THE UNIVERSITY OF HULL

Compactifications of a Uniform Space and the *LUC*-Compactification of the Real Numbers In terms of the Concept of Near Ultrafilters

being a Thesis submitted for the Degree of

Doctor of Philosophy

in the University of Hull

by

Mahmut Koçak B.Sc. (The University of Anatolia ,Turkey),

M.Sc. (The University of Leeds)

September 1994

To my Dear Family

CONTENTS

Summary of Thesis submitted for PhD degree

by MAHMUT KOÇAK

on

COMPACTIFICATIONS OF A UNIFORM SPACE AND THE LUC-COMPACTIFICATION OF THE REAL NUMBERS IN TERMS OF THE CONCEPT OF NEAR ULTRAFILTERS

Compactifications of topological spaces and semigroup compactifications of topological semigroups have been studied since 1930's. This thesis contributes to the new concept of near ultrafilters on a uniform space X, the compactification $\tilde{X}$ of X and the empactification $\tilde{\mathbb{R}}$ of the real numbers $\mathbb{R}$ in terms of near ultrafilters.

The concept of near ultrafilters on a uniform space X is introduced, some of their properties are investigated and the set of all near ultrafilters is made into a topological space $\tilde{X}$. It is shown that this space is a compact Hausdorff space containing X as a dense subspace. Furthermore, it is proved that any uniformly continuous function from X into a uniform space Y has a continuous extension from $\tilde{X}$ to $\tilde{Y}$.

The compactification $\tilde{\mathbb{R}}$, the set of all near ultrafilters on $\mathbb{R}$ with respect to the usual uniformity on $\mathbb{R}$, is constructed and it is shown that the semigroup operation + extends to a semigroup operation + on $\tilde{\mathbb{R}}$ which makes $\tilde{\mathbb{R}}$ into

a compact right topological semigroup $(\tilde{\mathbb{R}},+)$. Many topological and algebraic properties of the compactification $(\varphi,\tilde{\mathbb{R}})$ including the fact that $(\varphi,\tilde{\mathbb{R}})$ is the maximal semigroup compactification of $\mathbb{R}$ among those having the property that the mapping $(x,y)\longrightarrow\varphi(x)y:\mathbb{R}\times\tilde{\mathbb{R}}\longrightarrow\tilde{\mathbb{R}}$ is jointly continuous. Therefore, it is the *LUC*-compactification of $\mathbb{R}$.

Non-homogeneity of $\tilde{\mathbb{R}}$ is proved and the Rudin-Keisler and the Rudin-Frolik orders are defined on $\tilde{\mathbb{R}}$, some of the results concerning with them are obtained.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr.Dona Strauss for her help, her extraordinary patience and guidance during the preparation of this thesis. I would also like to thank all the members of the Department of Pure Mathematics for their hospitality and generosity shown to me during my study in the University of Hull. Finally, my thanks also to the Turkish Government providing me with the Research Fellowship which has enabled me to complete this thesis.

NOTATION

Symbol	Meaning
CHAPTER I	
(Y,e)	Compactification of a topological space.
$C(X)$	The set of all continuous real-valued (complex-valued) functions on X.
$C^*(X)$	The set of all continuous real-valued (complex-valued) bounded functions on X.
$\mathcal{F}$	A subset of C(X)
$\prod_{\alpha\in I} X_\alpha$	The product of family $\{X_\alpha\}_{\alpha\in I}$ of topological spaces.
$e_{\mathcal{F}}$	The evaluation mapping determined by $\mathcal{F}$.
f^{-1}	Inverse of f.
$x\in F$	x is an element of F.

Symbol	Meaning
$x \notin F$	x is not an element of F.
$Cl_X A = \bar{A}$	Closure of A in X with respect to the topology on X.
$X_{\mathcal{F}}$	The compactification of X determined by $\mathcal{F}$.
Z(X)	The set of all zero sets of X.
βX	The Stone-Čech compactification of X.
$X^* = \beta X \setminus X$	The remainder of X.
f^β	The Stone extension of f.
$\mathcal{W}(X)$	The Wallman compactification of X with respect to Wallman base W.
c	Cardinal number of the continuum.
$\aleph_0$	Cardinal number of a countable infinite set.

Symbol	Meaning
λ_x	Left translation by x.
ρ_x	Right translation by x.
$\Lambda = \Lambda(X)$	Topological centre of a compact right topological semigroup.
$K = K(X)$	Minimal ideal of a semigroup X.
$E(X)$	The set of idempotents of a compact right topological semigroup.
$M(F)$	The set of all means on F.
$MM(F)$	The set of all multiplicative means on F (the spectrum of F).
e^*	The adjoint of e.
F^*	The dual space of F.
$\sigma(F^*, F)$	Weak* topology on F^* induced by F.

Symbol	Meaning
$L_x f$	The left translates of f by x.
$R_x f$	The right translates of f by x.
(e,Y)	Semigroup compactification of a semitopological semigroup X.
$LUC(X)=LUC$	The set of left continuous functions on X.
X^{LUC}	The LUC-Compactification of X.

Chapter II

Symbol	Meaning
$(X,\mathcal{U})$	Uniform space with uniformity $\mathcal{U}$.
$\tau_{\mathcal{U}}$	Topology generated by the uniformity $\mathcal{U}$.
$\mathcal{C}$	The set of all near ultrafilters on a uniform space X.
$Y^*=X\setminus Y$	The complement of Y.
$\hat{x}$	The near ultrafilter $\{Y\subseteq X : x\in\bar{Y}\}$ on X.

Symbol	Meaning
$\tilde{X}$	The compactification of uniform space $(X,\mathcal{U})$.
$\tilde{f}$	The extension of a uniformly continuous function f on X to $\tilde{X}$.

Chapter III

Symbol	Meaning
$\mathcal{B}(0)$	The set of (symmetric) neighbourhoods of 0 in $\mathbb{R}$.
$d(x,y)$	The distance between x and y.
$\tilde{\mathbb{R}}$	The compactification of $\mathbb{R}$ with respect to the usual uniformity on $\mathbb{R}$.
$\rho\mathbb{R}$	$\tilde{\mathbb{R}}\backslash\mathbb{R}$.
$\tilde{G}$	Compactification of a topological group with respect to the right uniformity.

Chapter IV

Symbol	Meaning
$\mathbb{R}^n$	n-dimensional Euclidean space

Symbol	Meaning
$\tilde{\mathbb{R}}^n$	Compactification of $\mathbb{R}^n$

Chapter V

Symbol	Meaning
$\approx$	Type equivalent relation on βX and uniform type equivalence relation on $\tilde{\mathbb{R}}$.
$t(\eta)$	Type of η on βX and uniform type on $\tilde{\mathbb{R}}$.
$T(\beta X)$	The set of type equivalence classes on βX.
$T(\tilde{\mathbb{R}})$	The set of uniform type equivalence on $\tilde{\mathbb{R}}$.
$\equiv$	An equivalence relation on $\tilde{\mathbb{R}}$.
$\cong$	An equivalence relation on $\tilde{\mathbb{R}}$.
$t_1(\eta)$	Type of η in $\tilde{\mathbb{R}}$ with respect to $\cong$.
$\lvert A\rvert$	Cardinality of set A.

Symbol	Meaning
$T_1(\tilde{\mathbb{R}})$	The set of equivalent classes with respect to $\cong$.
$\leq$	The Rudin-Keisler pre-order (partial order) on βX $(T(\beta X))$ and the Rudin-Keisler pre-order on $\tilde{\mathbb{R}}$.
$\eta<\xi$	$\eta\leq\xi$ and η and ξ are not type equivalent on βX.
$\delta<\tau$	$\delta\leq\tau$ and $\delta\neq\tau$ on $T(\beta X)$.
$\underline{[}$	The Rudin-Frolik pre-order (partial order) on βX $(T(\beta X))$ and the Rudin-Frolik pre-order (partial order) on $\tilde{\mathbb{R}}$ $(T_1(\tilde{\mathbb{R}}))$.
$\eta[\xi$	$\eta\underline{[}\xi$ and η and ξ are not uniform type equivalent in $\tilde{\mathbb{R}}$.
$\delta[\tau$	$\delta\underline{[}\tau$ and $\delta\neq\tau$ on $T_1(\tilde{\mathbb{R}})$.

CHAPTER I

INTRODUCTION AND BACKGROUND

Section 1.1. Introduction.

A compactification of a topological space X is a compact space K that contains a dense homeomorphic image of X; and a semigroup compactification of a semigroup X which is also a topological space is a compact right topological semigroup K that contains a dense continuous homomorphic image of X in the topological centre of K. These compactifications have long been a major area of study in general topology.

In 1930, Tychonoff [60] discovered that completely regular (Hausdorff) spaces are precisely those topological spaces which can be embedded in a compact Hausdorff space. This was the beginning of the general study of compactifications of a topological space, since one can obtain a compactification of a completely regular space by embedding it in a compact Hausdorff space and taking its closure.

Compactifications of a uniform space can be obtained by using the same methods to obtain compactifications of a completely regular space because of the fact that every uniform space is a completely regular space and, as proved by Tychonoff [60], every completely regular space is a dense subspace of a compact space.

Compactifications of a completely regular space can be obtained in a variety of ways. One of the ways is the use of the subset F of $C^*(X)$ (the set of all bounded continuous real-valued functions on X) which separates points. This method was first used by Tychonoff [60] to get a Hausdorff compactification of a completely regular space X by using $C^*(X)$. This technique was studied extensively by Čech [11] and much work was done to improve this idea by Stone [60], Hewitt [25] and many authors. Now it is a well known fact that every compactification K of a completely regular space X can be described completely by the C^*-algebras of all continuous real-valued functions on X which can be extended over K, and many properties of the compactification K can be established more easily with the help of C^*-algebras.

Another way of obtaining a compactification of a completely regular space X is by the use of filters and ultrafilters on a non-empty collection of subsets of X. In 1939, Wallman [63] constructed a T_1-compactification of a T_1-space using the lattice of the closed sets. This construction was extended by Banaschewski [1,2], Frink [18], Sanin [52,53]. In 1948, for a given uniform space X, Samuel [54] defined an equivalence relation on the space of ultrafilters and obtained a compactification of X in terms of equivalence classes.

Like compactifications of a completely regular space, semigroup compactifications of a topological semigroup can be obtained in different ways. One of them is the use of the operator theory. This method was used by de Leeuw and Glicksberg [22,23] to construct the almost periodic and weakly almost periodic compactifications of any semigroup with identity.

The second method is the use of the spectrum of C^*-algebras of functions to construct compactifications. In this method, the compactification of a semitopological semigroup X appears as the spectra of certain C^*-algebras of functions on X. This method was used by Loomis [43] to obtain the almost periodic compactifications of a topological group.

One of the important compactification of a semitopological semigroup X is the *LUC*-compactification (φ, X^{LUC}) which is the spectrum of the C^*-algebra $\{f \in C(X) : x \longrightarrow L_x f : X \longrightarrow C(X)$ is norm continuous$\}$ of left uniformly continuous functions on X. It is a well known fact that (φ, X^{LUC}) has a natural semigroup multiplication $(x,y) \longrightarrow xy$ which is continuous in x for fixed y. This compactification is maximal among those having the property that the mapping $(x,y) \longrightarrow \varphi(x)y : X \times X^{LUC} \longrightarrow X^{LUC}$ is jointly continuous, where φ is the continuous homomorphism from X into X^{LUC} with $\varphi(X)$ dense in X^{LUC}.

Additional information about the compactifications of semitopological semigroups and topological spaces can be found in the references.

This thesis contains five chapters. The first chapter is on general information about various kinds of compactifications of completely regular topological spaces and compactifications of topological semigroups.

The second chapter introduces the concept of uniform spaces and their compactifications in terms of the new concept of near ultrafilters.

The third chapter is about the compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$ with respect to the usual uniformity on $\mathbb{R}$ and investigates its topological properties.

The fourth chapter is about the extension of a semigroup operation on a semigroup X which is also a uniform space to a semigroup operation on $\tilde{X}$ and the algebraic properties of the compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$.

Chapter five is about the non-homogeneity of $\tilde{\mathbb{R}}$ and the Rudin-Keisler, and the Rudin-Frolik orders on the compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$.

The first chapter contains two sections. In the first one, we shall introduce the concept of compactification of a completely regular topological space and some of the techniques to obtain compactifications of such spaces. In the second section, we will give some basic information about semigroups, topological semigroups and one of the way to produce a compactification of a topological semigroup X as a spectrum of some certain subalgebras of C(X).

Chapter two is divided into three sections. The first section is about uniform spaces and some basic properties of a uniform space. In the second section, we will introduce the new concept of near ultrafilters on a uniform space and investigate their properties. In section three, we will give the construction of a compactification $\tilde{X}$ of a uniform space X by the use of near ultrafilters on X.

Chapter three is on the compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$ with respect to the usual uniformity on $\mathbb{R}$ and we will investigate the topological properties of $\tilde{\mathbb{R}}$.

Chapter four contains two sections the first one is about the extension of a semigroup operation on a uniform topological semigroup X to a semigroup operation on $\tilde{X}$. In section two, we will investigate the algebraic properties of the semigroup compactification $\bar{\mathbb{R}}$ of $\mathbb{R}$ and show that this is the LUC-Compactification of $\mathbb{R}$.

Chapter five is divided into two sections, in the first section we will prove that $\tilde{\mathbb{R}}$ is not homogeneous. The second section is about the Rudin-Keisler and the Rudin-Frolik orders on $\tilde{\mathbb{R}}$.

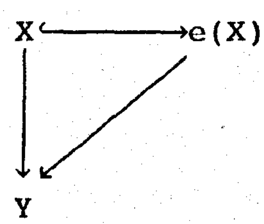
Section 1.2 Background

In this section, we shall first give the general construction for the compactification of a completely regular space. Then we will introduce the Stone-Čech compactification and Wallman's compactification.

1.2.1. Compactifications-Constructions

In this thesis all topological spaces are assumed to be completely regular Hausdorff spaces, unless specifically stated otherwise.

We mean by a compactification of a topological space X is a pair (Y,e) such that Y is a compact Hausdorff space and e is an embedding from X into Y with e(X) dense in Y



If (Y_1,e_1) and (Y_2,e_2) be two distinct compactifications of X and, if there is a continuous function f from Y_1 to Y_2 for which $f \circ e_1 = e_2$, then we write $Y_1 \geq Y_2$. We say Y_1 is equivalent to Y_2 if there is a homeomorphism h from Y_1 onto Y_2 such that $h \circ e_1 = e_2$ and is the identity on X. According to proposition g of [48, chap.4] Y_1 is equivalent to Y_2 if and only if $Y_1 \geq Y_2$ and $Y_2 \geq Y_1$.

In future we shall refer to Y itself as a compactification of X and simply regard X as embedded in Y as a dense subspace.

General Construction of a Compactification of X

Let F be a subset of $C^*(X)$, the set of all continuous real-valued bounded functions on X, and let I_f be a closed interval in $\mathbb{R}$ containing f(X) for each $f \in F$. Then the mapping e_F from X into $\prod_{f \in F} I_f$ defined by $\prod_f \circ e_F = f$ is called the evaluation mapping determined by F, where $\prod_f$ is the projection mapping from $\prod_{f \in F} I_f$ into I_f.

It is obvious from the definition of e_F that e_F is continuous, since each $f \in F$ is continuous.

A subset F of $C^*(X)$ is said to separate points and closed sets of X if for each $x\in X$ and every closed subset F of X with $x\notin F$ there is $f\in\mathcal{F}$ such that $f(x)\notin Cl_{\mathbb{R}}f(F)$.

1.2.1.1.Theorem [12]. Suppose that $\mathcal{F}\subseteq C^{*}(X)$ separates points and closed subsets of X, then the evaluation mapping $e_{\mathcal{F}}$ is an embedding of X onto $e_{\mathcal{F}}(X)$.

Proof.To see that $e_{\mathcal{F}}$ is an embedding, we will show that $e_{\mathcal{F}}$ is a one to one mapping.Let $x,y\in X$ with $x\neq y$,then since X is a completely regular Hausdorff space $\{y\}$ is a closed set, hence there exists $f\in\mathcal{F}$ such that $f(x)\notin Cl_{I_f}f(\{y\})$ which implies that $f(x)\neq f(y)$, so $\prod_f\circ e_{\mathcal{F}}(x)\neq\prod_f\circ e_{\mathcal{F}}(y)$.Therefore, $e_{\mathcal{F}}(x)\neq e_{\mathcal{F}}(y)$.

Let U be an open set in X and $e_{\mathcal{F}}(x)$ be an arbitrary point in $e_{\mathcal{F}}(U)$.Since $\mathcal{F}$ separates points of X and closed sets,there is a function $f\in\mathcal{F}$ such that $f(x)\notin Cl_{I_f}f(X\setminus U)$.Let $V=\prod_f^{-1}[I_f\setminus Cl_{I_f}f(X\setminus U)]$.Clearly,$e_{\mathcal{F}}(x)\in V$. We claim that $y\in U$ whenever $e_{\mathcal{F}}(y)\in V$.To see this suppose that $e_{\mathcal{F}}(y)\in V$ but $y\notin U$, then $f(y)\in f(X\setminus U)\subseteq Cl_{I_f}f(X\setminus U)$.However, $e_{\mathcal{F}}(y)\in V$ which implies that $f(y)\in\prod_f\circ e_{\mathcal{F}}(y)\in I_f\setminus Cl_{I_f}f(X\setminus U)$,so $f(y)\notin Cl_{I_f}f(X\setminus U)$ which is a contradiction. Thus $e_{\mathcal{F}}(y)\in e_{\mathcal{F}}(U)$ if $e_{\mathcal{F}}(y)\in V$. Hence we have that $V\cap e_{\mathcal{F}}(X)$ is a neighbourhood of $e_{\mathcal{F}}(x)$ in $e_{\mathcal{F}}(U)$.Since $e_{\mathcal{F}}(x)$ was an arbitrary point of $e_{\mathcal{F}}(U)$,$e_{\mathcal{F}}(U)$ is open∎

Let $Cl_{\prod_{f\in \mathcal{F}} I_f} e_{\mathcal{F}}(X)=X_{\mathcal{F}}$. Then $X_{\mathcal{F}}$ is a compactification of X.

1.2.1.2.Theorem [12]. Let Y be a compactification of X. Then there is a subset $\mathcal{F}$ of $C^*(X)$ separating points of X and closed sets for which $X_{\mathcal{F}}$ is homeomorphic to Y.

Proof. Let e be the embedding of X into Y with $Cl_Y e(X)=Y$, and let $\mathcal{F}=\{f\in C^*(X)\,|\,\text{there is a continuous map } f_Y:Y\longrightarrow\mathbb{R} \text{ with } f_Y\circ e=f\}$.

To see that $\mathcal{F}$ separates points and closed sets, assume that F is a closed set in X and x be any point with $x\in X\setminus F$. There is a closed set $D\subseteq Y$ such that $D\cap X=F$. Hence $x\notin D$. There is a function $f_Y:Y\longrightarrow\mathbb{R}$ such that $f_Y(x)\notin Cl_{\mathbb{R}}f_Y(D)$. Let $f=f_Y\circ e$, then $f\in C^*(X)$ and $f(x)\notin Cl_{\mathbb{R}}f(F)$.Hence $f\in\mathcal{F}$. By previous theorem $X_{\mathcal{F}}$ is a compactification of X.

We define $h:Y\longrightarrow \prod_{f\in\mathcal{F}} I_f$ as follow

$$[h(y)](f)=f_Y(y).$$

Then h is continuous,since $\prod_f \circ h=f_Y$ and each f_Y is continuous. Also $[h(e(x))]=f_Y(e(x))=f(x)$. This implies that h(Y) is a compact subset of $\prod_{f\in\mathcal{F}} I_f$ which contains $e_{\mathcal{F}}(X)$. Thus $h(Y)=X_{\mathcal{F}}$. If h(x)=h(y) then (h(x))(f)=f(x)=(h(y))(f)=f(y) for all $f \in \mathcal{F}$. Since $\mathcal{F}$ separates points, it must follow that x=y;i.e, h is one to one.Therefore,h is a homeomorphism from X onto $X_{\mathcal{F}}$, since any one to one continuous mapping from a compact space onto a Hausdorff space is homeomorphism∎

If $\mathcal{F}=C^{*}(X)$, then $X_{\mathcal{F}}$ is called the Stone-Čech compactification of X.

We will give the construction of the Stone-Čech compactification,and Wallman's compactification of X in terms of the $\mathcal{W}$-ultrafilters on X.

We first introduce the concept of $\mathcal{W}$-ultrafilters on X.

Let $\mathcal{W}$ be a non- empty collection of subsets of X.Then a non-empty subset η of $\mathcal{W}$ is called a $\mathcal{W}$-filter on X if the following conditions hold:

a) $\emptyset$ does not belong to η.

b) For F and G in η, $F \cap G \in \eta$.

c) For any H in $\mathcal{W}$ and $G \in \eta, G \subseteq H$ implies that $H \in \eta$.

A $\mathcal{W}$-filter η is said to be a $\mathcal{W}$-ultrafilter if η is not contained in any other $\mathcal{W}$-filter,i.e, if ξ is a $\mathcal{W}$-filter with $\eta\subseteq\xi$, then $\eta=\xi$.

If $\mathcal{W}$ is the collection Z(X) of all zero-sets in X, then a $\mathcal{W}$-ultrafilter is called a z-ultrafilter or zero-set ultrafilter.

The Stone-Čech Compactification:

Let βX be the set of all z-ultrafilters on X with the topology determined by the subbase

$$\mathcal{B}=\{\{\eta\in\beta X\,|\,A\notin\eta\}A\in Z(X)\}$$

for the closed sets in βX.

For each $x\in X$, the collection $\{A\in Z(X)\,|\,x\in A\}$ is a z-ultrafilter.

Let e be the mapping from X into βX defined by the rule

$$e(x)=\{A\in Z(X)\,|\,x\in A\}.$$

By Lemma 2.4 of [13], βX is a compact Hausdorff space and e is an embedding of X into βX with e(X) dense in βX and it is the Stone-Čech compactification of X.

The mapping e is called the canonical embedding of X into βX.

According to theorem 6.5 of [20] the space βX is characterized by the following three properties:

i) Each continuous mapping f from X into a compact Hausdorff space Y has a unique extension to a continuous mapping $g:\beta X \longrightarrow Y$ such that $f=g|_X$.

ii) If Z is a compactification of X having property (i), Z is homeomorphic to βX.

iii) βX is the largest compactification of X in the sense that any other compactification of X is a quotient space of X.

The continuous extension of a function f to βX is called the Stone extension of f and denoted by f^{β}.

Wallman's Compactification.

A Wallman base is a collection $\mathcal{W}$ of subsets of X with the following properties:

a) $A\cup B\in W$ and $A\cap B\in W$ whenever $A,B\in W$.

b) $\emptyset, X\in W$.

c) W is a closed base for X.

d) If $A\in W$ and $x\notin A$, there is a $B\in W$ such that $x\in B$ and $A\cap B=\emptyset$.

e) If $A,B\in W$ are such that $A\cap B=\emptyset$ then there are $C,D\in W$ with $A\subset X\setminus C$, $B\subset X\setminus D$ and $(X\setminus C)\cap(X\setminus D)=\emptyset$.

Let W be a Wallman base and let $W(X)$ be the set of all W-ultrafilters on X. Then $W(X)$ is made into a topological space by taking the collection

$$\{\{Z^{W} \mid Z\in\xi\}, Z\in W\}$$

as a base for the closed sets in $W(X)$, where $Z^{W}=\{\eta\in W(X) \mid Z\in\eta\}$.

Let e be the mapping from X into $W(X)$ defined by the rule

$$e(x)=\{A\in W \mid x\in A\}.$$

By section 9 of [64] , $W(X)$ is a compact Hausdorff space and e is an embedding of X into $W(X)$ with e(X) dense in $W(X)$.

A compactification Y is said to be of Wallman type if there is a Wallman base W on X which generates Y, that is, $W(X)=Y$.

Clearly, the Stone-Čech compactification βX of X is a Wallman type compactification,since the set Z(X) of all zero-sets of a completely regular space X is a Wallman base by theorem h of [48,chap.4] and generates βX.

By corollary 2 of [61],for each cardinality α such that $2^{\alpha}\geq\chi_2$, there is a compactification of a discrete space of cardinality α which is not of Wallman type.But if the continuum hypothesis holds,that is, $2^{\chi_0}=\chi_1$, then by theorem 2 of [3] every compactification of every separable completely regular Hausdorff space is of Wallman type.

As a result of the above discussion,if the cotinuum hypothesis holds,every compactification of the real line $\mathbb{R}$ is of Wallman type since $\mathbb{R}$ is a separable completely regular space.

Section 1.2.2 Semigroups and Compactifications of semitopological semigroups

In this section we will give some basic definitions and well-known facts about semigroups,compact right topological semigroups and semigroup compactifications.

We will begin this section by giving some definitions.

A semigroup is a non-empty set X together with an associative binary operation $(x,y)\longmapsto xy:X\times X\longrightarrow X$, called multiplication.

X is said to be commutative if xy=yx for each $x,y\in X$.

If X is a semigroup, then for each $x\in X$, the mappings

$$\lambda_x:X\longrightarrow X,\ \lambda_x(y)=xy,$$

and

$$\rho_x:X\longrightarrow X,\rho_x(y)=yx$$

are called,respectively,left and right multiplications or left translations and right translations by x.

From now on X denotes a semigroup.

Let A be a non-empty subset of X, then A is said to be:

1) A subsemigroup of X if $AA \subseteq A$.

2) A right ideal of X if $AX \subseteq A$.

3) A left ideal of X if $XA \subseteq A$.

4) A (two-sided) ideal of X if it is both a right and left ideal of X.

If $A \neq X$ in any of these definitions then A is said to be proper.

A left (respectively, right ideal, ideal) of X said to be a minimal left (respectively, right ideal, ideal) of X if it properly contains no left ideal (respectively, right ideal, ideal) of X.

X is called left simple (respectively, right simple) if it has no proper left (respectively, right) ideals.

Let $e \in X$, then e is said to be

5) An idempotent if $e^2=e$.

6) A left identity if $ex=x$ for all $x \in X$.

7) A right identity if $xe=x$ for all $x \in X$.

8) An identity if e is both a left and right identity.

9) A left zero if $ex=e$ for all $x \in X$.

10) A right zero if $xe=e$ for all $x \in X$.

11) A zero element if e is both a left and right zero.

12) A right (left) zero semigroup is one consisting entirely of right (left) zeros.

Let $x \in X$, then x is said to be

13) Left cancellative if $xz=xy$ if and only if $z=y$.

14) Right cancellative if $zx=yx$ if and only if $z=y$.

15) Cancellative if x is both left and right cancellative.

16) If every element of X is left (right) cancellative then X is called a left (right) cancellative semigroup. X is said to be cancellative if X is both left and right cancellative.

Let X be a semigroup with a Hausdorff topology. Then X is called

a) A right topological semigroup if for each $x \in X$, the mapping

$$\rho_x : X \longrightarrow X, \ \rho_x(y) = yx$$

is continuous.

b) A left topological semigroup if for each $x \in X$, the mapping

$$\lambda_x : X \longrightarrow X, \lambda_x(y) = xy$$

is continuous.

c) A semitopological semigroup if the both maps $\lambda_x : X \longrightarrow X$ and $\rho_x : X \longrightarrow X$ are continuous.

d) A topological semigroup if the multiplication $(x,y) \longmapsto xy : X \times X \longrightarrow X$ is (jointly) continuous.

If X is a right topological semigroup, then the set $\Lambda=\Lambda(X)=\{x\in X \mid \lambda_x:X\longrightarrow X$ is continuous$\}$ is called the topological centre of X.

A homomorphism from a semigroup X_1 into a semigroup X_2 is a mapping $\psi:X_1\longrightarrow X_2$ such that

$$\psi(xy)=\psi(x)\psi(y)$$

for each $x,y\in X_1$. ψ is called an isomorphism if ψ is one to one and onto.If X_1,X_2 are also topological spaces and ψ is a homeomorphism then ψ is called a topological isomorphism, in this case, X_1 and X_2 are said to be topologically isomorphic.

1.2.2.1.Proposition [5]. If the semigroup X contains a minimal right ideal,then it contains a minimum ideal K=K(X) which is the union of all the minimal right ideal of X.

The set of all idempotents of a compact right topological semigroup X is denoted by E(X).

Now we state some important properties of compact right topological semigroups.

1.2.2.2.Proposition [5]. Let X be a compact right topological semigroup. Then

i) Every left ideal of X contains a minimal left ideal. The minimal left ideals of X are closed.

ii) X has a smallest two-sided ideal $K=K(X)$.

iii) K contains idempotents and for an idempotent $e\in X$, the following are equivalent:

a) $e\in K$.

b) $K=XeX$.

c) Xe is a minimal left ideal.

d) eX is a minimal right ideal.

e) eXe is a subgroup of X.

f) Every minimal left ideal is of the form Xe for some idempotent $e\in K$; every minimal right ideal is of the form eX for some idempotent $e\in K$.

g) $$\begin{aligned} K &= \bigcup\{eXe \mid e\in E(K)\} \\ &= \bigcup\{eX \mid e\in E(K)\} \\ &= \bigcup\{Xe \mid e\in E(K)\}. \end{aligned}$$

Note that every minimal right ideal and every minimal left ideal is contained in K.

1.2.2.3.Theorem [5]. Let X be a compact right topological semigroup.Then

i) Every right ideal contains a minimal right ideal.

ii) Every closed right ideal contains a minimal closed right ideal.

1.2.2.4.Proposition [6]. The topological centre $\Lambda(X)=\{x\in X \mid \lambda_x : X\longrightarrow X$ is continuous$\}$ of a compact right topological semigroup is void or a subsemigroup of X. If X is a group , then Λ is a subgroup of X.

Now we will give some definitions before we introduce compactifications of a semitopological semigroup.

Let X be a topological space and let F be a conjugate closed,norm closed linear subspace of C(X) containing the constant function 1.Then a mean μ is a member of F^*,the dual space of F, such that $\mu(1)=\|\mu\|=1$. The set of all means on F is denoted by $M(F)$. If F is closed under pointwise multiplication then a mean μ is called multiplicative if $\mu(fg)=\mu(f)\mu(g)$ for every $f,g\in F$. The set of all multiplicative means on F is denoted by $MM(F)$ and is called the spectrum of F.

For each $x\in X$, the mean $e(x)$ defined by $e(x)(f)=f(x)$, $f\in F$ is called the evaluation at x,and the mapping $e:X\longrightarrow M(F)$ is called the evaluation mapping.

By proposition 2.5 of [5], $M(F)$ is convex and $\sigma(F^*,F)$ compact and the evaluation mapping $e:X\longrightarrow M(F)$ is $\sigma(F^*,F)$ continuous. Furthermore, by proposition 3.9 of [5], $MM(F)$ is $\sigma(F^*,F)$ compact and is the $\sigma(F^*,F)$ closure of $e(X)$.

Let F be a conjugate closed ,norm closed subspace of $C(X)$ containing the constant functions. Let $f\in C(X)$ and let $x\in X$. Then the functions

$$L_x f=f\circ\lambda_x,\ R_x f=f\circ\rho_x$$

are called the left and right translates of f by x, respectively.

F is said to be

i) Left translation invariant if $L_x F\subseteq F$ for each

$x \in X$.

ii) Right translation invariant if $R_x F \subseteq F$ for each $x \in X$.

iii) Translation invariant if F is both right and left translation invariant.

Let F be left translation invariant and let T_μ be the mapping from F into C(X) defined by

$$(T_\mu f)(x)=\mu(L_x f) \quad f \in F,\ x \in X,$$

for each $\mu \in F^*$. Then F is said to be

i) Left introverted if $T_\mu F \subseteq F$ for each $\mu \in M(F)$.

ii) Left m-introverted if $T_\mu F \subseteq F$ for each $\mu \in MM(F)$.

An admissible subalgebra of C(X) is a norm closed, conjugate closed translation invariant, left introverted subspace of C(X) containing the constant functions,and an m-admissible subalgebra of C(X) is a translation invariant, left m-introverted C^*-subalgebra of C(X) containing the constant functions.

A semigroup compactification of a semitopological semigroup X is a pair (e,Y), where Y is a compact Hausdorff right topological semigroup and $e:X\longrightarrow Y$ is a continuous homomorphism such that e(X) is dense in Y and $e(X)\subseteq\Lambda(Y)=\{y\in Y \mid \text{the map } x\longmapsto yx:Y\longrightarrow Y \text{ is continuous}\}$.

Notice that the definition of semigroup compactification (e,Y) of a semitopological semigroup X differs from the definition of a topological compactification in two ways.The first difference is that Y is required to be a compact right topological semigroup and the second one is that the mapping e is not required to be a homeomorphism onto e(X).

If (e,Y) is a compactification of a semigroup X, then by proposition 1.3 of [6,chap.3] the following assertions hold:

i) If φ is a continuous homomorphism from a semitopological semigroup Z onto a dense subsemigroup of X, $(e\circ\varphi,Y)$ is a compactification of Z.

ii) If $\varphi:Y\longrightarrow Z$ is a continuous homomorphism from Y onto a compact right topological semigroup Z, then $(\varphi\circ e,Z)$ is a compactification of X.

An $\mathcal{F}$-compactification of X is a pair (e,Y), where Y is a compact Hausdorff right topological semigroup and $e:X\longrightarrow Y$ is a continuous homomorphism with the following properties:

i) $Cl_Y(e(X))=Y$.

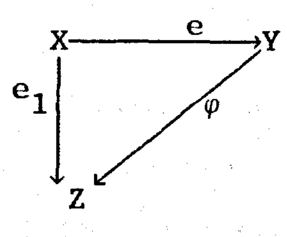
ii) $\lambda_{e(x)}:Y\longrightarrow Y$ is continuous for each $x\in X$.

iii) $e^*C(Y)=F$, where $e^*:C(Y)\longrightarrow C(X)$ is the adjoint of e.

Let P be a set of properties which compactifications (e,Y) of X may possess or may not possess. Then (e,Y) is said to be maximal with respect to P if

(a) (e,Y) possesses properties P, and

(b) whenever (e_1,Z) possesses properties P then there exists a continuous homomorphism $\varphi:Y\longrightarrow Z$ such that the following diagram commutes:



Now we state an important theorem that shows the relation between F-compactifications and the m-admissible subalgebras of C(X).

1.2.2.Theorem [6]. If (e,Y) is a compactification of a semitopological semigroup X, then $e^{*}C(Y)$ is an m-admissible subalgebra of C(X). Conversely, if F is an m-admissible subalgebra of C(X), then there exists a unique (up to isomorphism) compactification (e,Y) of X such that $e^{*}C(Y)=F$.

Now we will introduce one of the important compactifications of a semitopological semigroup X which we shall study in the next chapters.

1.2.2.6. Definition. A function $f \in C(X)$ is said to be left uniformly continuous if the mapping $x \longrightarrow L_x f : X \longrightarrow C(X)$ is norm continuous. The set of all left uniformly continuous functions on X is denoted by LUC(X) or simply LUC when there is no confusion.

If G is a topological group then according to proposition 1 of [65] $f \in LUC(G)$ if and only if f is uniformly continuous with respect to the right uniformity generated by entourages of the form $\{(x,y) |\ xy^{-1} \in V\}$, where V is a neighbourhood of the identity of G.

By the lemma 1 of [45], LUC(X) is a translation invariant left introverted C^{*}-subalgebra of C(X) containing the constant functions. In particular LUC(X) is admissible. Thus by the theorem 1.2.2.5 every semitopological semigroup X has a canonical LUC-compactification which is denoted by (e,X^{LUC}). According to the theorem 5.5 of [5], the compactification (e,X^{LUC}) is maximal in the following sense:If Y is a compact right topological semigroup and φ is a continuous homomorphism with the following properties :

a) $Cl_Y\varphi(X)=Y$,

b) $(x,y)\longrightarrow\varphi(x)y:X\times Y\longrightarrow Y$ is continuous,

then there exists a continuous homomorphism ϕ from X^{LUC} onto X such that $\varphi=\phi\circ e$.

1.2.2.7.Theorem [39]. If X be a locally compact group then the topological centre of X^{LUC} is X,that is, $\Lambda(X^{LUC})=X$.

CHAPTER II

UNIFORM SPACES, NEAR ULTRAFILTERS AND COMPACTIFICATION OF UNIFORM SPACES

In the first section of this chapter, we shall define what is meant by a uniformity and by a uniform space. In section 2, we will introduce the new concept of "near ultrafilters" on a uniform space and investigate their properties.In the last section of this chapter We shall describe the compactification of a uniform space in terms of the "near ultra filters"

Sectoin 1. Uniform Spaces

Let X be a non-empty set. By $X^2=X\times X$, we mean the set of all ordered pairs $\{(x,y): x,y\in X\}$, X^2 is the product of X by itself. For any subset U of X^2 , we define

$$U^{-1}=\{ (x,y)\in X^2 : (y,x)\in U \}.$$

If U,V are two subsets of X^2, we define $U\circ V$ as the collection of pairs $(x,y)\in X^2$ such that $(x,z)\in U$ and $(z,y)\in V$ for some $z\in X$. We put $U\circ U=U^2$; and,if U^n has been defined, we define U^{n+1} as $U \circ U^n$.

A subset U of X^2 is said to be symmetric if $U=U^{-1}$.

The set $\{(x,y)\in X^2 : x=y\}$ is called the diagonal of X^2 and is sometimes denoted by $\Delta(X)$ or simply Δ.

Let A be a subset of X and U a subset of X^2 .Then we define

$$\tilde{U}(A)=\{y\in X : (x,y)\in U \text{ for some } x\in A\}$$

$$\tilde{U}^{-1}(A)=\{y\in X : (y,x)\in U \text{ for some } x\in A\}$$

$$=\{y\in X : (x,y)\in U^{-1} \text{ for some } x\in A\}.$$

If A is a singleton {x}, then

$$\tilde{U}(x)=\tilde{U}(\{x\})=\{y\in X : (x,y)\in U\}.$$

$$\tilde{U}^{-1}(x)=\tilde{U}(\{x\})=\{y\in X : (y,x)\in U\}.$$

Thus $U=U^{-1}$ if and only if $\tilde{U}(x)=\tilde{U}^{-1}(x)$ for each $x\in X$.

2.1.1. Definition. A non-empty subfamily $\mathcal{U}$ of subsets of XxX is said to be a uniformity for X or to define a uniform structure on X if the following conditions are satisfied:

(a) $\emptyset$ does not belong to $\mathcal{U}$.

(b) For U_1 and U_2 in $\mathcal{U}$, $U_1 \cap U_2 \in \mathcal{U}$.

(c) For any U_1 in $X \times X$ with $U_2 \subseteq U_1$ for some U_2 in $\mathcal{U}$, then U_1 is also in $\mathcal{U}$.

(d) Each U in $\mathcal{U}$ contains the diagonal Δ.

(e) For each U in $\mathcal{U}$, $U^{-1} \in \mathcal{U}$.

(f) For each U in $\mathcal{U}$ there exists V in $\mathcal{U}$ such that $V \circ V \subset U$.

A member U of $\mathcal{U}$ is called a vicinity and the uniformity $\mathcal{V}$ is called separating if $\bigcap \{ V : V \in \mathcal{U} \} = \Delta$.

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is called a base of $\mathcal{U}$, if each U in $\mathcal{U}$ contains a member B of $\mathcal{B}$.

2.1.2. Example. The standard uniformity or usual uniformity on $\mathbb{R}$ is the uniformity having as base the collection of sets $U_\varepsilon = \{ (x,y) \in \mathbb{R}^2 : |x-y| < \varepsilon \}$ ∎

2.1.3.Definition.A non-empty set X with a uniformity $\mathcal{U}$ is called a uniform space and is denoted by $(X,\mathcal{U})$. If $\mathcal{U}$ satisfies only (a),(b),(c),(d),(f) and then $(X,\mathcal{U})$ is called a quasiuniform space.

Every uniformity generates a topology in a natural way, while different uniformities may produce the same topology.

2.1.4.Definition. Let $(X,\mathcal{U})$ be a separated uniform space. The topology defined by the uniformity $\mathcal{V}$ is the collection of all subsets T of X such that for each $x \in T$ there exists $U \in \mathcal{V}$ such that $\tilde{U}(x) \subseteq T$.

By theorem 1 of [30,chap 5], X is a completely regular Hausdorff space with this topology.

The topology associated with a uniformity $\mathcal{U}$ will be called the uniform topology $\tau_{\mathcal{U}}$ generated by $\mathcal{U}$. Whenever the topology on a topological space X can be obtained in this way from a uniformity , X is called a uniformizable topological space.

It is a well known fact that every compact space X has a unique uniformity compatible with the topology of X.

2.1.5.Proposition [30]. Each metric space is uniformizable.

There is a concept of uniform continuity for mapping between uniform spaces, which we now define.

2.1.6.Definition.Let X and Y be non-empty sets provided with the uniformities $\mathcal{U}, \mathcal{V}$, respectively. A function $f:X \longrightarrow Y$ is uniformly continuous iff for each $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ such that $(x,y) \in U$ implies that $(f(x),f(y)) \in V$. If f is one to one ,onto and both f and f^{-1} are uniformly continuous, then f is called a uniform isomorphism and X and Y are said to be uniformly isomorphic.

2.1.7.Theorem [30]. Every continuous function f from a compact space X to a uniform space Y is uniformly continuous.

Section 2. Near Ultrafilters

In this section, we will construct a compactification $\bar{X}$ of a given Hausdorff uniform space $(X,\mathcal{U})$ by means of "near ultrafilters " and we will prove some properties of this compactification.

2.2.1.Definition.Let $(X,\mathcal{U})$ be a uniform Hausdorff space.

A family η of subsets of X will be said to have the near finite intersection property if, for every finite subset φ of η and every U∈$\mathcal{U}$,

$$\bigcap_{Y\in\varphi}\tilde{U}(Y)\neq\emptyset.$$

We shall call η a " near ultrafilter " if it is maximal with respect to this property.

2.2.2.Remark. If X is discrete, $\mathcal{U}$ can be taken to be the family of supersets of the diagonal in X^2. Then a "near ultrafilter " is simply an ultrafilter.

2.2.3.Definition. Let $\mathcal{C}$ be the set of all near ultrafilters on X. We define a topology on $\mathcal{C}$ by stating that sets of the form

$$\mathcal{C}_Y = \{ \eta\in\mathcal{C} : Y\in\eta \},$$

where Y is a subset of X, is a base for the closed sets.

2.2.4.Proposition. For any $\eta\in\mathcal{C}$ and any finite subset φ of η

$$\bigcap_{Y\in\varphi}\tilde{U}(Y)\in\eta$$

for every $U\in\mathcal{U}$.

Proof. Suppose that $\bigcap_{Y\in\varphi}\tilde{U}(Y)\notin\eta$ for some finite subset φ of η and some $U\in\mathcal{U}$. Then there is a vicinity W in $\mathcal{U}$ and a finite subset γ of η such that

$$[\tilde{W}(\bigcap_{Y\in\varphi}\tilde{U}(Y))] \cap [\bigcap_{Z\in\gamma}\tilde{W}(Z)]=\emptyset.$$

There is a vicinity $W'\in\mathcal{U}$ such that $W'\subseteq U\cap W$. Then

$$[\bigcap_{Y\in\varphi}\tilde{W}'(Y)] \cap [\bigcap_{Z\in\gamma}\tilde{W}'(Z)]=\emptyset.$$

But this is a contradiction ∎

2.2.5.Proposition. For any $\eta\in\mathcal{C}$ and any subset Y of X, $Y\notin\eta$ implies that

$$\tilde{U}(Y) \cap \tilde{U}(Z) = \emptyset$$

for some $Z\in\eta$ and some $U\in\mathcal{U}$.

Proof. Since $Y\notin\eta$, there is a finite subset φ of η and $W\in\mathcal{U}$ such that

$$\tilde{W}(Y) \cap [\bigcap_{T\in\varphi}\tilde{W}(T)] = \emptyset.$$

We can choose a symmetric vicinity $U \in \mathcal{U}$ such that $U \circ U \subseteq W$. Put $Z = \bigcap_{T \in \varphi} \bar{U}(T)$. Then $Z \in \eta$, by proposition 2.2.4, it is easy to see that $\bar{U}(Y) \cap \bar{U}(Z) = \emptyset$; for, if $(x,y) \in U$ and $(x,z) \in U$, with $y \in Y$ and $z \in Z$, we have

$$z \in \bar{U}^2(Y) \cap Z \subseteq \bar{W}(Y) \cap Z = \emptyset,$$

contradiction∎

2.2.6.**Proposition.** Let $\eta \in \mathcal{C}$ and $Y \subseteq X$. Then, $Y \notin \eta$ if and only if $\bar{U}(Y) \cap Z = \emptyset$ for some $U \in \mathcal{U}$ and some $Z \in \eta$.

Proof. To see this suppose that $Y \notin \eta$, then there exists $Z \in \eta$ and $W \in \mathcal{U}$ such that $\tilde{W}(Z) \cap \tilde{W}(Y) = \emptyset$ which implies that $Z \cap \tilde{W}(Y) = \emptyset$. Conversely, if $Z \cap \tilde{W}(Y) = \emptyset$, then $\bar{U}(Z) \cap \bar{U}(Y) = \emptyset$ for every symmetric vicinity $U \in \mathcal{U}$ such that $U \circ U \subseteq W$∎

2.2.7.**Remark.** Let $Y \subseteq X$. Then $Y \in \eta$ if and only if $Cl_X Y \in \eta$ because, $W(y) \cap Z = \emptyset$ implies that $U(Cl_X Y) \cap Z = \emptyset$ for every vicinity $U \in \mathcal{U}$ such that $U \circ U \subseteq W$ and $Y \subseteq Cl_X Y$ where $Cl_X Y = \bigcap \{\tilde{U}(Y) : U \in \mathcal{U}\}$∎

2.2.8.**Proposition.** For given $\eta \in \mathcal{C}$ and any subsets Y_1, Y_2 of X, $Y_1, Y_2 \notin \eta$ implies that $Y_1 \cup Y_2 \notin \eta$. Hence,

$$\mathcal{C}_{Y_1 \cup Y_2} = \mathcal{C}_{Y_1} \cup \mathcal{C}_{Y_2}.$$

Proof. Since $Y_1 \notin \eta$, there exists $U \in \mathcal{U}$ and $Z_1 \in \eta$ such that

$$\bar{U}(Y_1) \cap \bar{U}(Z_1) = \emptyset \qquad (1)$$

and since $Y_2 \notin \eta$, there exists $V \in \mathcal{V}$ and $Z_2 \in \eta$ such that

$$\tilde{V}(Y_2) \cap \tilde{V}(Z_2) = \emptyset \qquad (2)$$

But there exists a vicinity $W \in \mathcal{U}$ such that $W \subset U \cap V$. Hence

$$\tilde{W}(Y_1 \cup Y_2) \cap (\tilde{W}(Z_1) \cap \tilde{W}(Z_2)) = \emptyset,$$

and so $Y_1 \cup Y_2 \notin \eta$ ■

It is clear from proposition.2.2.8 that for any subset Y of X, $\mathcal{C} = \mathcal{C}_Y \cup \mathcal{C}_{Y^*}$, where $Y^* = X \setminus Y$.

Section 3. Compactification of Uniform Spaces

2.3.1. Proposition. The space $\mathcal{C}$ is a compact Hausdorff space.

Proof. Let $\xi, \eta \in \mathcal{C}$ with $\xi \neq \eta$. We shall show that there exist closed subsets $\mathcal{C}_Y, \mathcal{C}_{Y^*}$ of $\mathcal{C}$ such that $\xi \notin \mathcal{C}_{Y^*}$ and $\eta \notin \mathcal{C}_Y$. Since $C_Y \cup C_{Y^*} = C$, it will follow that $\mathcal{C}$ is Hausdorff.

If $\xi \neq \eta$, then there exist $Z_1 \in \eta$ and $Z \in \xi$ and $V \in \mathcal{U}$ such that

$$\tilde{V}(Z_1) \cap \tilde{V}(Z)=\emptyset.$$

We can choose $W\in \mathcal{U}$ such that $W\circ W\subseteq V$, and put $Y=\tilde{W}(Z)$. Then $\tilde{W}(Z_1)\cap\tilde{W}(Y)=\emptyset$ which implies that $Y\notin\eta$. We also have $\tilde{W}'(Z)\cap\tilde{W}'(Y^*)=\emptyset$ for any symmetric vicinity W' satisfying $W'^2\subseteq W$. For, if $x\in\tilde{W}'(Z) \cap \tilde{W}'(Y^*)$, $(x,z)\in W'$ and $(x,t)\in W'$ for some $z\in Z$ and some $t\in Y^*$. But then $t\in\tilde{W}(Z)=Y$. This is a contradiction. So $Y^*\notin\xi$.

For the compactness, suppose that $(\mathcal{C}_{U_i})_{i\in I}$ has the finite intersection property Then $\{ U_i: i\in I \}$ has the near finite intersection property. Hence, by Zorn's lemma, there is a point ξ of $\mathcal{C}$ such that $\{U_i: i\in I \}\subseteq \xi$. Since $\xi \in \bigcap_{i\in I} \mathcal{C}_{U_i}$, it follows that $\bigcap_{i\in I} \mathcal{C}_{U_i}\neq\emptyset$ ∎

For each $x\in X$, we denote by $\hat{x}$ the set of subsets Y of X for which $x\in Cl_X Y$. Then $\hat{x}$ is an element of $\mathcal{C}$.

2.3.2. **Proposition.** The mapping $e:X\longrightarrow\mathcal{C}$ for which $e(x)=\hat{x}$ embeds X in $\mathcal{C}$ as a dense subspace.

Proof. clearly e is one to one, because if $x\neq y$, $\{x\}\in\hat{x}$ and $\{x\}\notin\hat{y}$.

Furthermore, $e(X)$ is dense in $\mathcal{C}$. To see this suppose that $Y\subset X$ and that $e(X)\cap(\mathcal{C}\setminus\mathcal{C}_Y)=\emptyset$. Then $x\in X$ implies that

$Y\in e(x)$, i.e, that $x\in\bar{Y}$. Thus $Cl_X Y=X$. It follows that $\mathcal{C}_Y=\mathcal{C}$; for, if $\xi\in\mathcal{C}\setminus\mathcal{C}_Y$, then $\tilde{V}(Z)\cap\tilde{V}(Y)=\emptyset$ for some $Z\in\xi$ and some $V\in\mathcal{U}$. This is impossible, since $Cl_X Y\subseteq\tilde{V}(Y)=X$.

To show that e is continuous, let $x\in X$ and let $\mathcal{C}\setminus\mathcal{C}_U$ be an open neigbourhood of e(x) in $\mathcal{C}$. Then the set $X\setminus Cl_X U$ is an open neigbourhood of x in X, and $e(X\setminus Cl_X U)\subset\mathcal{C}\setminus\mathcal{C}_U$. To see this, let $\eta\in e(X\setminus Cl_X U)$; then there exists $z\in X\setminus Cl_X U$ such that $e(z)=\eta$. Since $z\notin Cl_X U$, $U\notin e(z)=\eta$. Hence $\eta\in\mathcal{C}\setminus\mathcal{C}_U$.

To see that e is a closed map, let U be a closed set in X. Then we claim that $e(U)=e(X)\cap\{\xi\in\mathcal{C}|U\in\xi\}$. To see this let $\eta\in e(U)$, hence there is $x\in U$ such that $\eta\in e(x)=\hat{x}$. Since $x\in U$, $U\in\hat{x}=e(x)=\eta$ and so $\eta\in\{\xi\in\mathcal{C}|U\in\xi\}\cap e(X)$. Therefore,

$$e(U)\subseteq\{\xi\in\mathcal{C}|U\in\xi\}\cap e(X) \qquad (1).$$

For the reverse inclusion let $\eta_1\in\{\xi\in\mathcal{C}|U\in\xi\}\cap e(X)$, then there exists $y\in X$ such that $\eta_1=e(y)$, and so $U\in e(y)$ since $U\in\hat{y}$, $y\in Cl_X U$ and $Cl_X U=U$ since U is closed and so $\eta=e(y)\in e(U)$. Therefore,

$$\{\xi\in\mathcal{C}|U\in\xi\}\cap e(X)\subseteq e(X) \qquad (2).$$

Hence the result follows from (1) and (2)∎

2.3.3. Proposition. For any $Y\subseteq X$ and $\xi\in\mathcal{C}$, $Y\in\xi$ if and only if $\xi \in Cl_{\mathcal{C}} e(Y)$.

Proof. Let $Y \subseteq X$. Then $\mathcal{C}_Y = \{\eta \in \mathcal{C} : Y \in \eta\}$ is a closed set in $\mathcal{C}$ and $e(y) \in \{\eta \in \mathcal{C} : Y \in \eta\}$ for each $y \in Y$, and so

$$e(Y) \subseteq \{ \eta \in \mathcal{C} : Y \in \eta \} \quad \text{Thus } Cl_{\mathcal{C}} e(Y) \subseteq \{\eta \in \mathcal{C} : Y \in \eta \}.$$

For the reverse inclusion, let $\eta \in \mathcal{C}_Y$ and suppose that $\eta \notin Cl_{\mathcal{C}} e(Y)$. Then there is an open neigbourhood

$$\mathcal{C}'_Z = \{ \xi \in \mathcal{C} : Z \notin \xi \}$$

of η for some $Z \subseteq X$, such that

$$e(Y) \cap \{ \xi \in \mathcal{C} : Z \notin \xi \} = \emptyset.$$

So

$$e(Y) \subset e(X) \cap \{ \xi \in \mathcal{C} : Z \in \xi \} = e(Cl_X Z),$$

and hence $Y \subset Cl_X Z$. Since $Y \in \eta$, $Cl_X Z \in \eta$ and so $Z \in \eta$. However, this contradicts the assumption that $\eta \in \mathcal{C}'_Z$ ■

We denote the set $\mathcal{C}$ of all near ultrafilters by $\tilde{X}$.

2.3.4. Remark. We shall in future regard X as embedded in $\tilde{X}$, by identifying X with e(X).

2.3.5. Remark. It is clear that two equivalent uniformities define same compactification.

2.3.6. Theorem. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces. Any uniformly continuous function $f:X \longrightarrow Y$ has a continuous extension $\tilde{f}:\tilde{X} \longrightarrow \tilde{Y}$.

Proof. We first show that, for each $\xi \in \tilde{X}$, there is a unique $\eta \in \tilde{Y}$ with the property that $f(S) \in \eta$ whenever $S \in \xi$.

If we establish that $\{f(S):S \in \xi\}$ has the near finite intersection property. It will follow that there is at least one element $\eta \in \tilde{Y}$ with the above property.

Let φ be a finite subset of ξ, for which

$$\bigcap_{S \in \varphi} \tilde{V}(f(S)) = \emptyset$$

for some $V \in \mathcal{V}$. Then if,

$$U = \{ (x_1,x_2) \in X x X : (f(x_1),f(x_2)) \in V \},$$

We have $U \in \mathcal{U}$ and

$$\bigcap_{S \in \varphi} \tilde{U}(S) = \emptyset.$$

This is a contradiction. Thus $\{ f(S) : S \in \xi \}$ does have the near finite intersection property.

Suppose now that η_1 and η_2 are distinct elements of $\tilde{Y}$ which both contain $\{ f(S) : S\in\xi \}$. There will be sets $Y_1\in\eta_1$ and $Y_2\in\eta_2$ for which

$$\tilde{W}(Y_1) \cap \tilde{W}(Y_2)=\emptyset$$

for some $W\in\mathcal{V}$. Let $V\in\mathcal{V}$ be a symmetric vicinity for which $V^2\subseteq W$ and let U be defined as in the preceding paragraph. Now $f^{-1}(\tilde{V}(Y_1))\in\xi$; for if,

$$f^{-1}(\tilde{V}(Y_1))\cap S=\emptyset$$

for some $S\in\xi$, we have $\tilde{V}(Y_1) \cap f(S) = \emptyset$. This is a contradiction to the assumption that $Y_1\in\eta_1$ and $f(S)\in\eta_1$. Similarly, $f^{-1}(\tilde{V}(Y_2))\in\xi$. However,

$$\tilde{U}(f^{-1}(\tilde{V}(Y_1)))\cap\tilde{U}(f^{-1}(\tilde{V}(Y_2)))=\emptyset;$$

for, if $(x,x_1)\in U$ and $(x,x_2)\in U$, where $x_1\in f^{-1}(\tilde{V}(Y_1))$ and $x_2\in f^{-1}(\tilde{V}(Y_2))$, we have $f(x)\in\tilde{W}(Y_1) \cap \tilde{W}(Y_2)$. This contradicts our assumption that $\tilde{W}(Y_1) \cap \tilde{W}(Y_2) = \emptyset$. Thus the element $\eta\in Cl_{\tilde{Y}} Y$ which contains $\{f(S) : S\in\xi \}$ is unique.

We can now define $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ by stating that $\tilde{f}(\xi)=\eta$. As we have just seen, if $T\in\eta$, then $f^{-1}(\tilde{V}(T))\in\xi$ for every $V\in\mathcal{V}$. Conversely, if $f^{-1}(\tilde{V}(T))\in\xi$ for every $V\in\mathcal{V}$, it follows that $T\in\eta$. Otherwise, $\tilde{V}(T) \cap \tilde{V}(T') = \emptyset$ for some $T'\in\eta$ and some $V\in\mathcal{V}$. As in the preceding paragraph, we put

$U = \{(x_1,x_2)\in X\times X : (f(x_1),f(x_2))\in V\}$, and find that $\tilde{U}(f^{-1}(\tilde{V}(T))) \cap \tilde{U}(f^{-1}(\tilde{V}(T'))) = \emptyset$. This is a contradiction, as both $f^{-1}(\tilde{V}(T))$ and $f^{-1}(\tilde{V}(T'))$ are in ξ.

Thus we have shown that

$$\tilde{f}^{-1}(\mathcal{C}_T) = \cap\{\mathcal{C}_{f^{-1}(\tilde{V}(T))} : V\in \mathcal{V}\},$$

and hence that $\tilde{f}$ is continuous. It is obvious that $\tilde{f}$ is an extension of f∎

We have the following corollary as a consequence of above theorem.

2.3.7.Corollary. Let $(X,\mathcal{V})$ be a uniform space and $\tilde{X}$ be the compactification of X and suppose that Z is any compactification of X having the property that every uniformly continuous function f from X into a uniform space $(Y,\mathcal{U})$ has a continuous extension $\tilde{f}$ from $\tilde{Z}$ to $\tilde{Y}$. Then Z is homeomorphic to $\tilde{X}$.

Proof. Since Z is a compactification of X there is a homeomorphism g from X onto g(X) with g(X) dense in Z .Thus we can regard X as a subspace of Z and $\tilde{X}$. If *Id* is the identity map $X\longrightarrow X\subseteq Z$, then by the assumption there is a unique continuous map $\tilde{f}:\tilde{X}\longrightarrow Z$ such that $\tilde{f}|_X = Id$. Similarly there is a unique continuous map $\tilde{g}:Z\longrightarrow\tilde{X}$ such that $\tilde{g}|_X=Id^{-1}=Id$. Since X is dense in both $\tilde{X}$ and Z, and $\tilde{f}\circ\tilde{g}|_X=Id$ and $\tilde{g}|_X\circ\tilde{f}=Id^{-1}=Id$, $\tilde{f}\circ\tilde{g}=Id_Z$, $\tilde{g}\circ\tilde{f}=Id_{\tilde{X}}$. Therefore, both $\tilde{f}$ and $\tilde{g}$ are homeomorphisms∎

2.3.8.Proposition. The continuous real-valued functions on $\bar{X}$ are precisely the extension to $\bar{X}$ of bounded uniformly continuous real-valued functions defined on X.

Proof. We know from proposition 2.3.6 that every bounded uniformly continuous real-valued function defined on X has a continuous extension to $\bar{X}$.

The set A of all such extensions forms a uniformly closed algebra which contains the constant functions. If we can show that it separates the point of $\bar{X}$, it will follow from the Stone-Weierstrass theorem that it consist of all the continuous real-valued functions defined on $\bar{X}$. Suppose then that η_1 and η_2 are distinct elements of $\bar{X}$.We shall have $\bar{U}(Y_1)\cap\bar{U}(Y_2)=\emptyset$ for some $Y_1\in\eta_1$,some $Y_2\in\eta_2$ and some $U\in\mathcal{U}$. Then there will be a uniformly continuous bounded function $f:X\longrightarrow\mathbb{R}$ for which $f(Y_1)=\{0\}$ and $f(Y_2)=\{1\}$ according to [36].

It will follow that $\bar{f}(\eta_1)=0$ and $\bar{f}(\eta_2)=1$. Hence A does separate the points of $\bar{X}$ ∎

In future, we will denote the extension of a uniformly continuous function f from X to $\tilde{X}$ by $\tilde{f}$.

2.3.9. Proposition. Let $\xi \in \tilde{X}$. For any $Y \in \xi$ and any $U \in \mathcal{U}$, the set $\mathcal{C}_{\tilde{U}(Y)}$ is a neigbourhood of ξ, and the sets of this form provide a basis for the neihbourhoods of ξ.

Proof. Since $\mathcal{C} \setminus \mathcal{C}_{(\tilde{U}(Y))^*} \subseteq \mathcal{C}_{\tilde{U}(Y)}$, $\mathcal{C}_{\tilde{U}(Y)}$ is a neigbourhood of ξ by proposition 2.2.8.

Now suppose that $T \subseteq X$ and that $T \notin \xi$. Then $T \cap \tilde{V}(Y) = \emptyset$ for some $Y \in \xi$ and some $V \in \mathcal{U}$. Let $U \in \mathcal{U}$ satisfy $U^2 \subseteq V$ and $U = U^{-1}$. Then $\mathcal{C}_{\tilde{U}(Y)} \subseteq \mathcal{C} \setminus \mathcal{C}_T$, because $\tilde{U}(Y) \cap \tilde{U}(T) = \emptyset$.

This shows that the sets of the form $\mathcal{C}_{\tilde{U}(Y)}$ are a base for the neighbourhoods of ξ ∎

CHAPTER III.

COMPACTIFICATION $\tilde{\mathbb{R}}$ OF THE REAL NUMBERS $\mathbb{R}$ WITH RESPECT TO THE USUAL UNIFORMITY

Section I. Topological Properties of $\tilde{\mathbb{R}}$

We have seen in chapter 2 that any uniform space X has a compactification $\tilde{X}$, whose points are the near ultra filters, which has the property that any uniformly continuous function f from X to any compact space K can be extended to a continuous function $\tilde{f}$ from $\tilde{X}$ to K. Since $\mathbb{R}$ is a uniform space, $\mathbb{R}$ has such a compactification which we shall denote by $\tilde{\mathbb{R}}$. We remind the reader that this is defined in the following way:

Let $\mathcal{B}(0)$ denote the set of neigbourhoods of 0 in $\mathbb{R}$.

A point ξ of $\tilde{\mathbb{R}}$ is a family of subsets of $\mathbb{R}$ which is maximal with respect to the property that, for every finite subset φ of ξ and every neigbourhood W of 0,

$$\bigcap_{Y\in\varphi}(Y+W)\neq\emptyset.$$

The topology of $\tilde{\mathbb{R}}$ is defined by choosing sets of the form

$$\mathcal{C}_Y=\{\eta \in \tilde{\mathbb{R}} : Y \in \eta\},$$

where $Y\subseteq\mathbb{R}$, as a base for the closed sets.

We will denote $\tilde{\mathbb{R}}\setminus\mathbb{R}$ by $\rho\mathbb{R}$.

3.1.1.Theorem. Let $\xi\in\mathbb{R}^{+}\setminus\mathbb{R}$ and suppose that $Y\in\xi$. Then for any $k>0$, there is a sequence $(y_n)\subseteq Y$ such that $y_{n+1}-y_n>k$ for every n, and $(y_n)\in\xi$.

Proof. Since $\xi\in\tilde{\mathbb{R}}^{+}\setminus\mathbb{R}$, we may suppose $Y\subseteq[0,\infty)$. Now choose $m>k$. Then either

$$\bigcup_{n\in 2\mathbb{N}-1}[nm,(n+1)m]\in\xi$$

or

$$\bigcup_{n\in 2\mathbb{N}}[nm,(n+1)m]\in\xi.$$

Let X_1 denote the first of these sets which is a member of ξ. Let A_1 denote $2\mathbb{N}-1$ or $2\mathbb{N}$.If

$$X_1 = \bigcup_{n \in A_1} [nm, (n+1)m],$$

then

$$\bigcup_{n \in A_1} [nm, (n+\tfrac{1}{2})m) \in \xi$$

or

$$\bigcup_{n \in A_1} [(n+\tfrac{1}{2})m, (n+1)m] \in \xi.$$

Let X_2 denote the first of these sets which is a member of ξ. Let A_2 denote A_1 or $A_1 + \frac{1}{2}$. If

$$X_2 = \bigcup_{n \in A_2} [nm, \ (n+\tfrac{1}{2})m] \in \xi.$$

Then

$$\bigcup_{n \in A_2} [nm, (n+\tfrac{1}{4})m] \in \xi$$

or

$$\bigcup_{n \in A_2} [(n+\tfrac{1}{4})m, (n+\tfrac{1}{2})m] \in \xi.$$

Let X_3 denote the first of these sets which is a member of ξ.

Proceeding in this way, we can define a sequence of sets (X_n) with the following properties:

i) $X_n \in \xi$;

ii) Each X_n can be written as $\bigcup_{r=1}^{\infty} I_{n,r}$, where $I_{n,r}$ is a closed interval of length $\frac{m}{2^{n-1}}$;

iii) For each n, $d(I_{n,r}, I_{n,r'}) \geq m$ if $r \neq r'$;

iv) For each n and r, $I_{(n+1),r} \subseteq I_{n,r}$.

For each $r=1,2,3,\ldots$, there will be a unique point $x_r \in \bigcap_{n=1}^{\infty} I_{n,r}$.

Let $X=\{x_r : r \in \mathbb{N}\}$.

We claim that $X \in \xi$. To see this, let $Z \in \xi$ and let $\varepsilon > 0$. Choose n so that $\frac{m}{2^{n-1}} < \frac{\varepsilon}{2}$. Since $X_n \in \xi$, there will be a point $x \in X_n$ such that $d(x,Z) < \frac{\varepsilon}{2}$. If $x \in I_{n,r}$, then $d(x_r, x) < \frac{\varepsilon}{2}$. Hence $d(x_r, Z) < \varepsilon$. Thus $(X+(-\varepsilon,\varepsilon)) \cap Z \neq \emptyset$ and so $X \in \xi$.

Now, for each r, choose $y_r \in Y$ with $d(x_r, y_r) < d(x_r, Y) + \frac{1}{r}$. We shall show that $\{y_r : r \in \mathbb{N}\} \in \xi$. As before, let $Z \in \xi$ and let $\varepsilon > 0$. Then $V=\{x_r : d(x_r, Y) < \frac{\varepsilon}{2}\} \in \xi$; for otherwise we should have $V'=\{x_r : d(x_r, Y) \geq \frac{\varepsilon}{2}\} \in \xi$. This is impossible, since $d(V', Y) \geq \frac{\varepsilon}{2}$. Since $\{x_r : x_r \in V \text{ and } \frac{1}{r} < \frac{\varepsilon}{2}\} \in \xi$, we can choose $x_r \in V$ satisfying $d(x_r, Z) < \frac{\varepsilon}{2}$ and $\frac{1}{r} < \frac{\varepsilon}{2}$. We than have $d(y_r, Z) < \varepsilon$. Thus $\{y_r : r \in \mathbb{N}\} \in \xi$, as claimed.

A repetition of the preceding argument will show that $T=\{x_r : d(x_r,Y)<\frac{m-k}{2}$ and $\frac{1}{r}<\frac{m-k}{2}\}\in\xi$, and that $\{y_r : x_r\in T\}\in\xi$. We replace the sequence (y_n) by the sequence $(y_r)_{x_r\in T}$. This will clearly have the property that $|y_r-y_{r'}|>k$ if $r\neq r'$, since $|x_r-x_{r'}|\geq m$ ∎

A point η of a topological space X is called a remote point if it is not in the closure of a discrete subset of X.

As a consequence of above theorem, $\tilde{\mathbb{R}}$ has no remote points. But under the continuum hypothesis $\beta\mathbb{R}$ has remote points and the set of remote points of $\beta\mathbb{R}$ is dense in $\mathbb{R}^*$ by theorem 2.5 of [16].

Now we have a corollary which shows that the cardinality of a neigbourhood of a point ξ in $\rho\mathbb{R}$ is at least 2^c.

3.1.2.Corollary. Every neighbourhood of every point ξ in $\rho\mathbb{R}$ contains a copy of $\mathbb{N}^*$.

Proof. As we have seen in proposition 2.3.9 that a base of the neighbourhoods of the element ξ in $\rho\mathbb{R}$ is provided by choosing any closed set Y in ξ and any neighbourhood W of 0, and forming the set G of elements η in

$\rho\mathbb{R}$ for which $Y+W\in\eta$. For k=1, from the preceding theorem we have a sequence $(x_n)\subseteq Y$ such that $x_{n+1}-x_n\geq 1$ and $(x_n)\in\xi$.

We may suppose that $X=(x_n)$. Then G contains $Cl_{\tilde{\mathbb{R}}}(X)\setminus X$.

Now we define a mapping φ from $\mathbb{N}$ to $\mathbb{R}$ such that $\varphi(n)=x_n$. Clearly, the mapping φ is continuous and it extends to a continuous mapping φ^β of $\beta\mathbb{N}$ onto $Cl_{\tilde{\mathbb{R}}}(X)$. Let $\xi,\eta\in\beta\mathbb{N}$, with $\xi\neq\eta$. Then there exist $U\in\xi$ and $V\in\eta$ such that $U\cap V=\emptyset$ and so

$$\varphi^\beta(U)\cap\varphi^\beta(V)=\emptyset,$$

since φ is one to one on $\mathbb{N}$. Now if we take $W=(-1\backslash 2,\ 1\backslash 2)$

$$(\varphi(U)+W)\cap(\varphi(V)+W)=\emptyset,$$

since for any $n\in U$ and $m\in V$,

$$|\varphi(n)-\varphi(m)|\geq 1.$$

And so

$$Cl_{\tilde{\mathbb{R}}}\varphi^\beta(U)\cap Cl_{\tilde{\mathbb{R}}}\varphi^\beta(V)=\emptyset.$$

Since $\xi\in Cl_{\beta\mathbb{N}}U$, $\varphi^\beta(\xi)\in Cl_{\tilde{\mathbb{R}}}\varphi(U)$ and since $\eta\in Cl_{\beta\mathbb{N}}\varphi(V)$,

$\varphi^{\beta}(\eta)\in Cl_{\tilde{\mathbb{R}}}\varphi(V)$. Therefore, $\varphi^{\beta}(\xi)\neq\varphi^{\beta}(\eta)$. So φ^{β} is one to one.

Since a one to one, continuous mapping of a compact space onto a Hausdorff space is a homeomorphism, φ^{β} is a homeomorphism between $Cl_{\tilde{\mathbb{R}}}(X)\setminus X$ and $\mathbb{N}^{*}$ ∎

As a result of the above corollary, we have the result that the cardinality of a neighbourhood of ξ in $\rho\mathbb{R}$ is 2^{c}, since the cardinality of $\mathbb{N}^{*}$ is 2^{c} according to [61].

3.1.3.Proposition. No sequence in $\mathbb{R}$ can converge to a point of $\rho\mathbb{R}$.

Proof. Suppose that there is a sequence (x_n) in $\mathbb{R}$ which does converge to a point ξ in $\rho\mathbb{R}$. The sequence (x_n) cannot be bounded otherwise it would have a subsequence (x_{n_r}) converging to a real number k. It follows that (x_n) will have a subsequence (x_{n_r}), which satisfies $x_{n_{r+1}}-x_{n_r}>1$. But (x_{n_r}) also converges to ξ.

Now we define a uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$f(x_{n_r})=\begin{cases} 0, & \text{if } r \text{ is even} \\ 1, & \text{if } r \text{ is odd;} \end{cases}$$

and we complete the definition by piecewise linearity. Then $\tilde{f}(\xi)=\mathrm{Lim}f(x_{2n})=0$ and $\tilde{f}(\xi)=\mathrm{Lim}f(x_{2n+1})=1$. But this is a contradiction ■

3.1.4. Proposition. The space $\tilde{\mathbb{R}}$ is not metrizable.

Proof. Suppose on the contrary that $\tilde{\mathbb{R}}$ is metrizable, then every point η of $\tilde{\mathbb{R}}$ has a countable base of neighbourhoods. Let $\eta\in\rho\mathbb{R}$, then since $\mathbb{R}$ is dense in $\tilde{\mathbb{R}}$ there is a sequence of points of $\mathbb{R}$ which converges to η which contradicts to the proposition 3.1.3 ■

As a result of the above proposition $\tilde{\mathbb{R}}$ has not have a countable base.

3.1.5. Lemma. Let α be an ultrafilter of subsets of $\mathbb{R}$. Then, if $\xi=\{X\subseteq\mathbb{R} : X+W\in\alpha \text{ for every } W\in\mathcal{B}(0)\}$, ξ is a near ultrafilter.

Proof. Obviously, ξ has the near finite intersection property. To show that ξ is maximal with respect to this property, suppose that $Y \subseteq \mathbb{R}$ and $Y \notin \xi$. Then $Y+W \notin \alpha$ for some $W \in \mathcal{B}(0)$, and so $(Y+W) \cap Z = \emptyset$ for some $Z \in \alpha$. However, $Z \in \alpha$ implies that $Z \in \xi$. So $\xi \cup \{Y\}$ does not have the near finite intersection property. ξ is the unique near ultrafilter which contains α, since any near ultrafilter which contains α must contain ξ, and must therefore be equal to ξ ∎

3.1.6. Proposition. Every ξ in $\rho\mathbb{R}$ is the limit of a convergent sequence of distinct points of $\rho\mathbb{R}$.

Proof. It is trivial, since ξ is the limit of the sequence $((1/n)+\xi)$ ∎

Now we have the following proposition immediately:

3.1.7. Proposition. Every point ξ of $\rho\mathbb{R}$ is a limit point of a countable subset of $\rho\mathbb{R}$, which does not contain ξ.

3.1.8. Definition. A set G in a topological space is called a G_δ-set if it is the intersection of at most countably many open sets and a point of a topological space is called a P-point if every G_δ-set containing the point is a neighbourhood of the point.

It is a well known fact that the set of P-points of $\mathbb{N}^*$ is dense in $\mathbb{N}^*$ and its cardinality is 2^c under the continuum hypothesis according to corollary 4.30 of [62] and the set of P-points of $\beta\mathbb{R}$ is dense in $\mathbb{R}^*$ and its cardinality is 2^c according to theorem 6.2 of [47], assuming the continuum hypothesis.

We have a corollary of proposition 3.1.6, since no P-point can be a non-trivial limit of any sequence.

3.1.9.Corollary. $\rho\mathbb{R}$ has no P-point.

3.1.10. Definition. An F-space is a topological space X such that, if f∈C(X), the set of continuous functions from X to $\mathbb{R}$, then Pos(f) and Neg(f) are completely separated; that is, there exists a function g∈C(X) such that g(x)=1 if x∈Pos(f) and g(y)=0 if x∈Neg(f). X is called locally compact if each point of X has a basis of compact neighbourhood and and X is said to be σ-compact if it is the union of at most countably many compact subspaces.

It is a well known fact that the spaces $\beta\mathbb{N}\backslash\mathbb{N}$, $\beta\mathbb{R}\backslash\mathbb{R}$ and $\beta\mathbb{R}^+\backslash\mathbb{R}^+$,where $\mathbb{R}^+$ denotes the space of nonnegative real numbers $(0,\infty)$, are F-spaces. In fact, for any locally compact, σ-compact Hausdorff space X, $\beta X\backslash X$ is a compact F-space according to theorem 2.7 of [19].

Now we have the following corollary as a consequence of proposition 3.1.6, since no point of an F-space is the limit point of a sequence of distinct points.

3.1.11.Corollary. $\rho\mathbb{R}$ is not an F-space.

3.1.12.Definition. A point of a subset A of a topological space X is called an isolated point of A if it has a neighbourhood which contains no other points of the subset A.

It is a well known fact that $\mathbb{N}^*$ is the only commpact space with weight c with the property that it has no isolated points by corollary 3.11 of [61], and every non-empty G_δ-set has a non-empty interior by corollary 3.27 of [62]. Hence, we have the following property of $\rho\mathbb{R}$ immediately, since $\rho\mathbb{R}$ has no isolated points by the preceding proposition.

3.1.13. Proposition. It is not true that every non-empty G_δ-set in $\rho\mathbb{R}$ has non-empty interior.

3.1.14. Proposition. No point in $\rho\mathbb{R}$ has a countable base of neighbourhoods in $\rho\mathbb{R}$.

Proof. If $\xi\in\rho\mathbb{R}$, there is a subset X of $\rho\mathbb{R}$ such that $\xi\in X$ and X is homeomorphic to $\mathbb{N}^*$, since every neighbourhood of ξ contains a copy of $\mathbb{N}^*$ to which ξ belongs by corollary 3.1.2. Hence ξ can not have a countable base of neighbourhoods of (U_n) in $\rho\mathbb{R}$; because $(\bigcap U_n)\bigcap X$ cannot be a singleton since $(\bigcap U_n)\bigcap X$ is homeomorphic to a non-empty G_δ-set of $\mathbb{N}^*$, and in $\mathbb{N}^*$ every non-empty G_δ-set has non-empty interior.

3.1.15. Corollary. $\rho\mathbb{R}$ is not metrizable.

Proof. It is obvious, since every point of a metrizable space has a countable base of neighbourhoods∎

3.1.16. Corollary. $\rho\mathbb{R}$ has not have a countable base.

Proof. It is obvious.

3.1.17.Examples (1).The space $\tilde{\mathbb{Q}}$,the compactification of rational numbers: Since every uniformly continuous function on $\mathbb{Q}$ can be extended to a uniformly continuous function on $\bar{\mathbb{R}}$, the space $\bar{\mathbb{Q}}$ is as same as $\bar{\mathbb{R}}$.

It is always true that, for any dense subspace Y of a uniform space X, $\tilde{Y}$ and $\tilde{X}$ are homeomorphic since every uniformly continuous function on Y can be extended to a uniformly continuous function on X, where Y has the uniform structure induced by that of X.

(2). Let X=(0,1). Then the compactification $\bar{X}$ of X is the closed interval [0,1], since every uniformly continuous function on (0,1) can be extended to a uniformly continuous function on [0,1]. But [0,1] is not the Stone-Čech compactification of (0,1), since the continuous function $f(x)=\sin(\frac{1}{x})$ from (0,1) cannot be extended to a continuous function on [0,1].

Let X be a topological space then X is separable if it contains a countable dense subset.If every real-valued continuous function on X is bounded then X is called pseudocompact.

3.1.18. Corollary. The space $\tilde{\mathbb{R}}$ is separable.

Proof. It is obvious, since $\mathbb{Q}$ is dense in $\tilde{\mathbb{R}}$ ∎

We mean by 'Lg' the following function f from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x)=\begin{cases}0, & x<1\\ \log x, & x\geq 1\end{cases} .$$

It is easy to see that f is a uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$.

3.1.19. Theorem. Let $\check{[1,\infty)}$ be the completion of $[1,\infty)$ with the uniform structure associated with the group structure $([1,\infty),.)$. Then $\tilde{[0,\infty)}$ is homeomorphic to $\check{[1,\infty)}$.

Proof. The 'Lg' function from $([1,\infty),.)$ to $([0,\infty),+)$ is uniformly continuous and the function $f(x)=e^x$ from $([0,\infty),+)$ to $([1,\infty),.)$ is uniformly continuous with respect to the uniformities generated by the group operations (.), and (+), respectively. Hence $([0,\infty),+)$ and $([1,\infty),.)$ are isomorphic, and so $\tilde{[0,\infty)}$ is homeomorphic to $\check{[1,\infty)}$ ∎

A topological space X is called connected if it is not the union of two non-empty disjoint open sets and it is called locally connected if it has a basis consisting of connected sets.

By theorem b of [48,chap.6,2] a topological space X is connected if and only if βX is connected. In particular the space $\beta\mathbb{R}$ is connected.

3.1.20.Proposition. $\tilde{\mathbb{R}}$ is connected.

Proof. Since the image of a connected set under a continuous function is connected $\tilde{\mathbb{R}}$.

3.1.21. Theorem. The space $\rho\mathbb{R}\cap Cl_{\tilde{\mathbb{R}}}[0,\infty)$ is a connected space.

Proof. If not, there is a continuous function f from this space to $\mathbb{R}$ that assumes precisely the values 0 and 1. Then the function f has an extension to $g:[o,\tilde{\infty}) \longrightarrow \mathbb{R}$ (since $[0,\tilde{\infty})$ is compact) and g must assume values near 0 and 1, at arbitrary large $x \in \mathbb{R}^+$. Since $\mathbb{R}^+$ is connected, g must assume the value (1\2) on an unbounded set in $\mathbb{R}^+$ and hence at some point of $\rho\mathbb{R}\cap Cl_{\tilde{\mathbb{R}}}[0,\infty)$. This contradiction shows that

$\rho\mathbb{R}\cap Cl_{\tilde{\mathbb{R}}}[0,\infty)$ is connected∎

By proposition e of [48,chap.4,3], a completely regular space is locally compact if and only if X is open in every compactification of X. Therefore, $\mathbb{R}$ is open in $\tilde{\mathbb{R}}$ and similarly, $\mathbb{R}^+$ and $\mathbb{R}^-$ are open in $Cl_{\tilde{\mathbb{R}}}\mathbb{R}^+$ and in $Cl_{\tilde{\mathbb{R}}}\tilde{\mathbb{R}}^-$, where $\mathbb{R}^-=(-\infty,0)$, respectively. Hence $Cl_{\tilde{\mathbb{R}}}\mathbb{R}^+\backslash\mathbb{R}^+$ and $Cl_{\tilde{\mathbb{R}}}\mathbb{R}^-\backslash\mathbb{R}^-$ are compact. Thus $\rho\mathbb{R}$ is the union of two disjoint, homeomorphic connected sets.

CHAPTER IV

EXTENSION OF SEMIGROUP OPERATION ON X TO A SEMIGROUP OPERATION ON $\tilde{X}$ AND ALGEBRAIC PROPERTIES OF THE COMPACT RIGHT TOPOLOGICAL SEMIGROUP $(\tilde{\mathbb{R}},+)$

In the last two chapters, we studied the topological properties of the compactification $\tilde{X}$ of a uniform space X. In this chapter, we shall investigate whether a semigroup operation on X can be extended to a semigroup operation on $\tilde{X}$. We shall then study some algebraic properties of the compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$ and $\rho\mathbb{R}$.

Section 1. Extension of Semigroup Operation on X to a Semigroup Operation on $\tilde{X}$

4.1.1.Theorem. Suppose that $(X,\cdot)$ is a uniform topological semigroup with uniformity $\mathcal{V}$. Then under the following assumptions the semigroup operation $\cdot$ can be extended to a binary operation on $\tilde{X}$,

Assumption 1. The function λ_x is uniformly continuous for each x in X.

Assumption 2. For each $U\in\mathcal{V}$, there is a vicinity $V\in\mathcal{V}$

such that, for every $y \in X$, $(x_1 y, x_2 y) \in U$ if $(x_1, x_2) \in V$.

Proof. From assumption 1, for each $x \in X$, the mapping λ_x extends to a mapping from $\bar{X}$ into $\bar{X}$.

We denote the image of η under the extension mapping by $x\eta$. So

$$x\eta = \lim_{y \longrightarrow \eta} xy.$$

For fixed $\eta \in \bar{X}$, we want the mapping $x \longmapsto x\eta$ from X into $\bar{X}$ to be uniformly continuous. In other words, given a continuous real-valued function φ defined on $\bar{X}$ we require:

Given $\varepsilon > 0$, there exists a vicinity V in $\mathcal{V}$ such that $(x, x') \in V$ implies that

$$|\varphi(x\eta) - \varphi(x'\eta)| < \varepsilon.$$

By assumption 2, there will be a vicinity $V \in \mathcal{V}$ such that $|\varphi(xy) - \varphi(x'y)| < \varepsilon/3$ for every $y \in X$, if $(x, x') \in V$. Choose $(x, x') \in V$. Since

$$\varphi(x\eta) = \lim_{y \longrightarrow \eta} \varphi(xy),$$

and

$$\varphi(x'\eta) = \lim_{y \longrightarrow \eta} \varphi(x'y),$$

there exists $y \in X$ such that

$$|\varphi(x\eta)-\varphi(xy)|<\varepsilon/3 \qquad (1)$$

and

$$|\varphi(x'\eta)-\varphi(x'y)|<\varepsilon/3 \qquad (2).$$

We also have:

$$|\varphi(x'y)-\varphi(xy)|<\varepsilon/3 \qquad (3).$$

From (1),(2) and (3), we have

$$|\varphi(x\eta)-\varphi(x'\eta)|<\varepsilon.$$

Hence, under assumption 1 and assumption 2 the mapping $x \longmapsto x\eta$ is uniformly continuous for each η in $\tilde{X}$, so extends to a continuous mapping from $\tilde{X}$ to $\tilde{X}$.

The image of ξ under this mapping is denoted by $\xi\eta$ and so

$$\xi\eta = \underline{\mathrm{Lim}}_{x\to\xi}\ \underline{\mathrm{Lim}}_{y\to\eta}(xy)$$

It is easy to see that $\cdot$ is also associative. Hence $(\hat{X},\cdot)$ is a compact right topological semigroup.

4.1.2 Remark. If X has a uniform structure invariant under left and right translation, the semigroup structure extends to $\tilde{X}$, since assumption 1 and assumption 2 are automatically satisfied. In other words, the extension is possible if X has a base of vicinities U with the property that if, $(x,x')\in U$ then $(xy,x'y)\in U$ and $(yx,yx')\in U$ for every $y\in X$. This is the case if X has a metric d invariant under left and right translations.

4.1.3.Remark. For any topological group G, the group operation defined on G extends to a semigroup operation on $\tilde{G}$, when G has the right uniformity generated by entourages of the form $\{(x,y):xy^{-1}\in V\}$,where V is a neighbourhood of the identity of G.

4.1.4.Example. The following example will show that the operation of addition can not be extended to a semigroup operation on $\beta\mathbb{R}$.

Suppose that, for each element $\xi\in\beta\mathbb{R}$ and $y\in\mathbb{R}$, we define

$$y+\xi=\underline{\operatorname{Lim}}_{x\longrightarrow\xi}(y+x).$$

We shall show that there are elements of ξ of $\beta\mathbb{R}$ for which the mapping $y\longrightarrow y+\xi$ is discontinuous. In fact, let ξ be any element in $\mathrm{Cl}_{\beta\mathbb{R}}(\mathbb{N})\setminus\mathbb{R}$.

Choose a continuous real valued function $f:\mathbb{R}\longrightarrow[0,1]$ with the following properties: $f(n)=1$ if $n\in\mathbb{N}$, and $f(n+x)=0$ if $1/n<x<1-1/n$ for each $n\in\mathbb{N}$. Then for each $m\in\mathbb{N}$ $f((1/m)+n)=0$ for every $n\in\mathbb{N}$ satisfying $n>m$. Hence

$$f^{\beta}(1/m+\xi)=\operatorname{Lim}_{n\longrightarrow\xi}f(1/m+n)=0.$$

However,

$$f^{\beta}(\xi)=\operatorname{Lim}_{n\longrightarrow\xi}f(n)=1.$$

So

$$f^{\beta}(\xi)\neq\operatorname{Lim}_{m\longrightarrow\infty}f^{\beta}(1/m+\xi),$$

and so

$$\xi\neq\operatorname{Lim}_{m\longrightarrow\infty}(1/m+\xi)$$

in $\beta\mathbb{R}$.

4.1.5. Example. The semigroup structure on $(\mathbb{R},+)$ and on $(\mathbb{R}^n,+)$ can be extended to $\tilde{\mathbb{R}}$ and $\tilde{\mathbb{R}}^n$, respectively.

4.1.6. Theorem. The map $y\longmapsto xy$ from $\mathbb{R}$ to $\tilde{\mathbb{R}}$ for a fixed x in $\mathbb{R}$ extends to a mapping from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$, but this map does not extend further.

Proof. To see this, we shall show that the mapping is uniformly continuous.

Let $\varepsilon>0, x\neq 0$ and let $\delta=\varepsilon/|x|$. If

$$|y-y_1|<\delta$$

then

$$|xy_1-xy_2|\leq|x|\,|y_1-y_2|<|x|(\varepsilon/|x|)=\varepsilon.$$

Hence the map extends from $\mathbb{R}$ to a mapping from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$. We denote the image of ξ in $\tilde{\mathbb{R}}$ under this map by

$$x\xi=\underset{y\longrightarrow\xi}{\mathrm{Lim}}\, xy.$$

Now we will show that for a fixed $\eta\in\hat{\tilde{\mathbb{R}}}$, the map $x\longrightarrow x\eta$ is not uniformly continuous and hence it does not extend to $\hat{\tilde{\mathbb{R}}}$. To see this let A denote the set of positive integers of the form 1x3x5x7x...(2n+1) for some $n\in\mathbb{N}$ and let $\xi\in Cl_{\tilde{\mathbb{R}}}A\backslash\mathbb{R}$. Put $f(x)=\mathrm{Sin}(\frac{\pi x}{2})$. Hence $f^{\beta}(\xi)\in\{-1,1\}$. For any $m\in\mathbb{N}$ and any $n\in A$,

$$f(\frac{2m}{2m+1}n)=\mathrm{Sin}(\frac{2\pi m}{2m+1}n)=0$$

if n is a multiple of 2m+1.

Hence

$$f^{\beta}(\frac{2m}{2m+1}\xi)=\underset{n\longrightarrow\xi}{\mathrm{Lim}}\, f(\frac{2m}{2m+1}n)=0.$$

But $f^{\beta}(\xi)\neq 0$.

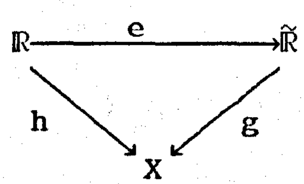
4.1.7.Theorem.The mapping $(x,\xi)\longmapsto x+\xi$ is a continuous mapping from $\mathbb{R}\times\bar{\mathbb{R}}$ to $\tilde{\mathbb{R}}$.

Proof.Let $Y\in x+\xi$ and $W\in\mathcal{B}(0)$. So $\mathcal{C}_{Y+W}$ is a basic neighbourhood of $x+\xi$, by proposition 2.3.9.

Choose any $V\in\mathcal{B}(0)$ for which $V+V\subseteq W$. We shall show that, if $t\in x+V$ and $\eta\in\mathcal{C}_{-x+Y+V}$, then $t+\eta\in\mathcal{C}_{Y+W}$. Now $-x+Y+V\in\eta$ which implies that $t-x+Y+V\in t+\eta$ and so $V+Y+V\in t+\eta$,since $t-x\in V$.Hence $Y+W\in t+\eta$ as claimed ∎

The next theorem shows that $\bar{\mathbb{R}}$ is the maximal compactification of $\mathbb{R}$ which has the continuity property described in the preceding theorem.

4.1.8.Theorem. Suppose that X is a compact right topological semigroup and that $h:\mathbb{R}\longrightarrow X$ is a continuous homomorphism for which $h(\mathbb{R})$ is dense in X. Suppose also that the mapping $(x,\xi)\longmapsto h(x)+\xi$ is a continuous mapping from $\mathbb{R}\times X$ into X. Then there is a continuous homomorphism $g:\bar{\mathbb{R}}\longrightarrow X$ for which the following diagram commutes:



Proof. If we can show that h is uniformly continuous, it will follow that there is a continuous function $g:\tilde{\mathbb{R}}\longrightarrow X$ for which $ge=h$.

Let $\varphi:X\longrightarrow[0,1]$ be a continuous function and let ε be a positive real number. We must show that, for some $W=(-\delta,\delta)\in\mathcal{B}(0)$ $(\delta\geq 0)$, $|x-y|<\delta$ implies that $|\varphi h(x)-\varphi h(y)|<\varepsilon$.

Now for each $\xi\in X$, there is a neighbourhood $N(\xi)$ of ξ and a set $W(\xi)\in\mathcal{B}(0)$ such that $|\varphi(h(w)+\eta)-\varphi(\xi)|<\frac{\varepsilon}{2}$ whenever $w\in W(\xi)$ and $\eta\in N(\xi)$. X will be covered by a finite number of the neighbourhoods $N(\xi)$. Choose $\xi_1,\xi_2,\xi_3,\ldots..\xi_n$ such that $X=\bigcup_1^n N(\xi_i)$, and put $W=\bigcap_1^n W(\xi_i)$, and say $W=(-\delta,\delta)$.

Suppose that $|x-y|<\delta$. If $h(y)\in N(\xi_i)$, we have

$$|\varphi(h(x-y)+h(y))-\varphi(\xi_i)|<\frac{\varepsilon}{2}$$

and

$$|\varphi(h(y))-\varphi(\xi_i)|<\frac{\varepsilon}{2}\ .$$

Thus

$$|\varphi(h(x))-\varphi(h(y))|<\varepsilon,$$

as required.

This shows that h is uniformly continuous, and hence the continuous function g exists.

We must still show that g is a homomorphism. This can be seen as follows:

For any $\lambda,\mu\in\tilde{\mathbb{R}}$,

$$g(\lambda+\mu)=g\Big(\lim_{e(l)\to\lambda}\ \lim_{e(m)\to\mu}\ g(e(l)+e(m))\Big)$$

$$=\lim_{e(l)\to\lambda}\ \lim_{e(m)\to\mu}\ ge(l+m)$$

$$=\lim_{e(l)\to\lambda}\ \lim_{e(m)\to\mu}\ h(l+m)$$

$$=\lim_{e(l)\to\lambda}\ \lim_{e(m)\to\mu}\ (h(l)+h(m))$$

$$=\lim_{e(l)\to\lambda}\ \lim_{e(m)\to\mu}\ (ge(l)+ge(m))$$

$$=g(\lambda)+g(\mu)\ \blacksquare$$

4.1.9.Remark. $\tilde{\mathbb{R}}$ is the maximal compactification of $\mathbb{R}$ with respect to this property.

4.1.10.Remark. The preceding two theorems have been stated for $\bar{\mathbb{R}}$. It is obvious, however, that they would apply to the compactification $\bar{G}$ of any topological group G. We remind the reader that as noted in the introduction the LUC-compactification of G is the maximal compactification of G with respect to the above property and therefore, $\tilde{G}$ is the LUC-compactification of G.

Section 2. Algebraic Properties of the Compact Right Topological Semigroup $(\tilde{\mathbb{R}},+)$

In section 1, we have seen that the semigroup structure of $(\mathbb{R},+)$ can be extended to $\bar{\mathbb{R}}$, which makes $\bar{\mathbb{R}}$ a right topological semigroup. We shall now study some properties of the compact right topological semigroup $(\bar{\mathbb{R}},+)$. Throughout this section we shall use $\mathcal{B}(0)$ to denote the set of symmetric neighbourhoods of 0 in $\mathbb{R}$.

4.2.1. **Proposition.** Let $\sigma=\xi+\eta$ in $\bar{\mathbb{R}}$. Then $Z\in\sigma$ if and only if, for every $W\in\mathcal{B}(0)$, there exists $X\in\xi$ such that $-x+Z+W\in\eta$ for every $x\in X$.

Proof. Suppose that there is a $W\in\mathcal{B}(0)$ such that

$X=\{x\in\mathbb{R} \mid -x+W+Z\in\eta\}\notin\xi$. Then there is a member V of ξ such that $X\cap V=\emptyset$. If $v\in V$, $-v+W+Z\notin\eta$. So $-v+(W+Z)^*\in\eta$, where $(W+Z)^*=\mathbb{R}\setminus(W+Z)$. Hence $v+\eta\in Cl_{\tilde{\mathbb{R}}}(W+Z)^*$; it follows that $\xi+\eta\in Cl_{\tilde{\mathbb{R}}}(W+Z)^*$. This contradicts the assumption that $Z\in\sigma$.

Conversely, suppose that, for every $W\in\mathcal{B}(0)$, $X_W=\{x\in\mathbb{R} \mid -x+W+Z\in\eta\}\in\xi$. Let $x\in X_W$ and let $y\in -x+W+Z$. So $x+y\in W+Z$. We can choose a net of values of y converging to η. Then $x+\eta\in Cl_{\tilde{\mathbb{R}}}(W+Z)$. Hence $\xi+\eta\in Cl_{\tilde{\mathbb{R}}}(W+Z)$. Therefore $W+Z\in\sigma$ for every $W\in\mathcal{B}(0)$, and so $Z\in\sigma$ ∎

4.2.2.Proposition. Let $\xi\in\rho\mathbb{R}\cap Cl_{\tilde{\mathbb{R}}}(x_n)$, where $(x_n)\subset\mathbb{R}$ and $x_{n+1}-x_n\longrightarrow\infty$ as $n\longrightarrow\infty$. Then ξ is right cancellable in $(\tilde{\mathbb{R}},+)$.

Proof. Suppose that ξ is not right cancellable. Then there are η,ζ in $\tilde{\mathbb{R}}$ such that $\eta\neq\zeta$ and $\eta+\xi=\zeta+\xi$. There is $Y\in\eta$ and $Z\in\zeta$ such that $d(Y,Z)=\delta>0$. We may suppose that $x_{n+1}-x_n>\delta$ for all n. If $y\in Y$ and $z\in Z$ with $y,z>0$ and if $y,z<\frac{1}{2}(x_{r+1}-x_r)$ for all $r>n$, then for any $r,s>n$,

$$|(y+x_r)-(z+x_s)|\geq\delta.$$

To see this,we may suppose that $s\geq r+1$, since the inequality clearly holds if $r=s$. Then

$$y+x_r<x_r+\frac{1}{2}(x_{r+1}-x_r),\ z+x_s>x_{r+1}+\frac{1}{2}(x_{r+1}-x_r)$$

and so

$$(z+x_s)-(y+x_r)>(x_{r+1}-x_r)>\delta.$$

For each $y\in Y$ and each $x\in Z$, let

$$X_y=\{x_{n_r}|x_{r+1}-x_r>2y \text{ for every } r\geq n\}$$

and let

$$X_z=\{x_{n_r}|x_{r+1}-x_r>2z \text{ for every } r\geq n\}.$$

Then

$$\bigcup_{y\in Y}(y+X_y)\in\eta+\xi,$$

and

$$\bigcup_{z\in Z}(z+X_z)\in\zeta+\xi.$$

However, let $x_r\in X_y$ and $x_s\in X_z$. Then

$$x_s<z+x_s<\frac{1}{2}(x_{s+1}-x_s)$$

and

$$x_r < y + x_r < \frac{1}{2}(x_{r+1} - x_r).$$

So by the preceding argument,

$$|(z+x_s)-(y+x_r)| \geq \delta.$$

Hence

$$d(\bigcup_{y \in Y}(x+X_y), \bigcup_{z \in Z}(z+X_z)) \geq \delta.$$

Contradiction ■

4.2.3.Proposition. Let $\xi, \eta \in \rho\mathbb{R}$ and $x \in \mathbb{R}$, then

$$x(\xi+\eta) = x\xi + x\eta.$$

Proof. It is obvious since

$$x(\xi+\eta) = \lim_{y \to \xi} \lim_{z \to \eta} x(y+z)$$

$$= \lim_{y \to \xi} \lim_{z \to \eta} (xy+xz)$$

and

$$x\xi+x\eta=\lim_{y\longrightarrow\xi}\ \lim_{z\longrightarrow\eta}(xy+xz)$$

4.2.4.Theorem. Let $\xi\in\rho\mathbb{R}$ and let $x\in\mathbb{R}$. Then

(a) There is $\eta\in\rho\mathbb{R}$ such that $x+\eta=\xi$.

(b) There is $\eta\in\rho\mathbb{R}$ such that $x\eta=\xi$ if $x\neq 0$.

Proof. (a) Put $\eta=-x+\xi$.

(b) Put $\eta=\frac{1}{x}\xi$ ∎

4.2.5.Proposition. Let $\eta\in\rho\mathbb{R}$. Then for $x,y\in\mathbb{R}$, $x+\eta=y+\eta$ implies that $x=y$.

Proof. Suppose that there is an $\eta\in\rho\mathbb{R}$ such that $x+\eta=y+\eta$, when $x\neq y$. We may suppose that $x>y$, and let $k=x-y$. Since $x+\eta=y+\eta$, $-y+x+\eta=\eta$, that is $k+\eta=\eta$. Thus if $A\in\eta$, $A\in k+\eta$ and $k+A\in k+\eta$. Since $k+A\in k+\eta$ and $A\in k+\eta$, $(k+A+W)\cap(A+W)\neq\emptyset$ for all $W\in\mathcal{B}(0)$.

By proposition.3.1.1,we can choose a sequence $(x_n)\subseteq\mathbb{R}$ such that $x_{n+1}-x_n>2k$ and $(x_n)\in\eta$. Let $W=(-k/4,k/4)$. Now we claim that $(\{x_n\}+W)\cap((k+\{x_n\})+W)=\emptyset$. If not there exists t in the intersection such that $t=x_{m_1}+w_1=k+x_{m_2}+w_2$, where $w_1,w_2\in W$ and $x_{m_1},x_{m_2}\in(x_n)$. So $x_{m_1}-x_{m_2}=k+w_1-w_2$. But this implies that $|x_{m_1}-x_{m_2}|\le k+(k/2)=3k/2$. But this is a contradiction since $x_{n+1}-x_n>2k$. Hence the intersection is empty, which is not possible since $(x_n)\in k+\eta$ and $(x_n)+k\in k+\eta$.

4.2.6. Proposition. For any uniformly continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ with the property that $|f(a+x)-f(x)|\longrightarrow 0$ as $x\longrightarrow\infty$, for each fixed $a\in\mathbb{R}$, we have $\tilde{f}(\xi+\eta)=\tilde{f}(\eta)$ for all $\xi,\eta\in\tilde{\mathbb{R}}^+$.

Proof. We can choose a net (y_α) in $\mathbb{R}$ which converges to η in $\tilde{\mathbb{R}}$. We want to show that for any uniformly continuous function $\varphi:\mathbb{R}\longrightarrow[0,1]$, $(\varphi(f(a+y_\alpha))$ and $(\varphi(f(y_\alpha))$ are nets with the same limit.

Since

$$|f(a+y_\alpha)-f(y_\alpha)|\longrightarrow 0 \quad \text{as } y_\alpha\longrightarrow\infty$$

for a given $\epsilon>0$, there exists $\alpha(\varepsilon)$ such that

$$|f(a+y_\alpha)-f(y_\alpha)|<\varepsilon$$

when $\alpha>\alpha(\varepsilon)$. Hence, by the uniform continuity of φ, for a given $\varepsilon>0$, there exists $\beta(\alpha)$ such that

$$|\varphi(f(a+y_\alpha)-\varphi(f(y_\alpha)|<\varepsilon$$

when $\alpha>\beta(\alpha)$. Therefore,

$$\operatorname{Lim}_\alpha \varphi(f(a+y_\alpha))=\operatorname{Lim}_\alpha \varphi(f(y_\alpha)).$$

Therefore, if $\tilde{\varphi}:\tilde{\mathbb{R}}\longrightarrow[0,1]$ denotes the extension of φ, then $\tilde{\varphi}(\tilde{f}(a+\eta))=\tilde{\varphi}(\tilde{f}(\eta))$. Hence $\tilde{f}(a+\eta)=\tilde{f}(\eta)$ for every $a\in\mathbb{R}$, and so $\tilde{f}(\xi+\eta)=\tilde{f}(\eta)$ ■

4.2.7. **Proposition.** Let $x,y,z\in\mathbb{R}$ such that $x\geq 0, y,z\geq 1$ and $\xi\in\rho\mathbb{R}\cap\mathrm{Cl}_{\tilde{\mathbb{R}}}[0,\infty)$. Then $x\xi+y\xi=z\xi$ implies that $y=z$.

Proof. We may suppose that $[1,\infty)\in\xi$. By the above proposition $Lg(\xi+\eta)=Lg(\eta)$ for $\eta\in\rho\mathbb{R}\cap\mathrm{Cl}_{\tilde{\mathbb{R}}}[1,\infty)$. If we apply the Lg function to the equation $x\xi+y\xi=z\xi$, we get

$$Lg(y\xi)=Lg(z\xi).$$

Hence

$$Lg(y)+Lg(\xi)=Lg(z)+Lg(\xi).$$

By proposition 4.2.5,

$$Lg(y)=Lg(z).$$

Therefore, $y=z$ since Lg is one-to one on $[1,\infty)$ ■

4.2.8. Proposition. Let $\xi\in\rho\mathbb{R}\cap Cl_{\tilde{\mathbb{R}}}[0,\infty)$, and let $x,y\in\mathbb{R}$ such that $x>0, y\geq 1$. Then $x\xi+y\xi\neq(x+y)\xi$.

Proof. This follows from proposition 4.2.7, since it implies $y=x+y$ if $x\xi+y\xi=(x+y)\xi$.

4.2.9. Theorem. Let $\xi\in Cl_{\tilde{\mathbb{R}}}[0,\infty)\cap\tilde{\mathbb{R}}$, and let $x,y\in\mathbb{R}$ such that $x,y\geq 1$. Then, $x\xi=y\xi$ if and only if $x=y$.

Proof. Clearly $x\xi=y\xi$ if $x=y$.

Now suppose that $x\xi=y\xi$, then $\lg x\xi=\lg y\xi$ and so $\lg x+\lg\xi=\lg y+\lg\xi$. By proposition 4.2.5, $\lg x=\lg y$ and so $x=y$ ■

4.2.10. Proposition. The semigroup $(\beta\mathbb{N},+)$ can be embedded in $(\tilde{\mathbb{R}},+)$.

Proof. Let φ be the mapping from $\mathbb{N}$ to $\mathbb{R}$ such that $\varphi(n)=n$. Since φ is continuous it extends to a continuous mapping φ^β from $\beta\mathbb{N}$ into $\tilde{\mathbb{R}}$. We will show that φ^β is one to one. To see this, take $\xi,\eta\in\beta\mathbb{N}$ with $\xi\neq\eta$. Then there are two subsets X and Y of $\mathbb{N}$ such that $X\in\xi$, $Y\in\eta$ and $X\cap Y=\emptyset$. We define a continuous function $f:\mathbb{R}\longrightarrow[0,1]$, stating that

$$f(x)=\begin{cases} 1 \text{ , if } x\in\varphi(X) \\ 0 \text{ , if } y\in\varphi(Y) \end{cases}$$

and we extend f to $\tilde{\mathbb{R}}$ by piecewise linearity. Then for any two real numbers r,s,

$$|f(r)-f(s)|\leq|r-s|.$$

Hence f is uniformly continuous and so extends to a mapping $\tilde{f}$ from $\tilde{\mathbb{R}}$ into [0,1]. Since $\tilde{f}(x)=1$ on $Cl_{\tilde{\mathbb{R}}}\varphi(X)$ and $\tilde{f}(x)=0$ on $Cl_{\tilde{\mathbb{R}}}\varphi(Y)$, $Cl_{\tilde{\mathbb{R}}}\varphi(X)\cap Cl_{\tilde{\mathbb{R}}}\varphi(Y)=\emptyset$. Hence $\varphi^\beta(\xi)\neq\varphi^\beta(\eta)$, since $\varphi^\beta(\xi)\in Cl_{\tilde{\mathbb{R}}}\varphi(X)$ and $\varphi^\beta(\eta)\in Cl_{\tilde{\mathbb{R}}}\varphi(Y)$. Also, $\varphi^\beta(\xi)+\varphi^\beta(\eta)=\varphi^\beta(\xi+\eta)$; that is, φ^β is a homomorphism. This can be seen as follows:

$$\xi+\eta=\underline{\mathrm{Lim}}_{m\longrightarrow\xi}\ \underline{\mathrm{Lim}}_{n\longrightarrow\eta}(m+n)$$

So

$$\varphi^\beta(\xi+\eta)=\underline{\mathrm{Lim}}_{m\longrightarrow\xi}\ \underline{\mathrm{Lim}}_{n\longrightarrow\eta}\varphi((m+n))$$

$$=\underset{m\longrightarrow\xi}{\mathrm{Lim}}\ \underset{n\longrightarrow\eta}{\mathrm{Lim}}(\varphi(m)+\varphi(n))$$

$$=\varphi^{\beta}(\xi)+\varphi^{\beta}(\eta)_{\blacksquare}$$

We have the following corollary immediately, since the cardinality of $\beta\mathbb{N}$ is 2^{c} by theorem 1.3 of [50].

4.2.11.Corollary. $\tilde{\mathbb{R}}$ has 2^{c} points.

Now we will prove the following property that will be used in future propositions.

4.2.12.Proposition. Let (x_n) be a sequence in $\mathbb{R}$ with the property that $x_{n+1}-x_n \longrightarrow \infty$. Then, if $\{x_n\}\in\xi$, $\xi\notin\rho\mathbb{R}+\rho\mathbb{R}$.

Proof. Suppose that $\xi=\lambda+\mu$, where $\lambda,\mu\in\rho\mathbb{R}$. From proposition 4.2.1, for every $W\in\mathcal{B}(0)$,

$$\{x\in\mathbb{R}\,|\,-x+(x_n)+W\in\mu\}\in\lambda.$$

Let $W=(-1/4,1/4)$ and choose two distinct values $x,x'\in\mathbb{R}^{+}$ such that

$$-x+(x_n)+W\in\mu,$$

$$-x'+(x_n)+W\in\mu,$$

and $x'-x>1$. Now suppose that, if $n>n_0$, $x_{n+1}-x_n>x+x'+1$. Then

$$-x+(x_n)_{n>n_0}+W\in\mu \text{ and } -x'+(x_n)_{n>n_0}+W\in\mu.$$

if $m,n>n_0$ and $m\neq n$, then

$$|(x_m-x)-(x_n-x')|\geq|x_m-x_n|-x-x'>1.$$

If $m=n$,

$$|(x_m-x)-(x_n-x')|=|x'-x|>1,$$

that is, $-x+(x_n)_{n>n_0}$ and $-x'+(x_n)_{n>n_0}$ have a distance apart at least equal to 1. Hence

$$(-x+(x_n)_{n>n_0}+W+W)\cap(-x'(x_n)_{n>n_0}+W+W)=\emptyset.$$

But this contradicts the assumption that

$$-x+(x_n)_{n>n_0}+W\in\mu \text{ and } -x'+(x_n)_{n>n_0}+W\in\mu_\blacksquare$$

4.2.13.Proposition. Let (x_n) be a sequence in $\mathbb{R}$ and let $\xi\in\rho\mathbb{R}+\rho\mathbb{R}$. If $(x_n)\in\xi$, then there is a real number $b\in\mathbb{R}$ such that $|x_{n+1}-x_n|\leq b$ infinitely often.

Proof. Suppose that $\xi\in\rho\tilde{\mathbb{R}}+\rho\tilde{\mathbb{R}}$ and $\{x_n\}\in\xi$, but that for any $b\in\mathbb{R}$

$$|x_{n+1}-x_n|>b$$

for all but a finite number of values of n. Then $x_{n+1}-x_n \longrightarrow \infty$ or $x_{n+1}-x_n \longrightarrow -\infty$; but this contradicts the assumption that ξ is in $\rho\mathbb{R}+\rho\mathbb{R}$, by proposition 4.2.12.∎

4.2.14. **Proposition.** $(\tilde{\mathbb{R}},+)$ has 2^c disjoint left ideals.

Proof. To prove this, we will first show that the function Lg assumes 2^c distinct values on $\tilde{\mathbb{R}}$.

Consider the Lg function from $\mathbb{R}^+$ into $\tilde{\mathbb{R}}^+$. Clearly $\mathbb{R}^+\subseteq Lg\mathbb{R}^+\subseteq lg\tilde{\mathbb{R}}^+\subseteq\tilde{\mathbb{R}}^+$. since $\mathbb{R}^+$ is dense in $\tilde{\mathbb{R}}^+$, $Lg\tilde{\mathbb{R}}^+$ is dense in $\tilde{\mathbb{R}}^+$ and since $Lg\tilde{\mathbb{R}}^+$ is compact, $Lg\tilde{\mathbb{R}}^+$ is the space $\tilde{\mathbb{R}}^+$, therefore, the cardinality of $Lg\tilde{\mathbb{R}}^+$ is 2^c, since the cardinality of $\tilde{\mathbb{R}}^+$ is 2^c. Hence the Lg function assumes 2^c distinct values on $\tilde{\mathbb{R}}$.

Now let $\xi_1,\xi_2\in\tilde{\mathbb{R}}\cap Cl_{\tilde{\mathbb{R}}}(1,\infty)$ be such that $Lg\xi_1\neq Lg\xi_2$. Then $(\tilde{\mathbb{R}}+\xi_1)\cap(\tilde{\mathbb{R}}+\xi_2)=\emptyset$. If not, there exists $\eta_1,\eta_2\in\tilde{\mathbb{R}}$ such that $\eta_1+\xi_1=\eta_2+\xi_2$. By proposition 4.2.6, $Lg\xi_1=Lg\xi_2$, which is a contradiction. So the intersection is empty. Since the Lg function assumes 2^c distinct values on $Cl_{\tilde{\mathbb{R}}}(1,\infty)\cap\tilde{\mathbb{R}}$, there are 2^c distinct points ξ_1,ξ_2 in $\tilde{\mathbb{R}}$ with the property that $lg\xi_1 \neq Lg\xi_2$. Hence $(\tilde{\mathbb{R}},+)$ has 2^c disjoint left ideals ■

In the following theorem, we shall regard $\mathbb{Z}$ as embedded in $\tilde{\mathbb{R}}$. We shall use $\mathbb{Z}^*$ to denote $\beta\mathbb{Z}\setminus\mathbb{Z}$. Our theorem gives a decomposition for $\tilde{\mathbb{R}}$ which is analogous to the decomposition of a real number as the sum of a fractional part and an integer.

4.2.15. Theorem [16]. Each $\xi\in\rho\mathbb{R}$ can be expressed uniquely as $\xi=x+\mu$ for some $x\in[0,1)$ and some $\mu\in\mathbb{Z}^*$.

This establishes a bijection between $\rho\mathbb{R}$ and $[0,1)x\mathbb{Z}^*$. The mapping $\xi\longmapsto(x,\mu)$ from $\rho\mathbb{R}$ to $[0,1)x\mathbb{Z}^*$ defined in this way, is continuous on $\rho\mathbb{R}\setminus\mathbb{Z}^*$.

Proof. Let $\xi\in\rho\mathbb{R}$. We may suppose that $\xi\in Cl_{\tilde{\mathbb{R}}}[0,\infty)$. We have seen that there is a sequence (x_n) in $\mathbb{R}$ for which $(x_n)\in\xi$ and $x_{n+1}-x_n\geq 1$ for every n. The mapping $n\longmapsto\{x_n\}$, where $\{x_n\}$ denotes the fractional part of x_n, extends to a continuous mapping $\varphi:\beta\mathbb{N}\longrightarrow[0,1]$.

We have also seen that the mapping $n\longmapsto x_n$ extends to a homeomorphism f from $\beta\mathbb{N}$ onto $Cl_{\tilde{\mathbb{R}}}\{x_n:n\in\mathbb{N}\}$.We choose x to be $\varphi f^{-1}(\xi)$.

Given $W\in\mathcal{B}(0)$, $\{n\in\mathbb{N}:\varphi(n)\in x+W\}\in f^{-1}(\xi)$; and so $\{x_n:\{x_n\}\in x+W\}$ being the image under f of this set,must be in ξ. Hence $\{x_n:x_n-x\in\mathbb{Z}+W\}$ is in ξ, because $x_n-\{x_n\}$ is , of course, in $\mathbb{Z}$ for each n.

If $X\in\xi$, X+W will contain a number x_n for which $x_n-x\in\mathbb{Z}+W$. This shows that $(X-x)\cap(\mathbb{Z}+W)\neq\emptyset$ and hence that $\mathbb{Z}\in\xi-x$. So $\xi-x\in\mathbb{Z}^*$, as required, and we have $\xi=x+\mu$ for some $x\in[0,1]$ and $\mu\in\mathbb{Z}^*$. In the case in which x=1, we replace x by 0 and μ by $1+\mu$. Hence we can assume that $x\in[0,1)$.

We shall show that this expression for ξ is unique. Suppose that $x+\mu=y+\zeta$, where $x,y\in[0,1)$ and $\mu,\zeta\in\mathbb{Z}^*$. If $x\neq y$, $0<|x-y|<1$. However, for any $W\in\mathcal{B}(0)$,

$$(x+\mathbb{Z}+W)\cap(y+\mathbb{Z}+W)\neq\emptyset$$

and so

$$x-y\in\mathbb{Z}+W+W.$$

It follows that $x-y\in\mathbb{Z}$ which is impossible. This establishes that x=y and hence that $\mu=\zeta$.

It is now clear that the mapping $\xi \longmapsto (x,\mu)$ from $\tilde{\mathbb{R}}$ to $[0,1) \times \mathbb{Z}^*$ is bijective.

To show that this is continuous on $\tilde{\mathbb{R}} \setminus \mathbb{Z}^*$, let $\xi \in \tilde{\mathbb{R}}$ satisfy $\xi = x + \mu$ for some $x \in (0,1)$ and some $\mu \in \mathbb{Z}^*$. Let $A \in \mu$ and $W \in \mathcal{B}(0)$. Choose $V \in \mathcal{B}(0)$ satisfying $x+V+V \subseteq (0,1)$, $V \subseteq (-1/4, 1/4)$ and $V+V \subseteq W$. Suppose that η is in the neighbourhood $\tilde{C}_{V(x+A)}$ of ξ and that $\eta = y + \zeta$, where $y \in [0,1)$ and $\zeta \in \mathbb{Z}^*$. Since $y + \mathbb{Z} \in \eta$ and $x+A+V \in \eta$,

$$(y+\mathbb{Z}+V) \cap (x+A+V) \neq \emptyset$$

Thus $y-x \in \mathbb{Z}+V+V$. Now

$$1+x+V+V \subseteq (1,\infty)$$

and

$$-1+x+V+V \subseteq (-\infty,0).$$

So y is not in either of these sets, and so $y-x \in V+V \subseteq W$.

Now, if $B \in \zeta$, where $B \subseteq \mathbb{Z}$ we have $y+B \in \eta$ and hence

$$(y+B+V) \cap (x+A+V) \neq \emptyset.$$

thus

$$A \cap (B+V+V+V+V) \neq \emptyset$$

and so $A\cap B\neq\emptyset$. It follows that $A\in\zeta$. Since $x+W$ is a basic neighbourhood of x and $\{\zeta\in\mathbb{Z}^*:A\in\zeta\}$ is a basic neighbourhood of μ, we have shown that the mapping $\xi\longmapsto(x,\mu)$ is continuous at ξ ■

4.2.16.Corollary. All the idempotents of $\tilde{\mathbb{R}}$ other than 0 lie in $\mathbb{Z}^*$.

Proof. Suppose that ξ is an idempotent in $\tilde{\mathbb{R}}$ and $\xi\neq 0$. Then, clearly $\xi\in\rho\mathbb{R}$. By theorem 4.2.15, ξ can be expressed uniquely as $x+\mu$, Where $x\in[0,1)$ and $\mu\in\mathbb{Z}^*$. If $x=0$ then the proof is obvious. Suppose that $x\neq 0$, then

$$(x+\mu)+(x+\mu)=x+\mu$$

which implies that

$$\mu+x+\mu=\mu,$$

Since $\mathbb{R}$ is in the center of $\tilde{\mathbb{R}}$, $\mu+x=x+\mu$ for every $x\in\mathbb{R}$, therefore, $x+\mu+\mu=\mu$. Hence $\mathbb{Z}\in x+\mu+\mu$ as $\mathbb{Z}\in\mu$, and so $\mathbb{Z}-x\in\mu+\mu$. Therefore, by the definition of addition on $\tilde{\mathbb{R}}$, for every $W\in B(0)$, there exists $T\in\mu$ such that $-t+\mathbb{Z}-x+W\in\mu$ and so $\mathbb{Z}-x+W\in\mu$ since $-t+\mathbb{Z}=\mathbb{Z}$. Let $W=(\frac{-x}{4},\frac{x}{4})$, then $((\mathbb{Z}-x)+W+W)\cap(\mathbb{Z}+W)=\emptyset$ which is is a contradiction. Hence $x=0$.

4.2.17. Corollary. Let $\xi\in\rho\mathbb{R}$ and $x\in\mathbb{R}\setminus\{0\}$ and suppose that ξ is an idempotent. Then $(x+\xi)$ is not an idempotent.

Proof. Since $\mathbb{R}$ is in the centre of $\tilde{\mathbb{R}}$, for all $x\in\mathbb{R}$ and $\xi\in\rho\mathbb{R}$, $x+\xi=\xi+x$. So

$$(x+\xi)+(x+\xi)=x+(\xi+x)+\xi$$

$$=x+(x+\xi)+\xi$$

$$=x+x+\xi+\xi$$

$$=x+x+\xi.$$

Suppose that $(x+\xi)$ is an idempotent. Then $x+\xi=x+x+\xi$ and so by the proposition 4.2.5, $x=x+x$ which is impossible since $x\neq 0$ ∎

4.2.18. Corollary. Every left (right, two-sided) ideal of $\tilde{\mathbb{R}}$ has the form $(\mathbb{R}+L)$ $(\mathbb{R}+R,\ \mathbb{R}+I)$, where L (R,I) is a left (right, two-sided) ideal in $\mathbb{Z}^*$.

If M denotes the minimum ideal of $\mathbb{Z}^*$, the minimum ideal of $\tilde{\mathbb{R}}$ will be $\mathbb{R}+M$.

Proof. Let K be a left ideal in $\tilde{\mathbb{R}}$. Then if $\xi\in K$, $\xi=x+\eta_1$

for some $x\in[0,1)$ and $\eta_1\in\mathbb{Z}^*$. Let

$$L=\{\eta\in\mathbb{Z}^*: x+\mu\in K \text{ for some } x\in\mathbb{R}, \mu\in\mathbb{Z}^*\}$$

Then L is a left ideal of $\mathbb{Z}^*$. To see this let $\eta_1\in\mathbb{Z}^*$. If $\eta\in L$, $x+\eta\in K$ for some $x\in[0,1)$ and so

$$\eta_1+(x+\eta)=x+\eta_1+\eta\in K,$$

since K is a left ideal and $x+\eta\in K$. Hence $\eta_1+\eta\in L$ so $\mathbb{Z}^*+L\subset L$, and so $K\subseteq\mathbb{R}+L$ (1)

To see that $\mathbb{R}+L\subseteq K$. Let $x+\mu\in\mathbb{R}+L$, There is $x_1\in\mathbb{R}$ such that $x_1+\mu\in K$. Since K is a left ideal, for every $y\in\mathbb{R}$, $y+x_1+\mu\in K$ and so $(x-x_1)+x_1+\mu\in K$, so $x+\mu\in K$. Hence $\mathbb{R}+L\subseteq K$ (2). From (1) and (2), we obtain $\mathbb{R}+L=K$.

If M' is the minimum ideal of $\tilde{\mathbb{R}}$. Then $M'=\bigcup L'$, L' is a minimal left ideal of $\tilde{\mathbb{R}}$. But $L'=\mathbb{R}+L$ for some left ideal of $\mathbb{Z}^*$. Hence $M'=\bigcup(\mathbb{R}+L)=\mathbb{R}+\bigcup L=\mathbb{R}+M$, where M is the minimum ideal of $\mathbb{Z}^*$ ■

4.2.19. **Theorem.** Let M be a left (right) ideal in $\tilde{\mathbb{R}}$ such that $M=\mathbb{R}+M_1$, where M_1 is a left (right) ideal in $\mathbb{Z}^*$. Then M is a minimal left ideal of $\tilde{\mathbb{R}}$ if and only if M_1 is a minimal left ideal of $\mathbb{Z}^*$.

Proof. Suppose that M is a minimal left ideal of $\tilde{\mathbb{R}}$ and suppose on the contrary that M_1 is not a minimal left ideal of $\mathbb{Z}^*$. Then there is a left ideal L_1 in $\mathbb{Z}^*$ such that $L_1 \subset M_1$. then $\mathbb{R}+L_1 \subset \mathbb{R}+M_1 = M$. Now $\mathbb{R}+L_1$ is a left ideal in $\tilde{\mathbb{R}}$. To see this let $\eta \in \tilde{\mathbb{R}}$ and let $\eta_1 \in \mathbb{R}+L_1$, where $\eta = x+\eta'$, $\eta_1 = x_1+\eta_1'$, $x \in [0,1)$, $x_1 \in \mathbb{R}$ and $\eta', \eta_1' \in \mathbb{Z}^*$. Then

$$\eta+\eta_1 = x+\eta' + x_1 + \eta_1$$

$$= x+x_1+\eta'+\eta_1'.$$

Since L_1 is a left ideal in $\mathbb{Z}^*$ and $\eta' \in \mathbb{Z}^*$, $\eta'+\eta_1' \in L_1$. So $x+x_1+\eta'+\eta_1' \in L_1$. Hence $\mathbb{R}+L_1$ is a left ideal in $\tilde{\mathbb{R}}$. This is a contradiction since M is a minimal left ideal in $\tilde{\mathbb{R}}$.

Now suppose that M_1 is a minimal left ideal in $\mathbb{Z}^*$. Then it is easy to see that $\mathbb{R}+M_1$ is a left ideal in $\tilde{\mathbb{R}}$. To see that M is minimal in $\tilde{\mathbb{R}}$, suppose that there is a left ideal L in $\tilde{\mathbb{R}}$ such that $L = \mathbb{R}+L_1 \subset M = \mathbb{R}+M_1$, where L_1 is a left ideal in $\mathbb{Z}^*$. Then $\mathbb{R}+L_1 \subseteq \mathbb{R}+M_1$ which implies that $L_1 \subseteq M_1$, but this contradicts the assumption that M_1 is a minimal left ideal since L_1 is a left ideal in $\mathbb{Z}^*$ ∎

4.2.20.Corollary. Every minimal left (respectively, right) ideal of $\tilde{\mathbb{R}}$ contains 2^c idempotents and so $\tilde{\mathbb{R}}$ contains 2^c minimal left and right ideals.

Proof.If L is a minimal left (right) ideal in $\tilde{\mathbb{R}}$ and $L=\mathbb{R}+L_1$, by the preceding theorem L_1 is a minimal left (right) ideal of $\mathbb{Z}^*$ and since $\mathbb{Z}^*$ is an ideal of $(\beta\mathbb{Z},+)$ by lemma 7.2 of [26], L_1 is a minimal left (right) ideal of $\beta\mathbb{Z}$.Since any minimal left (right) ideal of $\beta\mathbb{Z}$ contains 2^c idempotents according to corollary 2.6 in [29], L_1 contains 2^c idempotents and since $L_1\subseteq\mathbb{R}+L_1=L$, L contains 2^c idempotents.

Since the intersection of a minimal left and a minimal right ideal is a group with only one idempotent according to theorem in [5],the conclusion about the number of minimal left and minimal right ideals follows$_\blacksquare$

4.2.21.**Proposition.** Let M be the minimum ideal of $\tilde{\mathbb{R}}$. Then, for every $x\in\mathbb{R}$,$x+M=M+x=M$.

Proof.Since $M=\mathbb{R}+M_1$, where M_1 is a minimal ideal of $\mathbb{Z}^*$,

$$x+M=x+\mathbb{R}+M_1=\mathbb{R}+M_1=M$$

and since $\mathbb{R}$ is in the center of $\tilde{\mathbb{R}}$, $x+M=M+x$ so $M+x=\mathbb{R}+M_1=M_\blacksquare$

4.2.22.**Corollary.**If μ,ν are elements of $\mathbb{Z}^*$ which define disjoint principal left (right) ideals in $\mathbb{Z}^*$, then they also define disjoint principal left (right) ideals in $\tilde{\mathbb{R}}$.

Thus $\tilde{\mathbb{R}}$ has 2^c disjoint left ideals and 2^c disjoint right ideals.

Proof. Suppose that μ and ν define disjoint principal left ideals in $\mathbb{Z}^*$ and suppose that

$$(\tilde{\mathbb{R}}+\mu)\cap(\tilde{\mathbb{R}}+\nu)\neq\emptyset.$$

Then there exists $\xi_1, \xi_2\in\tilde{\mathbb{R}}$ such that $\xi_1+\mu=\xi_2+\nu$. Therefore, $\xi_1=x_1+\zeta_1$, $\xi_2=x_2+\zeta_2$, $x_1,x_2\in[0,1),\zeta_1,\zeta_2\in\mathbb{Z}^*$. Hence

$$x_1+\zeta_1+\mu=x_2+\zeta_2+\nu \qquad (1)$$

which implies that

$$x_1-x_2+\zeta_1+\mu=\zeta_2+\nu\in\mathbb{Z}^*.$$

So $x_1-x_2=0$, that is $x_1=x_2$. From (1) we have

$$\zeta_1+\mu=\zeta_2+\nu$$

But this is a contradiction since

$$(\mathbb{Z}^*+\mu)\cap(\mathbb{Z}^*+\nu)\neq\emptyset_{\blacksquare}$$

4.2.23. Corollary. If two principal left ideals of $\rho\mathbb{R}$ are not disjoint, one is contained in the other.

Proof. Suppose that $(\rho\mathbb{R}+\xi)\cap(\rho\mathbb{R}+\eta)\neq\emptyset$ and let $\xi=x+\xi'$, $\eta=y+\eta'$, where $y,x\in[0,1),\xi',\eta'\in\mathbb{Z}^*$. Then there exists $\xi_1,\eta_1\in\rho\mathbb{R}$ such that $\xi_1+\xi=\eta_1+\eta$. Choose $x_1,y_1\in[0,1)$ and $\xi_1',\eta_1'\in\mathbb{Z}^*$ such that

$$\xi_1=x_1+\xi_1',\eta_1=y_1+\eta_1'.$$

Then

$$(x_1+x)+\xi_1'+\xi'=(y_1+y)+\eta_1'+\eta'$$

and so

$$(x_1+x)-(y_1+y)+\xi_1'+\xi'=\eta_1'+\eta'\in\mathbb{Z}^*.$$

But this implies that $(x_1+x)-(y_1+y)\in\mathbb{Z}$. Say $z=(x_1+x)-(y_1-y)$, then

$$z+\xi_1'+\xi'=\eta_1'+\eta'.$$

Let $\zeta=z+\xi_1'$, then $\zeta\in\mathbb{Z}^*$ and $\zeta+\xi'=\eta_1'+\eta'$. Therefore, according to corollary in [58]

$$(\mathbb{Z}^*+\xi')\cap(\mathbb{Z}^*+\eta')\neq\emptyset.$$

Hence $\xi' \in \mathbb{Z}^*+\eta'$ or $\eta' \in \mathbb{Z}^*+\xi'$, and so $\xi \in \tilde{\mathbb{R}}+\eta$ or $\eta \in \tilde{\mathbb{R}}+\xi$ ∎

4.2.24.Corollary. $\rho\mathbb{R}+\rho\mathbb{R}$ is nowhere dense in $\rho\mathbb{R}$.

Proof. Suppose that there is a non-empty open subset $\tilde{U}$ contained in $Cl_{\tilde{\mathbb{R}}}(\rho\mathbb{R}+\rho\mathbb{R})$. Let $\xi \in \tilde{U}$, then $\tilde{U}$ is a neighbourhood of ξ, hence by proposition 2.3.9 there is a subset U of $\mathbb{R}$, $U \in \xi$ and $W \in \mathcal{B}(0)$ such that

$$\tilde{U}=\{\eta \in \rho\mathbb{R} \mid U+W \in \eta\}$$

By theorem 4.2.15 we can write ξ uniquely as $x+\xi'=\xi$, where $x \in [0,1)$ and $\xi' \in \mathbb{Z}^*$. Let

$$\tilde{U}_1=\{\eta' \in \rho\mathbb{R} \mid (U-x)+W \in \eta'\}$$

then $\tilde{U}_1$ is open neighbourhood of ξ' in $\rho\mathbb{R}$ since $(U-x) \in \xi'$.

Let $V=\tilde{U}_1 \cap \mathbb{Z}^*$, then V is open in $\mathbb{Z}^*$ and is not empty since $\xi' \in V$. We claim that $V \subset \mathbb{Z}^*+\mathbb{Z}^*$. To see this let $\eta' \in V$ and so $\eta' \in \tilde{U}_1$ and $\eta' \in \mathbb{Z}^*$. Therefore, $(U-x)+W \in \eta'$ and so $U+W \in x+\eta'$ which implies that $x+\eta' \in \tilde{U} \subset \rho\mathbb{R}+\rho\mathbb{R}$. Then there are $\eta_1,\eta_2 \in \rho\mathbb{R}$ such that $x+\eta'=\eta_1+\eta_2$. By theorem 4.2.15, we can write $\eta_1=x_1+\eta_1'$, $\eta_2=x_2+\eta_2'$, $x_1,x_2 \in [0,1)$ and $\eta_1',\eta_2' \in \mathbb{Z}^*$. Hence

$$x+\eta'=x_1+\eta_1'+x_2+\eta_2'$$

$$=x_1+x_2+\eta'_1+\eta'_2$$

$$\eta'=(x_1+x_2-x)+\eta'_1+\eta'_2.$$

Let $t=x_1+x_2-x$, then $\eta'=t+\eta'_1+\eta'_2$. There is an infinite subset A of $\mathbb{Z}$ such that $A\in\eta'$ and so $A\in t+\eta'_1+\eta'_2$ and so $A-t\in\eta'_1+\eta'_2$ which implies that $A-t\subseteq\mathbb{N}$. Hence $t+\eta'_1\in\mathbb{Z}^*$. Therefore, $\eta'=\eta'_1+\eta'_2\in\mathbb{Z}^*+\mathbb{Z}^*$, this shows that $\tilde{U}_1\subseteq\mathbb{Z}^*+\mathbb{Z}^*$. But this is a contradiction since $\mathbb{Z}^*+\mathbb{Z}^*$ is nowhere dense in $\mathbb{Z}^*$ according to theorem 4.2 in [58].

4.2.25. Corollary. Suppose that $\xi\in\rho\mathbb{R}$ can be written as $\xi=x+\mu$, where $x\in[0,1)$ and $\mu\in\mathbb{Z}^*$. Then ξ is right (left) cancellable in $\rho\mathbb{R}$ if and only if μ is right (left) cancellable in $\mathbb{Z}^*$.

Proof. Suppose that $\xi=x+\mu$ is right cancellable in $\rho\mathbb{R}$ and suppose that μ is not right cancellable in $\mathbb{Z}^*$. There are $\mu_1,\mu_2\in\mathbb{Z}^*$ such that $\mu_1+\mu=\mu_2+\mu$, $\mu_1\neq\mu_2$. Since $x+\mu_1+\mu_2=x+\mu_1+\mu_2$ $\mu_1+x+\mu=\mu_2+x+\mu$ and this implies that $\mu_1+\xi=\mu_2+\xi$. Since $\mu_1,\mu_2\in\mathbb{Z}^*\subset\rho\mathbb{R}$, this is a contradiction.

Conversely suppose that ξ is not right cancellable in $\rho\mathbb{R}$. Then there exist $\xi_1,\xi_2\in\rho\mathbb{R}$ such that $\xi_1\neq\xi_2$ and $\xi_1+\xi=\xi_2+\xi$. Let $\xi_1=x_1+\mu_1$ and $\xi_2=x_2+\mu_2$, $\xi=x+\mu$. So

$$x_1+\mu_1+x+\mu=x_2+\mu_2+x+\mu.$$

But this implies

$$x_1+\mu_1+\mu=x_2+\mu_2+\mu$$

$$(x_1-x_2)+\mu_1+\mu=\mu_2+\mu$$

and so $x_1-x_2\in\mathbb{Z}$. Since $o\leq|x_1-x_2|<1$, $x_1-x_2=0$ and so $\mu_1+\mu=\mu_2+\mu$. Since $\xi_1\neq\xi_2$ and $x_1= x_2$, $\mu_1\neq\mu_2$. This contradicts the assumption that μ is right cancellable in $\mathbb{Z}^*$∎

4.2.26.Corollary. If $\xi\in\rho\mathbb{R}\setminus(\rho\mathbb{R}+\rho\mathbb{R})$, ξ is right cancellable in $\rho\mathbb{R}$.

Proof. Let $\xi\in\rho\mathbb{R}\setminus(\rho\mathbb{R}+\rho\mathbb{R})$. Then $\xi=x+\mu$, $x\in[0.1)$, $\mu\in\mathbb{Z}^*$.Since $\xi\in\rho\mathbb{R}\setminus(\rho\mathbb{R}+\rho\mathbb{R})$, $\mu\in\mathbb{Z}^*\setminus(\mathbb{Z}^*+\mathbb{Z}^*)$. Since the set of right cancellable elements of $\mathbb{Z}^*$ contains $\mathbb{Z}^*\setminus(\mathbb{Z}^*+\mathbb{Z}^*)$ by [58], μ is right cancellable and so by the corollary 4.2.25, is right cancellable∎

4.2.27.Corollary. Let L (respectively,R,C) denote the set of left (respectively,right,cancellable) cancellable element of $\rho\mathbb{R}$, then $L=[0,1)+L_1$ (respectively,$R=[0,1)+R_1$,

$C=[0,1)+C_1$), where L_1 (respectively,R_1,C_1) is the set of left (respectively,right,cancellative) elements of $\mathbb{Z}^*$.

Proof. We will give the proof for the set L of left cancellable elements. Let $\eta\in L$, then since $\eta\in\rho\mathbb{R}$, η can be written uniquely as $x+\eta_1$, where $x\in[0,1)$ and $\eta_1\in\mathbb{Z}^*$. Therefore, η_1 is left cancellative in $\mathbb{Z}^*$ by corollary 4.2.25, and so $\eta\in[0,1)+L_1$.

For the reverse containment let $\eta\in[0,1)+L_1$ then $\eta=x+\eta_1$ $x\in[0,1)$, $\eta_1\in L_1$. Then by corollary 4.2.25, η is left cancellable and so $\eta\in L$. Hence $L=[0,1)+L_1$∎

4.2.28.Theorem. The set of left (respectively,right, cancellative) cancellative elements L (respectively,R,C) of $\rho\mathbb{R}$ is dense in $\rho\mathbb{R}$.

Proof. We will give the proof for the set L of left cancellative elements. Let $\eta=x+\eta_1$, where $x\in[0,1)$ and $\eta_1\in\mathbb{Z}^*$. We will show that if $\eta\notin Cl_{\rho\mathbb{R}}L$, then $\eta_1\notin Cl_{\mathbb{Z}^*}L_1$. Suppose on the contrary that $\eta_1\in Cl_{\mathbb{Z}^*}L_1$. Then $x+\eta_1\in x+Cl_{\mathbb{Z}^*}L_1$ and so $x+\eta_1\in x+Cl_{\rho\mathbb{R}}L_1\subseteq Cl_{\rho\mathbb{R}}(x+L_1)\subset Cl_{\rho\mathbb{R}}([0,1)+L_1)$. Hence $\eta=x+\eta_1\in Cl_{\rho\mathbb{R}}L$. This contradicts the assumption that $\eta\notin Cl_{\rho\mathbb{R}}L$. Hence if L is not dense in $\rho\mathbb{R}$, L_1 is not dense in $\mathbb{Z}^*$ but it is a well known fact that the set L_1 of left cancellative elements is dense in $\mathbb{Z}^*$ [58]∎

CHAPTER V.

NON-HOMOGENEITY OF $\tilde{\mathbb{R}}$, THE RUDIN-KEISLER AND THE RUDIN-FROLIK ORDERS

In the first section of this chapter, we shall give the definition of a homogeneous space and we shall show that $\rho\tilde{\mathbb{R}}$ is not homogeneous. In the second section, we shall give the definition of the Rudin-Keisler and Rudin-Frolik orders on $\tilde{\mathbb{R}}$ and study some of their properties.

Section I. Non-Homogeneity of $\tilde{\mathbb{R}}$

5.1.1.Definition.A topological space is called homogeneous if,for every pair of the points of the space, there is an automorphism of the space which exchanges the pair of points.

5.1.2.Lemma. For every uniformly continuous function $g:\mathbb{R}\longrightarrow\mathbb{R}$, there is a positive real number c such that $|g(x)-g(y)|\le c|x-y|$ whenever $|x-y|\ge 1$.

Proof. Since g is uniformly continuous there is a $\delta\ge 0$

such that $|x-y|\leq\delta$ implies that $|g(x)-g(y)|\leq 1$.

Suppose that $y\geq x+1$ and let n be the natural number for which $x+n\delta\leq y\leq x+(n+1)\delta$. Then

$$|g(x)-g(y)|\leq\sum_{r=1}^{n-1}|g(x+(r+1)\delta-g(x+r\delta)|+|g(y)-g(x+n\delta)|\leq n+1.$$

Since $y-x\geq n\delta$ and $y-x\geq 1$,

$$g(y)-g(x)|<(\frac{y-x}{\delta})+y-x<(\frac{1}{\delta}+1)(y-x).$$

Hence if we choose $c=\frac{1}{\delta}+1$, the result follows ■

5.1.3. **Theorem.** There are points ξ,η in $\rho\mathbb{R}$ such that for any homeomorphism

$$h:\tilde{\mathbb{R}}\longrightarrow\tilde{\mathbb{R}},\qquad h(\xi)\neq\eta$$

Proof. Suppose that $(x_n)\in\xi$, where $x_{n+1}-x_n\longrightarrow\infty$ as $n\longrightarrow\infty$. By theorem 3.1.1, we may suppose that $h(x_{n+1})-h(x_n)>1$. Then $(h(x_n))$ has the same property as (x_n), because the restriction $h|_{\mathbb{R}}$ of h to $\mathbb{R}$ is a homeomorphism of $\mathbb{R}$ into itself for which h and h^{-1} are uniformly continuous. By the lemma 5.1.2, we have that

$$|x_{n+1}-x_n|\le c|h(x_{n+1})-h(x_n)|.$$

Hence $(h(x_n))$ has the same property as (x_n); that is $h(x_{n+1})-h(x_n)\longrightarrow\infty$ and

$$h(\xi)\in Cl_{\tilde{\mathbb{R}}}\{h(x_n)\}\cap\rho\tilde{\mathbb{R}}.$$

Hence if $\eta\in\rho\mathbb{R}+\rho\mathbb{R}$, then $(h(x_n))\notin\eta$, that is $\eta\notin Cl_{\tilde{\mathbb{R}}}(h(x_n))$ But $\xi\in Cl_{\tilde{R}}(x_n)$ and so $h(\xi)\in Cl_{\tilde{\mathbb{R}}}(h(x_n))$. Hence $h(\xi)\neq\eta$ ∎

5.1.4. **Proposition.** Let $\xi\in\rho\mathbb{R}$ and choose $\{x_n\}\in\xi$ with $x_n\in\mathbb{R}$ and $x_{n+1}-x_n>1$. Let

$$G=\{\eta\in\rho\mathbb{R}\,|\,\bigcup_{n\in\mathbb{N}}(x_n-\varepsilon,x_n+\varepsilon)\in\eta\}$$

be the neighbourhood of ξ in $\rho\mathbb{R}$ defined by some ε satisfying $0<\varepsilon<1/4$.Then, for each $\eta\in G$, there is an ultrafilter $U_\eta\in\beta\mathbb{N}$ defined by stating that $A\in U_\eta$ if and only if

$$\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon)\in\eta.$$

Proof. Clearly, $\mathbb{N}\in U_\eta$ and $\emptyset\notin U_\eta$. If $A\in U_\eta$, $B\subseteq\mathbb{N}$ and $A\subseteq B$, then

$$(\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon))\subseteq(\bigcup_{n\in B}(x_n-\varepsilon,x_n+\varepsilon)),$$

Therefore, since

$$\bigcup_{n\in A}(x_n-\varepsilon,x_n-\varepsilon)\in\eta,\quad \bigcup_{n\in B}(x_n-\varepsilon,x_n+\varepsilon)\in\eta,$$

and so $B\in U_\eta$. Now to show that $A\cap B\in U_\eta$ whenever $A,B\in U_\eta$, we need to show that for any $A\subseteq\mathbb{N}$, either

$$\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon)\notin\eta$$

or

$$\bigcup_{n\in\mathbb{N}\setminus A}(x_n-\varepsilon,x_n+\varepsilon)\notin\eta.$$

Let $A\subseteq\mathbb{N}$. Since $A\cap(\mathbb{N}\setminus A)=\emptyset$ and $x_{n+1}-x_n>1$ and $0<\varepsilon<1/4$, there exists $W=(-1/4,1/4)$ in $\mathcal{B}(0)$ such that

$$(\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon)+W)\cap(\bigcup_{n\in\mathbb{N}\setminus A}(x_n-\varepsilon,x_n+\varepsilon)+W)=\emptyset.$$

This implies that

$$\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon)\notin\eta$$

or

$$\bigcup_{n\in\mathbb{N}\setminus A}(x_n-\varepsilon,x_n+\varepsilon)\notin\eta.$$

Now let $A,B\in U_\eta$ and suppose that $A\cap B\notin U_\eta$. Then $A\setminus(A\cap B)\in U_\eta$ and $B\setminus(A\cap B)\in U_\eta$. This is not possible as these two sets are disjoint ∎

5.1.5. **Proposition.** The mapping $h:\eta\longrightarrow U_\eta$ from G into $\mathbb{N}^*$ is continuous.

Proof. Let $\eta\in G$ and let $\bar{A}$ be a basic neighbourhood of U_η in $\mathbb{N}^*$. Then $A\in U_\eta$, since $\bar{A}=\{p\in\mathbb{N}^*|A\in p\}$. But this implies that

$$\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon)\in\eta$$

by proposition 5.1.4.Now let

$$B=\{\xi\in\rho\mathbb{R}|(\bigcup_{n\in A}(x_n-\varepsilon,x_n+\varepsilon))+(-\delta,\delta)\in\xi\},$$

where $\delta>0$ and $\varepsilon+\delta<1/4$.B is a neighbourhood of η by proposition 2.3.9 and $h(B\cap G)\subseteq\bar{A}$ ∎

5.1.6. **Proposition.** $\rho\mathbb{R}$ is not homogeneous.

Proof. We know from theorem 4.2.15 that each $\xi\in\rho\mathbb{R}$ can be expressed uniquely as $\xi=x+\xi'$,where $x\in[0,1)$ and $\xi'\in\mathbb{Z}^*$.Let U denote $\rho\mathbb{R}\setminus\mathbb{Z}^*$. Suppose that $\xi\in U$ and that $^*f:\rho\mathbb{R}\longrightarrow\rho\mathbb{R}$ is a homeomorphism for which $f(\xi)\in U$. We shall first show that ξ has a neighbourhood W in $\rho\mathbb{R}$ with the property that, for any $\zeta_1,\zeta_2\in W$, $\zeta_1'=\zeta_2'$ implies that $(f(\zeta_1))'=(f(\zeta_2))'$.

Let $\phi:U\longrightarrow(0,1)x\mathbb{Z}^*$ be defined by $\phi(\xi)=(x,\xi')$, where ξ has the decomposition given above. By theorem 4.2.15 and theorem 4.1.7, ϕ is a homeomorphism. It follows that each point in U has a basis of neigbourhoods of the form $\phi^{-1}((a,b)xV)$, where (a,b) is an open interval in (0,1) and V is an open subset of $\mathbb{Z}^*$.

We now observe that two points ζ_1,ζ_2 of U belong to the same component of U if and only if $\zeta_1'=\zeta_2'$. To see this, suppose that $\phi(\zeta_1)=(\mu_1,\zeta_1')$ and that $\phi(\zeta_2)=(\mu_2,\zeta_2')$. If $\zeta_1'=\zeta_2'$, ζ_1 and ζ_2 are connected by the path in U which is defined as $t\longrightarrow(1-t)\mu_1+t\mu_2+\zeta_1'$,where $t\in[0,1]$. On the other hand, if $\zeta_1'\neq\zeta_2'$, ζ_1 and ζ_2 cannot belong to any connected subset C of U. If they did, C' would be a connected subset of $\mathbb{Z}^*$ containing the distinct points ζ_1' and ζ_2' which is impossible.

We can choose a neigbourhood W of ξ which has the form $\phi^{-1}((a,b)xV)$ and satisfies $W\subseteq U\cap f^{-1}(U)$. Let $\zeta_1\zeta_2\in W$. If $\zeta_1'=\zeta_2'$, then the path P defined in the preceding paragraph will lie in W. It follows that $f(\zeta_1)$ and $f(\zeta_2)$ will belong to the connected subset f(P) of U, and hence that $(f(\zeta_1))'=(f(\zeta_2))'$.

Now choose $\xi,\eta\in U$ with the property that ξ' is a weak P-point in $\mathbb{Z}^*$, but η' is not. There will then be a sequence (η_n) in U having η as a limit point, for which none of the points η_n' are equal to η'. Suppose that $f:\rho\mathbb{R}\longrightarrow\rho\mathbb{R}$ is a homeomorphism for which $f(\xi)=\eta$. Let W be a neigbourhood of ξ with the property defined in the preceding paragraph. Since ξ is a limit point of $(f^{-1}(\eta_n))\cap W$, we must have $(f^{-1}(\eta_n))'=\xi'$ for some n for which $f^{-1}(\eta_n)\in W$. But this implies that $\eta_n'=(f(\xi))'=\eta'$ which is a contradiction ∎

According to theorem 4.35 of [62] if X is non-pseudocompact, $\beta X\backslash X$ is not homogeneous under the continuum hypothesis. Therefore, $\beta\mathbb{R}\backslash\mathbb{R}$ is not homogeneous.

Section II. The Rudin-Keisler and The Rudin-Frolik Orders

Let $\xi,\eta\in\beta X$, then ξ and η are said to be type equivalent if there is a homeomorphism h of βX onto itself such that $h(\xi)=\eta$, equivalently, there is a one to one continuous function Π from X onto X for which Π^{-1} is also continuous and $\Pi^{\beta}:\beta X\longrightarrow\beta X$ has the property that $\Pi^{\beta}(\xi)=\eta$. We write $\xi\approx\eta$ when ξ and η are type equivalent.

It is clear that $\approx$ is an equivalence relation on βX, and the set of equivalence classes is denoted by $\mathbb{T}(\beta X)$.

The quotient equivalence function is denoted by

$$t:\beta X \longrightarrow \mathbb{T}(\beta X)$$

and $t(\xi)$ is called the type of ξ, for $\xi\in\beta X$.

The Rudin-Keisler pre-order $\leq$ on βX is the binary relation given by $\eta\leq\xi$ if there is a continuous $f\in X^X$ such that $f^\beta(\xi)=\eta$. It is clear that $\leq$ is reflexive and transitive.

The Rudin-Keisler partial order on $\mathbb{T}(\beta X)$ also denoted by $\leq$ defined by

$\delta\leq\tau$, if there are $\eta,\xi\in\beta X$ such that $t(\xi)=\delta$, $t(\eta)=\tau$ and $\xi\leq\eta$.

According to [13, chap. 9], this order is well-defined, reflexive, transitive. It is also anti-symmetric that is if $\delta\xi\leq\tau$ and $\tau\leq\delta$ then $\delta=\tau$.

The notation $\xi<\eta$ means that $\xi\leq\eta$ and ξ is not type equivalent with η, and for partial order $\delta\leq\tau$ and $\delta\neq\tau$.

A subset D of βX is called strongly discrete if there is a family $\{A_d:d\in D\}$ of open subsets of X such that $A_d\cap A_{d'}=\emptyset$ for $d,d'\in D$, $d\neq d'$ and $d\in A_d$ for $d\in D$.

The Rudin-Frolik pre-order $\sqsubseteq$ on βX is the binary relation given by

$\xi \underline{[} \eta$ if there is $f:X \longrightarrow \beta X$ such that $f(X)$ is a (strongly) discrete subset of βX and $\tilde{f}(\xi)=\eta$.

The Rudin-Frolik partial order also denoted by $\underline{[}$ is binary relation on $\mathbb{T}(\beta X)$ denoted by

$\delta \underline{[} \tau$ if there are $\xi,\eta \in \beta X$ such that $t(\xi)=\delta, t(\eta)=\tau$ and $\xi \underline{[} \eta$.

According to [13,chap 9,16] $\underline{[}$ is well-defined partial order on $\mathbb{T}(\beta X)$ and $\eta \underline{[} \xi$ always implies that $\eta \leq \xi$.

Let $\eta,\xi \in \hat{\mathbb{R}}$, then we say η and ξ are equivalent if there are uniformly continuous functions $f,g:\mathbb{R} \longrightarrow \mathbb{R}$ such that $\tilde{f}(\eta)=\xi$ and $\tilde{g}(\xi)=\eta$, and when ξ and η are equivalent we write $\xi \equiv \eta$.

It is easy to see that $\equiv$ is an equivalent relation on $\hat{\mathbb{R}}$.

5.2.1. Definition. The Rudin-Keisler pre-order $\leq$ on $\bar{\mathbb{R}}$ is the binary relation on $\rho\mathbb{R}$ given by $\xi \leq \eta$ if there is a uniformly continuous function $f:\mathbb{R} \longrightarrow \mathbb{R}$ for which $\bar{f}(\eta)=\xi$. It is easy to see that $\leq$ is reflexive and transitive.

In the above definitions, $\bar{f}$ is the unique extension of f to $\bar{\mathbb{R}}$.

We say $\xi<\eta$ when $\xi\leq\eta$ and ξ and η are not equivalent, that is, there is no uniformly continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ for which $\tilde{f}(\xi)=\eta$.

5.2.2. Theorem. If $(x_n)\subseteq\mathbb{R}$ and $x_{n+1}-x_n\longrightarrow\infty$, then no element ξ in $Cl_{\tilde{\mathbb{R}}}\{x_n\}\cap\rho\mathbb{R}$ can be equivalent in the Rudin-Keisler order to an element in $\rho\mathbb{R}+\rho\mathbb{R}$.

Proof. Let $\xi\in Cl_{\tilde{\mathbb{R}}}(x_n)\cap\rho\mathbb{R}$, where $x_{n+1}-x_n\longrightarrow\infty$, $(x_n)\subset\mathbb{R}$ and suppose that ξ is equivalent to an element η in $\rho\mathbb{R}+\rho\mathbb{R}$ and let $X=\{x_n:n\in\mathbb{N}\}$. By assumption there are uniformly continuous functions $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ such that $\tilde{f}(\xi)=\eta$ and $\tilde{g}(\eta)=\xi$. Let $y_n=f(x_n)$ and $Y=\{y_n\}$, by theorem 3.1.1 we may suppose that $y_{n+1}-y_n\geq1$. For each n, let $\bar{g}(y_n)$ be the point of X closest to $g(y_n)$, if there is more than one such point we choose one withsmallest n.

For each $\varepsilon>0$, let $Y_\varepsilon=\{y_n:d(g(y_n),X)<\varepsilon\}$. Then $Y_\varepsilon\in\eta$, because $\{g(y_n):d(g(y_n),X)\geq\varepsilon\}=g(Y\setminus Y_\varepsilon)\notin\xi$ and hence $Y\setminus Y_\varepsilon\notin\eta$. Also $f^{-1}(Y_\varepsilon)\in\xi$, because $f(X\setminus f^{-1}(Y_\varepsilon))\notin\eta$ and hence $X\setminus f^{-1}(Y_\varepsilon)\notin\xi$. So we can replace X by $f^{-1}(Y_\varepsilon)$.

Let $h=\bar{g}f|_X$. Since we can identify $\tilde{X}$ with βX we can extend h to h^β on βX and $h^\beta(\xi)=\xi$. Hence by proposition 9.2 of [13], $X'=\{x\in X: h(x)=x\}\in\xi$. If we replace X by X', we have that $\bar{g}f\,(x_n)=x_n$ for all n and so $\bar{g}(y_n)=x_n$. Since

$d(\bar{g}(y_n),g(y_n))<\varepsilon$ for each n, $g(y_{n+1})-g(y_n)>x_{n+1}-x_n-2\varepsilon$. Hence $g(y_{n+1})-g(y_n)\longrightarrow\infty$. By proposition 4.2.13, there is a positive number b and infinitely many values of n for which $y_{n+1}-y_n<b$. This contradicts the lemma 5.1.2∎

It is a well-known fact that there are points in $\mathbb{N}^*$ which contain a sequence with the property that $x_{n+1}-x_n\longrightarrow\infty$ and which are equivalent to some points of $\mathbb{N}^*+\mathbb{N}^*$ in the Rudin-Keisler order. In fact, if ξ is a point in $\mathbb{N}^*$ which contains such a sequence and if η is in $\mathbb{N}^*$, then ξ is equivalent to $\xi+\eta$ [28].

Let $\xi,\eta\in\tilde{\mathbb{R}}$, then we say ξ and η are uniform type equivalent if there is a uniformly continuous homeomorphism h of $\mathbb{R}$ onto itself with $\tilde{h}(\eta)=\xi$ such that h^{-1} is also uniformly continuous.If η and ξ are uniform type equivalent, we write $\eta\approx\xi$.

It is clear that uniform type equivalence is reflexive,symmetric and transitive and therefore $\approx$ is an equivalence relation on $\tilde{\mathbb{R}}$.We will denote the set of the uniform equivalence classes by $T(\tilde{\mathbb{R}})$.We call an equivalence class a uniform type of $\tilde{\mathbb{R}}$,and we denote the quotient equivalent function by

$$t:\tilde{\mathbb{R}}\longrightarrow T(\tilde{\mathbb{R}})$$

and $t(\eta)$ is called the uniform type of η, for $\eta\in\tilde{\mathbb{R}}$.

5.2.3. Theorem. Let $\eta,\xi\in\tilde{\mathbb{R}}$. Then if η contains a sequence $(x_n)\subseteq\mathbb{R}$ such that $x_{n+1}-x_n\longrightarrow\infty$ as $n\longrightarrow\infty$ and if $\xi\in\rho\mathbb{R}+\rho\mathbb{R}$, then η and ξ are not uniform type equivalent.

Proof. Suppose on the contrary that η and ξ are uniform type equivalent, then there is a uniformly continuous homeomorphism of $\mathbb{R}$ onto itself such that f^{-1} is also uniformly continuous and $\tilde{f}(\eta)=\xi$. As in theorem 5.1.3 $f(x_{n+1})-f(x_n)\longrightarrow\infty$ as $n\longrightarrow\infty$ and $(f(x_n))\in\xi$, but this contradicts to proposition 4.2.12.

5.2.4. Definition. The Rudin-Frolik pre-order $\underline{[}$ on $\tilde{\mathbb{R}}$ is the binary relation on $\tilde{\mathbb{R}}$ given by $\eta\,\underline{[}\,\xi$ if there is a monotone uniformly continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that $\tilde{f}(\xi)=\eta$.

5.2.5. Remark. It is clear from the definition of $\leq$ and $\underline{[}$ that $\eta\,\underline{[}\,\xi$ always implies that $\eta\leq\xi$ for $\eta,\xi\in\tilde{\mathbb{R}}$.

For the Rudin-Frolik pre-order, we use the notation $\eta\,[\,\xi$ to mean that $\eta\,\underline{[}\,\xi$ and η and ξ are not uniform type equivalent.

5.2.6.Lemma.If $\eta' \approx \eta, \xi' \approx \xi$ and $\eta \underline{[} \xi$, then $\eta' \underline{[} \xi'$ for $\eta', \eta, \xi', \xi \in \tilde{\mathbb{R}}$.

Proof. Since $\eta \underline{[} \xi$, there is a monotone uniformly continuous function $f:\mathbb{R} \longrightarrow \mathbb{R}$ such that $\tilde{f}(\xi)=\eta$ and since $\eta' \approx \eta$ and $\xi' \approx \xi$ there are uniformly continuous homeomorphisms $\Pi_1, \Pi_2:\mathbb{R} \longrightarrow \mathbb{R}$ such that Π^{-1}, Π^{-2} are also uniformly continuous and $\Pi_1(\eta')=\eta, \Pi_2(\xi')=\xi$

Let $g=\Pi_1^{-1} \circ f \circ \Pi_2$, then g is monotone and uniformly continuous.Hence g has an extension $\tilde{g}$ on $\tilde{\mathbb{R}}$ such that

$$\tilde{g}(\xi')=\tilde{\Pi}_1^{-1} \circ f \circ \tilde{\Pi}_2(\xi')$$

$$=\tilde{\Pi}_1^{-1} \circ f(\xi)=\tilde{\Pi}_1^{-1}(\eta)=\eta' .$$

Hence,$\eta' \underline{[} \xi'$.

5.2.7.Lemma [13]. Let X be a set with the discrete topology and let D be a subset of X. Suppose that $f:D \longrightarrow X$ has the property that $f^{\beta}:\beta D \longrightarrow \beta X$ has a fixed point λ. Then there is a set $A \in \lambda$ such that $f(a)=a$ for every $a \in A$.

Proof. We first consider the case in which $D=X$, and show that f must have a fixed point. If f has no fixed point, it follows from lemma 9.1 of [13], that X can be expressed as the union of three disjoint sets, X_0, X_1, X_2, with the property that $X_i \cap f(X_i)=\emptyset$ for $i=0,1,2$. However, if $X_i \in \lambda$, then $f(X_i) \in f(\lambda)^{\beta}$. This is a contradiction.

If $D \neq X$, we again deduce that f must have a fixed point by applying the result in the preceding paragraph to the extension of f which maps every point in $X \setminus D$ to some chosen point of D.

Finally, let $A=\{x \in D \mid f(x)=x\}$. If $Y \in \lambda$, we know that $f|_Y$ must have a fixed point. Hence $A \cap Y \neq \emptyset$. Thus $A \in \lambda$ ∎

5.2.8. Corollary. Let $X=\{x_n\}$, where (x_n) is a sequence of real numbers satisfying $x_{n+1}-x_n>\delta$ for every n and some $\delta>0$. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a uniformly continuous function for which $f(X) \subseteq X$ and $\tilde{f}$ has a fixed point ξ in $\bar{X}$. Then there is a subset A of X satisfying $A \in \xi$, for which $f(x)=x$ for every $x \in A$.

Proof. We can identify $\bar{X}$ with βX and $\tilde{f}|_{\bar{X}}$ with $(f|_X)^{\beta}$. The result follows from the lemma ∎

5.2.9. Theorem. The Rudin-Frolik pre-order $\sqsubseteq$ on $\rho\mathbb{R}$ is reflexive and transitive. Furthermore if $\xi\sqsubseteq\eta$ and $\eta\sqsubseteq\xi$, then ξ and η are uniform type equivalent.

Proof. Let Id be the identity mapping on $\mathbb{R}$. Since Id is monotone and uniformly continuous, it extends to a mapping $\tilde{Id}$ on $\tilde{\mathbb{R}}$ for which $\tilde{Id}(\xi)=\xi$ for all $\xi\in\tilde{\mathbb{R}}$. Hence $\sqsubseteq$ is reflexive. To see that $\sqsubseteq$ is transitive, let $\xi,\eta,\gamma\in\rho\mathbb{R}$ and $\xi\sqsubseteq\eta$ and $\eta\sqsubseteq\gamma$. Then there are monotone uniformly continuous functions $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ such that $\tilde{f}(\eta)=\xi$ and $\tilde{g}(\gamma)=\eta$. The composite function $f\circ g$ is uniformly continuous monotone and $(\tilde{f\circ g})(\gamma)=\tilde{f}(\eta)=\xi$. To see that $\sqsubseteq$ is anti-symmetric, let $\xi,\eta\in\rho\mathbb{R}$ satisfying $\xi\sqsubseteq\eta$ and $\eta\sqsubseteq\xi$. Then there are uniformly continuous monotone functions f and g from $\mathbb{R}$ to itself, for which $\tilde{f}(\xi)=\eta$ and $\tilde{g}(\eta)=\xi$.

We choose sequences (x_n) and (y_n) of real numbers such that $\{x_n\}\in\xi$, $\{y_n\}\in\eta$, $x_{n+1}-x_n>1$ and $y_{n+1}-y_n>1$ for every n.

Put $X=\{x_n\}$ and $Y=\{y_n\}$. Let p and q denote the ultrafilters on $\mathbb{N}$ defined as follows: $A\in p$ if and only if $\{x_n \mid n\in A\}\in\xi$; and $B\in q$ if and only if $\{y_n \mid n\in B\}\in\eta$. Note that, if $A\in p$, then $\{f(x_n) \mid n\in A\}\in\eta$.

Now $\{n\in\mathbb{N} \mid d(f(x_n),Y)<1/4\}\in p$. Thus if we replace X by a suitable subsequence, we may suppose that $d(f(x_n),Y)<1/4$ for every n. Given n, let m_n be the integer for which y_{m_n} is the element of Y closest to $f(x_n)$. Then, if $A\in p$, $\{y_{m_n}|n\in A\}\in\eta$; for $f(x_n)$ is at a distance at least 3/4 from $Y\setminus\{y_{m_n}|n\in A\}$, for each n and so $Y\setminus\{y_{m_n}|n\in A\}\notin\eta$.

Now, consider the function which maps each x_n to y_{m_n}, and which is defined piecewise linearly otherwise. It is easy to see that this function is uniformly continuous and monotone, and that its extension to $\tilde{\mathbb{R}}$ maps ξ to η. So we may replace f by this function and suppose that $f(X)\subseteq Y$.

Similarly, we can now replace g by a suitable function and suppose that Y has a subsequence Y' satisfying $Y'\in\eta$ and $g(Y')\subseteq X$.

Let $X'=\{x_n|f(x_n)\in Y'\}$. Then $X'\in\xi$. Since $(g\circ f)\tilde{}(\xi)=\xi$, it follows from the corollary to the lemma, that there is a subset A of X such that $A\in\xi$ and $(g\circ f)(x)=x$ for all $x\in A$.

Now define $h:\mathbb{R}\longrightarrow\mathbb{R}$ by stating that $h(x_n)=f(x_n)$ if $x_n\in A$, and that h is piecewise linear otherwise.It is easy to check that h is a homeomorphism, that h and h^{-1} are uniformly continuous and that $\tilde{h}(\xi)=\eta$ ∎

The followings are clear from the fact that there are at most 2^c function from $\mathbb{R}$ to $\mathbb{R}$

a) $|t(\eta)|\le 2^c$ for $\eta\in\tilde{\mathbb{R}}$.

b) $|\{t(\eta)\in T(\tilde{\mathbb{R}})\,|\,t(\eta)\underline{[}t(\xi)|\le 2^c$.

Since there are c continuous function from $\mathbb{R}$ to itself we have

c) $|\{t(\eta)|\eta\in\tilde{\mathbb{R}}\}|=2^c$;

d) $|\{t(\eta)\in T(\tilde{\mathbb{R}})\,|\,t(\eta)\underline{[}t(\xi)\}|=2^c$
for every $\xi\in\tilde{\mathbb{R}}$.

We now give an example to show that there are some poits in $\rho\mathbb{R}$ which are equivalent in the Rudin-Keisler order but they are not equivalent in the Rudin-Frolik order.

5.2.10.Example. Define $X=\{2^n+r\ |n\in\mathbb{N},r\in\mathbb{Z}$ and $0\le r\le 2^{n-1}\}$.

We define $f:\mathbb{R}\longrightarrow\mathbb{R}$ by stating that $f(2^n+r)=2^n+2^{n-1}-r$ if $2^n+r\in X$, and extending f by piecewise linearity. Note that $f|_X=(f|_X)^{-1}$. It is easy to check that f is uniformly continuous .

Let $\lambda\in\tilde{\mathbb{R}}\backslash\mathbb{R}$ be such that $\{2^n\}\in\lambda$, and let $\xi=\lambda+\lambda$. Put $\eta=\tilde{f}(\xi)$. Then ξ and η are equivalent in the Rudin-Keisler order, because $\xi=\tilde{f}(\eta)$.

We claim that there is no uniformly continuous homeomorphism $h:\mathbb{R}\longrightarrow\mathbb{R}$ for which h^{-1} is also uniformly continuous and $\tilde{h}(\xi)=\eta$.

To see this, suppose that, on the contrary, such a homeomorphism h exists.

We know that $X\in\xi$, because the set of integers of the form 2^k+2^n $(k<n)$ will contain a net converging to ξ, since we can allow 2^n to converge to λ and then allow 2^k to do so. so $X=f(X)\in\eta$.

Proceeding as in the proof of the last theorem, we may suppose that $h(X)\subseteq X$, because we can construct another homeomorphism which will have this property, as well as the other properties assumed for h.

Now $\tilde{f}\tilde{h}(\eta)=\eta$.

It follows from the corollary 5.2.8 to the lemma 5.2.7 that there will be a subset Y of X such that $Y\in\eta$ and $fh(y)=y$ for every $y\in Y$. However, if $Y\in\eta$, $h(Y)\in\xi$. So,for some $n\in\mathbb{N}$, h(Y) must contain at least two elements from the set $\{2^n+r|0\leq r\leq 2^{n-1}\}$, for otherwise ξ could not be in $\rho\mathbb{R}+\rho\mathbb{R}$ by proposition 4.2.12. Thus $Y=f(h(Y))$ must contain at least two elements from this set, and fh cannot therefore be the identity on Y. This is because fh is order-reversing on this set, this is a contradiction■

Let $\eta,\xi\in\tilde{\mathbb{R}}$. Then we say η and ξ are equivalent if $\eta\sqsubseteq\xi$ and $\xi\sqsubseteq\eta$ and we write $\xi\cong\eta$ if ξ and η are equivalent. It is easy to see that $\cong$ is an equivalence relation on $\tilde{\mathbb{R}}$. We denote the set of all equivalence classes on $\tilde{\mathbb{R}}$ by $T_1(\tilde{\mathbb{R}})$ and the quotient equivalent function by

$$t_1:\tilde{\mathbb{R}}\longrightarrow T_1(\tilde{\mathbb{R}})$$

and $t_1(\eta)$ is called the type of η, for $\eta\in\tilde{\mathbb{R}}$.

5.2.11. Definition. The Rudin-Frolik partial order (we also denote it by $\sqsubseteq$) on the set of equivalent classes $T_1(\tilde{\mathbb{R}})$ is defined by

$\delta\sqsubseteq\tau$ if there are $\eta,\xi\in\tilde{\mathbb{R}}$ such that $t_1(\eta)=\delta$ and $t_1(\xi)=\tau$ and $\eta\sqsubseteq\xi$.

5.2.12. Lemma. Let $\eta_1,\eta_2,\xi_1,\xi_2\in\tilde{\mathbb{R}}$ and suppose that $\eta_1\cong\xi_1,\eta_2\cong\xi_2$ and $\xi_2\sqsubseteq\eta_1$, $\xi_1\sqsubseteq\eta_2$ then $\xi_1\cong\xi_2$.

Proof. It is obvious since $\sqsubseteq$ is transitive ■

It is clear that $\sqsubseteq$ is reflexive, transitive, and anti-symmetric ■

We write $\delta\sqsubset\tau$ when $\delta\sqsubseteq\tau$ and $\delta\neq\tau$.

REFERENCES:

1 B.Banaschewski; Normal Systems of Sets, Math. Nachr. 24(1962), 53-57.

2 B. Banaschewski; on Wallman's Method of Compactification, Math. Nachr. 27 (1963), 105-114.

3 C.Bandt; On Wallman Shanin Compactifications, Math.Nachr 77 (1977), 333-351.

4 J. F. Berglund and P. Milnes; Algebras of Functions on Semitopological Left Groups, Trans. Amer. Math. Soc. 222 (1976), 157-178.

5 J.F.Berglund, H.Junghenn and P.Miles; Compact Right Topological Semigroups and Generalizations of Almost Periodicity, Lecture Notes in Math. 663,Springer-Verlag,Berlin 1979.

6 J.F.Berglund; H.D.Junghenn and P.Milnes, Analysis On Semigroups, Wiley,New York 1989.

7 C. M. Biles; Wallman-Type Compactifications, Proc. Amer. Math. Soc. 25 (1970), 363-366.

8 C. M. Biles; Gelfand and Wallman-Type Compactifications, Proc. Amer. Math. Soc. 35 (1970), 267-278.

9 A. Blass; The Rudin-Keisler Ordering of P-Points, Amer. Math. Soc. 179 (1973), 145-166.

10 A. Blass and N. Hindman; Sums of Ultrafilters and The

Rudin-Keisler and Rudin-Frolik Orders, In General Topology and Aplications, R. Shortt ed., Lecture Notes in Pure and Applied Math. 123 (1990), 59-70.

11 E.Čech; On Bicompact Spaces, Ann. of Math. 38(4), 1937, 823-844.

12 R.E.Chandler; Hausdorff Compactifications, Marcel Dekker,Inc. 1976,New York,Basel.

13 W.W.Comfort and N.Negrepontis; The Theory of Ultrafilters, Springer-Verlag,Berlin Heidelberg New York 1974.

14 E. K. van Douwen, The Čech-Stone Compactifications of a Discrete Groupoid, Topology and its Applications,to appear.

15 R. Ellis; Locally Compact Transformation Groups, Duke. Math.J. 24 (1957), 119-126.

16 M. Filali; The Uniform Compactification of a Locally Compact Abelian Group, Camb.Phil.Soc.Math. Proceedings 108(1990), 527-538.

17 N.J.Fine and L.Gillman; Remote Points in $\beta\mathbb{R}$, Proc.Amer.Mat.Soc. 17 (1971), 299-304.

18 O. Frink; Compactification and Semi-normal Spaces, Amer. J.Math.,86 (1964), 602-607.

19 L.Gillman and M.Henriksen; Rings of Continuous Functions in Which Every Finitely Generated Ideal is Principal, Trans.Amer.Math.Soc. 82 (1956), 366-391.

20 L.Gillman and M.Jerison; Rings of Continuous Functions, D.Van Nostrand Campany, Inc.

Princeton New Jersey Toronto New York London 1960.

21 L.Gillman and M.Henriksen, Concerning Rings of Continuous Functions, Trans.Amer. Math. Soc.77 (1954), 340-362.

22 I. Glicksberg; Convolution Semigroups of Measures, Pac.j.of Math. 9 (1959), 51-67.

23 I. Glickberg; Weak Compactness and Separate Continuity, Pac. J. of Math. 11 (1961), 205-214.

24 M.Henriksen and Isbell; Locally connectedness in The Stone-Čech Compactification, Illinois J.Math.,vol. 1 (1957), 574-582.

25 E. Hewitt; Rings of Continuous Functions I, Trans. Amer. Math. Soc. 64 (1948), 54-59.

26 N.Hindman; Sums Equal the Products in $\beta\mathbb{N}$, Semigroup Forum 21 (1980), 221-255.

27 N. Hindman; Minimal Ideals and Cancellation in $\beta\mathbb{N}$, Semigroup Forum 25 (1982), 291-310.

28 N. Hindman and D. Strauss; Cancellation in the Stone-Čech Compactification of a Discrete Semigroup, to Appear.

29 N. Hindman and J.Pym;Free Groups and Semigroups in $\beta\mathbb{N}$, Semigroup Forum 30 (1984), 177-193.

30 T. Husain, Topology and Maps, Plenum Press, New York, London.

31 H. D. Junghenn, Extension of Continuous Functions on Dense Subsets, Illinois.J. of Math. 27(3) (1983), 421-435.

32 M. Katêtow; On Real-Valued Functions in Topological Spaces, Fundementa Mathematicae. 38 (1951-1952), 38-39.

33 J.E.Keesling; Open and Closed Mappings and Compactification; Fundementa Matematicae LXV (1969), 73-81.

34 A. W. Knapp; Distal Functions on Groups, Trans. Amer. Math. soc. 128 (1967), 1-40.

35 A. W. Knapp; Decomposition Theorem for Bounded Uniformly Continuous Functions on Groups, Amer. J.Math.88 (1966), 902-914.

36 H.J. Kowalsky; Topological Spaces, Academic Press New York and London 1964.

37 K. de Leeuw and Glicksberg; Application of Almost Periodic Compactifications, Acta Math. 105 (1961),63-97.

38 K. de Leeuw and Glicksberg; The Composition of Certain Group Representations, J, Analyse Math. 15 (1965),135-192.

39 A.T.Lau and J.Pym; The Topological Centre of The LUC-Compactification of a Locally Compact Group, Manuscript, to appear.

40 A. T. Lau, P. Milnes and J. Pym; Compactification of Locally Compact Groups and quotients, Trans. Amer. Math. Soc.320(1992), 97-115.

41 A. T. Lau; Continuity of Arens Multiplications on the Dual space of Bounded uniformly continuous Functions on Locally Compact Groups and Topological Semigroups, Proc. Camb. Phil.

Soc. 99 (1986), 273-283.

42 R. Levy and R. H. Mcdowell; Dense Subset of βX,Proc.of Amer. Math. Soc.50 (1975), 426-430.

43 L.H. Loomis; An Introduction to Harmonic Analysis (1953), Van Nostrand, Princeton, N.J.

44 P. Milnes; On the Extension of Continuous and Almost Periodic Functions, Pac. J. of Math.56(1) (1975), 187-193.

45 T.Mitchell; Topological Semigroups and Fixed Points, Illinois Journal of Mathematics 14 (1970), 630-641.

46 I. Nomioka; Separate Continuity and Joint Continuity, Pac.J. of Math. 51(2) (1974), 515-531.

47 D.Plank; On a Class of Subalgebras of C(X) with Applications to $\beta X\backslash X$, Fundamenta Matematicae LXIV (1969),41-54.

48 J.R.Porter and G.Woods, Extensions and Absolutes of Hausdorff Spaces, Springer-Verlag,New York 1988.

49 J.S.Pym; Compact Semigroups with One-sided Continuity, In the Analytical and Topological Theory of Semigroups K.H. Hofmann J. Lawson and J. Pym, eds, Walterde Gruyter, Berlin (1990), 197-217.

50 W.Rudin; Homogeneity Problems in The Theory of Čech Compactifications, Duke Math. J. 23, 663 (1956),409-419.

51 M. E. Rudin; Partial Orders on the Types in $\beta\mathbb{N}$, Trans. Amer. Math. Soc. 155(2) (1971),

353-362.

52 N.A. Šanin; On Special Extensions of topological Spaces, C.R. (Doklady) Akad. Sc.USSR (N.S) 38 (1943), 6-9.

53 N.A. Šanin; On the theory of Bicompact Extensions of topological Spaces,Ibid.,38 (1943), 110-113.

54 P. Samuel; Ultrafilters and Compactification of Uniform Spaces, Trans. Amer. Math. Soc.64 (1948), 100-132.

55 H. Sedaghat; Boolean Lattices of Function Algebras on Rectengular Semigroups, Semigroup Forum 47 (1993) 231-249.

56 H. Sedaghat; A New Semi Lattice of Function Algebras and Its Boolean Form on a Lattice of Groups, Semigroup Forum 46 (1993), 307-321.

57 M.H.Stone; Applications of the Theory of Boolean Rings to General Topology; Trans. Amer. Math. Soc. 41 (1937), 375-481.

58 D.Strauss; Semigroup Structure on $\beta\mathbb{N}$, Semigroup Forum 44 (1992), 238-244.

59 D. Strauss, $\mathbb{N}^*$ Does Not Contain an Algebraic and Topological Copy of $\beta\mathbb{N}$, J.London Math. Soc.(2) 46(1992), 463-470.

60 A. Tychonoff; Uber die Topologische Erweiterung Van Räumen, Math. Ann. 102 (1930), 544-561.

61 V.M.Ul'janov; Solution of Basic Problem on Compactifications of Wallman type, Dokl.Akad.Nauk SSSR 223 (1977), 1056-1059

=Soviet Math. Dokl 18 (1977), 567-571.

62 R.C. Walker; The Stone-Čech Compactification, Springer-Verlag, New York,Heidelberg Berlin 1974.

63 H. Wallman; Lattice and Topological Spaces; Ann. of Math. (2) 39 (1938), 112-126.

64 M.D. Weir; Hewitt-Nachbin Spaces, North-Holland Publishing Company 1975.

65 S.J.Wiley; Extension of Left Uniformly Continuous Functions on a Topological Semigroup, Proc. of the American Matematical Soc. 33(2) (1972), 572-575.