that (1) and (2) above are satisfied, follows from the discussion in [GL] §11; this concludes the proof of the lemma.

We remark that the asymptotic behaviour as $\lambda \to \infty$ of a function $\rho(\lambda)$ satisfying conditions (i)-(iv) of Lemma 5.11 is such that $h \neq \infty$ (cf. [GL], loc.cit.) or, equivalently, $\alpha \neq 0$ (cf.(2.3.9)); this fact will be used in Example 5.12.

It follows from Lemma 5.11 that if $\rho(\lambda)$ satisfies conditions (i)-(iv) of the hypothesis, then a potential $\mathring{V}(r)$ and a boundary condition h exist such that $\rho(\lambda)$ is the spectral function of the associated operator H_1 in $L_2(0,1]$. If we retain $\mathring{V}(r)$ but alter the boundary conditions to $u(1,\lambda) = 0$, $u'(1,\lambda) = 1$ (that is, equivalently, to $h = \infty$), the essential spectrum of the modified operator H_1 is the same as that of $\mathring{H}_1$ ([DS] Ch.XIII, §6.6). Moreover, absolutely continuous spectrum is preserved under a change of boundary condition (see Thm. 2.21) so the spectrum of H_1 is also purely singular. It follows that H_1 has dense singular spectrum on [0,1] (note that H_1 is defined here in accordance with the notation of Chapter IV).

Now let

$$V(r) = \begin{cases} \mathring{V}(r) & 0 < r \leq 1 \\ 0 & r > 1 \end{cases}$$

and let H be the unique self-adjoint operator arising from the differential expression $L = \dfrac{-d^2}{dr^2} + V(r)$ in $L_2(0,\infty)$. Note that L is limit point at 0 since the spectrum of H_1 is dense in [0,1] (see [CL],Ch.9, Thm.4.1), and since V(r) is continuous on (0,1] ([GL] §10), V(r) is in $L_1[a,\infty)$ for each $a > 0$. Moreover, as we have noted above, the spectrum of H_1 is singular, so the wave operators exist and are complete in the partial wave subspace $l = 0$ by Theorem 5.8.

We now use these facts, together with the result of Lemma 5.11, to construct a specific example where the scattering amplitude is a discontinuous function of energy while the theory is asymptotically complete in the partial wave subspace $l = 0$. We shall subsequently deduce that, for this example, generalised asymptotic completeness holds; that is, in every partial wave subspace the wave operators exist and are complete.

5.12 Example: Let $\rho(\lambda)$ satisfy the hypothesis of Lemma 5.11. Then there exists a potential V(r) which vanishes for $r > 1$, and an α in $(0,\pi)$ such that $\rho(\lambda)$ is the spectral function of $\frac{-d^2}{dr^2} + V(r)$ in $L_2(0,1]$ with boundary condition α at $r = 1$ (cf. (2.3.9)). For such a $\rho(\lambda)$, by (2.3.4),

$$\begin{aligned} m(z) &= \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - z)} + \cot\alpha \\ &= \sum_{i=1}^{\infty} \frac{\gamma_i}{(x_i - z)} + \cot\alpha \end{aligned} \tag{5.4.3}$$

where $\{x_i\}$ are the points of discontinuity of $\rho(\lambda)$, and $\gamma_i = \mu(\{x_i\})$ for each i in $\mathbb{N}$, μ being the spectral measure generated by $\rho(\lambda)$. Moreover,

$$\operatorname{Im} m(z) = \sum_{i=1}^{\infty} \frac{y\,\gamma_i}{(x - x_i)^2 + y^2} \longrightarrow 0$$

as $y \downarrow 0$ Lebesgue almost everywhere on $\mathbb{R}$, where $z = x+iy$, $x, y \in \mathbb{R}$ (cf. Ch.II §3, esp. Cor. 2.7, and Lemma 2.13). Note that this function m(z) is not only equal in absolute value to the function m(z) associated with the analogous operator $\tilde{H}_1$, but also has the same sign. This is because, although the sign of the boundary condition at $r = 1$ is opposite to that of the boundary condition of $\tilde{H}_1$ at $r = 0$, the regular endpoint is to the right of the singular endpoint which has a further sign reversing effect (cf. [CL] Ch.9, § 5, Ex.1)

For every potential V(r) arising in this way, the wave operators exist and are complete in the partial wave subspace $l = 0$, as we noted earlier. To show that a potential of this kind exists for which the scattering amplitude is discontinuous we prove:

(i) the $\{\gamma_i\}$ may be chosen so that as $y \downarrow 0$

$$m(z) \to \sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x} + \cot\alpha$$

Lebesgue almost everywhere on $\mathbb{R}$. We deduce

(ii) the $\{\gamma_i\}$ may be chosen so that the phase shift $\delta(x)$ is not an essentially continuous function on (0,1).

Note that for $x_i > 1$, $\gamma_i = 2$ by condition (iii) of Lemma 5.11; however, $S = \{\gamma_i : x_i \in [0,1]\}$ may be chosen quite freely, subject only to $\sum_{\gamma_i \in S} \gamma_i < \infty$.

Proof of (i):

We first show that the $\{\gamma_i\}$ may be chosen so that $\frac{1}{\lambda - x}$ is in $L_1^{\rho}(-\infty,\infty)$ for Lebesgue almost all x in $\mathbb{R}$, and to this end we prove that if

$$\gamma_i < \frac{1}{2^{2i+3}} \tag{5.4.4}$$

for each i such that $x_i \in [0,1]$, then

$$\int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{|\lambda - x|} = \sum_{i=1}^{\infty} \frac{\gamma_i}{|x_i - x|} < \infty \tag{5.4.5}$$

for Lebesgue almost all x in $\mathbb{R}$.

Let X denote $\{i \in \mathbb{N} : x_i \in [0,1]\}$.

Then if $i \in X$,

$$\kappa\left(\left\{x : \frac{\gamma_i}{|x_i - x|} > \frac{1}{2^{i+1}}\right\}\right) = \kappa(\{x : |x_i - x| < \gamma_i\, 2^{i+1}\})$$
$$< \frac{1}{2^{i+1}}$$

by (5.4.4), where κ denotes Lebesgue measure. Hence

$$\begin{aligned} &\kappa\left(\left\{x : \sum_{i \in X,\, i \geq k} \frac{\gamma_i}{|x_i - x|} > \frac{1}{2^k}\right\}\right) \\ &\leq \sum_{i \in X,\, i \geq k} \kappa\left(\left\{x : \frac{\gamma_i}{|x_i - x|} > \frac{1}{2^{i+1}}\right\}\right) \\ &< \sum_{i=k}^{\infty} \frac{1}{2^{i+1}} \\ &< \frac{1}{2^k} \end{aligned} \tag{5.4.6}$$

Moreover, for each fixed x in $\mathbb{R} \setminus \{x_i\}$, and for each k in $\mathbb{N}$,

$$\sum_{i=1}^{k-1} \frac{\gamma_i}{|x_i - x|} < \infty$$

and, by (5.4.6),

$$\sum_{i \in X,\, i \geq k} \frac{\gamma_i}{|x_i - x|} \leq \frac{1}{2^k}$$

except on a set whose Lebesgue measure is less than $\frac{1}{2^k}$. Hence

$$\sum_{i \in X} \frac{\gamma_i}{|x_i - x|} < \infty \tag{5.4.7}$$

except on a set whose Lebesgue measure is less than $\frac{1}{2^k}$. Since $k \in \mathbb{N}$ may be chosen arbitrarily, it follows that

(5.4.7) is true for Lebesgue almost all x in $\mathbb{R}$.

To deduce (5.4.5), we need only note that

$$\sum_{i \in \mathbb{N} \setminus X} \frac{\gamma_i}{|x_i - x|} = \frac{\mu(\{-1\})}{|-1 - x|} + \sum_{n=1}^{\infty} \frac{2}{|\pi^2 n^2 - x|} < \infty \tag{5.4.8}$$

for Lebesgue almost all x in $\mathbb{R}$.

Now since $\operatorname{Im} m(z) \to 0$ as $y \downarrow 0$ for Lebesgue almost all x in $\mathbb{R}$,

$$\lim_{y \downarrow 0} m(z) = \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{(\lambda - x)}{(\lambda - x)^2 + y^2} \, d\rho(\lambda) + \cot \alpha \tag{5.4.9}$$

for Lebesgue almost all x in $\mathbb{R}$. If V denotes the set of all x in $\mathbb{R}$ for which (5.4.7) and (5.4.8) hold simultaneously, then $\kappa(\mathbb{R} \setminus V) = 0$ and, since $\left| \frac{\lambda - x}{(\lambda - x)^2 + y^2} \right| < \frac{1}{|\lambda - x|}$ for each $y \neq 0$, $\frac{(\lambda - x)}{(\lambda - x)^2 + y^2}$ is integrable with respect to μ for each x in V and each $y > 0$ by (5.4.5). Therefore, the Lebesgue Dominated Convergence Theorem may be applied to the right hand side of (5.4.9) to give

$$\begin{aligned} \lim_{y \downarrow 0} m(z) &= \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - x)} + \cot \alpha \\ &= \sum_{i=1}^{\infty} \frac{\gamma_i}{(x_i - x)} + \cot \alpha \end{aligned}$$

for Lebesgue almost all x in $\mathbb{R}$: this completes the proof of (i).

Proof of (ii)

We prove that a sequence $\{\gamma_i\}$ exists such that, if $\delta(x)$ were to be essentially continuous, then $m+(x) \equiv \pm\infty$ Lebesgue almost everywhere on [0,1] which is impossible since m(z) converges to a finite limit Lebesgue almost everywhere on $\mathbb{R}$ by Theorem 2.12(i).

Our strategy will be achieved if we choose $\{x_i\}$ and $\{\gamma_i\}$ in such a way that $\{x_i\}$ is dense in [0,1] and for each K > 0, every neighbourhood of each $x_i \in [0,1]$ contains a subset of positive Lebesgue measure on which $|m(z)| \geqslant K$. In view of (5.4.8) and the fact that $\cot\alpha < \infty$ (see remarks following Lemma 5.11) it suffices to prove that every neighbourhood of each x_i in [0,1] contains a subset of positive Lebesgue measure on which

$$\left|\sum_{i\in X} \frac{\gamma_i}{x_i - x}\right| \geqslant K.$$

Consider $\{\gamma_i\}$ for which

$$\gamma_i = \frac{1}{(2^8)^i} \tag{5.4.10}$$

for each $i \in X$. Clearly $\frac{1}{(2^8)^i} < \frac{1}{2^{2i+3}}$ for each $i \in X$, so the conclusion of (i) holds.

Let $x_j \in \{x_i\} \cap [0,1]$ and a neighbourhood N_j of x_i be fixed, and suppose K > 0 is chosen arbitrarily. Since $\sum_{i\in X,\, i\leqslant j-1} \frac{\gamma_i}{x_i - x}$

is continuous and hence bounded on every sufficiently small neighbourhood of x_j, we may choose $C \geqslant K$ and $U_j = [x_j - \delta, x_j + \delta] \subseteq N_j$ such that $\frac{\gamma_j}{\delta} = 2C$ and

$$\left|\sum_{i\in X,\, i\leqslant j-1} \frac{\gamma_i}{x_i - x}\right| < \frac{C}{2} \quad \text{on} \quad U_j \tag{5.4.11}$$

Then

$$\left|\frac{\gamma_j}{x_j - x}\right| \geqslant 2C \quad \text{on} \quad U_j \tag{5.4.12}$$

$$\left|\frac{\gamma_j}{x_j - x}\right| < 2C \qquad \text{on} \quad \mathbb{R} \setminus U_j \tag{5.4.13}$$

and by our choice (5.4.10) of $\{\gamma_i\}$,

$$\kappa\left(\left\{x : \left|\frac{\gamma_i}{x_i - x}\right| \geq \frac{C}{2^{i-j+1}}, i \in X\right\}\right) = \kappa\left(\left\{x : |x_i - x| \leq \frac{2}{2^{7i+j}C}, i \in X\right\}\right) \leq \frac{2^2}{2^{7i+j}C}$$

It follows that

$$\kappa\left(\left\{x : \left|\sum_{i \in X, i \geq j+1} \frac{\gamma_i}{x_i - x}\right| \geq \frac{C}{2}\right\}\right) \leq \sum_{i \in X, i \geq j+1} \kappa\left(\left\{x : \frac{\gamma_i}{|x_i - x|} \geq \frac{C}{2.2^{i-j}}\right\}\right) \leq \sum_{i=j+1}^{\infty} \frac{2^2}{2^{7i+j}C} < \frac{1}{2^{8j+4}C} \tag{5.4.14}$$

and, by (5.4.12) and (5.4.13)

$$\kappa\left(\left\{x : \left|\frac{\gamma_j}{x_j - x}\right| \geq 2C\right\}\right) = \kappa(U_j) = \frac{\gamma_j}{C} = \frac{1}{2^{8j}C} \tag{5.4.15}$$

Therefore, by (5.4.11), (5.4.14) and (5.4.15),

$$\kappa\left(\left\{x \in U_j : \left|\sum_{\substack{i \in X \\ i \neq j}} \frac{\gamma_i}{x_i - x}\right| \geq C\right\}\right) < \frac{1}{2^4}\kappa(U_j)$$

so that, using (5.4.15) again,

$$\kappa\left(\left\{x \in N_j : \left|\sum_{i \in X} \frac{\gamma_i}{x_i - x}\right| \geq C\right\}\right) \geq \left(1 - \frac{1}{2^4}\right)\kappa(U_j) > 0$$

Since $C \geq K$, N_j contains a subset of positive Lebesgue measure on which

$$\left|\sum_{i \in X} \frac{\gamma_i}{x_i - x}\right| \geq K \text{, as required.}$$

The argument above refers to the function m(z) related by (5.4.3) to the spectral function $\rho(\lambda)$ of $L = \frac{-d^2}{dr^2} + V(r)$ in $L_2\ (0,1]$ with boundary condition α at r = 1. So that we may avail ourselves of the formula for the phase shift $\delta(x)$ in the case where V(r) = 0 on $[1,\infty)$, viz,:

$$\tan(\delta(x) + \sqrt{x}) = \frac{\sqrt{x}}{m_0(x)} \tag{5.4.16}$$

(see (5.3.29)), we require Lemma 2.18 which relates the functions m(z) associated with distinct boundary conditions.

Now the function $m_0(z) = m_0(z,0)$ in (5.4.16) is the function m(z) associated with L in $L_2(0,1]$ with boundary condition 0 at r = 1 (see Ch.IV, §1). It follows from Lemma 2.18 (applied to the interval (0,1], L being limit point at 0) and our conclusions above concerning $m(z) = m_0(z,\alpha)$, that if $x_j \in \{x_i\} \cap [0,1]$ and $\varepsilon > 0$ are given, then every neighbourhood of x_j contains a subset S_ε of positive Lebesgue measure on which $m_0(x) = \lim_{y\downarrow 0} m_0(z)$ exists and is real and

$$|m_0(x) + \cot\alpha| < \varepsilon \text{ on } S_\varepsilon \tag{5.4.17}$$

Suppose now that $\delta(x)$ is an essentially continuous function. Then (5.4.16) and (5.4.17) together imply that

$$m_0(x) = -\cot\alpha$$

Lebesgue almost everywhere on [0,1], from which it follows by Lemma 2.18 that

$$m(x) = m(x,\alpha) = \pm\infty \tag{5.4.18}$$

Lebesgue almost everywhere on [0,1]. Since (5.4.18) is impossible by Lemma 2.12(i), we have proved by contradiction that $\delta(x)$ cannot be essentially continuous on [0.1]; this completes the proof of the lemma.

Thus we have used the inverse method to construct an example showing that an operator H exists for which the wave operators $\Omega_\pm(H, H_0)$ exist and are complete in the partial wave subspace $l = 0$ but the scattering amplitude is a discontinuous function of energy. We now show that, if in Example 5.12 $\mu(\{\lambda_0\})$ is chosen suitably, there exists a self adjoint extension H_l of

$$\hat{H}_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)$$

in $C_0^\infty(\mathbb{R}^+)$ for each $l = 1,2....$such that $\Omega_\pm(H_l, H_0)$ exist and are complete, where V(r) is the potential associated with H. We first require the following:

5.13 Lemma: Let V(r) be as in Example 5.12. Then the self-adjoint operator H arising from $L = \frac{-d^2}{dr^2} + V(r)$ in $L_2(0,\infty)$ is bounded below.

Proof:

By construction, the operator $\overset{\circ}{H}_1$ defined by L in $L_2(0,1]$ with boundary condition α at r = 1 has no spectrum for $\lambda < -1$. We now deduce that in $(-\infty,-1)$, the spectrum of the operator H_1 defined by L in $L_2(0,1]$ with a Dinchlet boundary condition at r = 1 consists at most of a single eigenvalue, and hence is bounded below.

Firstly, since essential spectrum is preserved under a change of boundary condition ([EK], Thm.2.5.2) the essential spectrum of H_1 for $\lambda < 0$ is empty since the same is true of $\overset{\circ}{H}_1$. Suppose that λ_1, λ_2 are two consecutive eigenvalues of H_1 with $\lambda_1 < \lambda_2 < -1$. Then $m_0(z,0)$ may be analytically continued across the open subinterval (λ_1, λ_2) of $\mathbb{R}^-$ ([CE] §5, Thm.), so that if $x_1, x_2 \in (\lambda_1, \lambda_2)$, $m_0(x_1,0) = \lim_{y\downarrow 0} m_0(x_1+iy,0)$ and $m_0(x_2,0) = \lim_{y\downarrow 0} m_0(x_2+iy,0)$ exist finitely, are real and

$$\frac{m_0(x_2,0) - m_0(x_1,0)}{(x_2 - x_1)} = \frac{\mu(\{\lambda_1\})}{(\lambda_1 - x_1)(\lambda_1 - x_2)} + \frac{\mu(\{\lambda_2\})}{(\lambda_2 - x_1)(\lambda_2 - x_2)} + \lim_{y\downarrow 0} \int_{\mathbb{R}\setminus\{\lambda_1,\lambda_2\}} \frac{d\rho_0(\lambda)}{[\lambda - (x_1+iy)][\lambda - (x_2+iy)]}$$

where ρ_0 is the spectral function of H_1. Since the final integral is bounded for all x_1, x_2 in (λ_1, λ_2) it follows that as $x_1 \to \lambda_1$, $m_0(x_1,0) \to -\infty$ and as $x_2 \to \lambda_2$, $m_0(x_2,0) \to +\infty$. Therefore, by the analyticity of

$m_0(x,0)$ across (λ_1,λ_2), $m_0(x,0)$ takes every value in $\mathbb{R}$ as x increases from λ_1 to λ_2. In particular, there exists λ_3 in (λ_1,λ_2) such that $m_0(\lambda_3,0) = -\cot\alpha$, so that by Lemma 2.18 (suitably adapted to the interval $(0,1]$), λ_3 is a pole of $m_0(z,\alpha)$. However, this is not possible since, because $\dot{H}_1$ has no spectrum in $(-\infty,-1)$, $m_0(z,\alpha)$ may be analytically continued across $(-\infty,-1)$. Therefore our supposition that two eigenvalues of H_1 exist in $(-\infty,-1)$ must be false, so we have proved by contradiction that H_1 is bounded below.

It follows that $K \in \mathbb{R}^+$ exists such that $m_0(z,0)$ may be analytically continued across the real axis for all $x < -K$. Since $V(r) = 0$ for $r > 1$, $m_\infty(z) = m_\infty(z,0) = i\sqrt{z}$ (see (3.1.2)) so that the negative spectrum of H_∞ (for notation see Ch.IV, §3) is empty by Lemma 2.13. Hence $m_\infty(z,0) - m_0(z,0)$ is an analytic function in the region $x < -K$. Moreover, since there are no negative eigenvalues of H_∞, the negative spectrum of H is concentrated on the set

$$\Sigma = \{x \in \mathbb{R}^- : m_0(x,0) = m_\infty(x,0)\}$$

by Proposition 4.4. To show that $\Sigma \cap (-\infty, L)$ is empty for some L in $\mathbb{R}^-$, it suffices therefore to prove that $m_\infty(x,0) - m_0(x,0)$ eventually has the same sign as $x \to -\infty$.

Since $m_\infty(z,0) = i\sqrt{z}$,

$$m_\infty(x,0) = -\sqrt{-x}$$

for all $x < 0$. Moreover, by Lemma 2.18, if $x < -K$,

$$\begin{aligned} m_0(x,0) &= \lim_{y \downarrow 0} m_0(z,0) \\ &= \lim_{y\downarrow 0} \frac{1 + \cot\alpha\, m_0(z,\alpha)}{\cot\alpha - m_0(z,\alpha)} \\ &= \frac{1 + \cot\alpha\, m_0(x,\alpha)}{\cot\alpha - m_0(x,\alpha)} \\ &= \frac{1 + \cot\alpha \left[\sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x} + \cot\alpha \right]}{-\sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x}} \end{aligned}$$

where we have used the result proved in Example 5.12(i). Hence

$$m_0(x,0) = \frac{-\operatorname{cosec}^2\alpha}{\left(\sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x}\right)} - \cot\alpha$$

Now, by construction, if μ is the spectral measure of H_1, $A = \mu(\{\lambda_0\})$ and $M = \mu([0,1])$, then for $x < -1$,

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x} < \frac{A}{(-1-x)} + \frac{M}{(-x)} + \sum_{n=1}^{\infty} \frac{2}{(n^2\pi^2 - x)}$$

$$< \frac{A+M}{(-1-x)} + \int_0^{\infty} \frac{2}{(n^2\pi^2 - x)}\, dn$$

$$= \frac{A+M}{(-1-x)} + \frac{1}{\sqrt{-x}}$$

Hence, as $x \to -\infty$,

$$\frac{m_0(x,0)}{m_\infty(x,0)} > \frac{-\operatorname{cosec}^2\alpha\,(1+x)}{(A+M)\sqrt{-x} - (1+x)} + \frac{\cot\alpha}{\sqrt{-x}} \longrightarrow \operatorname{cosec}^2\alpha$$

Similarly, for $x < -1$,

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{x_i - x} > \frac{A}{-1-x} + \frac{M}{1-x} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2 - x}$$

$$> \frac{A+M}{1-x} + \int_1^{\infty} \frac{2}{(n^2\pi^2 - x)}\, dn$$

$$= \frac{A+M}{1-x} + \frac{1}{\sqrt{-x}}\left[1 - \frac{2}{\pi}\tan^{-1}\frac{\pi}{\sqrt{-x}}\right]$$

Hence, as $x \to -\infty$,

$$\frac{m_0(x,0)}{m_\infty(x,0)} < \frac{\operatorname{cosec}^2\alpha\,(1-x)}{(A+M)\sqrt{-x} + (1-x)\left[1 - \frac{2}{\pi}\tan^{-1}\frac{\pi}{\sqrt{-x}}\right]} + \frac{\cot\alpha}{\sqrt{-x}}$$

$$\longrightarrow \operatorname{cosec}^2\alpha \qquad (5.4.19)$$

It follows that

$$\frac{m_0(x,0)}{m_\infty(x,0)} \longrightarrow \operatorname{cosec}^2\alpha \quad \text{as} \quad x \to -\infty$$

Since $m_0(x,0)$, $m_\infty(x,0)$ are strictly negative for large negative x, we deduce that if $\operatorname{cosec}^2\alpha \neq 1$, $m_\infty(x,0) - m_0(x,0)$ eventually has the same sign as $x \to -\infty$.

If $\operatorname{cosec}^2\alpha = 1$, $\cot\alpha = 0$ and we have from (5.4.19)

$$\frac{m_0(x,0)}{m_\infty(x,0)} < \frac{1}{1 + \frac{\sqrt{-x}}{(1-x)}\left((A+M) - \frac{2}{\pi}\frac{(1-x)}{\sqrt{-x}}\tan^{-1}\frac{\pi}{\sqrt{-x}}\right)}$$

Now as $x \to -\infty$,

$$(A+M) - \frac{2}{\pi}\frac{(1-x)}{\sqrt{-x}}\tan^{-1}\frac{\pi}{\sqrt{-x}} \to (A+M) - \frac{2}{\pi}\sqrt{-x}\tan^{-1}\frac{\pi}{\sqrt{-x}}$$
$$\to A + M - 2$$

Hence if $A \geqslant 2$, $m_\infty(x,0) - m_0(x,0)$ is eventually negative as $x \to -\infty$ if $\operatorname{cosec}^2\alpha = 1$.

Thus, whatever the value in $(0,\pi)$ of the boundary condition α associated with $\overset{\circ}{H}_1$, if $\mu(\{\lambda_0\})$ is sufficiently large, the spectrum of H is bounded below.

This completes the proof of the Lemma.

With V(r) as in Lemma 5.13, it is now possible to establish that a self-adjoint operator exists for which the theory is asymptotically complete in every partial wave subspace and the scattering amplitude is discontinuous.

5.14 Theorem: Let V(r) be as in Lemma 5.13 with $\mu(\{\lambda_0\}) > 2$ and for each $l = 1,2,\ldots$ let H_l be the Friedrich's extension of the symmetric semibounded operator

$$\hat{H}_l = \frac{-d^2}{dr^2} + V(r) + \frac{l(l+1)}{r^2}$$

acting in $C_0^\infty(\mathbb{R}^+)$. Then, if H_0 is the unique self-adjoint extension of $\frac{-d^2}{dr^2}$ in $C_0^\infty(\mathbb{R}^+)$ defined by a Dirichlet boundary condition at 0, the wave operators $\Omega_\pm(H_l, H_0)$ exist and are complete.

Proof:

Since $V(r) = O(r^{-(1+\varepsilon)})$ as $r \to \infty$, $\Omega_\pm(H_l, H_0)$ exist for each l ([KS]) and the range of $\Omega_\pm(H_l, H_0)$ is equal to the subspace of scattering states of H_l([P4] Thm.3). To establish completeness of the wave operators, therefore, it remains to show that no states are asymptotically absorbed. ([P4] Thm.2). This will be achieved if we show that no $k \in (0,\infty)$ and $f^{\pm} \in \mathcal{H}_{a.c.}(H_l)$ can exist for which

$$\lim_{R\to 0} \lim_{t\to\pm\infty} \| F_{r<R} e^{-iH_l t} f\pm \| = k \tag{5.4.20}$$

where $\mathcal{H}_{a.c.}(H_l)$ is the subspace of absolute continuity of H_l ([KA] Ch.X, §2) and $F_{r<R}$ is the projection operator defined by

$$F_{r<R}\, h(r) = \begin{cases} h(r) & 0 < r < R \\ 0 & r \geqslant R \end{cases}$$

To prove that no $f^{\pm} \in \mathcal{H}_{a.c.}(H_l)$ and $k > 0$ exist for which (5.4.20) is true, it suffices to show that for each $C \in (0,\infty)$ and $f\pm \in \mathcal{H}_{a.c.}(H_l)$

$$\lim_{R\to 0} \limsup_{t\to\pm\infty} \| F_{r<R}\, E_{|H_l|<C}\, e^{-iH_l t} f\pm \| = 0 \tag{5.4.21}$$

where $E_{|H_l|<C}$ denotes the spectral projection of H_l associated with the λ-interval $(-C,C)$.

Now, from Lemma 5.13 and the fact that $\frac{l(l+1)}{r^2}$ is a positive operator, it follows that the operator

$$\hat{H}_l = \frac{l(l+1)}{r^2} + g\left(-\frac{d^2}{dr^2} + V(r)\right)$$

with domain $C_0^\infty(\mathbb{R}^+)$ is symmetric and bounded below for all g in $\mathbb{R}^+$. Hence, if $g > 1$ is fixed, $a_g > 0$ may be chosen so that

$$\frac{l(l+1)}{r^2} + g\left(-\frac{d^2}{dr^2} + V(r)\right) + a_g \geqslant 0 \tag{5.4.22}$$

Moreover, since $\hat{H}_l$ is bounded below, there exists γ in $\mathbb{R}$ such that

$$\langle \hat{H}_l f, f\rangle \geqslant \gamma \langle f,f\rangle$$

for all f in $\mathcal{D}(\hat{H}_l) = C_0^\infty(\mathbb{R}^+)$. It follows that

$$\langle \hat{H}_l f, f\rangle + (1-\gamma)\langle f,f\rangle \geqslant \langle f,f\rangle$$

for all f in $C_0^\infty(\mathbb{R}^+)$, so defining the $\hat{H}_l$ - form norm in $C_0^\infty(\mathbb{R}^+)$ by

$$\|\cdot\|_s = \langle \cdot,\cdot\rangle_s^{1/2}$$

where $\langle f,g\rangle_s = \langle \hat{H}_l f, g\rangle + (1-\gamma)\langle f,g\rangle$

we have
$$\|f\|_s \geqslant \|f\|$$

for all f in $C_0^\infty(\mathbb{R}^+)$. Hence if $\{f_n\}$ is a Cauchy sequence in the $\hat{H}_l$-form completion of $C_0^\infty(\mathbb{R}^+)$ (see eg. [RN] §124, [WE1] §5.5), then $\{f_n\}$ is also a Cauchy sequence in $\mathcal{H} = L_2(0,\infty)$. Also, since H_l is the Friedrich's extension ([RN],[WE1], loc.cit.) of $\hat{H}_l$ by hypothesis, $\mathcal{D}(H_l)$ is contained in the $\hat{H}_l$-form completion of $C_0^\infty(\mathbb{R}^+)$; therefore, if $h \in \mathcal{D}(H_l)$, there exists a sequence $\{h_n\}$ in $C_0^\infty(\mathbb{R}^+)$ such that

$$\lim_{m,n\to\infty} \langle \hat{H}_l(h_m - h_n),(h_m - h_n)\rangle + (1-\gamma)\langle(h_m - h_n),(h_m - h_n)\rangle \to 0 \tag{5.4.23}$$

By our remarks above, $\{h_n\}$ is also a Cauchy sequence in $\mathcal{H}$, so from (5.4.23)

$$\lim_{m,n\to\infty} \langle (\hat{H}_l + \frac{a_g}{g})(h_m - h_n),(h_m - h_n)\rangle \to 0$$

where a_g,g are as in (5.4.22). From

$$\hat{H}_l + \frac{a_g}{g} = (1 - \frac{1}{g})\frac{l(l+1)}{r^2} + \frac{1}{g}\left[\frac{l(l+1)}{r^2} + g(-\frac{d^2}{dr^2} + V(r)) + a_g\right] \tag{5.4.24}$$

it follows that

$$\lim_{m,n\to\infty} \langle \frac{l(l+1)}{r^2}(h_m - h_n),(h_m - h_n)\rangle \to 0$$

since each term on the right hand side of (5.4.24), regarded as a bilinear form, is positive (see (5.4.22)). We deduce that $h \in \mathcal{D}\left(\frac{(l(l+1))^{1/2}}{r}\right)$ so that

$$\mathcal{D}(H_l) \subseteq \mathcal{D}\left(\frac{(l(l+1))^{1/2}}{r}\right) \tag{5.4.25}$$

To prove (5.4.21), we need only show that for each C in $(0,\infty)$,

$$F_{r<R}\, E_{|H_l|<C}$$

has arbitrarily small norm as $R \to 0$. Since $\mathrm{range}(E_{|H_l|<C}) \subseteq \mathcal{D}(H_l)$ for each C in $(0,\infty)$, it follows from (5.4.25) that

$$F_{r<R}\, E_{|H_l|<C} = F_{r<R}\frac{r}{[l(l+1)]^{1/2}}\,\frac{[l(l+1)]^{1/2}}{r}\, E_{|H_l|<C}$$

Now $\frac{[l(l+1)]^{1/2}}{r}E_{|H_l|<C}$ is a closed operator defined on all of $\mathcal{H} = L_2(0,\infty)$, so is bounded by the Closed Graph Theorem ([KA] Ch.III, §4), and, clearly,

$$\left\| F_{r<R}\frac{r}{[l(l+1)]^{1/2}}\right\| \leq \frac{R}{[l(l+1)]^{1/2}}$$

Hence for each C in $(0,\infty)$, and each $l = 1,2,\ldots$,

$$\| F_{r<R} E_{|H_l|<C} \| \to 0$$

as $R \to 0$, which proves (5.4.21) and hence the theorem.

Thus the extension in §3 to the class of potentials for which the phase shift formula (Thm.5.9) for the scattering operator holds has enabled the existence of a potential for which the theory is asymptotically complete and the scattering operator is a discontinuous function of energy to be demonstrated. The potential V(r) is of finite range, and is such that the spectrum of every self-adjoint operator H_1 arising from $\frac{-d^2}{dr^2} + V(r)$ in $L_2(0,1]$ is singular and has a dense singular subset. It seems likely that asymptotic completeness and discontinuity of the scattering amplitude can occur in conjunction under more general conditions, and that the nature of the spectrum of H_1 may be of considerable significance in this connection.

CHAPTER VI

THE CONSTRUCTION OF POTENTIALS WITH SINGULAR CONTINUOUS SPECTRA

§1 Introduction

Whether or no the mathematical phenomenon of singular continuous spectrum has a distinct counterpart in nature, its relevance as a probe for exploring the limits and the limitations of quantum theory remains. A considerable literature has been devoted to identifying classes of potentials which ensure such properties as absence of singular continuous spectrum, asymptotic completeness of the wave operators and continuity of the scattering amplitude (eg. [RS IV], Ch.XIII, [AM], [D]) but it is no less relevant to identify situations in which the familiar behaviour breaks down. So-called pathological behaviour not only reveals the existence of limits to established theory, but also raises important questions of interpretation and realisability which may lead to new predictions and a reappraisal of accepted ideas.

Such a re-evaluation was undertaken by D.Pearson in "Singular Continuous Measures in Scattering Theory" ([P1]). This paper challenges the prevailing view that singular continuous spectrum has no physical interpretation, and, in the light of supporting examples, suggests that this type of spectrum may be associated with a characteristic recurrent behaviour of particles in the appropriate energy bands. Crucial to the construction of Pearson's examples is a theoretical result concerning the generation of singular continuous measures from limiting sequences of absolutely continuous measures. ([P1], Thm.1). This enables certain types of potential to be constructed inductively in such a way that singular continuity of the limiting spectral measure is assured; Pearson considered potentials which consist of an infinite sequence of potential "bumps" whose separation increases rapidly with distance from the origin. Provided the shape and the width of each "bump" remains invariant throughout the sequence, and the heights either remain constant or decrease to zero at infinity, a sufficiently rapid increase in the separation between consecutive "bumps" ensures a purely singular continuous spectrum ([P1], Props.1,2)

This type of inductive construction affords an unusual and promising approach to the problem of identifying potentials for which the associated spectrum is singular continuous. With a view to further extending the class of potentials which can be considered in this way, we reformulate Pearson's Theorem 1 under more general assumptions in §3, and make a careful comparison of the original and modified conditions.

We note, however, that even without modification to Theorem 1, Pearson's method may be applied to demonstrate the presence of singular continuous spectrum in situations where the potential does not satisfy the hypotheses of Propositions 1 or 2. We illustrate this point in §2 by showing that such spectrum can arise when both the width of the "bumps" and the separation between them becomes arbitrarily large with increase in distance from the origin. Our example suggests that slowly oscillating continuous potentials may give rise to singular continuous spectrum provided the wavelength of the oscillations increases sufficiently rapidly with distance.

In order to give a more precise idea of the type of sequence of absolutely continuous measures which can converge to a singular continuous measure, we construct in §4 a simple example where the value of the limiting measure of intervals may be computed exactly. Careful choice of the elements of the sequence ensures that the convergence is, in a sense that will become apparent, optimal; and the explicit formulae involved show that a precise determination of a suitable sequence and of the limiting measure may be obtained in specific cases.

To appreciate more clearly the manner in which Theorem 5.9 extends the class of potentials for which the phase shift formula for the scottering operator holds, it is desirable to identify specific examples. In Chapter V, §4, we used the inverse method of Gel'fand and Levitan to show that a potential exists for which the spectrum of the Hamiltonian restricted to $L_2(0,1]$ is dense singular on (0,1) and otherwise pure point. However, although the inverse method has wide applicability, as in this case, where confirmation of the existence of a potential with specific spectral properties is required, it does not provide a straightforward method for determining the potential explicitly. It may be that a suitable adaptation of the method of inductive construction of potentials to the case of a finite interval can provide some insight into the type of potential which satisfies the hypothesis of Theorem 5.8 but is not of the class considered by Green and Lanford.

§2. A slowly oscillating potential with singular continuous spectrum

In this section we describe a type of potential V(r) for which the spectrum is singular continuous in the λ-interval (lim inf V(r), lim sup V(r)).

Consider a potential V(r) which alternates between the constant values 0 and 1 on successive intervals of $\mathbb{R}^+$; 0 and 1 are chosen for convenience, but any potential which alternates between two constant values on successive intervals of $\mathbb{R}^+$ can be reduced to this problem by change of origin and scaling.

The lengths of successive intervals $I_n^{\,0}$ on which V(r) takes the constant value zero are chosen inductively to ensure singular continuity on (0,1] of the spectral measure of the Hamiltonian H arising from $\frac{-d^2}{dr^2} + V(r)$ on $[0,\infty)$ with boundary condition $\alpha = 0$ (see (2.3.9)).
We shall show that the lengths of the intervals $\{I_n'\}$ on which the potential takes the value 1 do not affect the prospect of singular continuous spectrum on (0,1) provided they do not decrease with n. We may therefore choose $\kappa(I_n') = \kappa(I_n^{\,0})$ for each n, where κ denotes Lebesgue measure.

We shall adapt the method of Pearson ([P1], §3) to prove singular continuity of the spectrum on (0,1). For ease of reference here and later, we state without proof Pearson's Theorem 1.

6.1 <u>Theorem</u>: Let the functions $f_n(k,y)$ ($\alpha \leq k \leq \beta$, $-\infty < y < \infty$, $n = 1,2\ldots$)

be periodic in y, with period c, continuously differentiable, and satisfy

(i) $f_n(k,y) \geq$ const. > 0.

(ii) $\bar{f}_n(k) = \frac{1}{c}\int_0^c f_n(k,y)\,dy = 1$

(iii) $\sum_{n=1}^{\infty} -m_n(k) = +\infty \quad , \alpha \leq k \leq \beta$

where $m_n(k) = \overline{\text{Log}}\,(f_n(k,y)) = \frac{1}{c}\int_0^c \log f_n(k,y)\,dy$

(iv) For N sufficiently large, $f_n(k,Nk)$ is an analytic function of k, where $\alpha \leq k \leq \beta$.

Given a sequence $\{N_i\}$, i = 1,2,3..., of increasing positive numbers with $\lim_{i\to\infty} N_i = \infty$, define the Lebesgue-Stieltjes measures $\{\nu_n\}$ by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{i=1}^{n} f_i(k, N_i k)\,dk$$

for every subinterval Σ of $[\alpha,\beta]$.

Then the sequence $\{N_i\}$ may be chosen such that $\lim_{n\to\infty} \nu_n(\Sigma) = \nu(\Sigma)$ exists for every subinterval Σ of $[\alpha,\beta]$ and defines a singular continuous Lebesgue Stieltjes measure on Borel subsets of $[\alpha,\beta]$.

We require the following preliminary result:

6.2 Lemma: Let $\{N_k\}$ be an increasing sequence in $\mathbb{R}^+$ such that $N_k \to \infty$ as $k\to\infty$, and let ν_k denote the spectral measure of $\frac{-d^2}{dr^2} + V(r)$ in $[0,N_k]$ with Dirichlet boundary conditions $u(0,\lambda) = u(N_k,\lambda) = 0$. If the spectral measure μ of $\frac{-d^2}{dr^2} + V(r)$ in $[0,\infty)$ with boundary condition $u(0,\lambda) = 0$ is continuous, then $\{\nu_k\}$ converges uniformly to μ over subintervals of any fixed finite interval.

Proof:

We show that if μ is continuous and if $\varepsilon > 0$ and a compact interval I of $\mathbb{R}$ are given, then K in $\mathbb{N}$ exists such that for all subintervals Σ of I

$$|\nu_k(\Sigma) - \mu(\Sigma)| < \varepsilon$$

whenever $k \geqslant K$.

If μ is continuous, we may subdivide I into a finite number p of disjoint intervals $\{I_j\}$, $j = 1,\ldots p$, such that

$$\mu(I_j) < \frac{\varepsilon}{16} \tag{6.2.1}$$

for each $j = 1,\ldots,p$. Since $\nu_k(I_j)$ converges to $\mu(I_j)$ as $k \to \infty$ for each $j = 1,\ldots p$, ([CL], Thm. 3.1(i)), there exists K in $\mathbb{N}$ such that for each $j = 1,\ldots,p$,

$$|\nu_k(I_j) - \mu(I_j)| < \frac{\varepsilon}{4p} \tag{6.2.2}$$

whenever $k \geqslant K$

Let I_s be any subinterval of I. We may write

$$I_s = I_q \bigcup_{j=m+1}^{n-1} I_j \cup I_r \tag{6.2.3}$$

where $I_q \subseteq I_m$, $I_r \subseteq I_n$, $\{m,\ldots,n\} \subseteq \{1,\ldots,p\}$. Using (6.2.1) and (6.2.2) we have

$$\begin{aligned} |\nu_k(I_q) - \mu(I_q)| &\leqslant \nu_k(I_q) + \mu(I_q) \\ &\leqslant \frac{\varepsilon}{4p} + 2\mu(I_m) \\ &< \frac{3\varepsilon}{8} \end{aligned}$$

whenever $k \geqslant K$, and a similar inequality holds in respect of I_r. It now follows from (6.2.3), Minkowski's inequality and (6.2.2) that

$$|\nu_k(I_s) - \mu(I_s)| \leqslant \sum_{j=m+1}^{n-1} |\nu_k(I_j) - \mu(I_j)| + \frac{3\varepsilon}{4} < \varepsilon$$

whenever $k \geqslant K$. Since I_s is an arbitrary subinterval of I, the lemma is proved.

Before proving singular continuity of the spectrum on (0,1) for a suitably constructed slowly oscillating potential, we show that if $V(r) = 0$

on a sequence of intervals $\{I_n^o\}$ of $\mathbb{R}^+$, then $(0,\infty)$ lies in the spectrum of H provided that $\kappa(I_n^o) \to \infty$ as $n \to \infty$. This ensures that Proposition 6.4 is a non-trivial result.

6.3 Proposition: Let $\{a_n\}$, $\{b_n\}$ be increasing sequences in $\mathbb{R}^+$ with $a_n < b_n < a_{n+1}$ for each n, and let I_n^0 and I_n^1 denote $[a_n, b_n]$ and (b_n, a_{n+1}) respectively. Define

$$V(r) = \begin{cases} 0 & r \in \bigcup_{n=1}^{\infty} I_n^o \\ 1 & r \in \bigcup_{n=1}^{\infty} I_n^1 \end{cases}$$

Then if $\kappa(I_n^o) \to \infty$ as $n \to \infty$, $(0,\infty)$ lies in the spectrum of every self adjoint operator arising from $L = \frac{-d^2}{dr^2} + V(r)$.

Proof:

We show that $(H-k^2)^{-1}$ is unbounded for all $k > 0$.

Now if f is in $\mathcal{D}(H)$,

$$\|f\| = \|(H-k^2)^{-1}(L-k^2)f\| \leq \|(H-k^2)^{-1}\| \, \|(L-k^2)f\|$$

where $\|(H-k^2)^{-1}\| = \sup_{\{g : \|g\| = 1\}} \|(H-k^2)^{-1}g\|$ and $\|\cdot\|$ denotes $\left(\int_0^\infty |\cdot|^2 dr\right)^{1/2}$. Hence it suffices to show that for each $k > 0$, if $\varepsilon > 0$ is given, there exists $f(r,k,\varepsilon)$ in $\mathcal{D}(H)$ for which

$$\|f(r,k,\varepsilon)\| \geq 1 \quad , \quad \|(L-k^2)f(r,k,\varepsilon)\| < \varepsilon \qquad (6.2.1)$$

Consider the sequence of functions

$$f_N(r,k,P) = \begin{cases} \left(\frac{2N}{\pi}\right)^{\frac{1}{2}} \dfrac{r^2 e^{ik(r-P)}}{2(r-P+iN)} & , \quad r \in [0,1) \\ \left(\frac{2N}{\pi}\right)^{\frac{1}{2}} \left[1 - \frac{1}{2}(r-2)^2\right] \dfrac{e^{ik(r-P)}}{(r-P+iN)} & , \quad r \in [1,2) \\ \left(\frac{2N}{\pi}\right)^{\frac{1}{2}} \dfrac{e^{ik(r-P)}}{(r-P+iN)} & , \quad r \in [2,\infty) \end{cases}$$

Clearly $f_N(0,k,P) = 0$ for each $N \in \mathbb{N}$, $k, p \in \mathbb{R}^+$ and, since

$$\| f_N(r,k,P) \| < 2 \; \left\| \left(\frac{2n}{\pi}\right)^{\frac{1}{2}} \frac{e^{ikr}}{(r+iN)} \right\| = 2 \qquad (6.2.5)$$

$f_N(r,k,P)$ is in $L_2[0,\infty)$ for each $N \in \mathbb{N}$, $k,p \in \mathbb{R}^+$. Moreover, if $P \geqslant 2$.

$$\| f_N(r,k,P) \| > \left(\int_P^\infty \left| \left(\frac{2N}{\pi}\right)^{\frac{1}{2}} \frac{e^{ik(r-P)}}{(r-P+iN)} \right|^2 dr \right)^{\frac{1}{2}} = \left\| \left(\frac{2n}{\pi}\right)^{\frac{1}{2}} \frac{e^{ikr}}{(r+iN)} \right\| = 1 \qquad (6.2.6)$$

and, since $f_N(r,k,P)$ is a twice differentiable function of r for all $n \in \mathbb{N}$, $r, k, P \in \mathbb{R}^+$ it may be deduced that

$$\left\| \left(-\frac{d^2}{dr^2} - k^2 \right) f_N(r,k,P) \right\| \leqslant \frac{C_k}{N^{1/2}} \qquad (6.2.7)$$

for some C_k in $\mathbb{R}^+$ which is independent of P.

Let $k \in \mathbb{R}^+$ be fixed, $\varepsilon > 0$ be given, and choose $M \in \mathbb{N}$ such that $\frac{C_k}{M^{1/2}} < \frac{\varepsilon}{2}$. Using the properties of $f_M(r,k,P)$, in particular $\| f_M(r,k,P) \| \uparrow 2$ as $P \to \infty$ (cf.(6.2.5)), we see that if $\kappa(I_n^0) \to \infty$ as $n \to \infty$, then $P_\varepsilon, L_\varepsilon \in \mathbb{R}^+$ may be chosen with $P_\varepsilon > 2$, $L_\varepsilon < P_\varepsilon$,

$$4 > \int_{P_\varepsilon - L_\varepsilon}^{P_\varepsilon + L_\varepsilon} | f_M(r,k,P_\varepsilon)|^2 \, dr > 4 - \left(\frac{\varepsilon}{2}\right)^2 \qquad (6.2.8)$$

and $V(r) = 0$ on $[P_\varepsilon - L_\varepsilon, P_\varepsilon + L_\varepsilon]$.

Let S denote $\mathbb{R}^+ \setminus [P_\varepsilon - L_\varepsilon, P_\varepsilon + L_\varepsilon]$, and let χ_S denote the characteristic function of the set S. Then, using $V(r) \leqslant 1$ for all r in $[0,\infty)$, Minkowski's inequality, (6.2.7) and (6.2.8) we have

$$\begin{aligned} &\| (L-k^2) f_M(r,k,P_\varepsilon) \| \\ &\quad = \left\| \left(-\frac{d^2}{dr^2} - k^2 \right) f_M(r,k,P_\varepsilon) + V(r) f_M(r,k,P_\varepsilon) \chi_S \right\| \\ &\quad \leqslant \left\| \left(-\frac{d^2}{dr^2} - k^2 \right) f_M(r,k,P_\varepsilon) \right\| + \left(\int_S | f_M(r,k,P_\varepsilon)|^2 dr \right)^{\frac{1}{2}} \\ &\quad < \varepsilon \end{aligned}$$

It follows that $f_M(r,k,P_\varepsilon)$ is in $\mathcal{D}(H)$, so setting $f(r,k,\varepsilon) = f_M(r,k,P_\varepsilon)$

we see from (6.2.6) and the above inequality that $f(r,k,\varepsilon)$ satisfies (6.2.4). Since $k,\varepsilon \in \mathbb{R}^+$ were chosen arbitrarily, we deduce that $(H-k^2)^{-1}$ is unbounded for all $k > 0$.

We conclude that $(0,\infty)$ is contained in the spectrum of H, and since the essential spectrum is independent of the boundary condition at 0 ([CE], Thm.2.5.2), $(0,\infty)$ is contained in the spectrum of every self adjoint operator arising from $V(r)$.

This completes the proof of the proposition.

We now describe a class of potentials for which there is singular continuous spectrum on (0,1). (To avoid confusion, we should point out that our notation, though similar, does not coincide with that of [P1] §3).

6.4 <u>Proposition</u>: Let $V(r)$ be as in Proposition 6.3, and suppose $\kappa(I_n^1) = \kappa(I_n^0)$ for $n = 1,2,3\ldots$. Then provided $\kappa(I_n^0)$ increases sufficiently rapidly with n, the operator H arising from $V(r)$ with Dirichlet boundary condition $u(0,\lambda) = 0$ has singular continuous spectrum on (0,1).

<u>Proof</u>:

We consider only $\lambda > 0$ and set $\lambda = k^2$.

Let ν_n denote the spectral measure of $\frac{-d^2}{dr^2} + V(r)$ in $[0,b_n)$, with Dirichlet boundary conditions $u(0,k) = u(b_n,k) = 0$, and let $\rho_n(k)$, $\mu_n(k)$ denote the spectral function and corresponding spectral measure of $\frac{-d^2}{dr^2} + V(r)$ in $[0,\infty)$ with the same boundary condition at 0, where

$$V_n(r) = \begin{cases} V(r) & \text{on } [0,a_n) \\ 0 & \text{on } [a_n,\infty) \end{cases}$$

The sequences of intervals $\{I_n^0\}$ and $\{I_n^0\}$ are determined inductively as follows: Suppose the first (2n-1) intervals $I_1{}^0, I_1{}^1, \ldots I_{n-1}{}^1, I_n{}^0$ have been established, and consequently the potential $V(r)$ on $[0,b_n]$. Then a_{n+1} is chosen so that $\kappa(I_n^1) = \kappa(I_n^0)$ and b_{n+1} may be chosen to satisfy the following conditions, as we shall show below:

(i) $$|\nu_{n+1}(\Sigma) - \mu_{n+1}(\Sigma)| < \frac{1}{n+1}$$

for all subintervals Σ of $(0,1)$.

(ii) The rate of increase of $\kappa(I_n^0)$ as $n \to \infty$ is sufficiently rapid to ensure that, if $\Sigma \subseteq (0,1)$,

$$\mu(\Sigma) = \lim_{n\to\infty} \mu_n(\Sigma)$$

exists for all intervals $\Sigma \subseteq (0,1)$, and defines a singular continuous measure on Borel subsets of $(0,1)$.

To see that b_{n+1} may be chosen to satisfy (i), note that μ_{n+1} is determined once a_{n+1} is fixed, whereas ν_{n+1} is not determined until b_{n+1} is fixed. Hence, since μ_{n+1} is absolutely continuous on $(0,\infty)$, once a_{n+1} is fixed we may choose b_{n+1} so that (i) is satisfied by Lemma 6.2. We commence the inductive process by choosing $a_1 > 0$ arbitrarily, and setting $V(r) = 1$ on $[0,a_1)$.

We now adapt the method of Pearson ([P1] §3) to show that condition (ii) may be satisfied.

Let $\phi(r,k)$ be the solution of $-\frac{d^2u}{dr^2} + V(r)u = k^2u$ which satisfies $\phi(0,k) = 0$, $\phi'(0,k) = 1$. Let $R(r,k)$ and $\theta(r,k)$ be defined by the relations

$$\phi(r,k) = \frac{R\cos\theta}{k} \tag{6.2.9}$$

$$\phi'(r,k) = R\sin\theta \tag{6.2.10}$$

Clearly

$$R^2 = (\phi')^2 + k^2\phi^2$$

Moreover, applying the theory of Chapter IV to $V_n(r)$ with boundary condition $u(0,k) = 0$, and taking H_1, H_∞ to be the appropriate Hamiltonian operators in

$[0,a_n]$ and $[a_n,\infty)$ respectively, we deduce from (3.1.2) with $\alpha = 0$, and from (4.5.5) that for n = 1,2...,

$$\frac{d\rho_n(k)}{dk} = \frac{2k^2}{\pi\left([\phi'(a_n,k)]^2 + k^2[\phi(a_n,k)]^2\right)}$$
$$= \frac{2k^2}{\pi\,[R_n(k)]^2}$$

where $R_n(k) = R(a_n,k)$. Thus, if $\Sigma \subseteq (0,1)$ is an interval,

$$\mu_n(\Sigma) = \frac{2}{\pi}\int_\Sigma \frac{k^2}{[R_n(k)]^2}\,dk \tag{6.2.11}$$

Let $\theta_n(k)$ denote $\theta(a_n,k)$ and let N_n denote $\kappa(I_n^0) = \kappa(I_n^1)$. From (6.2.9) and (6.2.10),

$$\theta = \tan^{-1}\left(\frac{\phi'}{k\phi}\right)$$

so that

$$\frac{d\theta}{dr} = -k + \frac{k\phi^2 V}{R^2}$$

Since $V(r) = 0$ on $[a_n,b_n]$, we deduce that

$$\theta(b_n,k) = \theta_n(k) - N_n k$$

and, using $\frac{dR}{dr} = 0$ if $V(r) = 0$, we obtain

$$R(b_n,k) = R_n(k)$$

Combining these results with (6.2.9) and (6.2.10) yields

$$\phi(b_n,k) = \frac{R_n(k)\cos(\theta_n(k) - N_n k)}{k} \tag{6.2.12}$$

$$\phi'(b_n,k) = R_n(k)\sin(\theta_n(k) - N_n k) \tag{6.2.13}$$

Since $V(r) = 1$ on (b_n,a_{n+1}) we have

$$\begin{pmatrix} \phi(a_{n+1},k) \\ \phi'(a_{n+1},k) \end{pmatrix} = M_n(k) \begin{pmatrix} \phi(b_n,k) \\ \phi'(b_n,k) \end{pmatrix} \tag{6.2.14}$$

where the transfer matrix $M_n(k)$ satisfies

$$M_n(k) = \begin{pmatrix} \cosh(\sqrt{(1-k^2)}\,N_n) & \dfrac{\sinh(\sqrt{(1-k^2)}N_n)}{\sqrt{(1-k^2)}} \\ \sqrt{(1-k^2)}\sinh(\sqrt{(1-k^2)}N_n) & \cosh(\sqrt{(1-k^2)}N_n) \end{pmatrix} \tag{6.2.15}$$

for $0 < k \leq 1$.

(6.2.12), (6.2.13) and (6.2.14) together imply

$$\left(\frac{R_{n+1}(k)}{R_n(k)}\right)^2 = A_n(k) + B_n(k)\cos(2(\theta - y)) + C_n(k)\sin(2(\theta - y)) \tag{6.2.16}$$

where $\theta = \theta_n(k)$, $y = N_n k$, and omitting the arguments,

$$A_n = \frac{1}{2}\left(M_{n11}^2 + M_{n22}^2 + k^{-2}M_{n21}^2 + k^2 M_{n12}^2\right) \tag{6.2.17}$$

$$B_n = \frac{1}{2}\left(M_{n11}^2 + k^{-2}M_{n21}^2 - M_{n22}^2 - k^2 M_{n12}^2\right)$$

$$C_n = k M_{n11} M_{n12} - k^{-1} M_{n21} M_{n22}$$

where M_{nij} is the element in the ith row and the jth column of $M_n(k)$.

If $f_n(k, N_n k, \theta_n(k))$ denotes $\left(\frac{R_n(k)}{R_{n+1}(k)}\right)^2$ for $0 < k \leq 1$, and if $\Sigma \subseteq (0,1)$ is an interval, we have from (6.2.11)

$$\mu_n(\Sigma) = \frac{2}{\pi}\int_\Sigma k^2 \prod_{i=0}^{n} f_i(k, N_i k, \theta_i(k))\, dk$$

where we have taken $[R_0(k)]^2 = 1$.

We now show that the sequence $\{f_i\}$ satisfies the conditions of Theorem 6.1 (see Remarks 6.5(d)).

Firstly, for each j in $\mathbb{N}$, $R_j(k)$ is continuous and non-zero for all r in $[0,\infty)$, so for each i in $\mathbb{N}$ there exists $C_i > 0$ such that $f_i \geqslant C_i$. Moreover, for each a_n, $\phi(a_n,k)$ and $\phi'(a_n,k)$ are analytic functions of k (cf. [LS] pp. 3-5); the same is therefore true of $f_i(k,N_i k, \theta_i(k))$ for each i.

For each i in $\mathbb{N}$, set $N_i k = y$ so that $f_i(k,N_i k, \theta_i(k))$ may be written as $f_i(k,y,N_1,\ldots N_{i-1})$ since $\theta_i(k)$ depends on $\{N_1,\ldots N_{i-1}\}$. In this notation, f_i is analytic in k and y.

Reformulating the left hand side of (6.2.16) as $a(k,\theta) + b(k,\theta)\cos 2y$ and using

$$\int_0^{2\pi} \frac{1}{a + b\cos z}\, dz = \frac{2\pi}{\sqrt{(a^2 - b^2)}}$$

we obtain

$$\frac{1}{\pi}\int_0^{\pi} f_n(k,y,\theta)\,dy = \frac{1}{(A_n^2 - B_n^2 - C_n^2)} = \frac{1}{\det M_n} = 1$$

It remains therefore to show that if

$$m_n(k) = \frac{1}{\pi}\int_0^{\pi} \log f_n(k,y,\theta)\,dy$$

then $\sum_{n=1}^{\infty} m_n(k) = -\infty$. Using

$$\int_0^{2\pi} \log(a + b\cos z)\,dz = 2\pi \log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

we obtain

$$m_n(k) = \log\left(\frac{2}{A_n + 1}\right)$$

so that, if $0 < k \leq 1$,

$$A_n = 1 + \frac{1}{2k^2(1-k^2)} \sinh^2\left(\sqrt{(1-k^2)}\, N_n\right)$$

Hence for each k in (0,1), $A_n \gneqq 1$ and A_n increases with n if and only if N_n increases with n. If, therefore, N_n does not decrease with n, for each k in (0,1) there exists K_k in (0,1) such that

$$\log\left(\frac{2}{A_n(k) + 1}\right) < -K_k$$

for all n in $\mathbb{N}$, so that $\sum_{n=1}^{\infty} m_n(k) = -\infty$ for each k in (0,1), and the conditions of Theorem 6.1 are satisfied.

It follows that the sequences $\{I_n^0\}$ and $\{I_n^1\}$ may be chosen to satisfy conditions (i) and (ii) above, and hence, since the spectral measure of H is $\lim_{n\to\infty} \nu_n$, so that H has singular continuous spectrum on (0,1).

The proposition is now proved.

6.5 Remarks:

(a) If $k > 1$ the transfer matrix M_n becomes

$$M_n(k) = \begin{pmatrix} \cos\left(\sqrt{(k^2-1)}\, N_n\right) & \dfrac{\sin\left(\sqrt{(k^2-1)}\, N_n\right)}{\sqrt{(k^2-1)}} \\ -\sqrt{(k^2-1)}\, \sin\left(\sqrt{(k^2-1)}\, N_n\right) & \cos\left(\sqrt{(k^2-1)}\, N_n\right) \end{pmatrix}$$

which yields

$$A_n = 1 + \frac{1}{2k^2(k^2-1)} \sin^2(\sqrt{k^2-1}\; N_n)$$

Therefore, the divergence of $\sum_{n=1}^{\infty} m_n(k)$ is no longer immediate, since in general we may not assert that $A_n \geqslant$ const. > 1, or even that $A_n > 1$. However, it may still be the case that $\sum_{n=1}^{\infty} m_n(k)$ diverges, at least for almost all $k > 1$. To satisfy the requirements of the proof of Theorem 6.1, (see [P1], Thm.1, Step IV), it is also necessary to ensure that, for sufficiently large m,

$$\sum_{i=m+1}^{n} -m_i(k) \geqslant K > 0$$

for almost all k in the interval under consideration, where K is independent of k. These difficulties indicate that a modified, possibly statistical, approach is required to ascertain the nature of the spectrum for $k \geqslant 1$.

(b) Sufficient constraints may be extracted from the details of proof of Theorem 6.1 to construct particular examples of suitable sequences of intervals $\{I_n^0\}$.

(c) For an alternative derivation of $\frac{d\rho_n(k)}{dk}$, see [P3], Lemma 3.

(d) More precisely, we show that $\{f_i\}$ satisfies the conditions of the Corollary to Pearson's Theorem 1 ([P1], §2, Cor. to Thm. 1).

§3. On the Generation of Singular Continuous Measures

We now reformulate Theorem 6.1 in such a way that condition (i) may be replaced by the requirement that $f_n(k,y) \geqslant 0$ for each n, and condition (iv) may be removed entirely. In addition, we no longer require the sequence $\{f_n(k,y)\}$ to be continuously differentiable, or even continuous as a function of y. To achieve this generalisation, we replace condition (iii) by

$$\lim_{n \to \infty} \prod_{r=1}^{n} \beta_r = 0 \tag{6.3.1}$$

where $\beta_r = \sup_k \int_0^1 [f_r(k,y)]^\alpha \, dy$, for some fixed α in (0,1). (the precise value of α is immaterial).

As we shall show in the discussion after the proofs, our result is not, strictly speaking, stronger than that of Pearson, because the condition (6.3.1) implies condition (iii) of Theorem 6.1 but is not implied by it. However, the removal of condition (iv) and the significant improvement to condition (i) and to the continuity conditions on $\{f_n(k,y)\}$ means that the approach of Theorem 6.1 is now extended to a considerably wider range of absolutely continuous measures.

We develop our reformulation in three main stages. First we suppose $\{f_n\}$ to be a sequence of periodic step functions in one dimension, and then, using the fact that step functions are dense in $L_1[0,1]$, generalise this result to include sequences of arbitrary bounded periodic functions in one dimension. (By a step function we mean a function of the form $\sum_{i=1}^{n} \alpha_i \chi_i$, where for each $i = 1,\dots,n$, $\alpha_i \in \mathbb{R}$ and χ_i is the characteristic function of a bounded interval). We then proceed to consider the problem in the type of two dimensional situations envisaged in Theorem 6.1.

We remark that our methods do not depend on the somewhat probabilistic approach used by Pearson (see [P1], Thm.1, especially Step IV).

6.6 <u>Proposition</u>: Let $f(x) \geqslant 0$ be a periodic function with period 1 such that $\int_0^1 f(x)\,dx = 1$, the restriction of $f(x)$ to $[0,1]$ is a step function and $f(x)$ is not almost everywhere constant. Then there exists a sequence of natural numbers $\{N_n\}$ such that the limit as $n \to \infty$ of the sequence of measures defined on subintervals Σ of $[0,1]$ by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{k=1}^{n} f(N_k x)\,dx$$

exists and defines a singular continuous measure on Borel subsets of $[0,1]$.

<u>Proof</u>:

Let $f_k(x)$ denote $f(N_k x)$ for each k in $\mathbb{N}$, and set $D = \max\{y: f(x) = y\}$.

Let α such that $0 < \alpha < 1$ be fixed, and define $g(x) = [f(x)]^\alpha$, $g_k(x) = g(N_k x)$.

It is straightforward to show that, since f(x) is not almost everywhere constant on [0,1],

$$\int_0^1 g(x)dx = \beta \tag{6.3.2}$$

for some β such that $0 < \beta < 1$.

We shall prove the proposition in four main stages, as follows:

(i) We show that a sequence of natural numbers $\{N_n\}$ may be chosen inductively to ensure the inequality

$$\left| \int_\Sigma f_{n+1}(x)\, dx - \int_\Sigma dx \right| < \frac{1}{2^{M_{n+1}} D^n P_n} \tag{6.3.3}$$

holds for every n in $\mathbb{N}$ on each subinterval Σ of [0,1]. where P_n is the maximum number of steps in the step function $\prod_{k=1}^{n} f_k(x)$ on [0,1] and $M_{n+1} \gneq M_n$ is chosen inductively in stage (iv) to guarantee the singularity of the limiting measure.

We also show that, in addition, $\{N_n\}$ may be chosen to satisfy

$$\int_0^1 \prod_{k=1}^{n} g_k(x)\, dx \leqslant \beta^{(n+1)/2} \tag{6.3.4}$$

for all n in $\mathbb{N}$.

(ii) We prove that if $\{N_n\}$ satisfies (6.3.3), then $\nu_n(\Sigma)$ is uniformly Cauchy on all subintervals Σ of [0,1], and hence that $\lim_{n\to\infty} \nu_n$ defines a continuous measure on subintervals of [0,1].

(iii) We show that (6.3.4) implies that, on [0,1], $\prod_{k=1}^{n} g_k(x)$ converges to zero in measure as $n \to \infty$, and conclude that the same is true of $\prod_{k=1}^{n} f_k(x)$.

(iv) We deduce that, if $\{N_n\}$ increases sufficiently rapidly with n, and satisfies (6.3.3) and (6.3.4), then $\lim_{n\to\infty} \nu_n$ is singular and non-trivial.

<u>Proof of (i)</u>:

Let $M_1 \geqslant 2$ be fixed, and let $\Sigma \subseteq [0,1]$ be an interval with endpoints a and b, $a < b$.

Choose N_1 in $\mathbb{N}$ so that the length of period $\frac{1}{N_1} < \frac{1}{2^{M_1}(D+1)}$

and let q be the greatest integer such that $\frac{q}{N_1} < b - a$.

Then if S is an interval with endpoints a, $a+\frac{q}{N_1}$,

$$\kappa(\Sigma \setminus S) < \frac{1}{N_1} < \frac{1}{2^{M_1}(D+1)} \tag{6.3.5}$$

Moreover, using $f_1(x) = f(N_1 x)$ and the properties of $f(x)$, we obtain

$$\int_S f_1(x)\,dx = \frac{q}{N_1}$$

Hence, by Minkowski's inequality and (6.3.5)

$$\begin{aligned} &\left| \int_\Sigma f_1(x)\,dx - \int_\Sigma dx \right| \\ &\qquad \leq \left|\int_S f_1(x)\,dx - \int_S dx\right| + \left|\int_{\Sigma\setminus S} f_1(x)\,dx - \int_{\Sigma\setminus S} dx\right| \\ &\qquad \leq 0 + (D+1)\,\kappa(\Sigma\setminus S) \\ &\qquad \leq \frac{1}{2^{M_1}} \end{aligned} \tag{6.3.6}$$

We note that this result is independent of the particular subinterval Σ of [0,1] which is chosen.

Also,

$$\int_0^1 g_1(x)\,dx = \int_0^{N_1} [f(y)]^\alpha \frac{1}{N_1}\,dy = \beta \tag{6.3.7}$$

Similarly, we choose N_2 in $\mathbb{N}$ so that

$$\frac{1}{N_2} < \min\left\{ \frac{1}{2^{M_2}\,(D+1)\,D\,P_1},\ \frac{\varepsilon_1}{D^2\,P_1} \right\}$$

for some $\varepsilon_1 < \beta^{3/2} - \beta^2$. This yields

$$\left| \int_\Sigma f_2(x)\,dx - \int_\Sigma dx \right| < \frac{1}{2^{M_2}\,D\,P_1}$$

for every subinterval Σ of [0,1].

Moreover, as we now show,

$$\int_0^1 g_1(x)\,g_2(x)\,dx < \beta^{3/2} \tag{6.3.8}$$

For, since $g_1(x)$ is a step function on [0,1] we may write

$$g_1(x) = \sum_{i=1}^{P_1} \alpha_i \chi_i$$

where χ_i is the characteristic funtion of the interval $\Sigma_i \subseteq [0,1]$; there is no loss of generality if we assume $\Sigma_i \cap \Sigma_j = \phi$ whenever $i \neq j$. Hence

$$\int_0^1 g_1(x) g_2(x)\, dx = \sum_{i=1}^{P_1} \alpha_i \int_{\Sigma_i} g_2(x)\, dx$$

For each i in $\{1,\ldots,P_1\}$ let $S_i \subseteq \Sigma_i$ be that subinterval of Σ_i sharing the same left endpoint such that $\kappa(S_i) = \frac{q_i}{N_2}$ where q_i is the largest integer such that $\frac{q_i}{N_2} < \kappa(\Sigma_i)$. Then for each $i = 1,\ldots,P_1$,

$$\kappa(\Sigma_i \setminus S_i) < \frac{1}{N_2} < \frac{\varepsilon_1}{D^2 P_1}$$

so that, using $g \leqslant D^{\alpha} < D$ and $\alpha_i \leqslant D$,

$$\begin{aligned}
\alpha_i \int_{\Sigma_i} g_2(x)\, dx &= \alpha_i \int_{S_i} g_2(x)\, dx + \alpha_i \int_{\Sigma_i \setminus S_i} g_2(x)\, dx \\
&\leqslant \alpha_i \frac{q_i}{N_2} \beta + \alpha_i D \kappa(\Sigma_i \setminus S_i) \\
&\leqslant \alpha_i \beta \kappa(\Sigma_i) + \frac{D^2 \varepsilon_1}{D^2 P_1} \\
&= \beta \int_{\Sigma_i} g_1(x)\, dx + \frac{\varepsilon_1}{P_1}
\end{aligned}$$

for each $i = 1,\ldots P_1$. Hence, by (6.3.7) and our choice of ε_1 ,

$$\sum_{i=1}^{P_1} \alpha_i \int_{\Sigma_i} g_2(x)\, dx \leqslant \beta^2 + \varepsilon_1 < \beta^{3/2}$$

so that (6.3.8) is proved.

Continuing in this way, N_n is chosen so that

$$\frac{1}{N_n} < \min \left\{ \frac{1}{2^{M_n} (D+1) D^{n-1} P_{n-1}} , \frac{\varepsilon_{n-1}}{D^n P_{n-1}} \right\}$$

Using the first bound on $\frac{1}{N_n}$, we obtain the general result (6.3.3); using the second bound, and noting that $\prod_{k=1}^{n-1} g_k(x)$ is a step function bounded above by D^{n-1}, the method above gives the general result (6.3.4).

<u>Proof of (ii)</u>:

For a given subinterval Σ of [0,1],

$$\prod_{k=1}^{n} f_k(x) = \sum_{i=1}^{Q_n} \gamma_i \chi_i$$

where $Q_n \leq P_n$, and $\gamma_i \leq D^n$ for each $i = 1,\dots,Q_n$. Hence, using Minkowski's inequality and (6.3.3),

$$\begin{aligned} |\nu_{n+1}(\Sigma) - \nu_n(\Sigma)| &= \left| \sum_{i=1}^{Q_n} \gamma_i \int_{\Sigma_i} (f_{n+1}(x) - 1)\,dx \right| \\ &\leq D^n \sum_{i=1}^{Q_n} \left| \int_{\Sigma_i} f_{n+1}(x)\,dx - \int_{\Sigma_i} dx \right| \\ &\leq \frac{D^n Q_n}{2^{M_{n+1}} D^n P_n} \\ &\leq \frac{1}{2^{M_{n+1}}} \end{aligned} \tag{6.3.9}$$

Since this is true for all such interals Σ, and $\{M_n\}$ is an increasing sequence in $\mathbb{N}$, $\nu_n(\Sigma)$ is uniformly Cauchy on all subintervals Σ of $[0,1]$.

Hence $\nu(\Sigma) = \lim_{n\to\infty} \nu_n(\Sigma)$ exists finitely for each subinterval Σ of $[0,1]$, and since, for each n in $\mathbb{N}$, ν_n is a positive measure, $\nu(\Sigma) \geq 0$ for each interval $\Sigma \subseteq [0,1]$ and $\nu(\phi) = 0$.

To show that ν is countably additive on subintervals of $[0,1]$, we prove that if $\{\Sigma_k\}$ is a sequence of disjoint intervals in $[0,1]$ such that, for each p in $\mathbb{N}$, $\bigcup_{k=1}^{p} \Sigma_k$ is an interval, then

$$\nu\left(\bigcup_{k=1}^{\infty} \Sigma_k\right) = \sum_{k=1}^{\infty} \nu(\Sigma_k) \tag{6.3.10}$$

Since for each p in $\mathbb{N}$,

$$\nu\left(\bigcup_{k=1}^{p} \Sigma_k\right) = \lim_{n\to\infty} \nu_n\left(\bigcup_{k=1}^{p} \Sigma_k\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \nu_n(\Sigma_k) = \sum_{k=1}^{p} \nu(\Sigma_k)$$

and $\bigcup_{k=1}^{p} \Sigma_k$ is an interval, we have

$$\sum_{k=1}^{p} \nu_n(\Sigma_k) \longrightarrow \sum_{k=1}^{p} \nu(\Sigma_k) \tag{6.3.11}$$

uniformly over p as $n \to \infty$, by above. Moreover $\sum_{k=1}^{p} \nu(\Sigma_k)$ increases with p and is bounded above by $\nu(\Sigma)$, where $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k$, so there exists $a < \infty$ such that

$$\sum_{k=1}^{p} \nu(\Sigma_k) \to a \tag{6.3.12}$$

as $p \to \infty$. This together with (6.3.11) implies that if $\varepsilon > 0$ is given, then there exists $P_\varepsilon, N_\varepsilon$ in $\mathbb{N}$ such that whenever $p > P_\varepsilon$

$$\left| \sum_{k=1}^{p} \nu(\Sigma_k) - a \right| < \frac{\varepsilon}{3} \tag{6.3.13}$$

and, for every p in $\mathbb{N}$,

$$\left| \sum_{k=1}^{p} \nu_n(\Sigma_k) - \sum_{k=1}^{p} \nu(\Sigma_k) \right| < \frac{\varepsilon}{3} \tag{6.3.14}$$

whenever $n > N_\varepsilon$. Also, since for each n, ν_n is a measure, there exists Q_ε depending on n such that

$$\left| \nu_n\left(\bigcup_{k=1}^{\infty} \Sigma_k \right) - \sum_{k=1}^{p} \nu_n(\Sigma_k) \right| < \frac{\varepsilon}{3} \tag{6.3.15}$$

whenever $p > Q_\varepsilon$. Hence for each $n > N_\varepsilon$, we may choose $p > \max\{P_\varepsilon, Q_\varepsilon\}$ so that (6.3.13), (6.3.14) and (6.3.15) hold simultaneously, giving

$$\left| \nu_n\left(\bigcup_{k=1}^{\infty} \Sigma_k \right) - a \right| < \varepsilon$$

This together with (6.3.12) implies (6.3.10), so that ν is a measure on subintervals of [0,1].

Let $\eta > 0$ be given.

Since $\nu_n(\Sigma) \to \nu(\Sigma)$uniformly over all subintervals Σ of [0,1] as there exists N in $\mathbb{N}$ such that $|\nu_n(\Sigma) - \nu(\Sigma)| < \frac{\eta}{2}$ for all subintervals Σ if $n > N$; and since ν_n is absolutely continuous for each n, for a given fixed $n > N$ there exists $\delta > 0$ depending on n such that $\kappa(\Sigma) < \delta \Rightarrow \nu_n(\Sigma) < \frac{\eta}{2}$. It follows that

$$\kappa(\Sigma) < \delta \Rightarrow \nu(\Sigma) < \eta$$

so that ν is continuous on subintervals of [0,1].

<u>Proof of (iii)</u>:

From (6.3.4), $\prod_{k=1}^{n} g_k(x)$ converges to zero in $L_1[0,1]$, which implies that $\prod_{k=1}^{n} g_k(x)$ converges to zero in Lebesgue measure. That is, for given $\varepsilon, \eta > 0$ there exists N in $\mathbb{N}$ such that

$$\kappa\left(\left\{x \in [0,1] : \prod_{k=1}^{n} g_k(x) > \varepsilon^{\alpha}\right\}\right) < \eta$$

for all n > N, or, equivalently, such that

$$\kappa\left(\left\{x \in [0,1] : \prod_{k=1}^{n} f_k(x) > \varepsilon\right\}\right) < \eta$$

for all n > N; hence $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure as $n \to \infty$.

Proof of (iv):

Let η_1 such that $0 < \eta_1 < 1$ be given, and define

$$S_{n,\eta_1} = \left\{x \in [0,1] : \prod_{k=1}^{n} f_k(x) > \frac{\eta_1}{3}\right\}$$

Since we have chosen $M_1 \geqslant 2$, and $M_{k+1} \gneqq M_k$ for each k, it follows from (6.3.9) that $|\nu_k(\Sigma) - \nu_1(\Sigma)| \leq \frac{1}{2}$ for all k > 1 and every subinterval Σ of [0,1]. In particular, choosing $\Sigma = [0,1]$, we have

$$\nu_k([0,1]) \geqslant \frac{1}{2} \qquad (6.3.16)$$

for all k in $\mathbb{N}$, since $\nu_1([0,1]) = 1$. Therefore for each n in $\mathbb{N}$, $\kappa(S_{n,\eta}) \neq 0$ for all $\eta < \frac{1}{2}$.

Let the construction described in (i), with $\{M_k : k = 1, \ldots K_1\}$ chosen arbitrarily subject to $M_1 \geqslant 2$, $M_{k+1} \gneqq M_k$, be followed until $k = K_1$ is reached for which $\kappa(S_{K_1,\eta_1}) < \eta_1$. That such a K_1 exists follows from (iii).

Since $\prod_{k=1}^{K_1} f_k(x)$ is a step function, S_{K_1,η_1} consists of a finite number q_1 of subintervals of [0,1]. Let M_{K_1+1} be chosen to satisfy

$$\frac{1}{2^{M_{K_1+1}}} < \frac{\eta_1}{6q_1} \qquad (6.3.17)$$

so that, using (6.3.9),

$$|\nu(\Sigma) - \nu_{K_1}(\Sigma)| < \frac{\eta_1}{3q_1}$$

for all subintervals Σ of [0,1]. In particular,

$$|\nu([0,1]) - \nu_{K_1}([0,1])| < \frac{\eta_1}{3q_1} \leqslant \frac{\eta_1}{3}$$

and, by Minkowski's inequality,

$$|\nu(S_{K_1,\eta_1}) - \nu_{K_1}(S_{K_1,\eta_1})| < \frac{\eta_1}{3}$$

Also, by the definitions of ν_{K_1} and S_{K_1,η_1}

$$\nu_{K_1}([0,1] \setminus S_{K_1,\eta_1}) = \nu_{K_1}(\{x \in [0,1] : \prod_{k=1}^{K_1} f_k(x) \leq \frac{\eta_1}{3}\})$$
$$\leq \frac{\eta_1}{3}$$

Combining these results, we have

$$\begin{aligned} \nu(S_{K_1,\eta_1}) &> \nu_{K_1}(S_{K_1,\eta_1}) - \frac{\eta_1}{3} \\ &= \nu_{K_1}([0,1]) - \nu_{K_1}([0,1] \setminus S_{K_1,\eta_1}) - \frac{\eta_1}{3} \\ &\geq \nu([0,1]) - \eta_1 \end{aligned}$$

$\eta_2 < \eta_1$ is now chosen, and the procedure described above is repeated for $k = K_1+1,\ldots,K_2$ where K_2 is such that $\kappa(S_{K_2,\eta_2}) < \eta_2$. We then obtain

$\nu(S_{K_2,\eta_2}) \geq \nu([0,1]) - \eta_2$. Continuing in this way, if the decreasing sequence $\{\eta_m\}$ satisfies $\eta_m \to 0$ as $m \to \infty$ and $\{N_n\},\{M_n\}$ are chosen inductively at each stage to satisfy the construction described in (i) and the constraint (6.3.17), then as $m \to \infty$,

$$\nu(S_{K_m,\eta_m}) \to \nu([0,1])$$

and, by (iii),

$$\kappa(S_{K_m,\eta_m}) \to 0$$

That is, the limiting measure ν is singular.

The Hahn Extension Theorem ensures that ν may be extended to a measure on Borel subsets of $[0,1]$, and it follows from (6.3.16) that ν is non-trivial.

The proof of the proposition is now complete.

We now generalise Proposition 6.6 using the fact that step functions are dense in L_1. The proof follows a similar pattern to that of Proposition 6.6; parts (ii) and (iii) are unchanged, while part (iv) requires considerable modification.

6.7 <u>Theorem:</u> Let $f(x) \geq 0$ be a bounded periodic function with period 1 which is not almost everywhere constant and is such that $\int_0^1 f(x)\,dx = 1$. Then there exists a sequence of natural numbers $\{N_n\}$ such that the limit as $n \to \infty$ of the sequence of measures defined on subintervals Σ of [0,1] by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{k=1}^{n} f(N_k x)\,dx$$

exists and defines a singular continuous measure on Borel subsets of [0,1].

Proof:

Let the notation be as in Proposition 6.6, except that we now define $D = \sup\{ y: f(x) = y \}$. As before,

$$\int_0^1 g(x)\,dx = \beta$$

for some β such that $0 < \beta < 1$.

We first prove a modified version of part (i) of Theorem 6.6.

Choosing M_1, N_1 as before, (6.3.6) and (6.3.7) may be deduced as in Theorem 6.6.

Now let $\varepsilon_1 > 0$ be such that $\varepsilon_1 < \beta^{3/2} - \beta^2$.

Since the step functions are dense in $L_1([0,1])$ ([HS] Ch.IV, 13.23), there exists a step function $\sum_{i=1}^{P} \alpha_i \chi_i$ such that

$$\int_0^1 \left| g_1(x) - \sum_{i=1}^{P} \alpha_i \chi_i \right| dx < \frac{\varepsilon_1}{2D}$$

Thus, using $g_2(x) \leq D$, we have

$$\left| \int_0^1 g_1(x)\, g_2(x)\,dx - \int_0^1 g_2(x) \sum_{i=1}^{P} \alpha_i \chi_i\,dx \right| \leq \frac{\varepsilon_1}{2} \qquad (6.3.18)$$

Likewise, there exists a step function $\sum_{i=1}^{Q} \beta_i \chi_i$ such that

$$\int_0^1 \left| f_1(x) - \sum_{i=1}^{Q} \beta_i \chi_i \right| dx < \frac{1}{2D\,2^{M_2}}$$

Thus, using $|f_2(x)-1| \leq D$, we have

$$\left| \int_\Sigma (f_2(x)-1)\, f_1(x)\,dx - \int_\Sigma (f_2(x)-1) \sum_{i=1}^{Q} \beta_i \chi_i\,dx \right| \leq \frac{1}{2.2^{M_2}} \qquad (6.3.19)$$

for every subinterval Σ of [0,1].

Now choose N_2 in $\mathbb{N}$ sufficiently large to ensure that

$$\frac{1}{N_2} < \frac{\varepsilon_1}{2D^2 P} \qquad \text{and} \qquad \left| \int_\Sigma f_2(x)\,dx - \int_\Sigma dx \right| < \frac{1}{2.2^{M_2} D Q}$$

Using the first inequality, we may proceed as in the proof of part (i) of Theorem 6.6 to show that

$$\int_0^1 g_2(x) \sum_{i=1}^{P} \alpha_i \chi_i \, dx = \sum_{i=1}^{P} \alpha_i \int_{\Sigma_i} g_2(x)dx \leq \beta^2 + \frac{\varepsilon_1}{2}$$

which together with (6.3.18) implies

$$\int_0^1 g_1(x)\, g_2(x)\, dx < \beta^{3/2}$$

Using the second inequality and $\beta_i \leq D$ for i = 1,...Q,

$$\left| \int_\Sigma (f_2(x)-1) \sum_{i=1}^{Q} \beta_i \chi_i \, dx \right| \leq \sum_{i=1}^{Q} \beta_i \left| \int_{\Sigma_i} f_2(x)dx - \int_{\Sigma_i} dx \right| \leq \frac{1}{2.2^{M_2}}$$

so that, using (6.3.19), we have

$$|\nu_2(\Sigma) - \nu_1(\Sigma)| = \left| \int_\Sigma (f_2(x)-1) f_1(x) dx \right| \leq \frac{1}{2^{M_2}}$$

on all subintervals Σ of [0,1].

Continuing in this way, it is evident that a sequence ν_n may be constructed which is uniformly Cauchy on all subintervals Σ of [0,1], and for which

$$\int_0^1 \prod_{k=1}^{n} g_k(x)\, dx < \beta^{(n+1)/2}$$

for all n in $\mathbb{N}$. Note that, at the nth stage, the procedure is to approximate $\prod_{k=1}^{n-1} g_k(x)$ and $\prod_{k=1}^{n-1} f_k(x)$ respectively by step functions.

It now follows, as in the proof of Proposition 6.6, that if ν_n is constructed as above to satisfy (6.3.4) and (6.3.9), then $\nu(\Sigma) = \lim_{n\to\infty} \nu_n(\Sigma)$ exists and defines a continuous measure on subintervals Σ of [0,1] and $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure as $n \to \infty$.

We now prove that if $\{N_n\}$ increases sufficiently rapidly, then such a measure ν is singular.

Let η_1 such that $0 < \eta_1 < \frac{1}{2}$ be given.

Since $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure as $n \to \infty$, we may follow the construction described above until we reach n = K_1 for which

$$\kappa(\{x \in [0,1] : \prod_{k=1}^{K_1} f_k(x) > \frac{\eta_1}{8}\}) < \frac{\eta_1}{2} \qquad (6.3.20)$$

As in Proposition 6.6, we suppose that $M_1 \geq 2$ and $M_{k+1} \gneq M_k$ for $k = 1,\dots K_1-1$.

Since the step functions are dense in L_1, there exists a step function $\sum_{i=1}^{Q} \alpha_i \chi_i$ such that

$$\int_0^1 \left| \prod_{k=1}^{K_1} f_k(x) - \sum_{i=1}^{Q} \alpha_i \chi_i \right| dx < \frac{\eta_1^2}{64 D^{K_1}} \qquad (6.3.21)$$

Let S_{K_1,η_1} denote $\{x \in [0,1] : \sum_{i=1}^{Q} \alpha_i \chi_i > \frac{\eta_1}{4}\}$. Since (6.3.16) remains true under our present assumptions, it is evident from (6.3.21) that $\kappa(S_{K_1}, \eta_1) \neq 0$ for $\eta_1 < \frac{1}{2}$. We shall prove that

$$\nu(S_{K_1,\eta_1}) > \nu([0,1]) - \eta_1 \qquad (6.3.22)$$

and deduce that a continuation of this process will result in singularity of the limiting measure ν.

It follows from (6.3.21) that

$$\kappa(\{x \in [0,1] : \left|\prod_{k=1}^{K_1} f_k(x) - \sum_{i=1}^{Q} \alpha_i \chi_i\right| > \frac{\eta_1}{8}\}) < \frac{\eta_1}{8D^{K_1}} < \frac{\eta_1}{2} \qquad (6.3.23)$$

so that

$$\kappa(\{x \in [0,1] : \sum_{i=1}^{Q} \alpha_i \chi_i > \prod_{k=1}^{K_1} f_k(x) + \frac{\eta_1}{8}\}) < \frac{\eta_1}{2}$$

which, together with (6.3.20) implies

$$\kappa(S_{K_1,\eta_1}) < \eta_1$$

Since S_{K_1,η_1} is non-empty, and $\sum_{i=1}^{Q} \alpha_i \chi_i$ is a step function, S_{K_1,η_1} consists of a finite number q of intervals. Choosing M_{K_1+1} to satisfy

$$\frac{1}{2^{M_{K_1+1}}} < \frac{\eta_1}{16q}$$

we have

$$|\nu(\Sigma) - \nu_{K_1}(\Sigma)| < \frac{\eta_1}{8q}$$

for every subinterval Σ of $[0,1]$, since (6.3.9) holds under the construction we have described in this proof. It follows that

$$|\nu([0,1]) - \nu_{K_1}([0,1])| < \frac{\eta_1}{8} \tag{6.3.24}$$

and, by Minkowski's inequality,

$$|\nu(S_{K_1,\eta_1}) - \nu_{K_1}(S_{K_1,\eta_1})| < \frac{\eta_1}{8} \tag{6.3.25}$$

In order to deduce (6.3.22), we first relate S_{K_1,η_1} to the subset S_1 of [0,1], defined by

$$S_1 = \{ x \in [0,1] : \prod_{k=1}^{K_1} f_k(x) > \frac{\eta_1}{2} \}$$

Clearly,

$$\int_{[0,1]\setminus S_1} \prod_{k=1}^{K_1} f_k(x)\,dx \leq \frac{\eta_1}{2} \tag{6.3.26}$$

and, by definition of ν_{K_1},

$$\int_{S_1} \prod_{k=1}^{K_1} f_k(x)dx - \nu_{K_1}(S_{K_1,\eta_1}) \leq \int_{S_1\setminus S_{K_1,\eta_1}} \prod_{k=1}^{K_1} f_k(x)\,dx \tag{6.3.27}$$

Moreover,

$$\begin{aligned} \kappa(S_1 \setminus S_{K_1,\eta_1}) &= \kappa(\{x \in [0,1] : x \in S_1 , x \notin S_{K_1,\eta_1}\}) \\ &= \kappa(\{x \in [0,1] : \prod_{k=1}^{K_1} f_k(x) > \frac{\eta_1}{2}, \sum_{i=1}^{Q} \alpha_i \chi_i \leq \frac{\eta_1}{4} \}) \\ &\leq \kappa(\{x \in [0,1] : |\prod_{k=1}^{K_1} f_k(x) - \sum_{i=1}^{Q} \alpha_i \chi_i| > \frac{\eta_1}{8} \}) \\ &< \frac{\eta_1}{8 D^{K_1}} \end{aligned}$$

by (6.3.23). Using this inequality in (6.3.27), we have

$$\int_{S_1} \prod_{k=1}^{K_1} f_k(x)\,dx - \nu_{K_1}(S_{K_1,\eta_1}) < \frac{\eta_1}{8}$$

since $f_k(x) \leq D$ for each k. This, together with (6.3.24), (6.3.25) and (6.3.26) implies that

$$\begin{aligned} \nu(S_{K_1,\eta_1}) &> \nu_{K_1}(S_{K_1,\eta_1}) - \frac{\eta_1}{8} \\ &= \nu_{K_1}([0,1]) - \int_{S_1} \prod_{k=1}^{K_1} f_k(x)\,dx - \int_{[0,1]\setminus S_1} \prod_{k=1}^{K_1} f_k(x)\,dx \ldots \end{aligned}$$

$$\ldots + \nu_{K_1}(S_{K_1,\eta_1}) - \frac{\eta_1}{8}$$

$$> \nu([0,1]) - \eta_1$$

so we have proved (6.3.22).

Continuing in this way, a sequence of sets $\{S_{K_m,\eta_m}\}$ is constructed satisfying

$$\nu(S_{K_m,\eta_m}) > \nu([0,1]) - \eta_m$$

and

$$\kappa(S_{K_m,\eta_m}) < \eta_m$$

where $\{\eta_m\} \to 0$. Thus $\nu(S_{K_m,\eta_m}) \to \nu([0,1])$ and $\kappa(S_{K_m,\eta_m}) \to 0$ so that the measure ν must be singular.

This completes the proof of the theorem.

The results of Proposition 6.6 and Theorem 6.7 are readily extended to include the case in which the sequence of measures $\{\nu_n\}$ is defined by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{k=1}^{n} h_k(N_k x)\, dx \tag{6.3.28}$$

where for each k, $h_k(x)$ may be distinct. If we suppose that for each k, $h_k(x)$ is a non-constant periodic function with period 1 such that $\int_0^1 h_k(x)\,dx = 1$ and that $h_k(x)$ is a step function on $[0,1]$, or, respectively, an essentially bounded function, then the proofs of Proposition 6.6 and Theorem 6.7 may be simply adapted as follows:

Wherever D^n occurs, it is replaced by $\prod_{k=1}^{n} D_k$ where D_k = ess sup $h_k(x)$; and instead of choosing $\{\varepsilon_n\}$ to satisfy

$$\varepsilon_{n-1} < \beta^{(n+1)/2} - \beta^{(n+2)/2}$$

we now suppose that

$$\varepsilon_{n-1} < \beta_1^{1/2} \prod_{k=1}^{n+1} \beta_k^{1/2} (1 - \beta_{n+2}^{1/2})$$

where $\beta_k = \int_0^1 [h_k(x)]^\alpha dx$ for some fixed α such that $0 < \alpha < 1$. It may then be shown that, for a suitable sequence $\{N_n\}$,

$$\int_0^1 \prod_{k=1}^{n} [h_k(N_k x)]^\alpha dx \leq \beta_1^{1/2} \prod_{k=1}^{n} \beta_k^{1/2}$$

Evidently, to ensure that $\prod_{k=1}^{n} h_k(N_k x)$ converges to zero in L_1, it is sufficient to impose the additional constraint that $\prod_{k=1}^{n} \beta_k$ converge to zero as $n \to \infty$.

Using these modifications to the hypotheses and proofs of Proposition 6.6 and Theorem 6.7, it is straightforward to show that the sequence of measures defined by (6.3.28) will converge to a singular continuous measure on [0,1] for a sufficiently rapidly increasing sequence $\{N_n\}$.

We now use the denseness of the step functions in L_1 to deduce a two-dimensional generalisation of Theorem 6.7. We note that, although we do not require f(k,y) to be a continuous function of y for fixed k, we have found it necessary to retain a continuity condition in the k-direction. This is to ensure that on each sufficiently small k-interval I_k, f(k,y) is approximately constant for each fixed y, so that the two dimensional domain may be partitioned into a finite number of subdomains on each of which the behaviour of f(k,y) approximates that of a one-dimensional function. The method of proof of Theorem 6.7 may then be adapted without undue difficulty to this new situation.

6.8 <u>Theorem</u>: Let $f(k,y) \geq 0$ be a bounded function on $[0,1] \times (-\infty, \infty)$ which for each fixed k in [0,1] is a periodic function of y satisfying $\int_0^1 f_k(k,y)dy = 1$ and for each α with $0 < \alpha < 1$ is such that $\sup_k \int_0^1 [f(k,y)]^\alpha dy \lneq 1$.

Suppose also that for each y, f(k,y) is a continuous function of k, uniformly in y; that is, for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ which is independent of y such that $|f(k_1,y) - f(k_2,y)| < \varepsilon$ whenever $|k_1 - k_2| < \delta_\varepsilon$. Then there exists a sequence of natural numbers $\{N_n\}$ such that the limit as $n \to \infty$ of the sequence of measures $\{\nu_n\}$ defined on subintervals Σ of [0,1] by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{i=1}^{n} f(k, N_i k)\, dk$$

exists and defines a singular continuous measure on Borel subsets of [0,1].

Proof:

Let $D = \sup_{k,y} f(k,y)$, and let α with $0 < \alpha < 1$ be given.

We first show that for each n in $\mathbb{N}$, if $\eta_n > 0$ is given then B_n in $\mathbb{N}$ exists such that

$$\left| \int_\Sigma f(k, N_n k)\, dk - \int_\Sigma dk \right| < \eta_n \tag{6.3.29}$$

for every subinterval Σ of [0,1], whenever $N_n \geq B_n$.

Let Σ be an arbitrary subinterval of [0,1], with endpoints a and b, where $a < b$, and let $\eta_n > 0$ be given.

We may choose B_n in $\mathbb{N}$ such that

$$\frac{1}{B_n} < \frac{\eta_n}{2(D+1)} \tag{6.3.30}$$

and so that $|k' - k''| < \frac{1}{B_n}$ implies

$$|f(k', y) - f(k'', y)| < \frac{\eta_n}{2} \tag{6.3.31}$$

for every y, using the hypothesis of the theorem.

Let N_n in $\mathbb{N}$ be chosen so that $N_n \geq B_n$, and let q be the greatest integer such that $a + \frac{q}{N_n} \leq b$. Let I_r denote $(a + \frac{r-1}{N_n}, a + \frac{r}{N_n})$ for each $r = 1, \ldots, q$, and let I denote $(a + \frac{q}{N_n}, b)$. If for each $r = 1, \ldots, q$, k_r is some fixed element of I_r, then we have by Minkowski's inequality, (6.3.30) and (6.3.31),

$$\begin{aligned}
&\left| \int_\Sigma f(k, N_n k)\, dk - \int_\Sigma dk \right| \\
&\leq \sum_{r=1}^{q} \left| \int_{I_r} f(k, N_n k)\, dk - \int_{I_r} dk \right| + \int_I f(k, N_n k)\, dk + \int_I dk \\
&\leq \sum_{r=1}^{q} \int_{I_r} |f(k, N_n k) - f(k_r, N_n k)|\, dk \\
&\qquad + \sum_{r=1}^{q} \left| \int_{I_r} f(k_r, N_n k)\, dk - \int_{I_r} dk \right| + \frac{D+1}{N_n}
\end{aligned}$$

$$< \frac{\eta_n}{2} \sum_{r=1}^{q} \kappa(I_r) + 0 + \frac{\eta_n}{2}$$

$$\leq \eta_n$$

Since Σ was chosen arbitrarily, (6.3.29) is proved.

We now show that $\{N_n\}$ may be chosen so that, in addition to satisfying $N_n \geq B_n$ for each n, it also satisfies

$$\int_0^1 \prod_{i=1}^{n} g(k, N_i k)\, dk \leq \beta^{n/2} \tag{6.3.32}$$

for all n in $\mathbb{N}$, where $g(k, N_i k) = [f(k, N_i k)]^{\alpha}$ for each i, and $\beta = \sup_k \int_0^1 [f(k,y)]^{\alpha} dy$; by hypothesis, $\beta < 1$.

We first show that N_1 may be chosen so that $N_1 \geq B_1$, and

$$\int_0^1 g(k, N_1 k)\, dk \leq \beta^{1/2} \tag{6.3.33}$$

Choose $\varepsilon_1 > 0$ to satisfy $\varepsilon_1 \leq \beta^{1/2} - \beta$. Let N_1 be such that $N_1 > B_1$,

$$\frac{1}{N_1} < \frac{\varepsilon_1}{2D} \tag{6.3.34}$$

and such that, whenever $|k' - k''| < \frac{1}{N_1}$, then

$$|g(k', N_1 k) - g(k'', N_1 k)| < \frac{\varepsilon_1}{2} \tag{6.3.35}$$

for all k in [0,1]. That N_1 may be chosen to satisfy the last condition follows from the hypothesis and the inequality $|r^{\alpha} - s^{\alpha}| \leq |r - s|^{\alpha}$ for $0 < \alpha < 1$.

Let [0,1] be subdivided into q disjoint intervals $I_1, \ldots, I_q$ and one interval I, which may be vacuous, such that $\mu(I_r) = \frac{1}{N_1}$ for r = 1,...q, and $\mu(I) \lneq \frac{1}{N_1}$. If for each r = 1,...q, k_r is some fixed element of I_r, then we have by (6.3.34), (6.3.35) and our choice of ε_1,

$$\begin{aligned}\int_0^1 g(k, N_1 k)\, dk &= \sum_{r=1}^{q} \int_{I_r} g(k, N_1 k)\, dk + \int_I g(k, N_1 k)\, dk \\ &\leq \sum_{r=1}^{q} \int_{I_r} g(k_r, N_1 k)\, dk + \sum_{r=1}^{q} \int_{I_r} \frac{\varepsilon_1}{2}\, dk + D\,\kappa(I) \\ &\leq \beta \sum_{r=1}^{q} \kappa(I_r) + \varepsilon_1 \\ &\leq \beta^{1/2}\end{aligned}$$

so that (6.3.33) is proved.

To illustrate the method of proof of (6.3.32) for $n > 1$, we give details for $n = 2$.

Choose ε_2 to satisfy $0 < \varepsilon_2 \leq \beta - \beta^{3/2}$ and let [0,1] be partitioned into q equal intervals $I_1, \dots I_q$ each of length l where l is sufficiently small to ensure that

$$|g(k',y) - g(k'',y)| < \frac{\varepsilon_2}{4D} \tag{6.3.36}$$

for all y, whenever $|k' - k''| < l$.

Since $g(k,N_1 k)$ is in $L_1([0,1])$, there exists a step function $\sum_{i=1}^{Q} \alpha_i \chi_i$ such that

$$\int_0^1 |g(k,N_1 k) - \sum_{i=1}^{Q} \alpha_i \chi_i| \, dk < \frac{\varepsilon_2}{4Dq} < \frac{\varepsilon_2}{4} \tag{6.3.37}$$

where the χ_i are characteristic functions of intervals J_i, for $i = 1,\dots,Q$. There is no loss of generality if we suppose that each J_i is a subinterval of some I_r. Let $J_i \subseteq I_r$ be denoted by $J_{i,r}$.

We now choose N_2 to satisfy $N_2 \geq B_2$ and

$$\frac{1}{N_2} < \frac{\varepsilon_2}{4QD^2} \tag{6.3.38}$$

Then, if for each $r = 1,\dots q$, k_r is some fixed element of I_r, we have by (6.3.36)

$$\begin{aligned}
\int_0^1 \prod_{i=1}^{2} g(k,N_i k)\, dk &= \sum_{r=1}^{q} \int_{I_r} g(k,N_1 k)\, g(k,N_2 k)\, dk \\
&\leq \sum_{r=1}^{q} \int_{I_r} g(k,N_1 k)\, g(k_r,N_2 k)\, dk \\
&\qquad + \sum_{r=1}^{q} \int_{I_r} g(k,N_1 k) \frac{\varepsilon_2}{4D}\, dk \\
&\leq \sum_{r=1}^{q} \int_{I_r} g(k,N_1 k)\, g(k_r,N_2 k)\, dk + \frac{\varepsilon_2}{4}
\end{aligned} \tag{6.3.39}$$

Moreover, if the restriction of $\sum_{i=1}^{Q} \alpha_i \chi_i$ to I_r is denoted by $\sum_{i=P_r}^{Q_r} \alpha_{i,r} \chi_{i,r}$

for each $r = 1,\ldots,q$, where $P_r = Q_{r-1}$, then by (6.3.37)

$$\int_{I_r} g(k,N_1k)\, g(k_r,N_2k)\, dk$$
$$\leq \sum_{i=P_r}^{Q_r} \int_{J_{i,r}} \alpha_{i,r}\, g(k_r,N_2k)dk \;+\; \frac{\varepsilon_2}{4q} \qquad (6.3.40)$$

Let each $J_{i,r}$ be partitioned into a maximum number of disjoint intervals, each of length $\frac{1}{N_2}$, together with one remaining interval of strictly smaller length.

Then by (6.3.38)

$$\int_{J_{i,r}} \alpha_{i,r}\, g(k_r,N_2k)\, dk \;\leq\; \alpha_{i,r}\,\beta\,\kappa(J_{i,r}) \;+\; \frac{\varepsilon_2}{4Q}$$

for each $r = 1,\ldots q$, with $i \in \{P_r,\ldots Q_r\}$. Hence using (6.3.37), (6.3.40) implies

$$\sum_{r=1}^{q} \int_{I_r} g(k,N_1k)\, g(k_r,N_2k)\, dk$$
$$\leq \sum_{r=1}^{q}\left[\sum_{i=P_r}^{Q_r}\left(\alpha_{i,r}\,\beta\,\kappa(J_{i,r}) + \frac{\varepsilon_2}{4Q}\right) + \frac{\varepsilon_2}{4q}\right]$$
$$= \beta \int \sum_{i=1}^{Q} \alpha_i\, \chi_i\; dk \;+\; \frac{\varepsilon_2}{2}$$
$$\leq \beta \int_0^1 g(k,N_1k)\, dk \;+\; \frac{3\varepsilon_2}{4}$$

It follows from (6.3.33), (6.3.39) and our choice of ε_2 that

$$\int_0^1 g(k,N_1k)\, g(k,N_2k)\, dk \;\leq\; \beta$$

so that (6.3.32) is proved for $n = 2$.

In general, at the nth stage ε_n is chosen to satisfy

$$0 < \varepsilon_n \leq \beta^{n/2} - \beta^{(n+2)/2}$$

and [0,1] is partitioned into q equal intervals of sufficiently small length to ensure that for all y

$$|g(k',y) - g(k'',y)| < \frac{\varepsilon_n}{4D^{n-1}}$$

whenever k' and k'' are both elements of the same interval. $\prod_{r=1}^{n-1} g(k,N_rk)$ is approximated in L_1 norm by a step function so that the norm of the difference is less than $\frac{\varepsilon_n}{4QD^n}$, and N_n is now chosen to satisfy $N_n \geqslant B_n$ and $\frac{1}{N_n} < \frac{\varepsilon_n}{4QD^n}$ where Q is the number of steps in the approximating step function. The method of proof of (6.3.32) is then just as for $n = 2$, so the assertion is proved for all n.

We now show that the sequence $\{\eta_n\}$ in (6.3.29) may be chosen to ensure that $\{\nu_n\}$ is uniformly Cauchy on all subintervals Σ of [0,1].

Let $\{M_n\}$ be a sequence of natural numbers such that $M_{n+1} \geq M_n$ for each n, and let $\sum_{i=1}^{R_n} \gamma_i \chi_i$ be a step function approximation of $\prod_{i=1}^{n} f(k, N_i k)$ for which

$$\int_0^1 \left| \prod_{i=1}^{n} f(k, N_i k) - \sum_{i=1}^{R_n} \gamma_i \chi_i \right| dk < \frac{1}{2 \cdot 2^{M_{n+1}} D}$$

Let $\eta_1 = \dfrac{1}{2 \cdot 2^{M_1}}$, and for each n in $\mathbb{N}$, let $\eta_{n+1} = \dfrac{1}{2 \cdot 2^{M_{n+1}} D^n R_n}$

Then for every subinterval Σ of [0,1]

$$\begin{aligned} &\left| \int_\Sigma (f(k, N_{n+1} k) - 1)\left(\prod_{i=1}^{n} f(k, N_i k) - \sum_{i=1}^{R_n} \gamma_i \chi_i \right) dk \right| \\ &\quad \leq D \int_\Sigma \left| \prod_{i=1}^{n} f(k, N_i k) - \sum_{i=1}^{R_n} \gamma_i \chi_i \right| dk \\ &\quad < \frac{1}{2 \cdot 2^{M_{n+1}}} \end{aligned}$$

and, by (6.3.29),

$$\begin{aligned} &\left| \int_\Sigma (f(k, N_{n+1} k) - 1)\left(\sum_{i=1}^{R_n} \gamma_i \chi_i \right) dk \right| \\ &\quad \leq D^n \sum_{i=1}^{R_n} \left| \int_{\Sigma \cap \Sigma_i} (f(k, N_{n+1} k) - 1)\, dk \right| \\ &\quad \leq \frac{1}{2^{M_{n+1}}} \end{aligned}$$

where Σ_i is the interval on which χ_i takes the value 1. We deduce from these two inequalities

$$\begin{aligned} |\nu_{n+1}(\Sigma) - \nu_n(\Sigma)| &= \left| \int_\Sigma (f(k, N_{n+1} k - 1) \prod_{i=1}^{n} f(k, N_i k)\, dk \right| \\ &\leq \frac{1}{2^{M_{n+1}}} \end{aligned}$$

for all subintervals Σ of [0,1], so that $\{\nu_n\}$ is uniformly Cauchy on subintervals of [0,1].

The remainder of the proof is as in Theorem 6.7, with the sequence $\{M_n\}$ chosen to ensure singularity of the limiting measure.

The theorem is now proved.

6.9 Remarks:

(i) Proposition 6.6 and Theorem 6.7 may be deduced from Theorem 6.8 if we suppose that f(k,y) is a constant function of k for each fixed y. It would therefore be possible to amalgamate these three results into a single theorem; we have preferred to present them separately in order to emphasise the underlying ideas. Theorem 6.7 arises from Proposition 6.6, which is the fundamental result, because step functions are dense in L_1; and Theorem 6.8 owes its existence to the insight afforded by Theorem 6.7.

(ii) Theorem 6.8 is not the only possible two dimensional extension of Theorem 6.7. For example, suppose f(k,y) is a two dimensional step function on [0,1] x [0,1] so that f(k,y) has the form $\sum_{i=1}^{p} \alpha_i \chi_i$ where each χ_i is the characteristic function of a bounded rectangle whose edges are parallel to the rectangular co-ordinates. If f(k,y) is extended to a function on [0,1] x $(-\infty, \infty)$ which is periodic in y with period 1, then the conclusions of Theorem 6.8 hold. To see this, it is only necessary to divide the domain of f(k,y) into a finite number of strips with edges parallel to the y-axis, on each of which f(k,y) is a constant function of k for fixed y. The sequence $\{N_n\}$ may then be chosen inductively so that the conditions of Proposition 6.6 hold on each strip.

(iii) Theorem 6.8 may be extended to the more general case where the sequence of measures $\{\nu_n\}$ is defined by

$$\nu_n(\Sigma) = \int_\Sigma \prod_{i=1}^{n} h_i(k, N_i k)\, dk$$

where for each i, $h_i(k,y)$ may be distinct, but possesses the same general properties as f(k,y). In this case a sufficient condition which ensures that $\nu = \lim_{n \to \infty} \nu_n$ defines a singular continuous measure is

$$\lim_{n\to\infty} \prod_{i=1}^{n} \beta_i = 0$$

(this certainly holds unless $\beta_i \to 1$ as $i \to \infty$), where for each i, $\beta_i = \sup_k \int_0^1 [f_i(k,y)]^\alpha dy$ and α such that $0 < \alpha < 1$ is fixed.

(iv) Further generalisations may be deduced using the ideas of (iii) and (iv). For example, Theorem 6.8 may be modified to include the case where for

$k_r \in \{k_1,\ldots,k_n\} \subseteq [0,1]$, $f(k_r,y)$ is discontinuous at some or all y, provided a uniform condition on the continuity is retained within each k-interval (k_r,k_{r+1}), $r = 1,\ldots,n-1$.

Theorem 6.8, with the modifications described in Remark (iii), is analogous to Pearson's result (Theorem 6.1); however the continuity conditions are considerably weakened and the requirements that each of the sequence of functions $\{f_n(k,y)\}$ be bounded away from zero and that $f_n(k,N\,k)$ be analytic for large N have been removed. We now use some well known inequalities to determine some relationships between the remaining conditions.

For each k in [0,1], we shall consider the behaviour of the sequence of functions $\{f_n(k,y)\}$ on the y interval [0,1] in the more general context of sequences of positive functions $\{f_n\}$ on a measure space Ω with probability measure μ.

Let the expectation of f be denoted by E(f), so that $E(f) = \int f\,d\mu$ and suppose that for each n, $\int f_n\,d\mu = 1$.

It is a straightforward consequence of Hölder's inequality that if $a + b = r + s = 1$ and $a,b > 0$, $r,s \geq 0$ then

$$E(f) \leq (E(f^{r/a}))^a\,(E(f^{s/b}))^b \qquad (6.3.41)$$

Hence, setting $f = f_n^{\alpha_1}$, $r = 0$ and $b = \dfrac{\alpha_1}{\alpha_2}$,

$$(E(f_n^{\alpha_1}))^{1/\alpha_1} \leq (E(f_n^{\alpha_2}))^{1/\alpha_2} \qquad (6.3.42)$$

for $0 < \alpha_1 < \alpha_2 < 1$.

It may be shown that if $f \geq 0$ is non-constant μ-almost everywhere and $E(f) = 1$, then for $0 < \alpha < 1$

$$E(f^\alpha) \lneq 1$$

and for $\beta > 1$

$$E(f^\beta) \gneq 1$$

Hence for all γ in $(0,\infty)$,

$$\prod_{n=1}^{\infty} E(f_n^{\gamma}) = \lim_{k\to\infty} \prod_{n=1}^{k} E(f_n^{\gamma})$$

exists finitely or infinitely.

The following lemma is simply deduced.

6.10 Lemma: Let α_1, α_2 such that $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ be given.

Then

$$\prod_{n=1}^{\infty} E(f_n^{\alpha_1}) = 0 \iff \prod_{n=1}^{\infty} E(f_n^{\alpha_2}) = 0$$

Proof:

The result is trivial for $\alpha_1 = \alpha_2$.

If $\alpha_1 \neq \alpha_2$ there is no loss of generality if we suppose that $\alpha_1 < \alpha_2$.

Then

$$\prod_{n=1}^{\infty} E(f_n^{\alpha_2}) = 0 \Rightarrow \prod_{n=1}^{\infty} E(f_n^{\alpha_1}) = 0$$

by (6.3.42). Setting $f = f_n^{\alpha_2}$, $\frac{r}{a} = \frac{\alpha_1}{\alpha_2}$, $\frac{s}{b} = \frac{1}{\alpha_2}$ in (6.3.41) yields

$$E(f_n^{\alpha_2}) \leq (E(f_n^{\alpha_1}))^a (E(f_n))^b = (E(f_n^{\alpha_1}))^a$$

from which we deduce

$$\prod_{n=1}^{\infty} E(f_n^{\alpha_1}) = 0 \Rightarrow \prod_{n=1}^{\infty} E(f_n^{\alpha_2}) = 0$$

so the lemma is proved.

An analogous result is true for $\beta_1, \beta_2 > 0$ and the conditions $\prod_{n=1}^{\infty} E(f_n^{\alpha}) = 0$ and $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ are normally equivalent, as we now show.

6.11 Lemma: Let $\beta_1, \beta_2 > 1$ be given, and suppose that $\{f_n\}$ is bounded above uniformly in n. Then

$$\prod_{n=1}^{\infty} E(f_n^{\beta_1}) = \infty \iff \prod_{n=1}^{\infty} E(f_n^{\beta_2}) = \infty$$

and $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ for all $\beta > 1$ if and only if $\prod_{n=1}^{\infty} E(f_n^{\alpha}) = 0$ for all α in $(0,1)$.

Proof:

Let $\beta_1 > 1$ be fixed, and suppose $\beta > \beta_1$. Setting $f = f_n^{\beta_1}$, $\frac{r}{a} = \frac{1}{\beta_1}$, $\frac{s}{b} = \frac{\beta}{\beta_1}$ in (6.3.41), we obtain

$$E(f_n^{\beta_1}) \leq (E(f_n^{\beta}))^b$$

where $b = \left(\frac{\beta_1 - 1}{\beta - 1}\right)$. We deduce

(A): If $\prod_{n=1}^{\infty} E(f_n^{\beta_1}) = \infty$ for some $\beta_1 > 1$, then $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ for all $\beta > \beta_1$.

Setting $f = f_n$, $\frac{r}{a} = \alpha$ in (6.3.41), where $0 < \alpha < 1$ yields

$$1 \leq (E(f_n^{\alpha}))^a (E(f_n^{(1-a\alpha)/(1-a)}))^{1-a} \tag{6.3.43}$$

If a in (0,1) is fixed, then α may be chosen so that $\frac{1-a\alpha}{1-a}$ is as close to 1 as we please. Hence (6.3.43) implies that

$$1 \leq (E(f_n^{\alpha}))^a (E(f_n^{\beta}))^{1-a} \tag{6.3.44}$$

for all β in $(1, \frac{1}{1-a})$, with $\alpha = \frac{1 - \beta(1-a)}{a}$. It follows from (6.3.44) that if $\prod_{n=1}^{\infty} E(f_n^{\alpha}) = 0$ for all α in (0,1), then $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ for all β in $(1, \frac{1}{1-\alpha})$. Using Lemma 6.10 and (A) above we conclude:

(B): If $\prod_{n=1}^{\infty} E(f_n^{\alpha}) = 0$ for some α in (0,1), then $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ for every $\beta > 1$.

To complete the proof of the lemma we need only show that if $\{f_n\}$ is bounded above uniformly in n and if $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ for some $\beta > 1$, then $\prod_{n=1}^{\infty} E(f_n^{\alpha}) = 0$ for some $\alpha < 1$.

From (A) and Lemma 6.10, there is no loss of generality in supposing that β is an integer and $\alpha = \frac{1}{2}$.

Using the Taylor expansion of $(1 + y)^{\frac{1}{2}}$ it is straightforward to show that

$$\frac{1 + \frac{1}{2}y - (1+y)^{1/2}}{y^2} \geqslant \frac{1}{16}$$

whenever $-1 \leqslant y \leqslant 1$. Therefore, since $(1 + \frac{1}{2}y - (1+y)^{\frac{1}{2}})$ is increasing for $y \geqslant 1$, it follows that for $K > 1$,

$$\frac{1 + \frac{1}{2}y - (1+y)^{1/2}}{y^2} \geqslant \frac{1}{16K^2} \tag{6.3.45}$$

whenever $-1 \leqslant y \leqslant K$.

Similarly, if $K > 1$ and $-1 \leqslant y \leqslant K$, there exists $C > 0$ such that

$$0 \leqslant \frac{(1+y)^{\beta} - (1+\beta y)}{y^2} \leqslant C \tag{6.3.46}$$

Since β is an integer, the right hand inequality follows immediately from the Binomial Theorem and $y \leqslant K$, and to see that the left hand inequality is true, note that the minimum value of $((1+y)^{\beta} - (1+\beta y))$ on $[-1, \infty)$ occurs at $y = 0$ and is zero, so that $\dfrac{(1+y)^{\beta} - (1+\beta y)}{y^2}$ is positive for y in $[-1, K] \setminus \{0\}$. The case $y = 0$ may be shown using L'Hopital's rule.

Setting $f_n = 1+y$, we have from (6.3.45) and (6.3.46)

$$1 + \frac{1}{2}(f_n - 1) - f_n^{1/2} \geqslant k(f_n^{\beta} - 1 - \beta(f_n - 1))$$

where $k = \dfrac{1}{16K^2C}$ is constant. Integrating with respect to μ yields

$$1 - E(f_n^{1/2}) \geqslant k(E(f_n^{\beta}) - 1) \tag{6.3.47}$$

since $\int f_n \, d\mu = \int d\mu = 1$, and applying the inequality $x - 1 \geqslant \log x$ for $x > 0$ to (6.3.47), we obtain

$$-\log E(f_n^{1/2}) \geqslant k \log E(f_n^{\beta})$$

for each n in $\mathbb{N}$. It now follows that if $\prod_{n=1}^{\infty} E(f_n^{\beta}) = \infty$ then $\prod_{n=1}^{\infty} E(f_n^{1/2}) = 0$, where for convenience we have assumed $\beta > 1$ is an integer.

The first part of the lemma may now be deduced using (A) and (B), and the second part using (B) and Lemma 6.10.

The proof is now complete.

Lemma 6.11 fails if the hypothesis that $\{f_n\}$ is bounded above is removed. To see this, consider the sequence of functions defined by

$$f_n = \begin{cases} \dfrac{2^n-1}{2^n} & \text{on} \quad \left[0,\ \dfrac{2^{2n-1}-1}{2^{2n-1}}\right] \\ 2^{n-1}+1-\dfrac{1}{2^n} & \text{on} \quad \left(\dfrac{2^{2n-1}-1}{2^{2n-1}},\ 1\right] \end{cases}$$

Taking μ to be Lebesgue measure, $\int f_n \, d\mu = \int d\mu = 1$ for each n, and for α in (0,1), $\inf f_n < \int f_n^\alpha \, d\mu$, so that

$$E(f_n^\alpha) > \left(1-\frac{1}{2^n}\right)$$

$$\log \prod_{n=1}^{\infty} E(f_n^\alpha) > \sum_{n=1}^{\infty} \log\left(1-\frac{1}{2^n}\right)$$

Using the inequality $\log x \geqslant 2\,(\log \frac{1}{2})(1-x)$ for x in $[\frac{1}{2},1]$ yields

$$\log \prod_{n=1}^{\infty} E(f_n^\alpha) > \sum_{n=1}^{\infty} 2\left(\log\frac{1}{2}\right)\frac{1}{2^n} > -2$$

from which it follows that $\prod_{n=1}^{\infty} E(f_n^\alpha) \neq 0$

However, for $\beta \geqslant 3$, $E(f_n^\beta) \to \infty$ as $n \to \infty$, so that $\prod_{n=1}^{\infty} E(f_n^\beta) = \infty$ for some $\beta > 1$.

Thus, provided $\{f_n\}$ is bounded above, the conditions $\prod_{n=1}^{\infty} E(f_n^\alpha) = 0$ for α in (0,1) and $\prod_{n=1}^{\infty} E(f_n^\beta) = \infty$ for $\beta > 1$ are equivalent. However, these conditions are not equivalent to $\sum_{n=1}^{\infty} \int \log f_n \, d\mu = -\infty$ even if $\{f_n\}$ is bounded above and for each n there exists c_n in $\mathbb{R}$ such that $f_n \geqslant c_n > 0$.

6.12 <u>Lemma</u>: If $\prod_{n=1}^{\infty} E(f_n^\alpha) = 0$ for some α in (0,1), then $\sum_{n=1}^{\infty} \int \log f_n \, d\mu = -\infty$.

<u>Proof</u>:

The hypothesis implies that for each α in (0,1), $\prod_{n=1}^{\infty} (E(f_n^\alpha))^{1/\alpha} = 0$ using Lemma 6.10. Now for each n, $(E(f_n^\alpha))^{1/\alpha}$ is a decreasing function of by (6.3.42), and

$$\lim_{\alpha \downarrow 0} (E(f_n^\alpha))^{1/\alpha} = \exp \int \log f_n \, d\mu$$

(see [HLP] §6.18). Hence

$$0 \leq \prod_{n=1}^{\infty} \exp \int \log f_n \, d\mu \leq \prod_{n=1}^{\infty} (E(f_n^{\alpha}))^{1/\alpha} = 0$$

which gives the result.

To see that the converse of Lemma 6.12 is not true in general, consider the sequence of functions defined by

$$f_n = \begin{cases} \dfrac{1}{2^n} & \text{on } \left[0, \dfrac{1}{n^2}\right] \\[2ex] \dfrac{2^n n^2 - 1}{2^n n^2 - 2^n} & \text{on } \left(\dfrac{1}{n^2}, 1\right] \end{cases}$$

and suppose μ denotes Lebesgue measure. Then

$$E(f_n^{1/2}) = 1 + O\left(\frac{1}{n^2}\right)$$

so that

$$\sum_{n=1}^{\infty} (1 - E(f_n^{1/2})) < \infty \qquad (6.3.48)$$

To see that (6.3.48) implies that $\prod_{n=1}^{\infty} E(f_n^{1/2}) > 0$, note that if $\prod_{n=1}^{\infty} E(f_n^{1/2}) = 0$, then using $y < \exp(-(1-y))$ for $0 < y < 1$, we have $\sum_{n=1}^{\infty} \log \exp(-(1-E(f_n^{1/2})) = -\infty$, i.e. $\sum_{n=1}^{\infty} (1 - E(f_n^{1/2})) = \infty$

However,

$$\int \log f_n \, d\mu = -\frac{1}{n} \log 2 + O\left(\frac{1}{n^2}\right)$$

so that $\sum_{n=1}^{\infty} \int \log f_n \, d\mu = -\infty$. We note that $\lim_n \sup f_n = 1$, so that $\{f_n\}$ is a bounded sequence of functions.

As a result of the above discussion, we may state:

6.13 <u>Theorem</u>: Let the functions $f_n(k,y)$, $0 \leq k \leq 1$, $-\infty < y < \infty$, be periodic in y with period 1 and satisfy:

(i) $0 \leq f_n(k,y) \leq K$ where $K < \infty$ is independent of n.

(ii) $\int_0^1 f_n(k,y) \, dy = 1$ for each k in [0,1].

(iii) Either for some α in (0,1)

$$\prod_{n=1}^{\infty} \sup_{k} \int_0^1 (f_n(k,y))^{\alpha}\, dy = 0$$

or for some $\beta > 1$

$$\prod_{n=1}^{\infty} \inf_{k} \int_0^1 (f_n(k,y))^{\beta}\, dy = \infty$$

(iv) For each n, $f_n(k,y)$ is continuous in k for fixed y, and if $\varepsilon > 0$ is given there exists $\delta_{\varepsilon,n}$ which is independent of y such that

$$|f_n(k',y) - f_n(k'',y)| < \varepsilon \quad \text{whenever} \quad |k' - k''| < \delta_{\varepsilon,n} .$$

Then the conclusions of Theorem 6.1 hold.

Evidently conditions (i) and (iv) are considerably weaker than the corresponding conditions in Theorem 6.1; condition (ii) is unchanged, and, as the discussion above shows, condition (iii) is stronger than Pearson's condition (iii). Thus, in general, Theorem 6.13 does not include Theorem 6.1 as a special case; however, if there exists C in $\mathbb{R}$ which is independent of n such that $f_n(k,y) \geqslant C > 0$ for all n, then both conditions (iii) are equivalent (see [AS], Prop.A.3.3).

We have felt it worthwhile to include the alternative condition with $\beta > 1$ in Theorem 6.13(iii) since there are many situations where $\int f^2$, for example, may be evaluated analytically but $\int f^{\alpha}$ cannot, for any α in (0,1). Where the sequence $\{f_n(k,y)\}$ is known to be uniformly bounded away from zero, the logarithmic condition (iii) of Theorem 6.1, with c = 1, may be used instead of condition (iii) of Theorem 6.13, as convenient.

A discussion of Theorem 6.1 and its ramifications, due to J.Avron and B.Simon, deduces Theorem 6.7 using Kakutani's Theorem ([AS] Appendix 3). Their method of proof leads the authors to the conclusion that some special significance attaches to the value $\alpha = \frac{1}{2}$ (where α is used as in Theorem 6.13, condition (iii)); however, our approach suggests that this is not the case. Avron and Simon proceed to deduce a result which is similar to Theorem 6.13; however, the stronger condition that each of the functions $f_n(k,y)$ be continuous in both k and y is required.

The methods of Pearson, Avron and Simon, and our own proofs are an indication of the wealth of strategies that may be utilised in the generation of singular continuous measures from sequences of periodic functions. A further method, using the binomial distribution and Chebyshev's inequality, is used on the example of the following section.

§4 A dense singular continuous measure generated by a sequence of periodic step functions

So far we have confined our attention to the existence of sequences $\{N_n\}$ of natural numbers which ensure that a sequence $\{f_n(N_n k)\}$ (or $\{f_n(k, N_n k)\}$) generates a singular continuous measure. To give an idea of what rate of increase of the $\{N_n\}$ can be sufficient, we analyse a specific example in detail.

6.14 Example: of a dense singular continuous measure on $[0,1)$ which is the limit of a sequence of absolutely continuous measures, and is obtained from a sequence of periodic step functions.

Starting with the step function $f(x)$ which is periodic with period 1 and is defined on $[0,1)$ by

$$f(x) = \begin{cases} \frac{1}{2} & , \quad x \in [0,\frac{1}{2}) \\ 3/2 & , \quad x \in [\frac{1}{2},1) \end{cases}$$

we construct a sequence $\{f_k(x)\}$ on $[0,1)$ such that

$$\begin{aligned} f_1(x) &= f(x) \\ f_k(x) &= f(2^{k-1}x) \text{ for } k \geq 2 \end{aligned}$$

Note that the range of $\prod_{k=1}^{n} f_k(x)$ is $\left\{ \frac{3^r}{2^n} : r \in 0, \ldots, n \right\}$.

Defining the set function $\nu(\Sigma)$ on subintervals Σ of $[0,1)$ to be $\lim_{n \to \infty} \nu_n(\Sigma)$, where

$$\nu_n(\Sigma) = \int_{\Sigma} \prod_{k=1}^{n} f_k(x)\, dx \qquad (6.4.1)$$

we show that ν defines a singular continuous measure on Borel subsets of $[0,1)$, and that the ν-measure of subintervals of $[0,1)$ whose endpoints are diadic rationals may be determined exactly. The main steps of the proof are as follows:

(i) We show that for each n in $\mathbb{N}$, if $r \in \{0, \ldots, 2^n - 1\}$,

$$\nu\left(\left[\frac{r}{2^n}, \frac{r+1}{2^n}\right)\right) = \frac{3^{\alpha_r}}{2^{2n}} \tag{6.4.2}$$

where α_r is the sum of the coefficients in the binary expansion of r.

(ii) We deduce that ν is a unique, continuous and everywhere dense measure on $[0,1)$.

(iii) We use induction to prove that

$$\kappa\left(\left\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) = \frac{3^r}{2^n}, \ 0 \leq r \leq n, n \in \mathbb{N}\right\}\right) = \frac{1}{2^n}\binom{n}{r} \tag{6.4.3}$$

for all n in $\mathbb{N}$, where K denotes Lebesgue measure.

(iv) Using the theory of the binomial distribution and Chebyshev's inequality, we deduce that as $n \to \infty$, $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure on $[0,1)$.

(v) We show that for $n \geq r$, if $\eta > 0$ is given, then $\nu_n(\{x \in [0,1) : \prod_{k=1}^{r} f_k(x) < \eta\})$ is independent of n, and deduce that the measure ν is singular.

<u>Proof of (i)</u>:

We first prove by induction that for $n \geq p$

$$\nu_n\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) = \nu_p\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) \tag{6.4.4}$$

where $n, p \in \mathbb{N}$ and $r \in \{0, \ldots, 2^{p-1}\}$.

We suppose that (6.4.4) is true for $n = q \geq p$ and note that our construction of the sequence $\{f_k\}$ ensures that $\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)$ is a finite union of intervals of the form $\left[\frac{s}{2^q}, \frac{s+1}{2^q}\right)$, where $s \in \{0, \ldots, 2^{q-1}\}$, on each of which $\prod_{k=1}^{q} f_k(x)$ takes a fixed constant value. Consider one such constituent interval, $\left[\frac{t}{2^q}, \frac{t+1}{2^q}\right)$, where $t \in \{0, \ldots, 2^{q-1}\}$.

Since

$$f_{q+1}(x) = \begin{cases} \frac{1}{2} \quad , & x \in \left[\frac{2t}{2^{q+1}}, \frac{2t+1}{2^{q+1}}\right) \\ \frac{3}{2} \quad , & x \in \left[\frac{2t+1}{2^{q+1}}, \frac{2t+2}{2^{q+1}}\right) \end{cases}$$

we have by (6.4.1)

$$\nu_{q+1}\left(\left[\frac{t}{2^q}, \frac{t+1}{2^q}\right)\right) = \frac{1}{2} \prod_{k=1}^{q} f_k\left(\frac{t}{2^q}\right) \frac{1}{2^{q+1}} + \frac{3}{2} \prod_{k=1}^{q} f_k\left(\frac{t}{2^q}\right) \frac{1}{2^{q+1}}$$

$$= \nu_q\left(\left[\frac{t}{2^q}, \frac{t+1}{2^q}\right)\right)$$

so that

$$\nu_{q+1}\left(\left[\frac{r}{2^q}, \frac{r+1}{2^q}\right)\right) = \nu_q\left(\left[\frac{r}{2^q}, \frac{r+1}{2^q}\right)\right)$$

That is, if (6.4.4) is true for $n = q \geqslant p$, then it is also true for $n = q+1$. Since (6.4.4) is trivially true for $n = p$, it is proved for $n \geqslant p$ by induction.

It follows that if $p \in \mathbb{N}$,

$$\nu\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) = \nu_p\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) \qquad (6.4.5)$$

for each r in $\{0, \ldots, 2^p - 1\}$.

We are now in a position to prove (6.4.2) by induction, and suppose first that the assertion is true for $n = q$. It is required to prove that

$$\nu\left(\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}}\right)\right) = \frac{3^{\alpha_r}}{2^{2(q+1)}}$$

where $r = \sum_{i=0}^{q} a_{i_{q+1}} 2^i$ and $\alpha_r = \sum_{i=0}^{q} a_{i_{q+1}}$.

In the case where r is even, $\prod_{k=1}^{q} f_k(x)$ is constant on $\left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r}{2}+1}{2^q}\right)$ and $f_{q+1}(x) = \frac{1}{2}$ on $\left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r}{2}+\frac{1}{2}}{2^q}\right)$ so that by (6.4.5)

$$\nu\left(\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}}\right)\right) = \nu_{q+1}\left(\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}}\right)\right)$$

$$= \int_{\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}}\right)} \frac{1}{2} \prod_{k=1}^{q} f_k(x)\, dx$$

$$= \frac{1}{4} \int_{\left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r}{2}+1}{2^q}\right)} \prod_{k=1}^{q} f_k(x)\, dx$$

$$= \frac{1}{4}\, \nu_q \left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r}{2}+1}{2^q} \right)$$

$$= \frac{1}{4}\, \nu \left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r}{2}+1}{2^q} \right)$$

Since the sum of the coefficients in the binary expansion of $\frac{r}{2}$ and of r are equal, it follows that if (6.4.2) is true for $n = q$ and $r \in \{0, \ldots, 2^{q+1}-1\}$ is even, then (6.4.2) is true for $n = q+1$.

In the case where r is odd, $\prod_{k=1}^{q} f_k(x)$ is constant on $\left[\frac{\frac{r-1}{2}}{2^q}, \frac{\frac{r+1}{2}}{2^q} \right)$ and $f_{q+1}(x) = \frac{3}{2}$ on $\left[\frac{\frac{r}{2}}{2^q}, \frac{\frac{r+1}{2}}{2^q} \right)$, so that, arguing as above,

$$\nu\left(\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}} \right)\right) = \frac{3}{4}\, \nu\left(\left[\frac{\frac{r-1}{2}}{2^q}, \frac{\frac{r+1}{2}}{2^q} \right)\right) \tag{6.4.6}$$

If (6.4.2) is true for $n = q$, then

$$\nu\left(\left[\frac{\frac{r-1}{2}}{2^q}, \frac{\frac{r+1}{2}}{2^q} \right)\right) = \frac{3^{\alpha_{(r-1)/2}}}{2^{2q}}$$

where $\alpha_{(r-1)/2}$ is the sum of the coefficients in the binary expansion of $\frac{r-1}{2}$, and hence from (6.4.6)

$$\nu\left(\left[\frac{r}{2^{q+1}}, \frac{r+1}{2^{q+1}} \right)\right) = \frac{3^{\alpha_{(r-1)/2}+1}}{2^{2(q+1)}} \tag{6.4.7}$$

However, if α_r is the sum of the coefficients in the binary expansion of r, then

$$\alpha_r = \alpha_{(r-1)/2} + 1$$

if r is odd, so that from (6.4.7), (6.4.2) is true for $n = q+1$.

Using (6.4.5)

$$\nu\left(\left[0, \tfrac{1}{2}\right)\right) = \nu_1\left(\left[0, \tfrac{1}{2}\right)\right) = \frac{1}{4} = \frac{3^0}{2^2}$$

and

$$\nu\left(\left[\tfrac{1}{2}, 1\right)\right) = \nu_1\left(\left[\tfrac{1}{2}, \tfrac{2}{2}\right)\right) = \frac{3}{4} = \frac{3^1}{2^2}$$

so (6.4.2) is true for $n = 1$; this completes the proof of (6.4.2) for each n in $\mathbb{N}$, with $r \in \{0, \ldots, 2^n - 1\}$.

Proof of (ii):

If $\frac{r}{2^p} \in [0,1)$ then $\alpha_r \leq p$, where α_r is as in (i). Hence by (6.4.2),

$$\nu\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) \leq \left(\frac{3}{4}\right)^p \tag{6.4.8}$$

for all $r \in \{0, \ldots, 2^p - 1\}$, so that if $\varepsilon > 0$ is given

$$\nu\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) < \varepsilon \tag{6.4.9}$$

whenever p is sufficiently large.

We prove that ν is a measure on Borel subsets of [0,1). This will be achieved if we show that the function

$$f(x) = \lim_{n \to \infty} \nu_n [0, x)$$

is defined at all points x of [0,1) and is bounded, continuous and increasing. (see [H] §43, Thm.B).

By (6.4.2), f(x) is defined for all x of the form $\frac{r}{2^p}$, where

$r, p \in \mathbb{N}$, $r \in \{1, \ldots, 2^p - 1\}$. Consider therefore an arbitrary point a in [0,1) which is not of this form and let $\varepsilon > 0$ be given. By (6.4.9) there exist $p, r \in \mathbb{N}$ such that

$$\frac{r}{2^p} < a < \frac{r+1}{2^p}$$

and $\nu\left(\left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)\right) < \varepsilon$.

Hence by (6.4.4) and (6.4.5)

$$\begin{aligned}\nu\left(\left[0, \frac{r}{2^p}\right)\right) &= \nu_n\left(\left[0, \frac{r}{2^p}\right)\right) \leq \nu_n([0, a)) \leq \nu_n\left(\left[0, \frac{r+1}{2^p}\right)\right) \\ &= \nu\left(\left[0, \frac{r+1}{2^p}\right)\right) < \nu\left(\left[0, \frac{r}{2^p}\right)\right) + \varepsilon\end{aligned}$$

for all $n \geq p$, so that

$$\nu\left(\left[0, \frac{r}{2^p}\right)\right) \leq \liminf_{n \to \infty} \nu_n([0, a)) \leq \limsup_{n \to \infty} \nu_n([0,a)) < \nu\left(\left[0, \frac{r}{2^p}\right)\right) + \varepsilon$$

It follows from the arbitrariness of ε that f(x) is defined at a: hence f(x) is defined at all points of [0,1).

To see that $x_2 > x_1$ implies $f(x_2) > f(x_1)$ for all x_1, x_2 in $[0,1)$, note that for such x_1, x_2, there exist r, n in $\mathbb{N}$ such that

$$x_1 < \frac{r}{2^n} \qquad , \qquad \frac{r+1}{2^n} < x_2$$

so that by (6.4.2),

$$f(x_2) - f(x_1) \geqslant \nu\left(\left[\frac{r}{2^n}, \frac{r+1}{2^n}\right)\right)$$

By (6.4.5),

$$f(1) = \nu([0,1)) = \nu_1([0,1))$$

so that $f(x)$ is bounded on $[0,1)$.

We now show that $f(x)$ is continuous on $[0,1)$; let $a \in [0,1)$ be an arbitrary point.

For each n in $\mathbb{N}$, there exists an interval

$$I_{a,n} = \left[\frac{r_{a,n}}{2^n}, \frac{r_{a,n}+1}{2^n}\right).$$

with $r_{a,n} \in \{0, \ldots, 2^n - 1\}$ such that $a \in I_{a,n}$; clearly $I_{a,n} \subset I_{a,m}$ whenever $n > m$. Hence

$$\begin{aligned}\nu(\{a\}) &= \nu\left(\bigcap_{n \in \mathbb{N}} I_{a,n}\right) \\ &= \lim_{n \to \infty} \nu(I_{a,n}) \\ &= 0\end{aligned}$$

by (6.4.8). (see [BA], Lemma 3.4).

It follows that ν is a unique continuous measure on Borel subsets of $[0,1)$ ([H] §23 Thms.B,C); and since f is strictly increasing on $[0,1)$, ν is everywhere dense on $[0,1)$.

Proof of (iii):

Suppose that (6.4.3) is true for $n = p$, so that

$$\kappa\left(\left\{x \in [0,1) : \prod_{k=1}^{p} f_k(x) = \frac{3^r}{2^p}\right\}\right) = \frac{\binom{p}{r}}{2^p}$$

for all $r = 0, \ldots, p$.

Let ${}_rS_p$ denote $\{x \in [0,1) : \prod_{k=1}^{p} f_k(x) = \frac{3^r}{2^p}\}$ and let ${}_tI_p$ denote

$\left[\frac{t}{2^p}, \frac{t+1}{2^p}\right)$ for each $t = 0,\dots,2^p-1$.

Now $\prod_{k=1}^{p} f_k(x)$ takes constant values on each interval ${}_tI_p$, and by the construction of the sequence $\{f_k(x)\}$,

$$\kappa(\{x \in {}_tI_p : f_{p+1}(x) = \tfrac{1}{2}\}) = \tfrac{1}{2}\,\kappa({}_tI_p)$$

$$\kappa(\{x \in {}_tI_p : f_{p+1}(x) = \tfrac{3}{2}\}) = \tfrac{1}{2}\,\kappa({}_tI_p)$$

Therefore, since ${}_{r-1}S_p$, ${}_rS_p$ are finite unions of such intervals,

$$\kappa(\{x \in {}_rS_p : f_{p+1}(x) = \tfrac{1}{2}\}) = \tfrac{1}{2}\,\kappa({}_rS_p)$$

$$\kappa(\{x \in {}_{r-1}S_p : f_{p+1}(x) = \tfrac{3}{2}\}) = \tfrac{1}{2}\,\kappa({}_{r-1}S_p)$$

for each $r = 0,\dots,p$. Hence

$$\begin{aligned}\kappa(\{x \in [0,1) : \prod_{k=1}^{p+1} = \frac{3^r}{2^{p+1}}\}) &= \kappa({}_rS_{p+1})\\ &= \tfrac{1}{2}\,\kappa({}_rS_p) + \tfrac{1}{2}\,\kappa({}_{r-1}S_p)\\ &= \frac{1}{2}\,\frac{\left[\binom{p}{r} + \binom{p}{r-1}\right]}{2^p}\\ &= \frac{\binom{p+1}{r}}{2^{p+1}}\end{aligned}$$

for $r = 1,\dots p$, and if $r = p+1$,

$$\kappa({}_{p+1}S_{p+1}) = \tfrac{1}{2}\,\kappa({}_pS_p) = \frac{1}{2^{p+1}} = \frac{\binom{p+1}{p+1}}{2^{p+1}}$$

Thus if (6.4.3) is true for $n = p$, it is also true for $n = p+1$.

If $n = 1$,

$$\kappa(\{x \in [0,1) : f_1(x) = \tfrac{1}{2}\}) = \frac{1}{2} = \frac{\binom{1}{0}}{2^1}$$

$$\kappa(\{x \in [0,1) : f_1(x) = \tfrac{3}{2}\}) = \frac{1}{2} = \frac{\binom{1}{1}}{2^1}$$

so that (6.4.3) is true for $n = 1$.

This completes the proof by induction.

<u>Proof of (iv)</u>:

We first show that

$$\kappa(\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) \geq \frac{3^{\frac{5}{8}n}}{2^n}\}) \to 0 \qquad (6.4.6)$$

as $n \to \infty$.

Now, using (6.4.3) and $\binom{n}{k} = \binom{n}{n-k}$

$$\kappa(\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) \geqslant \frac{3^{5/8 n}}{2^n}\}) = \sum_{\{k \in \mathbb{N} : \frac{5}{8}n \leqslant k \leqslant n\}} \frac{\binom{n}{k}}{2^n}$$

$$= \sum_{\{k \in \mathbb{N} : 0 \leqslant k \leqslant \frac{3}{8}n\}} \frac{\binom{n}{k}}{2^n}$$

$$= \sum_{\{k \in \mathbb{N} : 0 \leqslant k \leqslant \frac{3}{8}n\}} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$= P\left(X \leqslant \frac{3}{8}n\right)$$

whereX is a random variable with binomial distribution and parameters n and $p = \frac{1}{2}$, and P denotes probability.

Using Chebyshev's inequality, where $V(X) = np(1-p)$ is the variance of X.

$$P\left(X \leqslant \frac{3}{8}n\right) = \frac{1}{2} P\left(|X - \tfrac{1}{2}n| \geqslant \tfrac{1}{8}n\right)$$

$$\leqslant \frac{1}{2} \frac{V(x)}{(n/8)^2}$$

$$= \frac{8}{n}$$

from which we deduce (6.4.10). (For discussion of the binomial distribution and Chebyshev's inequality, see for example [B] Ch.6 §38, Ch.7 §44).

Since $3^5 < 2^8$, for each $\varepsilon > 0$ there exists N_ε in $\mathbb{N}$ such that $\frac{3^{5/8 n}}{2^n} < \varepsilon$ whenever $n \geqslant N_\varepsilon$. Hence

$$\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) > \varepsilon\} \subseteq \{x \in [0,1) : \prod_{k=1}^{n} f_k(x) \geqslant \frac{3^{5/8 n}}{2^n}\}$$

for all $n \geqslant N_\varepsilon$, so that, by (6.4.10),

$$\kappa(\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) > \varepsilon\}) \to 0$$

as $n \to \infty$; that is, $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure on $[0,1)$.

<u>Proof of (v):</u>

Let $S_{n,\eta}$ denote $\{x \in [0,1) : \prod_{k=1}^{n} f_k(x) \leqslant \eta\}$.

We use induction to show that

$$\nu_r(S_{n,\eta}) = \nu_n(S_{n,\eta}) \qquad (6.4.11)$$

for all $r \geqslant n$.

Suppose it is true that $\nu_p(S_{n,\eta}) = \nu_n(S_{n,\eta})$ for some $p \geqslant n$.

By definition

$$\nu_{p+1}(S_{n,\eta}) = \int_{S_{n,\eta}} \prod_{k=1}^{p+1} f_k(x)\, dx$$

Now $S_{n,\eta}$ may be expressed as a finite union of disjoint intervals $\{I_r\}$, where each $I_r = \left[\frac{r}{2^p}, \frac{r+1}{2^p}\right)$ for some $r \in \{0, \ldots, 2^p - 1\}$, and on each of which $\prod_{k=1}^{p} f_k(x)$ takes a fixed constant value, p_r, say. Moreover,

for each r such that $I_r \subseteq S_{n,\eta}$

$$f_{p+1}(x) = \frac{1}{2} \qquad \text{on} \quad \left[\frac{r}{2^p}, \frac{r+1/2}{2^p}\right)$$

$$f_{p+1}(x) = \frac{3}{2} \qquad \text{on} \quad \left[\frac{r+1/2}{2^p}, \frac{r+1}{2^p}\right)$$

by the construction of the sequence $f_k(x)$. Hence for each such r

$$\begin{aligned}\int_{I_r} \prod_{k=1}^{p+1} f_k(x)\, dx &= \int_{\left[\frac{r}{2^p}, \frac{r+1/2}{2^p}\right)} \tfrac{1}{2}\, p_r\, dx + \int_{\left[\frac{r+1/2}{2^p}, \frac{r+1}{2^p}\right)} \tfrac{3}{2}\, p_r\, dx \\ &= p_r\, \kappa(I_r) \\ &= \int_{I_r} \prod_{k=1}^{p} f_k(x)\, dx\end{aligned}$$

which implies

$$\int_{S_{n,\eta}} \prod_{k=1}^{p+1} f_k(x)\, dx = \int_{S_{n,\eta}} \prod_{k=1}^{p} f_k(x)\, dx$$

i.e.

$$\nu_{p+1}(S_{n,\eta}) = \nu_p(S_{n,\eta})$$

Since it is trivially true that $\nu_r(S_{n,\eta}) = \nu_n(S_{n,\eta})$ for $r = n$, this completes the proof by induction of (6.4.11).

It follows immediately that

$$\nu(S_{n,\eta}) = \nu_n(S_{n,\eta}) \qquad (6.4.12)$$

from which we now deduce the singularity of the measure ν.

Define

$$\tilde{S}_{n,q} = \{ x \in [0,1) : \prod_{k=1}^{n} f_k(x) \leqslant \frac{1}{q} \}$$

Since $\prod_{k=1}^{n} f_k(x)$ converges to zero in Lebesgue measure as $n \to \infty$, there exists N_q in $\mathbb{N}$ such that

$$\kappa(\{ x \in [0,1) : \prod_{k=1}^{n} f_k(x) > \frac{1}{q} \}) = \kappa([0,1)) - \kappa(\tilde{S}_{n,q})$$
$$< \frac{1}{q}$$

whenever $n \geqslant N_q$; that is, such that

$$\kappa(\tilde{S}_{n,q}) > 1 - \frac{1}{q}$$

whenever $n \geqslant N_q$.

Fix $n = N_q$; then

$$\kappa(\tilde{S}_{N_q,q}) > 1 - \frac{1}{q}$$

On the other hand, by (6.4.12)

$$\begin{aligned} \nu(\tilde{S}_{N_q,q}) &= \nu_{N_q}(\tilde{S}_{N_q,q}) \\ &= \int_{\tilde{S}_{N_q,q}} \prod_{k=1}^{N_q} f_k(x)\,dx \\ &= \int_{[0,1)} \frac{1}{q}\,dx = \frac{1}{q} \end{aligned}$$

Thus we have determined a sequence of sets $\{\tilde{S}_{N_q,q}\}$ whose Lebesgue measure converges to that of [0,1], and whose ν-measure converges to zero as $q \to \infty$. It follows that the measure ν is singular, so the proof of (vi) is complete.

This example is intimately related to a class of monotonic continuous functions whose derivatives are zero almost everywhere, considered by F.Riesz and B. Sz.-Nagy. (see [RN] §24, in particular consider $t = \frac{1}{2}$, $F_n(x) = \nu_n([1-x, \cdot))$; also [HS] 18.8).

The method used in Example 6.14 does not have general application, even to sequences of step functions. However, apart from its intrinsic interest, the example shows that far slower sequences $\{N_n\}$ than those obtainable from the general theory may be sufficient to generate a singular continuous measure from a suitable sequence of periodic functions. Indeed, inspection of parts (i) and (iv) of the proof shows that such a slowly increasing sequence as $\{N_n : N_n = 2^n - 1\}$ could not have been obtained by the method of Proposition 6.6.

CONCLUSION

During the course of this work we have established a characterisation of each part of the spectrum in terms of solutions of the Schrödinger equation, demonstrated by example that asymptotic completeness does not imply continuity of the scattering amplitude as a function of energy, and extended and generalised a number of existing results of relevance to scattering theory and spectral analysis. As with every development, new questions and further problems arise; it seems appropriate, therefore, to conclude with a brief discussion of the advantages and limitations of our theories and a tentative consideration of how they might be applied and in what directions they could be extended.

The theory of subordinacy developed in Chapter III and extended in Chapter IV is attractive in several respects. Firstly, unlike many direct methods in spectral analysis, its validity is independent of the detailed behaviour of the potential; only very general conditions, as for example, that the spherically symmetric potential $V(r)$ be integrable at infinity and the spectrum of every self-adjoint operator arising from L on (0,1] be singular need to be met (see eg. Thms. 3.21, 4.10). Secondly the required estimates of the relative size of solutions of the Schrödinger equation at infinity (and / or 0) are comparatively crude; this information should be considerably easier to obtain than, for example, the detailed knowledge of $m(z)$ required by Titchmarsh ([T2] Ch.V), and only a consideration of solutions of $Lu = \lambda u$ for real values of λ is involved. Finally, our limited excursion into cases where there is singular behaviour of the potential at both ends of an interval (Thm.4.10), suggests that considerable extension of the theory, perhaps to all second order differential operators of the Sturm-Liouville type, may eventually be possible.

In the generalisation of the Weyl-Titchmarsh theory to differential equations of any even order by Kodaira ([KO2]), all the original features remain (for example, nesting circles are generalised to nesting hypersurfaces), which suggests that an extension of the theory of subordinacy in this direction might be relatively straightforward. The structural correspondence between the three dimensional Schrödinger operator and the one dimensional Schrödinger operator appears to be less exact; for example, there is nothing quite comparable to the spectral function ([T3] Ch.XII, §12.10). On the other hand, much of the theory which applies in the one dimensional case has been shown to apply in a modified form in the three dimensional case (cf. [T2], [T3]) and, of course, with central potentials the three dimensional problem effectively simplifies to a family of one dimensional problems ([AJS] Ch.11). It therefore seems not improbable that some adaptation of our theory might apply, and in view of the importance of the three dimensional Schrödinger operator in quantum mechanics, such an investigation would seem to be very worthwhile.

The most pressing immediate problem, which has not been tackled in this thesis, is to find ways of applying the theory developed so far to specific situations. In some cases sufficiently detailed knowledge of the solutions may be known already, so that an immediate application of Thm. 3.21 or of Thm. 4.10 is possible. However, it is likely that subordinacy will be of most value to spectral analysis when dense point or singular continuous spectrum is a possibility, and that in such cases suitable estimates of solutions - as of everything else - will be hard to obtain. The possibility of using the theory indirectly, for example, in conjunction with perturbation methods, should not therefore be overlooked.

Although the main application of subordinacy must surely be to spectral analysis, it is worth noting that, where details of the spectrum are known already, some new knowledge of the asymptotic behaviour of solutions of the Schrödinger equation is now immediately available. This may not only be of interest so far as properties of solutions are concerned; it also suggests a possible line of enquiry for spectral analysis. For, because the behaviour of solutions can now be related to that of the potential in the many cases that have been analysed, it may sometimes be possible to extrapolate causal connections between them from which further results can be obtained.

The simplified eigenfunction expansion of Chapter IV is of fundamental theoretical interest and further generalisations would be of value not only to quantum mechanics but in many branches of physical science. Again, there is the possibility of analogous results for ordinary differential operators of any even order. It seems quite likely that the Weyl-Kodaira Theorem can be simplified as in Theorem 4.9 whenever the spectrum is simple and that, in general, the simplified expansion is a natural extension of the well established expansion for the case where L is regular at 0 ([CL] Ch.9, §3). Further investigation of this problem might well clarify whether the theory of subordinacy applies under weaker assumptions (cf. Theorem 4.10, which is a by-product of the groundwork for Theorem 4.9). As noted earlier, some related results which were unavailable to the author have been obtained by Kac ([K1], [K2]), so before proceeding further it would be prudent to investigate the precise nature and scope of this work.

If only those potentials V(r) for which H_C is spectrally simple and the wave operators $\Omega_\pm(H_C, H_0)$ are complete for each C in $\mathbb{R}$ are considered, where H_C is any self-adjoint extension of $-\frac{d^2}{dr^2} + V(r) + C\chi[0,1]$, then no further weakening of the condition at r = 0 on the class of potentials for which the phase shift formula for the scattering operator (Thm.5.9) holds is possible. However, it may be that the condition at infinity can be weakened in certain respects; for example, it was shown by Kuroda that the condition $V(r) = O(r^{-(1+\varepsilon)})$ can be replaced by $V(r) \in L_1[1,\infty)$ if the potential is not too singular at 0 ([KU2]), so a similar improvement could be possible where the spectrum of each self-adjoint operation H_1 arising from L on (0,1] is singular. It may also be worth considering whether certain oscillatory potentials which are not integrable at infinity can be accommodated under our weaker conditions at the origin (cf. [RS III]. p.167).

The proof of the existence of a potential for which the wave operators are complete even though the scattering amplitude is a discontinuous function of energy in Chapter V demonstrates that completeness and discontinuity of the scattering amplitude can occur in conjunction when the potential is sufficiently singular at the origin but absorption does not occur. It may be that this phenomenon occurs quite generally whenever there is dense singular spectrum of H, and $V(r) = O(r^{-(1+\varepsilon)})$ as $r \to \infty$, and that whenever the spectrum of H_1 is isolated pure point, continuity of the scattering amplitude is assured given a suitable condition on V(r) at infinity. Certainly the relationship between the scattering amplitude as a function of energy and the spectral properties of H_1 seems worthy of further investigation.

The method of inductive construction of potentials, originally devised by Pearson ([P1]) and discussed in Chapter VI is an interesting alternative approach to spectral analysis which seems particularly promising where singular continuous spectrum is concerned. It is difficult to assess the likely future significance of the method given the rather limited class of problem to which it has been applied so far; in each case the constructed potential vanishes on successively larger intervals of $\mathbb{R}$ as $r \to \infty$ (see eg. Prop.6.4, [P1] Props. 1,2). Initially, an investigation into whether the method could also be applied when the potential is small, just touching zero glancingly, on successively larger intervals of $\mathbb{R}$ as $r \to \infty$ (as for example $(1 + \cos\sqrt{r})$) could lead to a useful extension of the approach.

While the material of this thesis has for the most part been motivated by problems in theoretical physics, we hope that some of its contents may also be of interest in other fields. Only Chapter V is exclusively quantum mechanical in its subject matter; mathematical topics occurring elsewhere include ordinary differential equations, eigenfunction expansions, measure and spectral theory and complex analysis. Of particular interest, perhaps, are the intimate connections that have been exposed between spectral properties of Sturm-Liouville operators, solutions of the associated equations, the theory of measure and boundary properties of analytic functions.

APPENDIX

We prove that the isometric Hilbert space isomorphism $\tilde{S}$ of Theorem 4.9 is surjective; that is, we show that for each $G(\lambda)$ in $L_2^{\tilde{\rho}}(-\infty,\infty)$ there exists f(r) in $L_2(0,\infty)$ such that $(\tilde{S}f)(\lambda) = G(\lambda)$ $\tilde{\mu}$-almost everywhere.

Now from the Weyl-Kodaira Theorem, for each given element $(\phi_1(\lambda), \phi_2(\lambda))$ of $L_2^{\rho ij}(-\infty,\infty)$ there exists f(r) in $L_2(0,\infty)$ for which $((Tf)_1(\lambda),(Tf)_2(\lambda))$ converges to $(\phi_1(\lambda), \phi_2(\lambda))$ in the topology of $L_2^{\rho ij}(-\infty,\infty)$. Moreover, for each $(\phi_1(\lambda), \phi_2(\lambda))$ in $L_2^{\rho ij}(-\infty,\infty)$ there is a corresponding element $G(\lambda)$ in $L_2^{\tilde{\rho}}(-\infty,\infty)$ such that $G(\lambda) = \phi_1(\lambda) + m_0(\lambda)\,\phi_2(\lambda)$ $= (\tilde{S}f)(\lambda)$ $\tilde{\mu}$-almost everywhere (see proof of Thm. 4.9). However, in general, there appears to be no obvious way in which we may associate a particular element $(\phi_1(\lambda), \phi_2(\lambda))$ of $L_2^{\rho ij}(-\infty,\infty)$ with an arbitrary given element $G(\lambda)$ of $L_2^{\tilde{\rho}}(-\infty,\infty)$. It seems, therefore, that the surjective property of $\tilde{S}$ cannot be deduced in a straightforward way from the surjective property in the general case.

To overcome the problem, we have adapted a proof due to Coddington and Levison ([CL] Ch.9, Thm.3.2). To illuminate the main steps of the proof, we present the preliminary stages as a sequence of lemmas.

Throughout this Appendix, $\|\cdot\|_1$ will denote the $L_2[0,1]$ norm, $\Delta, \Delta_1, \Delta_2$ compact subintervals of $\mathbb{R}$, and χ_I the characteristic function of an interval I.

A.1 Lemma: If $L = \dfrac{-d^2}{dr^2} + V(r)$ is in the limit circle case at 0, then

$$\|y_1(r,z)\|_1,\ \|y_2(r,z)\|_1 \text{ are continuous functions of } z \text{ on } \mathbb{C}$$

where $y_1(r,z)$ and $y_2(r,z)$ are defined as in Chapter IV, §1.

Proof:

We shall give the proof for $y_1(r,z)$; there is no difference of principle in the case of $y_2(r,z)$.

Using the "variation of constants" formula ([CL], Ch.3, Thm.6.4), we have

$$y_1(r,z) = y_1(r,z_0) + y_1(r,z_0)\int_r^1 y_2(v,z_0)(z-z_0)\,y_1(v,z)\,dv$$
$$\qquad - y_2(r,z_0)\int_r^1 y_1(v,z_0)(z-z_0)\,y_1(v,z)\,dv$$

so that, proceeding as in Lemma 3.2, we obtain

$$\| y_1^{(n+1)}(r,z) - y_1^{(n)}(r,z)\|$$
$$\leq [\,2\,|z-z_0|\,\|y_1(r,z_0)\|_1\,\|y_2(r,z_0)\|_1\,]^n\,\|y_1(r,z_0)\|_1$$

where

$$y_1^{(n+1)}(r,z) = y_1(r,z_0) + y_1(r,z_0)\int_r^1 y_2(v,z_0)(z-z_0)\,y_1^{(n)}(v,z)\,dv$$
$$\qquad - y_2(r,z_0)\int_r^1 y_1(v,z_0)(z-z_0)\,y_1^{(n)}(v,z)\,dv$$

Since L is limit circle at 0, $y_1(r,z_0)$ and $y_2(r,z_0)$ are in $L_2[0,1]$ for each z_0 in $\mathbb{C}$. Hence if z and z_0 are sufficiently close, the iterations converge to the solution $y_1(r,z)$ (cf. proof of Lemma 3.2), and

$$\begin{aligned}
\big|\,\|y_1(r,z)\|_1 - \|y_1(r,z_0)\|_1\big| &\leq \|y_1(r,z) - y_1(r,z_0)\|_1 \\
&= \Big\|\sum_{n=1}^{\infty}\big(y_1^{(n+1)}(r,z) - y_1^{(n)}(r,z)\big)\Big\|_1 \\
&\leq \sum_{n=1}^{\infty}\|y_1^{(n+1)}(r,z) - y_1^{(n)}(r,z)\|_1 \\
&< \delta
\end{aligned}$$

for some predetermined $\delta > 0$.

Thus $\|y_1(r,z)\|_1$ is a continuous function of z at z_0; the arbitrary choice of z_0 implies that $\|y_1(r,z)\|_1$ is a continuous function of z on $\mathbb{C}$.

The lemma is now proved.

A.2 <u>Lemma</u>: With the notation of Theorem 4.9, if $G(\lambda)$ is in $L_2^{\tilde\rho}(-\infty,\infty)$ and

$$y_\Delta(r) = \int_\Delta y_s(r,\lambda)\,G(\lambda)\,d\tilde\rho(\lambda)$$

then $y_\Delta(r)$ is in $L_2(0,\infty)$ for each $\Delta \subset (-\infty,\infty)$ and $\{y_\Delta(r)\}$ converges in the mean as $\Delta \to (-\infty,\infty)$.

<u>Proof</u>:

Let $G(\lambda)$ be in $L_2^{\tilde\rho}(-\infty,\infty)$.

Consider the integral

$$\int_0^\infty y_\Delta(r)\, P(r)\, dr$$

where $P(r) \in L_2(0,\infty)$ vanishes outside [a,b] for some a,b such that $0 < a < b < \infty$ and

$$y_\Delta(r) = \int_\Delta y_s(r,\lambda)\, G(\lambda)\, d\tilde{\rho}(\lambda)$$

If $Q(\lambda) = (\tilde{S} P)(\lambda)$ then by Theorem 4.9

$$Q(\lambda) = \int_a^b y_s(r,\lambda)\, P(r)\, dr \tag{A.2.1}$$

We prove that

$$\int_0^\infty y_\Delta(r)\, P(r)\, dr = \int_\Delta Q(\lambda)\, G(\lambda)\, d\tilde{\rho}(\lambda)$$

for each compact Δ in $(-\infty,\infty)$, and deduce that $\{y_\Delta(r)\}$ is Cauchy in $L_2(0,\infty)$.

Now

$$\int_0^\infty y_\Delta(r)\, P(r)\, dr = \int_a^b \int_\Delta y_s(r,\lambda)\, G(\lambda)\, P(r)\, d\tilde{\rho}(\lambda)\, dr \tag{A.2.2}$$

We show that $y_s(r,\lambda)\, G(\lambda)\, P(r)$ is integrable on $[a,b] \times \Delta$ so that the order of integration may be reversed.

By the Cauchy -Schwarz inequality

$$\int_\Delta |y_s(r,\lambda)\, G(\lambda)|\, d\tilde{\rho}(\lambda)$$

$$\leq \left(\int_\Delta |y_s(r,\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_\Delta |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \tag{A.2.3}$$

From (4.4.14),

$$\tilde{\mu} = \begin{cases} \mu_{22} & \text{on} \quad E \\ \mu_{11} & \text{on} \quad \mathbb{R} \setminus E \end{cases}$$

so that using Minkowski's inequality, (4.4.15) and Lemma 4.8 we have

$$\left(\int_\Delta |y_s(r,\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2} = \left(\int_\Delta |y_s(r,\lambda)\, \chi_{\Delta\setminus E} + y_s(r,\lambda)\, \chi_{\Delta\cap E}|^2 d\tilde{\rho}(\lambda)\right)^{1/2}$$

$$= \left(\int_\Delta |y_1(r,\lambda)\, \chi_{\Delta\setminus E} + m_0(\lambda)\, y_2(r,\lambda)\, \chi_{\Delta\setminus E} + y_2(r,\lambda)\, \chi_{\Delta\cap E}|^2 d\tilde{\rho}(\lambda)\right)^{1/2}$$

$$\leq \left(\int_{\Delta\setminus E} |y_1(r,\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2} + \left(\int_\Delta |m_0(\lambda)\, y_2(r,\lambda)\, \chi_{\Delta\setminus E} + y_2(r,\lambda)\chi_{\Delta\cap E}|^2 d\tilde{\rho}(\lambda)\right)^{1/2}$$

$$= \left(\int_{\Delta\setminus E} |y_1(r,\lambda)|^2 d\rho_{11}(\lambda)\right)^{1/2} + \left(\int_\Delta |y_2(r,\lambda)|^2 d\rho_{22}(\lambda)\right)^{1/2} \tag{A.2.4}$$

Now $y_1(r,\lambda)$, $y_2(r,\lambda)$ are bounded on $[a,b]\times\Delta$. To see this we use the "variation of constants" formula on the reformulated Schrödinger equation

$$(L-\lambda_0)\,y(r,\lambda) = (\lambda-\lambda_0)\,y(r,\lambda)$$

where λ_0 is some fixed point in Δ. This yields

$$\begin{aligned} y_1(r,\lambda) = y_1(r,\lambda_0) &+ y_1(r,\lambda_0)\int_r^1 y_2(v,\lambda_0)(\lambda-\lambda_0)\,y_1(v,\lambda)\,dv \\ &- y_2(r,\lambda_0)\int_r^1 y_1(v,\lambda_0)(\lambda-\lambda_0)\,y_1(v,\lambda)\,dv \end{aligned} \tag{A.2.5}$$

for r in (0,1]. Hence, since Δ is finite and $y_1(r,\lambda_0)$, $y_2(r,\lambda_0)$ are continuous functions of r on $(0,\infty)$, there exist K, M in $\mathbb{R}^+$ such that

$$|y_1(r,\lambda)| \leq K + \int_r^1 M\,|y_1(v,\lambda)|\,dv$$

for each r in $[a,b]\cap(0,1]$ and all λ in Δ. If a < 1, we may apply Lemma 5.2 to give

$$|y_1(r,\lambda)| \leq K\exp[M(1-a)]$$

for all r in [a,1] and all λ in Δ. Similarly, if b > 1, there exist K', M' in $\mathbb{R}^+$ such that

$$|y_1(r,\lambda)| \leq K'\exp[M'(b-1)]$$

for all r in [1,b] and all λ in Δ. Consequently, $y_1(r,\lambda)$ is bounded on $[a,b]\times\Delta$; likewise $y_2(r,\lambda)$ is bounded on $[a,b]\times\Delta$.

Since $\rho_{11}(\lambda),\rho_{22}(\lambda)$ are functions of bounded variation on compact subintervals of $\mathbb{R}$, it now follows from (A.2.4) that $\left(\int_\Delta |y_s(r,\lambda)|^2\,d\tilde{\rho}(\lambda)\right)^{1/2}$ is a bounded function of r on [a,b]. Hence, since P(r) is integrable on [a,b] and $G(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$

$$\int_a^b\int_\Delta |y_s(r,\lambda)\,G(\lambda)\,P(r)|\,d\tilde{\rho}(\lambda)\,dr < \infty$$

by (A.2.3), and so, from (A.2.1) and (A.2.2),

$$\begin{aligned}\int_0^\infty y_\Delta(r)\,P(r)\,dr &= \int_\Delta\left(\int_a^b y_s(r,\lambda)\,P(r)\,dr\right)G(\lambda)\,d\tilde{\rho}(\lambda) \\ &= \int_\Delta Q(\lambda)\,G(\lambda)\,d\tilde{\rho}(\lambda)\end{aligned} \tag{A.2.6}$$

To show that $y_\Delta(r)$ is in $L_2(0,\infty)$, we first prove that $y_\Delta(r)$ is continuous on [a,b].

Since $y_1(r,\lambda)$ is bounded on $[a,b] \times \Delta$ we deduce from (A.2.5) that, if a < 1, then $y_1(r,\lambda)$ is a continuous function of λ for each fixed r in [a,1]. Similarly if b > 1, $y_1(r,\lambda)$ is a continuous function of λ for each fixed r in [1,b]. Hence $y_1(r,\lambda)$, and similarly $y_2(r,\lambda)$, is continuous on $[a,b] \times \Delta$.

Now, by (4.4.14), (4.4.15) and Lemma 4.8,

$$\begin{aligned} y_\Delta(r) &= \int_\Delta (y_1(r,\lambda)\,\chi_{\Delta\setminus E} + m_0(\lambda)\, y_2(r,\lambda)\,\chi_{\Delta\setminus E} \\ &\qquad + y_2(r,\lambda)\,\chi_{\Delta\cap E})\, G(\lambda)\, d\tilde{\rho}(\lambda) \\ &= \int_\Delta y_1(r,\lambda)\, d\int^\lambda G(\lambda)\, d\rho_{11}(\lambda) + \int_\Delta y_2(r,\lambda)\, d\int^\lambda G(\lambda)\, d\rho_{12}(\lambda) \\ &\qquad + \sum_{\{i\,:\,e_i \in \Delta\cap E\}} y_2(r,e_i)\, G(e_i)\, \mu_{22}(\{e_i\}) \end{aligned} \qquad (A.2.7)$$

We show that $\int^\lambda G(\lambda)\, d\rho_{ij}(\lambda)$ is a function of bounded variation on Δ for j = 1,2. If V_Δ denotes the total variation on Δ, then (see [HS], proof of Thm. 18.1),

$$\begin{aligned} V_\Delta\left(\int^\lambda G(\lambda)\, d\rho_{11}(\lambda)\right) &\leq \int_\Delta |G(\lambda)|\, d\rho_{11}(\lambda) \\ &\leq \left(\int_{-\infty}^{\infty} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_\Delta d\rho_{11}(\lambda)\right)^{1/2} \\ V_\Delta\left(\int^\lambda G(\lambda)\, d\rho_{12}(\lambda)\right) &\leq \int_\Delta |G(\lambda)\, m_0(\lambda)|\, d\rho_{11}(\lambda) \\ &\leq \left(\int_\Delta |G(\lambda)|^2\, d\rho_{11}(\lambda)\right)^{1/2} \left(\int_\Delta |m_0(\lambda)|^2\, d\rho_{11}(\lambda)\right)^{1/2} \\ &< \left(\int_{-\infty}^{\infty} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_\Delta d\rho_{22}(\lambda)\right)^{1/2} \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and Theorem 4.8. Since $G(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$ by hypothesis, the total variations above are finite, and hence the continuity of $y_1(r,\lambda)$ and $y_2(r,\lambda)$ on $[a,b] \times \Delta$ implies that the first two terms on the right hand side of (A.2.7) are continuous functions of r on [a,b]. (see [AP] Thm.7.38).

To see that

$$\sum_{\{i\,:\,e_i \in \Delta\cap E\}} y_2(r,e_i)\, G(e_i)\, \mu_{22}(\{e_i\}) \qquad (A.2.8)$$

is a continuous function of r on [a,b] even if the number of constituent terms is infinite, note that, using the continuity of $y_2(r,\lambda)$ on $[a,b] \times \Delta$ there exists K in $\mathbb{R}^+$ such that

$$\int_{\Delta \cap E} |y_2(r,\lambda)\, G(\lambda)|\, d\rho_{22}(\lambda) \leq K \left(\int_{-\infty}^{\infty} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_{\Delta \cap E} d\rho_{22}(\lambda)\right)^{1/2}$$

It follows that each sequence of partial sums associated with (A.2.8) is uniformly convergent on [a,b], so that (A.2.8) is a continuous function of r on [a,b].

Thus we have proved that $y_\Delta(r)$ is a continuous function of r on [a,b], and so $y_\Delta(r)\,\chi_{[a,b]}$ is in $L_2(0,\infty)$ for each finite Δ in $(-\infty,\infty)$ and each a, b in $\mathbb{R}^+$.

Now suppose $\Delta_1 \supset \Delta_2$, and set

$$P(r) = y_{\Delta_1}(r)\,\chi_{[a,b]} - y_{\Delta_2}(r)\,\chi_{[a,b]}$$

Using (A.2.6) and the Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^{\infty} (y_{\Delta_1}(r) - y_{\Delta_2}(r))\, P(r)\, dr &= \int_{\Delta_1 \setminus \Delta_2} Q(\lambda)\, G(\lambda)\, d\tilde{\rho}(\lambda)\\ &\leq \left(\int_{-\infty}^{\infty} |Q(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_{\Delta_1 \setminus \Delta_2} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2}\\ &= \|P(r)\| \left(\int_{\Delta_1 \setminus \Delta_2} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2}\end{aligned}$$

Substituting for P(r), this yields

$$\left(\int_a^b |y_{\Delta_1}(r) - y_{\Delta_2}(r)|^2\, dr\right)^{1/2} \leq \left(\int_{\Delta_1 \setminus \Delta_2} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2}$$

for each a,b in $\mathbb{R}^+$. This implies

$$\|y_{\Delta_1}(r) - y_{\Delta_2}(r)\| \leq \left(\int_{\Delta_1 \setminus \Delta_2} |G(\lambda)|^2\, d\tilde{\rho}(\lambda)\right)^{1/2} \qquad \text{(A.2.9)}$$

Setting $\Delta_1 = \Delta$, $\Delta_2 = \emptyset$ we see that $y_\Delta(r)$ is in $L_2(0,\infty)$ for each Δ in $(-\infty,\infty)$, since $G(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$ by assumption. It therefore follows from (A.2.9) that $\{y_\Delta(r)\}$ is Cauchy in $L_2(0,\infty)$ as $\Delta \to (-\infty,\infty)$, and so the lemma is proved.

Lemma A.2 shows that for each $G(\lambda)$ in $L_2^{\tilde{\rho}}(-\infty,\infty)$ there exists u(r) in $L_2(0,\infty)$ such that

$$u(r) = \underset{\omega \to \infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} y_s(r,\lambda)\, G(\lambda)\, d\tilde{\rho}(\lambda)$$

To complete the proof that $\tilde{S}$ is surjective, we need to show that $(\tilde{S}u)(\lambda) = G(\lambda)$. Define $\xi(\lambda) = G(\lambda) - (\tilde{S}u)(\lambda)$, where we note that $\xi(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$. Then it suffices to show that

$$\int_{-\infty}^{\infty} |\xi(\lambda)|^2 d\tilde{\rho}(\lambda) = 0 \tag{A.2.10}$$

Now from Theorem 4.9 and Lemma A.2,

$$\underset{\omega \to \infty}{\mathrm{l.i.m.}} \int_{-\omega}^{\omega} y_s(r,\lambda)\, \xi(\lambda)\, d\tilde{\rho}(\lambda) = 0 \tag{A.2.11}$$

where the integral converges in $L_2(0,\infty)$. Defining

$$\tau_\Delta(r) = \int_\Delta y_s(r,\lambda)\, \xi(\lambda)\, d\tilde{\rho}(\lambda)$$

and

$$R_\Delta(r,l) = \int_\Delta y_s(r,\lambda) \frac{\bar{\xi}(\lambda)}{\lambda - l}\, d\tilde{\rho}(\lambda) \tag{A.2.12}$$

for $r > 0$ and l in $\mathbb{C}\setminus\mathbb{R}$, we shall show that for each l in $\mathbb{C}\setminus\mathbb{R}$, $R_\Delta(r,l)$ converges in $L_2(0,\infty)$ to the zero function as $\Delta \to (-\infty,\infty)$, and deduce that for each finite Δ in $(-\infty,\infty)$, $\tau_\Delta(r) = 0$. We shall then be in a position to prove (A.2.10).

A.3 <u>Lemma</u>: If l is in $\mathbb{C}\setminus\mathbb{R}$, $R_\Delta(r,l)$ is in $L_2(0,\infty)$ for each Δ in $(-\infty,\infty)$, and $\{R_\Delta(r,l)\}$ converges in the mean to 0 as $\Delta \to (-\infty,\infty)$.

<u>Proof</u>:

Let $\mathring{R}_\Delta(r,l)$ denote $R_\Delta(r,l)\,\chi_{[a,b]}$ for some a, b in $\mathbb{R}^+$. Then for fixed Δ in $(-\infty,\infty)$ and l in $\mathbb{C}\setminus\mathbb{R}$, we have for each r in [a,b]

$$|\mathring{R}_\Delta(r,l)| \leq \int_\Delta \left| y_s(r,\lambda) \frac{\xi(\lambda)}{(\lambda - l)} \right| d\tilde{\rho}(\lambda)$$

$$\leq \frac{1}{|\operatorname{Im} l|} \left(\int_\Delta |y_s(r,\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_\Delta |\xi(\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2} \tag{A.3.1}$$

so that for each such Δ and l, $\dot{R}_\Delta(r,l)$ is a bounded function of r on [a,b] (cf. (A.2.4)) and hence is in $L_2(0,\infty)$.

Now let $\Delta_1 \supset \Delta_2$. Using the method of Lemma A.2 we obtain for each fixed l in $\mathbb{C}\setminus\mathbb{R}$

$$\left(\int_a^b |R_{\Delta_1}(r,l) - R_{\Delta_2}(r,l)|^2 dr\right)^{1/2} \leq \left(\int_{\Delta_1\setminus\Delta_2} \left|\frac{\xi(\lambda)}{(\lambda - l)}\right|^2 d\tilde{\rho}(\lambda)\right)^{1/2}$$

and

$$\left(\int_a^b |R_\Delta(r,l)|^2 dr\right)^{1/2} \leq \left(\int_\Delta \left|\frac{\xi(\lambda)}{(\lambda - l)}\right|^2 d\tilde{\rho}(\lambda)\right)^{1/2}$$

for every a,b in $\mathbb{R}^+$. Since $\xi(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$, we deduce that $R_\Delta(r,l)$ is in $L_2(0,\infty)$ for each Δ in $(-\infty,\infty)$, and $\{R_\Delta(r,l)\}$ converges in $L_2(0,\infty)$ as $\Delta \to (-\infty,\infty)$.

We now show that $\{R_\Delta(r,l)\}$ converges in the mean to 0 as $\Delta \to (-\infty,\infty)$. We shall prove that $R_\Delta(r,l)$ is in $\mathbf{D}(H)$ and that

$$(H - l)\, R_\Delta(r,l) = \tau_\Delta(r) \qquad (A.3.2)$$

for each l in $\mathbb{C}\setminus\mathbb{R}$ and each Δ in $(-\infty,\infty)$.

From (4.4.14), (4.4.15) and Lemma 4.8,

$$\begin{aligned} R_\Delta(r,l) = &\int_\Delta y_1(r,\lambda)\, d\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)}\, d\rho_{11}(\lambda) \\ &+ \int_\Delta y_2(r,\lambda)\, d\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)}\, d\rho_{12}(\lambda) \\ &+ \sum_{\{i\,:\, e_i \in \Delta\cap E\}} y_2(r,e_i)\, \frac{\xi(e_i)}{(e_i - l)}\, \mu_{22}(\{e_i\}) \qquad (A.3.3) \end{aligned}$$

(cf. (A.2.7)). As in the proof of Lemma A.2, $\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)}\, d\rho_{ij}(\lambda)$ is a function of bounded variation on Δ for j = 1,2, and $y_1(r,\lambda)$, $y_2(r,\lambda)$, $y_1'(r,\lambda)$, $y_2'(r,\lambda)$ are continuous on $[a,b]\times\Delta$ for each a,b in $\mathbb{R}^+$ and each finite Δ in $(-\infty,\infty)$. Moreover, the final sum on the right hand side of (A.3.3) is a continuous function of r on [a,b]. Hence we may differentiate (A.3.3) with respect to r, and equate the derivatives of the integrals with the integrals of the derivatives (see [AP] Thm. 7.40), to give

$$R'_\Delta(r,l) = \frac{\partial}{\partial r} R_\Delta(r,l) = \int_\Delta \frac{\partial y_s(r,\lambda)}{\partial r} \frac{\xi(\lambda)}{(\lambda - l)} \, d\tilde{\rho}(\lambda) \qquad \text{(A.3.4)}$$

We deduce that, if p is in [a,b], and if $y_i'(r,\lambda) = \frac{\partial}{\partial r} y_i(r,\lambda)$, $i = 1,2$,

$$\left.\frac{\partial^2}{\partial r^2} R_\Delta(r,l)\right|_{r=p} = \lim_{h\to 0} \frac{R'_\Delta(p+h,l) - R'_\Delta(p,l)}{h}$$

$$= \lim_{h\to 0} \int_\Delta \frac{[y_1'(p+h,\lambda) - y_1'(p,\lambda)]}{h} \, d\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)} \, d\rho_{11}(\lambda)$$

$$+ \lim_{h\to 0} \int_\Delta \frac{[y_2'(p+h,\lambda) - y_2'(p,\lambda)]}{h} \, d\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)} \, d\rho_{12}(\lambda)$$

$$+ \lim_{h\to 0} \int_{\Delta\cap E} \frac{[y_2'(p+h,\lambda) - y_2'(p,\lambda)]}{h} \, d\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)} \, d\rho_{22}(\lambda) \qquad \text{(A.3.5)}$$

Let us consider the first term on the right hand side of this equality. For each fixed p in [a,b] and each h > 0,

$$\frac{y_1'(p+h,\lambda) - y_1'(p,\lambda)}{h} \qquad \text{(A.3.6)}$$

is a continuous function of λ, and hence is integrable with respect to $\int^\lambda \frac{\xi(\lambda)}{(\lambda - l)} \, d\rho_{11}(\lambda)$ on Δ. Moreover, for each λ in Δ and each h < K, (A.3.6) is dominated by

$$\Gamma_p(\lambda,h) = \sup_{0<\tilde{h}\leq h} \frac{|y_1'(p+\tilde{h},\lambda) - y_1'(p,\lambda)|}{\tilde{h}}$$

which is a continuous function of λ and h on $(0,K] \times \Delta$. If we extend the domain of $\Gamma_p(\lambda,h)$ so that

$$\Gamma_p(\lambda,0) = \left.\frac{\partial^2}{\partial r^2} y_1(r,\lambda)\right|_{r=p}$$

then $\Gamma_p(\lambda,h)$ is continuous, and hence bounded, on the compact set $[0,K] \times \Delta$. A similar argument applies to the second and third terms on the right hand side of (A.3.5), so that, by the Lebesgue Dominated Convergence Theorem, we may take the limits under the integral signs to give

$$\frac{\partial^2}{\partial r^2} R_\Delta(r,l) = \int_\Delta \frac{\partial^2}{\partial r^2} y_s(r,\lambda) \frac{\xi(\lambda)}{(\lambda - l)} \, d\tilde{\rho}(\lambda) \qquad \text{(A.3.7)}$$

for all r in [a,b] and each l in $\mathbb{C} \setminus \mathbb{R}$. Since a,b in $\mathbb{R}^+$ are arbitrary, we deduce that (A.3.7) holds for all r in $(0,\infty)$ and each l in $\mathbb{C}\setminus\mathbb{R}$.

Hence

$$L R_\Delta(r,l) = -\frac{\partial^2}{\partial r^2} R_\Delta(r,l) + V(r) R_\Delta(r,l)$$
$$= \int_\Delta \left[-\frac{\partial^2}{\partial r^2} y_s(r,\lambda) + V(r) y_s(r,\lambda)\right] \frac{\xi(\lambda)}{(\lambda - l)} d\tilde{\rho}(\lambda)$$
$$= \int_\Delta \lambda\, y_s(r,\lambda) \frac{\xi(\lambda)}{(\lambda - l)} d\tilde{\rho}(\lambda)$$
$$= l R_\Delta(r,l) + \tau_\Delta(r) \tag{A.3.8}$$

for all r in $(0,\infty)$ and each l in $\mathbb{C}\setminus\mathbb{R}$ (Note that we have derived (A.3.8) without assuming V(r) to be continuous). To deduce (A.3.2), we need to show that $R_\Delta(r,l)$ is in $\mathfrak{D}(H)$.

We first show that if L is in the limit circle case at 0, then $R_\Delta(r,l)$ satisfies the same boundary condition at 0 as $y_s(r,\lambda)$.

If L is in the regular limit circle case at 0, $y_s(r,\lambda)$ satisfies a boundary condition of the form

$$\cos\alpha\; y_s(0,\lambda) - \sin\alpha\; y_s'(0,\lambda) = 0$$

for some α in $[0, 2\pi)$. (see (4.1.1)). In this case the definition (A.2.12) of $R_\Delta(r,l)$ may be extended to include r = 0, so that, by (A.3.4)

$$\cos\alpha\; R_\Delta(0,l) - \sin\alpha\; R'_\Delta(0,l) = 0$$

since $R'_\Delta(0,l) = \lim_{r\downarrow 0} R'_\Delta(r,l)$.

If L is in the singular limit circle case at 0, then $y_s(r,\lambda)$ satisfies a boundary condition of the form

$$\lim_{r\downarrow 0} W(y_s(r,\lambda),\, y_1(r,z_0) + \hat{m}(z_0) y_2(r,z_0)) = 0$$

where z_0 is in $\mathbb{C}\setminus\mathbb{R}$ and $\hat{m}(z_0)$ is some point on the limit circle associated with z_0. Clearly $y_1(r,z_0) + \hat{m}(z_0) y_2(r,z_0) = y_s(r,z_0)$, and, using the Schrödinger equation,

$$W(y_s(r,\lambda), y_s(r,z_0)) = (\lambda - z_0)\int_0^r y_s(r,\lambda)\, y_s(r,z_0)\, dr$$

Hence, by (A.2.12) and (A.3.4),

$$W(R_\Delta(r,\iota), y_s(r,z_0)) = \int_\Delta \frac{(\lambda - z_0)}{(\lambda - \iota)} \xi(\lambda) \int_0^r y_s(r,\lambda)\, y_s(r,z_0)\, dr\, d\tilde{\rho}(\lambda)$$

Since z_0 and ι in $\mathbb{C} \setminus \mathbb{R}$ are fixed and $\lambda \in \mathbb{R}$, there exists K_Δ in $\mathbb{R}^+$ such that $\left|\frac{\lambda - z_0}{\lambda - \iota}\right| < K_\Delta$ on Δ. Hence, if $\| y(r,\lambda)\|_r$ denotes $(\int_0^r |y(r,\lambda)|^2 dr)^{1/2}$,

$$|W(R_\Delta(r,\iota), y_s(r,z_0))| \leq K_\Delta \int_\Delta |\xi(\lambda)| \, \|y_s(r,\lambda)\|_r \, \|y_s(r,z_0)\|_r \, d\tilde{\rho}(\lambda)$$

$$\leq K_\Delta \|y_s(r,z_0)\|_r \left(\int_\Delta |\xi(\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2} \left(\int_\Delta \|y_s(r,\lambda)\|_r^2 \, d\tilde{\rho}(\lambda)\right)^{1/2} \tag{A.3.9}$$

by the Cauchy-Schwarz inequality.

Now by Minkowski's inequality, (4.4.14), (4.4.15) and Lemma 4.8,

$$\left(\int_\Delta \|y_s(r,\lambda)\|_r^2 \, d\tilde{\rho}(\lambda)\right)^{1/2} \leq \left(\int_\Delta \|y_s(r,\lambda)\|_1^2 \, d\tilde{\rho}(\lambda)\right)^{1/2}$$

$$= \left(\int_\Delta \|y_1(r,\lambda)\, \chi_{\Delta\setminus E} + m_0(\lambda)\, y_2(r,\lambda)\, \chi_{\Delta\setminus E} + y_2(r,\lambda)\, \chi_{\Delta\cap E}\|_1^2 \, d\tilde{\rho}(\lambda)\right)^{1/2}$$

$$\leq \Big(\int_{\Delta\setminus E} \|y_1(r,\lambda)\|_1^2 \, d\rho_{11}(\lambda) + 2\int_{\Delta\setminus E} \|y_1(r,\lambda)\|_1 \|y_2(r,\lambda)\|_1 \, d\rho_{12}^+(\lambda)$$

$$+ 2\int_{\Delta\setminus E} \|y_1(r,\lambda)\|_1 \|y_2(r,\lambda)\|_1 \, d\rho_{12}^-(\lambda) + \int_\Delta \|y_2(r,\lambda)\|_1^2 \, d\rho_{22}(\lambda)\Big)^{1/2} \tag{A.3.10}$$

It follows from Lemma A.1 that $\|y_1(r,\lambda)\|_1$ and $\|y_2(r,\lambda)\|_1$ are bounded functions of λ on Δ; therefore, since $\xi(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$, (A.3.9) and (A.3.10) together imply that C_Δ in $\mathbb{R}^+$ exists such that

$$|W(R_\Delta(r,\iota), y_s(r,z_0))| \leq C_\Delta \|y_s(r,z_0)\|_r$$

for all $r < 1$. Since $y_s(r,z_0)$ is in $L_2(0,1]$, it follows that

$$\lim_{r \downarrow 0} W(R_\Delta(r,\iota), y_s(r,z_0)) = 0$$

Thus for each ι in $\mathbb{C} \setminus \mathbb{R}$ and each finite Δ, $R_\Delta(r,\iota)$ satisfies the same boiundary condition at 0 as $y_s(r,\lambda)$ if L is in the limit circle case at 0. That is, if L is limit circle at 0, $R_\Delta(r,\iota)$ satisfies the boundary condition required of all elements of $\mathcal{D}(H)$.

To complete the proof that $R_\Delta(r,\iota)$ is in $\mathcal{D}(H)$, we show that $L R_\Delta(r,\iota)$ is in $L_2(0,\infty)$, and that $R_\Delta(r,\iota)$ and $R'_\Delta(r,\iota)$ are absolutely continuous functions of r on each compact subinterval [a,b] of $(0,\infty)$.

Evidently it may be shown that $\tau_\Delta(r)$ is in $L_2(0,\infty)$ by the method used at the beginning of this lemma to prove that $R_\Delta(r,l)$ is in $L_2(0,\infty)$ for each l in $\mathbb{C}\setminus\mathbb{R}$. That $LR_\Delta(r,l)$ is in $L_2(0,\infty)$ for each l in $\mathbb{C}\setminus\mathbb{R}$ then follows from (A.3.8).

By (A.3.8), $R_\Delta(r,l)$ is a solution of $(L-l)u(r,l)=\tau_\Delta(r)$. Therefore, applying the "variation of constants" formula ([CL] Ch.3, Thm.6.4), we have for $0<r\leq 1$,

$$\begin{aligned} R_\Delta(r,l) &= y_1(r,l)\int_\Delta \frac{\xi(\lambda)}{(\lambda-l)}\,d\rho_{11}(\lambda) + y_2(r,l)\int_\Delta \frac{\xi(\lambda)}{(\lambda-l)}\,d\rho_{12}(\lambda) \\ &\quad + y_2(r,l)\int_{\Delta\cap E}\frac{\xi(\lambda)}{(\lambda-l)}\,d\rho_{22}(\lambda) \\ &\quad + y_1(r,l)\int_r^1 y_2(v,l)\tau_\Delta(v)\,dv - y_2(r,l)\int_r^1 y_1(v,l)\tau_\Delta(v)\,dv \end{aligned}$$

where we have used (4.4.14), (4.4.15), Lemma 4.8 and (A.3.8); a similar formula holds for r > 1. The absolute continuity of $R_\Delta(r,l)$ on each compact subinterval of $\mathbb{R}^+$ follows from these formulae since $y_1(r,l)\tau_\Delta(r)$, $y_2(r,l)\tau_\Delta(r)$ are integrable on [a,b], and $y_1(r,l)$, $y_2(r,l)$ are absolutely continuous functions of r on a,b.

We have now proved that $R_\Delta(r,l)$ is in $\mathbb{D}(H)$ for each l in $\mathbb{C}\setminus\mathbb{R}$, and each finite Δ in $(-\infty,\infty)$.

It therefore follows from (A.3.2) that

$$\|R_\Delta(r,l)\| \leq \|(H-l)^{-1}\|\,\|\tau_\Delta(r)\| \tag{A.3.11}$$

for each l in $\mathbb{C}\setminus\mathbb{R}$ and each Δ in $(-\infty,\infty)$, where

$$\|(H-l)^{-1}\| = \sup_{\{f\in L_2(0,\infty)\,:\,\|f\|=1\}} \|(H-l)^{-1}f\| \leq \frac{1}{|\operatorname{Im} l|}$$

(see [HE] §24 Proof of Thm.3). Taking limits as $\Delta\to(-\infty,\infty)$ in (A.3.11) it follows from (A.2.11) that $R_\Delta(r,l)$ converges in the mean to zero as $\Delta\to(-\infty,\infty)$.

The proof of the lemma is now complete.

A.4 <u>Corollary</u>: For each finite Δ in $(-\infty,\infty)$ whose endpoints are points of continuity of $\tilde{\rho}(\lambda)$, $\tau_\Delta(r) = \int_\Delta y_s(r,\lambda)\xi(\lambda)\,d\tilde{\rho}(\lambda) = 0$

for all r in $(0,\infty)$.

Proof:

Define

$$F_t(\lambda) = \int_a^t y_s(r,\lambda)\, dr \qquad \text{(A.4.1)}$$

where $0 < a < t < \infty$ so that $\tilde{F}_t(\lambda) = (\tilde{S} f_t)(\lambda)$ (see Theorem 4.9), where

$$f_t(r) = \begin{cases} 1 & a \leq r \leq t \\ 0 & \text{otherwise} \end{cases}$$

Since $f_t(r)$ is in $L_2(0,\infty)$ for each a and t, $\tilde{F}_t(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$.

Now from the final inequality in (A.3.1), we see that $y_s(r,\lambda)\,\dfrac{\xi(\lambda)}{(\lambda - l)}$ is integrable on $[a,t] \times \Delta$ for each a,t, Δ. Hence, using (A.4.1) and (A.2.12),

$$\begin{aligned} \left| \int_\Delta \frac{\xi(\lambda)}{(\lambda - l)} \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda) \right| &= \left| \int_a^t \int_\Delta y_s(r,\lambda) \frac{\xi(\lambda)}{(\lambda - l)}\, d\tilde{\rho}(\lambda)\, dr \right| \\ &= \left| \int_a^t R_\Delta(r,l)\, dr \right| \\ &\leq (t-a)^{1/2} \, \| R_\Delta(r,l) \| \end{aligned}$$

Taking limits as $\Delta \to (-\infty,\infty)$, we deduce that

$$\int_{-\infty}^{\infty} \frac{\xi(\lambda)}{(\lambda - l)} \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda) = 0 \qquad \text{(A.4.2)}$$

We now prove that, if t > a,

$$\int_{-\infty}^{\infty} \xi(\lambda)\, \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda) = 0 \qquad \text{(A.4.3)}$$

for all ν, μ which are points of continuity of $\tilde{\rho}(\lambda)$.

Let x, y denote $\operatorname{Re} l$ and $\operatorname{Im} l$ respectively.

Using $\xi(\lambda)\,\tilde{F}_t(\lambda) \in L^{\tilde{\rho}}(-\infty,\infty)$ and the Lebesgue Dominated Convergence Theorem, we deduce from (A.4.2)

$$\begin{aligned} 0 &= \lim_{y \downarrow 0} \int_\nu^\mu \operatorname{Im} \int_{-\infty}^{\infty} \frac{\xi(\lambda)}{(\lambda - l)} \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda)\, dx \\ &= \lim_{y \downarrow 0} \int_\nu^\mu \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} \tilde{F}_t(\lambda)\, \xi(\lambda)\, d\tilde{\rho}(\lambda)\, dx \\ &= \lim_{y \downarrow 0} \int_{-\infty}^{\infty} \left[\tan^{-1}\left(\frac{\lambda - \mu}{y}\right) - \tan^{-1}\left(\frac{\lambda - \nu}{y}\right) \right] \xi(\lambda)\, \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda) \\ &= \int_{-\infty}^{\infty} \pi\, \chi_{[\nu,\mu]}\, \xi(\lambda)\, \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda) \end{aligned}$$

$$= \pi \int_{\nu}^{\mu} \xi(\lambda) \tilde{F}_t(\lambda)\, d\tilde{\rho}(\lambda)$$

for all ν and μ which are points of continuity of $\tilde{\rho}(\lambda)$, so that (A.4.3) is proved.

Using (A.4.1), (A.4.3), (4.4.14), (4.4.15) and Lemma 4.8, we see that

$$0 = \int_{\nu}^{\mu} \xi(\lambda) \int_a^t y_s(r,\lambda)\, dr\, d\tilde{\rho}(\lambda)$$

$$= \int_{[\nu,\mu]\setminus E} \int_a^t y_1(r,\lambda) dr\, d\int^{\lambda} \xi(\lambda)\, d\rho_{11}(\lambda)$$

$$+ \int_{[\nu,\mu]\setminus E} \int_a^t y_2(r,\lambda) dr\, d\int^{\lambda} \xi(\lambda)\, d\rho_{12}(\lambda)$$

$$+ \int_{[\nu,\mu]\cap E} \int_a^t y_2(r,\lambda) dr\, d\int^{\lambda} \xi(\lambda)\, d\rho_{22}(\lambda)$$

Since each integrand in the expanded expression is a continuous function of t and λ on [a,b] x $[\nu,\mu]$ for each finite b > a, we may (cf. proof of Lemma A.3) differentiate under the integral sign with respect to t to give

$$\tau_{\Delta}(t) = 0 \tag{A.4.4}$$

for every t in [a,b] and each λ-interval $\bar{\Delta}$ whose endpoints are points of continuity of $\tilde{\rho}(\lambda)$. The arbitrary choice of a,b implies that (A.4.4) is true for all t in $(0,\infty)$.

The corollary is now proved.

A.5 Proposition: The isometric Hilbert space isomorphism $\tilde{S}$ of Theorem 4.9 is surjective.

Proof:

On account of Lemma A.2, we need only prove (A.2.10).

Now $y_1(1,z) = 1$ for each z in $\mathbb{C}$ (see Ch.IV, §1) so, setting r = 1 in the result of Corollary A.4, we have

$$\int_{\Delta} \xi(\lambda)\, d\tilde{\rho}(\lambda) = 0$$

for each finite Δ in $(-\infty,\infty)$ whose endpoints are points of continuity of $\tilde{\rho}(\lambda)$.

Therefore if $\sum_{i=1}^{n} \alpha_i \chi_i$ is any step function such that the endpoints of each interval I_i are points of continuity of $\tilde{\rho}(\lambda)$ then

$$\int_{\Delta} \left(\sum_{i=1}^{n} \alpha_i \chi_i\right) \xi(\lambda)\, d\tilde{\rho}(\lambda) = \sum_{i=1}^{n} \alpha_i \int_{\Delta\cap I_i} \xi(\lambda)\, d\tilde{\rho}(\lambda) = 0$$

for each finite Δ in $(-\infty, \infty)$.

Let $\varepsilon > 0$ be given. Since $\xi(\lambda)$ is in $L_2^{\tilde{\rho}}(-\infty,\infty)$ and the step functions are dense in $L_2^{\tilde{\rho}}(-\infty,\infty)$ there exists a step function $\sum_{i=1}^{n} \alpha_i \chi_i$ such that

$$\left(\int_{-\infty}^{\infty} |\xi(\lambda) - \sum_{i=1}^{n} \alpha_i \chi_i|^2 d\tilde{\rho}(\lambda)\right)^{1/2} < \frac{\varepsilon}{\left(\int_{-\infty}^{\infty} |\xi(\lambda)|^2 d\tilde{\rho}(\lambda)\right)^{1/2}}$$

There is no loss of generality if we suppose the endpoints of each interval I_i occur at points of continuity of $\tilde{\rho}(\lambda)$ since the points of discontinuity of $\tilde{\rho}(\lambda)$ are, at most, countably infinite. Hence for each finite Δ in $(-\infty,\infty)$

$$\begin{aligned} &\left| \int_{\Delta} |\xi(\lambda)|^2 d\tilde{\rho}(\lambda) - \int_{\Delta} \left(\sum_{i=1}^{n} \alpha_i \chi_i\right) \xi(\lambda)\, d\tilde{\rho}(\lambda) \right| \\ &= \left| \int_{\Delta} \xi(\lambda) \left[\xi(\lambda) - \sum_{i=1}^{n} \alpha_i \chi_i \right] d\tilde{\rho}(\lambda) \right| \\ &\leq \left(\int_{\Delta} |\xi(\lambda)|^2 d\tilde{\rho}(\lambda) \right)^{1/2} \left(\int_{\Delta} \left|\sum_{i=1}^{n} \alpha_i \chi_i - \xi(\lambda)\right|^2 d\tilde{\rho}(\lambda)\right)^{1/2} < \varepsilon \end{aligned}$$

from which may be deduced by (A.5.1) and the arbitrariness of ε

$$\int_{\Delta} |\xi(\lambda)|^2 d\tilde{\rho}(\lambda) = 0$$

Since $\mathbb{R}$ may be expressed as a disjoint union of finite intervals Δ, we deduce (A.2.10), and the proposition is proved.

Proposition A.5 completes the proof of Theorem 4.9.

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