THE UNIVERSITY OF HULL

Time Domain Threshold Crossing For Signals In noise

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by

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ABSTRACT

This work investigates the discrimination of times between threshold crossings for deterministic periodic signals with added band-limited noise. The methods include very low signal to noise ratio (one or less).

Investigation has concentrated on the theory of double threshold crossings, with especial care taken in the effects of correlations in the noise, and their effects on the probability of detection of double crossings. A computer program has been written to evaluate these probabilities for a wide range of signal to noise ratios, a wide range of signal to bandwidth ratios, and a range of times between crossings of up to two signal periods. Correlations due to the extreme cases of a Brickwall filter and a second order Butterworth filter have been included; other filters can easily be included in the program.

The method is simulated and demonstrated by implementing on a digital signal processor (DSP) using a TMS32020. Results from the DSP technique are in agreement with the theoretical evaluations.

Probability results could be used to determine optimum time thresholds and windows for signal detection and frequency discrimination, to determine the signal length for adequate discrimination, and to evaluate channel capacities.

The ability to treat high noise, including exact effects of time correlations, promises new applications in electronic signal detection, communications, and pulse discrimination neural networks.

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DECLARATION

I, Sam Nooh Kiryakos Al-Jajjoka, do declare that this work has not been submitted for any other degree at this University or any other Institution.

GLOSSARY OF ABBREVIATIONS AND MATHEMATICAL SYMBOLS

This section details all of the special abbreviations and common mathematical symbols used in this thesis.

A	Square matrix
B(x,y,z)	Bivariate normal probability.
$b_0, b_1 ., b_k$	(1)Numerical coefficients of series (4.53). (2)Matrix coefficient.
b.n.d.	Bivariate normal distribution.
C_1, C_2, C_k	Normalised correlation for Butterworth and Brickwall filter.
dB	Decibel.
DSP	Digital Signal Processor.
D/A	Digital-to-Analogue Converter.
D	Element of block square matrix.
D_T, D_1, D_2	Function of bivariate normal probability.
D_T	The integral over the bivariate normal distribution.
eg.	For example.
e_r	A parameter which specifies the correlation.
E	Square matrix.
$E[N\alpha]$	Expected number of threshold crossings in unit time.
E[No]	Expected number of zero crossings per unit time (both up and down).

$E[No^2(t)]$	Expectation values of the square of the number of crossings in an interval t.
FIR	Finite impulse response.
Fs	Sampling frequency.
$F(t)$	Cumulative probability function.
$f_2(X_1,X_2)$	Joint probability density function for two variables.
$f_4(X_1..X_4)$	Joint probability density function for four variables $=f_z(X_1,..X_2)$.
$F(x,u,a)$	Function defined as the sum of integrals P and Q.
FSK	Frequency Shift Keying.
G	Constant in section 2.8.
$g(x,y,\rho)$	Bivariate normal probability.
G_1	$D(r,\theta_k)$
G_2	$D(r,\theta_h)$
$H(z)$	Impulse response function for a linear system.
IIR	Infinite Impulse Response.
$I_n(x)$	Repeated integrals of the normal probability.
I_F	Approximation for the first crossing.
J	Jacobian density function.
k_1,k_2	Variables for total period of the signal.
$L(x,y,\rho)$	Bivariate normal variates.
M,M_z	Correlation Matrix.
MFSK	Multi-Frequency Shift Keyed.
$n(t)$	Stochastic Gaussian noise process.
$P\alpha(t)$	Probability density function for either upward or downward crossing.

$P_m(t)$	Probability density of the intervals between a zero and the (m+1)st later zero.
P_{2X}	Probability of detection of a double crossing.
$P_x^C(t_1,t_2)$	A simple high correlation approximation for the probability of a single crossing (P_xC).
$P_x^H(t_1,t_2)$	The high noise correlation approximation to the probability of crossing (PxH).
P_{xcor}^H	Correction to the high noise correlation approximation (PxHC).
PxH2	Corrected high noise correlation approximation to second order (PxHC+PxH).
$P_x^L(t_1,t_2)$	The low noise correlation approximation to the probability of crossing (PxL).
PxLC12	Correction to the low noise correlation approximation (PxLC1+PxLC2).
PxLcext	Extension to the correction of the low noise approximation.
PxLC	First order correction to the low noise correlation approximation (PxLCext+PxLC12).
PxL1	Corrected low noise correlation approximation (PxL+PxLC).
P(U)	Standard normal cumulative distribution function.
$P(y,\dot{y};t)$	Probability density function (PDF).
PDC	Probability of detection of a double crossing.
PCW	Probability of detection of double crossing within a time window.
PD	Pulse discriminate.

$Q(t)$	Complementary cumulative normal probability function.
$r(t)$	Normalised autocorrelation function, defined by $r(t) = \frac{R(t)}{\sigma^2}$
r	Variable integer used in chapter 4 section 4.7.1 to define the absolute magnitude of h & k.
$R(t)$	Correlation function.
Rs	The signal to bandwidth frequency ratio, defined as $(\frac{W_S}{W_B})$.
SNR	Signal to Noise Ratio.
$s(t)$	Deterministic signal.
$S(t)$	The joint probability of a pair of zeroes separated by an interval t, used in chapter 2.
$\dot{s}$	Derivative of the signal.
t	(1) Time period.
t_x	Time delay between the double crossing.
U_1, U_2 & U_3, U_4	Normalised Signal values of sampling point.
var[No]	The variance of the number of zero per unit time by definition is $=E[No^2]-E^2[No]$.
W	Angular Frequency.
W_S	Signal bandwidth.
W_B	Filter bandwidth.
X_1, X_2	Usually refers to random variables obtained by sampling functions of a random process $X(t)$.
$X(n)$	Filter input.
$y(t)$	Total signal.
$\dot{y}(t)$	Signal velocity (slope) defined as $(\frac{dy}{dt})$.

Y(n)	Filter output.
Z(x)	Standard normal probability density function.
ZC	Zero crossing.
ZCD	Zero crossing detector.
α	Threshold crossing.
β	Signal velocity (slope).
θ	Step function.
θ_h, θ_k	The angle used in chapter 4 to define the region of the bivariate normal probability.
ρ	Correlation Coefficient.
σ	Standard deviation.
σ^2	Normalised covariance (noise power).
i_r	The incremental signal to noise correlation ratio (ISNCR).
τ	Sampling period.
τ_1, τ_2	Observation time for first and second crossing.
$\bar{U}$	The difference between the signal in the clocking interval and the threshold, normalised by the square root of the noise power.
Δs	The difference between consecutive signal samples.
ΔU	The normalised increment in the signal over the interval.
Δt	The interval of the crossing.

LIST OF FIGURES

CONTENTS

Chapter 1

Introduction

1.1 Introduction

This work investigates the theory of detection of threshold crossings for deterministic periodic signals with additive band-limited Gaussian noise, including low signal to noise ratios.

The theory of threshold crossings for such stochastic processes is applicable over a wide range of electronic engineering, e.g.: signal processing [Rainal 1966, 1967; Dechamber 1985; Kedem 1986a, 1986b; Hinich 1982], speech processing [Done 1975; Waibel 1989; Niederjohn 1985], communication [Voelcker 1972; Ogmundson 1991; Blachman 1964; Higgins 1980], spectral analysis [Kay 1986; Niederjohn 1992], and image processing [Haralick 1984]. The theory is also applicable in many other areas of engineering and science including meteorology, reliability theory [Bendat 1958], medicine and biomedical engineering [Mackay 1982], optics [Ohtsubo 1982], fluid mechanics [Sreenivasan 1983], etc. This incomplete list of examples indicates the scientific and engineering importance of the study.

A particular novel and developing field, in which threshold crossing is important, is that of the pulse discrimination neuron. The possible application in this field was the initiating motivation for the study of this work (and was suggested by Dr.E.D.Chesmore). The importance of the zerocrossing mechanism in pulse discrimination neurons is discussed briefly in section

1.2.

An extensive number of theoretical and experimental works from early to recent years have treated the classic problem of threshold crossing. A fundamental article on theory by Rice[1945], initiated the study in this field. Rice considered a stationary random process, consisting of a single totally ergodic sine wave of fixed amplitude, phase and frequency plus normal random noise. Another study by Bendat[1958] allows the sine wave to be a representative member of a partially ergodic, stationary, random process, varying in amplitude and phase, and fixed in frequency only. However, both Rice and Bendat looked at problems associated with zero-crossing by finding the probability that a random noise record will pass through zero in the interval $(\tau, \tau + d\tau)$ with a negative slope when it is known that it passes through zero at $\tau=0$ with a positive slope; and to evaluate the length and expected number of zero-crossings in an extended time period.

Previous work has a background of analogue electronics and has mostly concentrated on purely stochastic signals. Previous work has also had restricted computational facilities. The primary interest was to determine statistical information on the length of time taken by a random process to reach, or return to a boundary in the plane. These previous works are mostly concerned with statistics and expectation values, such as average number of crossings within a given time period, average time crossings, and their variances. Their basic conclusions are presented in Chapter Two, in which the relevant problems are discussed.

Modern methods of signal processing, including digital signal processing (DSP), allow more convenient practical use of threshold crossing methods. They also extend the ways in which threshold crossing may be implemented. At the same time, improved computational facilities enable engineering theoretic evaluations to be extended from statistics and expectation values to probability distributions. This in turn allows for study of design related problems such as choice of time intervals and time windows, and minimisation of errors.

To adapt this problem to a more useful modern form it is necessary to consider signals consisting of deterministic signals with added noise, and sampled signal processing methods. A clocked digital detector samples the total signal at times separated by an interval τ. The actual crossing time can only be found to within the sampling period τ, which can, however, be small. The approximation to the probability of detection of a single threshold crossing is considered in this work. The effect of correlation in the noise between the pairs of samples is studied. Analogue processing results are deduced from the sampled signal expressions by taking the limit of infinitesimal intervals. The methods are independent of the signal function, and depend only on the value and incremental change (or rate of change) at the threshold crossing, and can therefore be adapted for use with any form of signal. The approximations are useful because they involve only simple functions, and are therefore easy to evaluate numerically, and to adapt in nearly all practical cases of multiple crossings.

The methods of single crossing can be used for the very short time correlation effects, which arise when sampled signal methods are used for detecting threshold crossings. and the basic methods used in analysing this specific problem are applicable to a wider range of probability distribution based on signal and fluctuation noise on electronic relays, triggers and other devices whose operation is based on the threshold effect.

The problem of the probability of the occurrence of a second crossing within a time window after some interval following an initial crossing is studied in detail (Chapters 4, 5, 6). This is studied for the case of a sine wave signal with additive Gaussian noise, and is evaluated numerically to a high degree of accuracy.

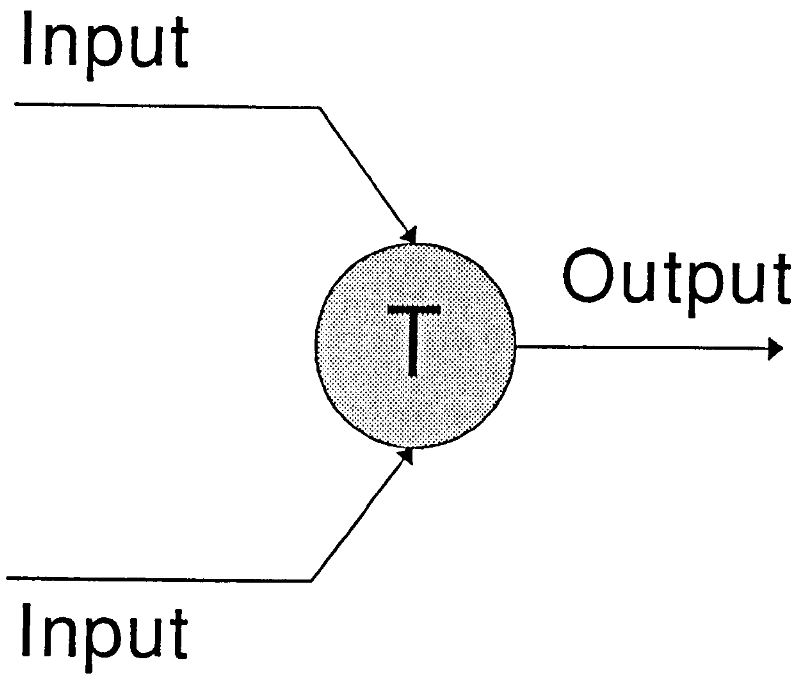
The study shows that, when the deterministic signal structure is distinguishable, the probability density of double crossing can be approximated by the no correlation case. Correlational calculations are only needed to determine the limits. Determination of performance parameters can be accomplished using the no correlation methods of single crossing. The correlations affecting the double crossing occur at worst at time scales of about one or two periods of the band frequency of the noise.

1.2 Pulse Discrimination (PD) Neural Networks

1.2.1 Introduction

As noted previously, pulse discrimination neural networks is a novel and developing field, and provided the initial motivation for this study of the threshold crossing problem. Such networks are therefore discussed briefly here.

It may be noted that real brain neurons almost certainly work by time domain processing of pulses. Pulse processing networks can be implemented for a variety of purposes, such as speech and image processing and more generally in signal processing for communications. Early studies on pulse processing networks by Reiss[1964], were under the name "resonant networks". Fig[1.1] shows the basic construction of a pulse discriminater (PD) neuron, consisting of two (or more) inputs and a single output [Chesmore 1990(May), 1990(July), 1989(July), 1989(March)-1989(October)]. The output will "fire" (produce a single narrow pulse), if and only if pulses arrive at the inputs within a "temporal threshold" of duration T seconds. Any pulses arriving with a time difference greater than T will result in no output. A pulse will be detected by threshold crossing. Thus, the basic neuron can be considered as a double crossing detector. Some real neurons may be inhibited if the second pulse arrives at too short an interval after the first.



Fig[1.1] Block diagram of basic construction of a PD.

1.2.2 Application To Speech Analysis and Recognition

It is possible for PD neural networks to perform a number of integrated functions relating to the analysis of speech. In particular, bandpass filtering and autocorrelation and crosscorrelation function, [Chesmore 1989(March)-1990(March)].

1.2.3 Application to FSK and MFSK Signal Demodulation

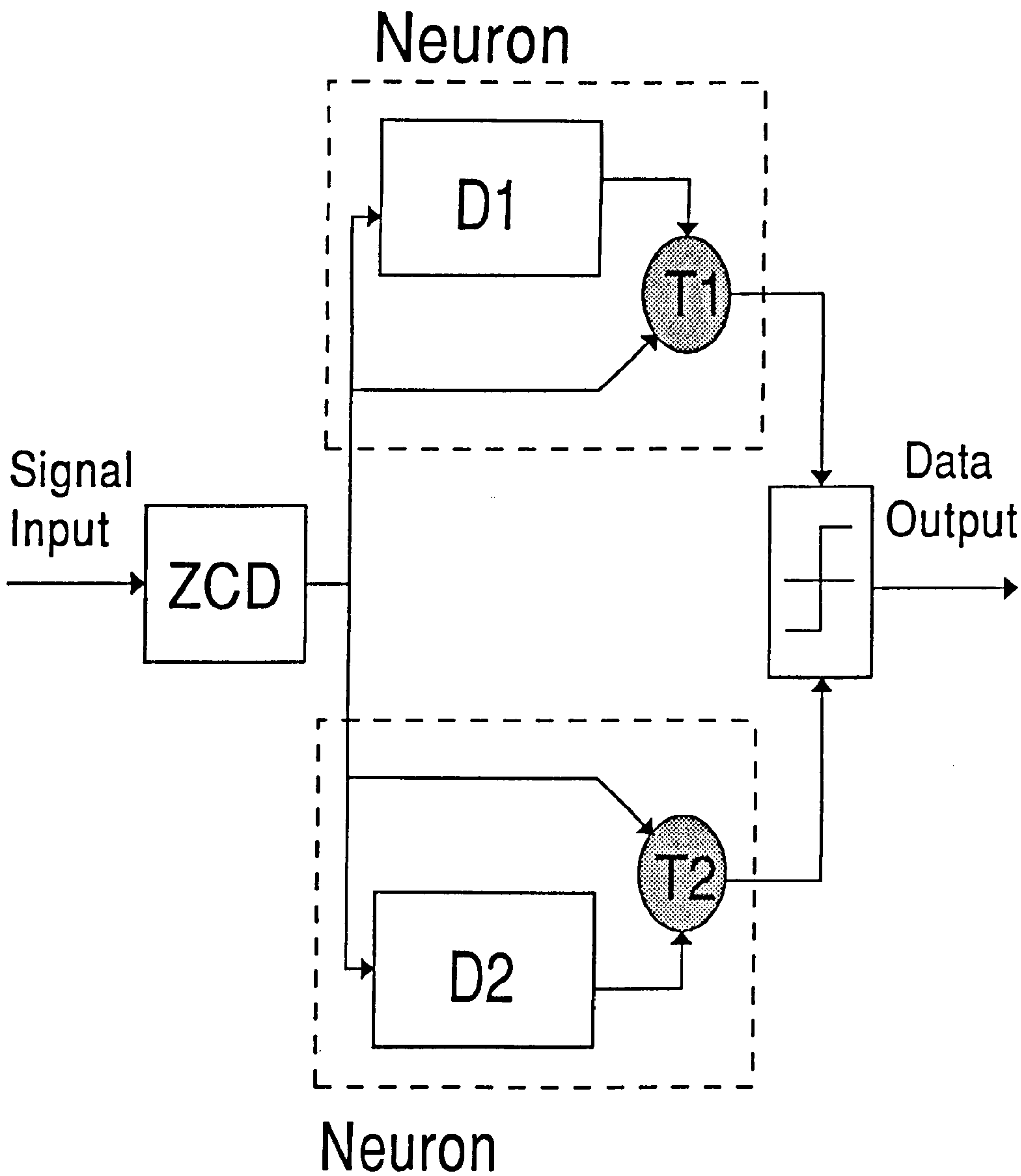
In general, non-coherent demodulation of M-ary frequency-shift keying(FSK and MFSK) schemes is carried out by applying the received signal to a number of bandpass filters, the outputs of which are rectified and low-pass filtered. The resultant dc levels are compared to a threshold and a decision made as to which bit was transmitted.

This section illustrates the application and performance of simple PD neural networks for detection and demodulation. This is achieved by configuring PD neurons as bandpass filters, as shown in Fig[1.2]. Here, input pulses are derived from a zero-crossing detector of the baseband demodulated signal and passed to N neurons, where N is the number of tones employed. Each neuron has two inputs, one directly from the zero-crossing detector and the other being delayed by a factor of D seconds from the same input. Any neuron will fire (produce pulses) at the zero-crossing rate only if input pulse intervals coincide with delayed pulses within the range $(D+T)^{-1}$ to $(D-T)^{-1}$. From these conditions, an equation for the equivalent "bandwidth" of the filter can be derived [Reiss 1964] as

$$B = 2T/(D^2-T^2)^{-1}$$

and the centre pulse repetition frequency (prf), F_c is given as

$$F_c = D^{-1}$$



Fig[1.2] Binary FSK neural demodulator.

It is important to note that the classical definition of bandwidth does not strictly apply here, since no amplitude information is utilised. The filter is effectively "Brickwall" to pulses. The selectivity Q, of the filter is defined by F_c/B and is

$$Q = 0.5\ ((TF_c)^{-1} - TF_c)$$

One apparent problem with the basic neural filter is that it will pass any pulses with intervals of D/2, D/3, etc, ie all harmonics. It is possible to overcome this problem by use of an inhibitory input whose action is to inhibit any output for an interval T_I. If the inhibitory input is connected directly to one of the other inputs and T_I is between D and D/2, then any pulse arriving with intervals less than D will automatically inhibit any output and hence suppress harmonics.

A generalised network for the demodulation of signals comprises one or more layers of PD neurons acting as bandpass filters together with general pulse processing units. The number of layers is termed the order of the netwok. It is possible to extend further the demodulator to an MFSK system by increasing the number of filters [Chesmore 1990(May)].

Work carried out in Hull by Dr. Chesmore has concentrated on first order networks as shown in Fig[1.2]. Demodulation of binary FSK was investigated by software simulation written in pascal. The results indicated that under certain condition (no noise in this case), PD neural demodulator has a performance several dB's worse than a conventional non-coherent FSK, demodulator due to the lack of amplitude information. The performance of PD demodulators can be increased substantially by increasing the number of tones to 8, 16, 32, thus resulting in a system with equivalent performance characteristics to MFSK demodulation systems.

1.2.4 PD Neuron Networks and Threshold Crossings

Pulse discrimination techniques rely on threshold detection, either single or double, for the action of the neurons. In addition, the inputs to the PD systems also rely on threshold crossing detections to initiate the pulses. When used for demodulation, or filtering, the input to the PD system will be a periodic signal together with inevitable noise, and the period (frequency) parameters of the periodic signal will be of prime concern. The work of this thesis is directly related to such a system.

Theoretical and design analysis of such systems should take into account the noise present and correlation effects due to limited bandwidth, as studied in this work, to evaluate the PD demodulator error performance. This is however, dependent on a number of factors including signal to noise ratio, channel bandwidth, and the channel noise correlation properties. Practical performance tests involving simulation over actual radio channels, and incorporating direct implementation of neural demodulators on a digital signal processor have not yet been carried out. The DSP simulation of this thesis could form the basis for such a study.

1.3 Organization of The Thesis

The second chapter reviews the threshold crossing problem as presented in the literature, and gives a brief indication of the techniques used in their solution. Material of a more or less historical nature has been included, discussed and reviewed. Included in the work reviewed are, the expected number of zeros per second, the variance of ZC intervals, and the distribution function of the ZC intervals. These are examined to determine the utility of these statistics in detecting a sinusoidal signal in the presence of Gaussian noise.

The third chapter deals with signal crossings in the more tractable case of a single crossing of a deterministic signal with added noise, and by considering a sampled signal detector at a high sampling rate. Under these assumptions the noise correlation will be important in evaluating the probability of crossing within a specific clocking interval. Approximations are studied using Mathcad.

Chapter Four is concerned with the joint probability of detection of an up-crossing and the next down-crossing. The method is an extension from Chapter Three for a single crossing. The general expression for detection of a double crossing is complicated, and is reduced, by theoretical manipulation, until expressions involving distribution functions for the normal and bivariate normal distribution are obtained. A polynomial approximation for integrals of the bivariate normal distribution is adopted for this work. This allows numerical calculation of the relevant probability whilst maintaining sufficient accuracy.

Chapter Five presents numerical evaluations for the probability density of detecting a second crossing at a specified interval after the first in the presence of correlation due to band limiting. The correlation and band limiting effects are normalised by a second order Butterworth filter and an ideal Brickwall filter. These are assumed to give the extreme cases. Evaluation is carried out over a wide range of SNR including very low values (to 0.1 amplitude SNR; -20dB). Evaluations with no correlation are included for comparison. The total probability of a second crossing within a time window centered about a delay interval after the first crossing is also evaluated.

In real zero-crossing situations, the noise may not be Gaussian and the precise correlation effects may not be known.

Chapter Six discusses the numerical results from Chapters Four & Five. These results are compared in detail. It would therefore be useful, for the design of zero-crossing systems, to have a simulation of such systems, which can be used for investigation of real signals. A DSP implementation of zero-crossing detection is presented in Chapter seven and its use demonstrated using a real signal like that of the theoretical investigation. Results of this simulation-demonstration are compared with the theoretically based evaluation of Chapter Six.

Chapter Eight reviews and summarises the main results and conclusions drawn from the body of the work. It also outlines some ideas for further work.

1.4 Original Aspects Of The Research

After the review of previous work in Chapter Two, the remainder of this thesis, up to Chapter eight, presents original work. Individual theoretical and numerical procedures may not be original (and this will be obvious from context and references) but their adaptation and application to this work is original.

Chapter 2

Zero-Crossing For Random Processes

2.1 Introduction

Theoretical studies of Rice[1945, 1948, 1958] have been influential in the study of the zero-crossing problem because of the scope of the studies: noise models, rectification of noise, effect of nonlinear devices; and because of the variety of the theoretical and mathematical methods used. More accessible treatments of this work and its development are Papoulis[1989], Bendat[1958], and Newland[1980]. More modern mathematical treatments are Ito[1964] and Ylvisaker[1965]. The literature on the ZC is extensive, and the previous work reviewed in this chapter is limited to that which is relevant to the present study.

2.2 The Threshold Crossing Problem

In order to cross a threshold, a signal $y(t)$ must take the threshold value, and pass through this value. Thus for a threshold α, a crossing will occur at time t' if either

$$y(t')=\alpha \text{ and } \left[\frac{dy(t)}{dt}\right]_{t=t'} >0; \text{or } y(t)=\alpha \text{ and } \left[\frac{dy(t)}{dt}\right]_{t=t'} <0 \tag{2.1}$$

for upwards or downwards crossing respectively. These criteria are certainly applicable in the case of a purely deterministic signal. However there is ambiguity with respect to time differentiation in the case of a stochastic process when ergodic averaging is used [Papoulis 1989, section 9.5]. Eq(2.2) are the classical threshold crossing criteria [Bendat 1958,section 3.5], applicable to analogue signal processing (e.g. detection by Schmitt trigger), but do not model the behaviour of a sampling threshold for bandlimited process [Papoulis 1989, chapter 11].

Because the crossing condition involves both the signal $y(t)$ and its slope, or signal velocity, $\dot{y}(t)$,a stochastic signal must be represented by a joint probability density function in y and $\dot{y}$. For a single crossing the appropriate probability density function is $P(y,\dot{y};t)$, such that

$$P(\alpha,\beta;t)\ d\alpha d\beta = \Pr[\alpha<y(t)<\alpha+d\alpha, \beta<\dot{y}(t)<\beta+d\beta] \tag{2.2}$$

This gives the probability, at time t, that the signal is in the range $[\alpha,\alpha+d\alpha]$ with signal velocity (slope)in the range $[\beta,\beta+d\beta]$. More crossings require higher order probability densities.

However $\dot{y}=\frac{dy}{dt}$; thus, for a threshold crossing, the increment $d\alpha$ in the level of the signal is related to the slope of the signal and the time interval dt taken for the crossing, vis:

$$d\alpha=\beta dt \tag{2.3}$$

Hence, substituting Eq(2.3)in Eq(2.2), the probability of crossing a threshold α with a signal velocity in the range $[\beta,\beta+d\beta]$ and within a time interval dt at time t is

[Rice 1945]

$$|\beta|P(\alpha,\beta;t)d\beta dt \tag{2.4}$$

An upward crossing occurs for any positive slope (c.f.2.1), so that the probability of an upward crossing of threshold α in interval dt is (replacing β by $\dot{y}$)

$$P_\alpha(t)dt = \left[\int_0^\infty \dot{y}\; P(\alpha,\dot{y};t)d\dot{y}\right]dt \tag{2.5}$$

and the probability of either an upward or downward crossing is

$$P_\alpha(t)dt = \left[\int_{-\infty}^\infty |\dot{y}|P(\alpha,\dot{y};t)d\dot{y}\right]dt \tag{2.6}$$

The expected number $E[N_\alpha(t_1,t_2)]$ of threshold crossings in an extended interval from t_1 to t_2 is then

$$E[N_\alpha(t_1,t_2)]=\int_{t_1}^{t_2} P_\alpha(t)dt=\int_{t_1}^{t_2}\int_b^\infty |\dot{y}|P(\alpha,\dot{y};t)d\dot{y}dt \quad b=\begin{Bmatrix}0 & \text{upward}\\ -\infty & \text{either}\end{Bmatrix} \tag{2.7}$$

2.3 The Zero-Crossing Problem

Most workers take the threshold as zero. It is anticipated that other thresholds can be treated by the methods of this work, in most cases by changes, which are in principle trivial, but tedious in evaluation. For the zero-crossing problem the substitution $\alpha=0$ is made in the above expression.

2.4 Stationary Stochastic Signals

Non-stationary stochastic processes are notoriously problematic. If time averages cannot be replaced by ensemble averages (ergodicity), then ultimately the use of probability might be questionable [Papoulis 1989,section 9.5 discusses ergodicity]. In practice, for non-stationary signals, it is usually assumed that stochastic signals are stationary over a sufficiently long local time interval, so that actual measurements can be interpreted in terms of stationary processes, whose parameters do not change significantly within the time taken to make the measurement. Under these conditions it is appropriate to assume stationarity and ergodicity in the theory.

For a purely stochastic signal, which is stationary, the probabilities are independent of time and

$$P(y,\dot{y};t)=P(y,\dot{y}) \tag{2.8}$$

$$P_{\alpha}(t)=P_{\alpha} \tag{2.9}$$

The time integral in Eq(2.7) gives just the time interval (t_2-t_1), and hence the number of crossings in unit time $N\alpha$, or crossing rate is

$$E[N\alpha] = P_{\alpha} = \int_{b}^{\infty} |\dot{y}| P(\alpha, \dot{y}) d\dot{y} \qquad b = \begin{Bmatrix} 0 & \text{upward} \\ -\infty & \text{either} \end{Bmatrix} \tag{2.10}$$

2.5 Joint Probability Densities for Normal (Gaussian) Stochastic Signals-Noise

To proceed, an expression for the joint probability density $P(y,\dot{y})$ is needed. In principle this can be obtained from the joint probability density function $f_2(y_1,y_2)$, for signal values y_1 and y_2 at times t_1 and t_2, by taking the limit $t_2-t_1 \longrightarrow 0$. Joint signal sample probability densities are used in the work presented in this thesis, and so are introduced first. In this work, purely stochastic signals are taken to be noise for which the standard model of a zero mean normal (Gaussian) process [Papoulis 1989, chapter 6] is used.

The normal distribution arises naturally from thermally generated processes at normal temperatures, and also where the process is result of many small variations (central limit theorem). However its unimodal symmetric shape also enables it to be used as an approximate model for other unimodal symmetric distributions. The engineering problem is then over what variable range the normal distribution is a good model. Where the normal distribution is used as an approximation, it is the larger deviations from the mean, and consequently the rarer events, for which the model is more approximate. The Gaussian process form for joint probabilities guarantees the normal form for each variable separately (marginal probabilities), and is simply an extension of the normal form. However the joint probabilities introduce correlation effects, and some further assumptions are needed on the correlation. In particular it is assumed that the autocorrelation function (defined in Eq.2.14) has the properties $\dot{R}(0)=0$, $\ddot{R}(0)<0$, these are certainly true for a band limited, stationary, stochastic process [Papoulis 1989, section 11.2], and in this work it is assumed that all signals are band limited; specific filters are introduced in chapter 5.

For a zero mean normal process the joint probability density of the noise signal values y_1 and y_2 at times t_1 and t_2 is [Bendat 1958, section 5.11]

$$f_2(y_1,y_2) = \frac{1}{(2\pi)(\det M)^{1/2}} \mathrm{EXP}(-\tfrac{1}{2}.Y^T.M^{-1}.Y) \qquad (2.11)$$

where

$$Y^T = (y_1, y_2) \qquad (2.12)$$

$$M = \begin{bmatrix} R(t_1,t_1) & R(t_1,t_2) \\ R(t_2,t_1) & R(t_2,t_2) \end{bmatrix} \qquad (2.13)$$

$$R(t_1,t_2) = E[y_1/y_2] \qquad (2.14)$$

$R(t_1,t_2)$ is the correlation function between signals at times t_1 and t_2. Such correlation between samples at different times is an important aspect of any stochastic process such as noise, and must be taken into account in any study of time domain methods of signal processing.

For a stationary process the probability distributions are invariant with respect to overall time and the correlation function becomes a function of the time difference [Bendat 1958, section 3.4-4 and 5.11]

$$R(t_1,t_2) = R(t_1-t_2) = R(t_2-t_1) \tag{2.15}$$

and in this case

$$R(0) = E[y^2] = \sigma^2 \tag{2.16}$$

is a constant, and is the square of the standard deviation of the stationary process. Thus for a Gaussian process the correlation matrix takes the form

$$M = \begin{bmatrix} R(0) & R(t) \\ R(t) & R(0) \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & r(t) \\ r(t) & 1 \end{bmatrix} \tag{2.17}$$

The normalised autocorrelation function

$$r(t) = \frac{R(t)}{R(0)} = \frac{R(t)}{\sigma^2} \tag{2.18}$$

It is possible to obtain the joint probability density function $P(y,\dot{y})$ from $f_2(y_1,y_2)$ by taking a Taylor expansion of $R(t)$, and the limit $t \longrightarrow 0$. However a more direct approach is usually taken [Newland 1980, chap 6]. For stationary processes it can be shown that [Newland 1980, chap 5].

$$E[y/\dot{y}] = R(0) \qquad E[\dot{y}^2] = \ddot{R}(0) \tag{2.19}$$

Therefore assuming a stationary Gaussian process

$$P(y,\dot{y}) = \frac{1}{(2\pi)(\det M_p)^{1/2}} EXP(-\frac{1}{2}.Y_p^T.M_p^{-1}.Y_p) \tag{2.20}$$

$$Y_p^T = (y,\dot{y})$$

$$M_p = \begin{bmatrix} R(0) & \dot{R}(0) \\ \dot{R}(0) & \ddot{R}(0) \end{bmatrix} \tag{2.21}$$

As already noted $\dot{R}(0)=0$, so that the stochastic signal and its slope are uncorrelated, and the joint probability density function decouples to give

$$P(y,\dot{y}) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{EXP}\left(-\frac{y^2}{2\sigma^2}\right) \frac{1}{\sqrt{-2\pi\,\ddot{R}(0)}}\ \mathrm{EXP}\left(\frac{\dot{y}^2}{2(\ddot{R}(0))}\right) \tag{2.22}$$

2.6 Expected Number Of Crossings

For a stationary, Zero-mean Gaussian process the expected number of threshold crossings per unit time is easily found by substituting Eq(2.22) in Eq(2.10),which gives

$$E[N\alpha] = \frac{1}{2\pi}\left[\frac{-\ddot{R}(0)}{\sigma^2}\right]^{1/2} \mathrm{EXP}(-\alpha^2/2\sigma^2) \tag{2.23}$$

$$E[N\alpha] = \frac{1}{2\pi}(-\dot{r}^{\cdot}(0))^{1/2}\ \mathrm{EXP}(-\alpha^2/2\sigma^2)$$

for one way crossings of threshold α, and double this for crossings either way. This result was originally given by Rice[1945], [see also Bendat 1958; Newland 1980]. The expected number of zero-crossings per unit time (both up and down) is

$$E[No] = \frac{1}{\pi}\left[\frac{-\ddot{R}(0)}{\sigma^2}\right]^{1/2} = \frac{1}{\pi}(-\dot{r}^{\cdot}(0))^{1/2} \tag{2.24}$$

Furthermore Ylvisaker[1965] shows that if $\dot{r}^{\cdot}(0)$ does not exist then $E[No] = \infty$.

2.7 Variance Of The Number Of Crossings

As usual the variance of the number of crossings gives an idea of the variation that can be expected in the crossing rate. For a stationary process, Bendat[1958] gives the expectation values $E[No^2(t)]$ of the square of the number of crossings in an interval t as

$$E[No^2(t)] = E\,[No]t + 2\int_0^t (t-t')\ S(t')\ dt' \qquad (2.25)$$

where S(t) is the joint probability of a pair of zero-crossings separated by an interval t and has been evaluated for the normal random variable case, by Bendat [1958], Leadbetter and Cryer[1965], Steinberg[1955] and Rice[1948]. By analogy with the single crossing case, Eq(2.10) this will be given by

$$S(t_2-t_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{y}_1|\,|\dot{y}_2|\ P(0,\dot{y}_1,0,\dot{y}_2)\ d\dot{y}_1 d\dot{y}_2 \qquad (2.26)$$

$P(y_1,\dot{y}_1,y_2,\dot{y}_2)$ is the fourfold joint probability density of the signal and its derivative $y_1,\dot{y}_1$ at time t_1 and $y_2,\dot{y}_2$ at time t_2.

If it is assumed that the signal is a Gaussian process with a multivariate normal distribution, then $P(y_1,\dot{y}_1,y_2,\dot{y}_2)$ will have a form like that of Eq(2.20) but with $Y_P^T=(y_1,\dot{y}_1,y_2,\dot{y}_2)$, and a 4.4 correlation matrix, [Rice 1945]

$$M=\begin{bmatrix} R(0) & 0 & R(t) & \dot{R}(t) \\ 0 & -\ddot{R}(0) & -\dot{R}(t) & -\ddot{R}(t) \\ R(t) & -\dot{R}(t) & R(0) & 0 \\ \dot{R}(t) & -\ddot{R}(t) & 0 & -\ddot{R}(0) \end{bmatrix} \tag{2.27}$$

This then gives

$$P(0,\dot{y}_1,0,\dot{y}_2)=\frac{1}{4\pi^2\ |M|^{1/2}}\mathrm{Exp}\left\{-\frac{1}{2|M|}\left[|M_{22}|(\dot{y}_1^2+\dot{y}_2^2)+2|M_{24}|\dot{y}_1.\dot{y}_2\right]\right\} \tag{2.28}$$

where $|M_{ij}|$ is the cofactor of the element in the row i and column j of M. Then it may be shown that using (2.28) in (2.26) yields

$$S(t)=\frac{1}{\pi^2}A(t)[1+B(t)\tan^{-1}B(t)] \tag{2.29}$$

where

$$A(t)=\frac{(|M_{22}^2|-|M_{24}^2|)^{1/2}}{(r(0)^2-r^2(t))^{3/2}} \tag{2.30}$$

$$B(t)=\frac{|M_{24}|}{(|M_{22}^2|-|M_{24}^2|)^{1/2}}$$

The variance of the number of zero per unit time is then given by

$$\begin{aligned} var[No] &=E[No^2]-E^2[No] \\ &=E[No]-E^2[No]+2\int_0^1(1-t)\ S(t)dt \end{aligned}$$

2.8 The Distribution Function of the Intervals Between Level Crossings

This section discusses approaches made in attempting to find the distribution function of zero-crossings when the total signal is a stochastic random process.

Let Po(t)dt be the conditional probability of having a zero-crossing in the interval (t,t+dt) given that it had a zero-crossing at the time of origin, regardless of what happens between 0 and t. The function S(t), given in (2.29) is just the probability density that zero-crossings occur at 0 and t. Therefore, to obtain Po(t)dt it is only necessary to divide S(t) by the expected number of zero-crossings (up or down) per unit time (2.24), to give

$$Po(t)dt = \frac{1}{\pi}\left[\frac{R(0)}{-\ddot{R}(0)}\right]^{1/2} A(t)[1+B(t)\tan^{-1} B(t)]\, dt \qquad (2.31)$$

The expression and integrals become progressively more difficult. A modification of the series (2.31) given by Longuet-Higgins[1962] apparently converges more rapidly.

Rice[1958] derives an approximate expression for the density of the lengths of time that a process spends above a level α, $P_\alpha(t)$ (i.e., between an upcrossing of the level and the next downcrossing). The approximation is valid for small values of t. The argument is similar to one used previously. The probability that $y(t)$ will have an upcrossing of level α in (t_1, t_1+dt_1) and a downcrossing in (t_2, t_2+dt_2) is

$$-dt_1\ dt_2 \int_0^{\infty} \dot{y}(t_1) \int_{-\infty}^{0} \dot{y}(t_2)\ P\left(\alpha, \dot{y}(t_1), \alpha, \dot{y}(t_2)\right) d\dot{y}(t_1)\ d\dot{y}(t_2) \tag{2.32}$$

If the level $\alpha=0$, then already (2.32) has an expression for the normal case, by [Rice 1945]

$$P\alpha(t) = \frac{1}{4\pi^2} \frac{(|M_{22}^2| - |M_{24}^2|)^{1/2}}{(r(0)^2 - r^2(t))^{3/2}} \left[1 + B(t)\cot^{-1}B(-t)\right] dt_1\ dt_2 \tag{2.33}$$

For $\alpha \neq 0$, no such closed form expression is available. If we condition (2.32) on the probability that there is an upcrossing in (t_1, t_1+dt_1), which in the normal case is given by Eq(2.10) and for the normal case is

$$E[N\alpha] = \frac{1}{2\pi} \left[\frac{-\ddot{R}(0)}{R(0)}\right]^{1/2} \mathrm{Exp}\left[-\alpha^2/2R(0)\right] dt_1 \tag{2.34}$$

then the ratio of (2.32) and (2.34) gives an approximation to $P_{\alpha}(t)(t=t_2-t_1)$, [Rice 1958, Eq.38]. It is only an approximation, as there are possibilities of level crossings between t_1 and t_2. A slightly better approximation involving three time instants, rather than two, is also given by Rice[1958, Eq.82]. Rice's result was also obtained by Volkonskii and Rozanov[1961], and under more general conditions, by Belyaev and Nosko[1969], McFadden[1958, 1961] and Kac and

Slepian[1959].

Another method of interest, uses a restrictive class of zero mean normal stationary Gaussian process, with a covariance function of the form

$$r(t) = 1 - \frac{t^2}{2} + \frac{G}{6}|t|^3 + 0(t^4). \tag{2.35}$$

The zero-crossing problem for r(t) in the above form expression has been studied by Longuet-Higgins[1962] Wong[1970]. If t_{00} is the random variable denoting the length of the interval between two successive zeros of y(t), then the cumulative probability function

$$F(t) = \Pr\left[(t_{00}) < t\right]$$

is given by

$$q(t) = \frac{dF(t)}{dt}$$

where it is known that

$$\lim_{t \to 0} q(t) = C.G$$

Where G is the constant in (2.35) and C is also a constant. Exact determination of this constant is difficult, and hence attempts were made to place upper and lower bounds on it. The best bounds were obtained by Longuet-Higgins[1962], as

$$\frac{1.1556}{6} < C < \frac{1.158}{6}$$

Wong[1970], obtained the exact result

$$C = \left(\frac{37}{32}\right)\frac{1}{6} = \frac{1.15625}{6}$$

Longuet-Higgins[1962] investigated the function $P_m(t)$, the probability density of the intervals between a zero and the (m+1)st later zero. Several bounds on the quantity $P_m(0)$ and the behaviour of $P_m(t)$ near t=0 were studied. Wong, by a another technique, was able to obtain an integral equation for $P_m(t)$ for the case where the correlation function is given as

$$r(t) = \frac{3}{2}\operatorname{Exp}\left[-\frac{|t|}{\sqrt{3}}\right]\left(1-\frac{1}{3}\operatorname{Exp}\left[-\frac{2}{\sqrt{3}}|t|\right]\right) \qquad (2.36)$$

The integral is

$$P_m(t) = \frac{1}{\pi}\,\dot{g}(t)\int_{\cosh^{-1}\sqrt{1+g(t)}}^{\infty} \frac{1}{\sqrt{\sinh^2(x)-g(t)}} \cdot (-1)\frac{d}{dx}f_m(x)dx \qquad (2.37)$$

where $f_m(x)$ is defined by

$$f_0(x) = \frac{3}{4}\frac{1}{\cosh((3/2)x)} \quad ; \quad f_1(x) = \frac{9x}{8\pi\sinh((3/2)x)}$$

$$f_m(x) = \frac{1}{m}\left[\left(\frac{3x}{2\pi}\right)^2 + \left(\frac{m-1}{2}\right)^2\right] f_{m-2}(x)$$

and $g(t) = \operatorname{Exp}\left(\frac{2}{\sqrt{3}}t\right)-1$

Unfortunately the integrals have so far only been evaluated explicitly for the case m=0.

2.9 Zero-Crossings For Deterministic Signals With Added Noise

As seen in the brief review above, most previous work on the threshold crossing problem for random signals, has concentrated on signals which are either purely stochastic, or have been treated as purely stochastic. A more useful problem is that of the determination of characteristics of a deterministic signal, which is part of a total signal comprising the deterministic signal and random noise. This is a problem of significance in communications, control and other branches of engineering. The effectiveness of zero-crossing statistics to detect a sinusoidal signal with added Gaussian noise has been established. A study by Blachman[1975], derived a mathematical expression to calculate the mean zero-crossing rate for a sum of two sinusoidal signals with random phases; Cobb[1963] evaluated the distribution of intervals between ZC of sine wave plus random noise, for the case of large signal to noise ratio; Higgins[1980] investigated the effectiveness of zero-crossing information to detect a sinusoidal signal combined with either sinusoidal or noise interferences.

All this previous work assumed analogue methods of signal processing. Sampled signal processing allows new and efficient methods for detecting crossings and processing the data. In order to detect a crossing precisely, using signal sampling, the sampling rate must be much higher than the highest frequency in the signal. The sampling rate will therefore be higher than the Nyquist rate for the signal, and higher than the bandwidth of the noise. With modern A/D converters and base signals, high sampling to band ratios are easily achievable.

As noted in section 2.5, Eqs(2.21 and 2.22), with the instantaneous detection of a crossing, the signal and its slope are uncorrelated. With sampling frequencies much greater than the bandwidth of the noise, successive samples of noise will be highly correlated, and the effects of this correlation must be determined.

2.10 Conclusions

The threshold crossing problem for random signals, and in particular the zero-crossing problem, is a classic problem of signal processing. Previous studies have been limited to analogue processing methods, and have mostly concentrated on purely stochastic signals. To adapt this problem to a more useful modern form it is necessary to consider signals consisting of deterministic signals with added noise, and sampled signal processing methods.

Because the signals considered were purely stochastic, previous studies have been limited to such quantities as average crossing rates and the probability of the size of intervals between crossings. With a signal comprising of a deterministic signal plus noise, the more relevant questions relate to the relationship of the probabilites of crossings, or pairs of crossings, to the times of crossings expected from the deterministic signal alone.

The remainder of this work considers the problem of threshold crossings for a total signal consisting of a deterministic signal with added noise, under the assumption that the deterministic signal is to be detected and characterised to some extent from the crossing data. In the first part (chapter 3) the probability of a single threshold crossing, and its relation to the crossing of the deterministic signal, is studied. The deterministic signal can take any form. Sampled signal methods are assumed initially. The continuous limit is also taken to obtain results applicable to the analogue case. In the second part (chapters 4, 5 and 6) the probability of pairs of crossings is studied. Analogue methods are assumed in order to make the problem tractable. However, it is shown that the single crossing expressions for sampled signal processing can be combined and used in all practical applications.

Chapter 3

Detection Of A Single Threshold Crossing By Signal Sampling

3.1 Introduction

In this chapter a single threshold crossing for a total signal consisting of a deterministic signal plus added noise is considered. The work on double crossings (Chapters 4, 5 and 6) shows that, in nearly all practically usable cases, joint probabilities for multiple crossings can be obtained simply by taking the product of the probabilities for the respective single crossings. The single crossing probability is therefore important for time domain methods of signal analysis.

The probability distribution with time of the detection of a crossing, and its relation to the crossing of the deterministic signal, are considered. In the first instance sampled signal processing is assumed, and the effect of correlation in the noise between the pairs of samples is taken into account. Analogue processing results are deduced from the sampled signal expressions by taking the limit of infinitesimal intervals.

In the general cases of a deterministic signal with noise, the total signal cannot be assumed to be stationary. However, it is assumed that the noise signal itself is. The total signal can then be taken as a stationary noise signal centered on the non-stationary deterministic signal. This allows the probability of crossing within a specified interval to be specified by integrals over noise probability functions. Because the detection process refers to two times, the probabilities of detection are double integrals over bivariate noise distributions. Assumption of Gaussian noise reduces the probability to one integral. However, even one integral would lead to two integrals for a double crossing, and more for multiple crossings. Therefore, closed form approximations, without integrals, are deduced. Limits of applicability for these are studied so that appropriate detectors may be designed.

3.2 Threshold Crossing Detection Using Sampling

The threshold detection process is taken to act on a total signal.

$$y(t)= s(t) + n(t) \tag{3.1}$$

Where $s(t)$ is a deterministic signal, and $n(t)$ is an additive zero mean, stochastic noise process, which is not correlated with the deterministic signal. It is assumed that a good approximation to the functional form of the signal $s(t)$ is known; and that numerical properties or parameters of this functional form are to be determined by detecting when the total signal passes through a given threshold α in a given sense, here taken to be positive. It is also assumed that the noise process $n(t)$ is also completely characterized in the statistical sense. In particular a multivariate, zero mean Gaussian process will be used.

A clocked digital detector samples the total signal $y(t)$ at times separated by the clock period interval τ. The A/D converter gives a digital value corresponding to the signal value at these times. Comparison of the digital outputs at successive times, then enables the threshold crossing to be detected. The quantisation of the A/D converter restricts the values precision and accuracy with which the threshold can be selected, but otherwise does not affect the crossing process. The actual crossing time, however, can only be found to within the sampling period τ .

A crossing of the threshold α occurs, in a positive sense, within the period t_1 to t_2 if both

$$\begin{aligned} & y(t_1) < \alpha \\ \text{and} \quad & y(t_2) > \alpha \end{aligned} \tag{3.2}$$

$$\text{where} \quad t_2 = t_1 + \tau \tag{3.3}$$

Eq(3.2) is a finite interval equivalent of Eq(2.1). This criterion avoids any problems with time differentiation, and models the action of the sampling detector correctly.

Since the signal $s(t)$ is deterministic, (3.2) can be expressed as a condition on the noise contribution by using Eq(3.1) to give

$$\begin{aligned} & n(t_1) < \alpha - s(t_1) \\ \text{and} \quad & n(t_2) > \alpha - s(t_2) \end{aligned} \tag{3.4}$$

Using this and treating $s(t)$ as simply a specified value of the random noise variable, the statistics of the crossing detection can be expressed entirely in terms of the statistics of the noise process.

For a zero mean process, and particularly for a Gaussian process, it is convenient to renormalise the signals with respect to the standard deviation of the noise, i.e. the noise power, and introduce the random noise variable

$$x(t)=\frac{n(t)}{\sigma} \tag{3.5}$$

The crossing criterion 3.4 then becomes

$$\begin{aligned} x(t_1) &< \frac{\alpha}{\sigma} - \frac{s(t_1)}{\sigma} \\ x(t_2) &> \frac{\alpha}{\sigma} - \frac{s(t_2)}{\sigma} \end{aligned} \tag{3.6}$$

3.3 Probability Of Detection Of A Single Crossing

A crossing will be detected in the interval t_1 to t_2 if the noise has any value x_1 satisfying the first condition from (3.6) at time t_1 together with any value x_2 satisfying the second condition at time t_2. Hence, with a joint probability density $f_2(x_1,x_2)$, the probability of detecting a crossing in the interval is

$$P_x(t_1,t_2) = \int_{-\infty}^{-U_1} dx_1 \int_{-U_2}^{+\infty} dx_2 \; f_2(x_1,x_2) \tag{3.7}$$

$$U_1 = \frac{s(t_1)-\alpha}{\sigma} \quad \text{and} \quad U_2 = \frac{s(t_2)-\alpha}{\sigma} \tag{3.8}$$

As with the classical crossing problem (chapter 2) it is necessary to use a particular joint probability density function in order to continue.

3.4 The High Correlation Approximation

When the sampling rate is much higher than the bandwidth of the noise, consecutive samples of the noise will be highly correlated. Taking the limit of total correlation gives a simple approximation to the probability of detection of a single threshold crossing.

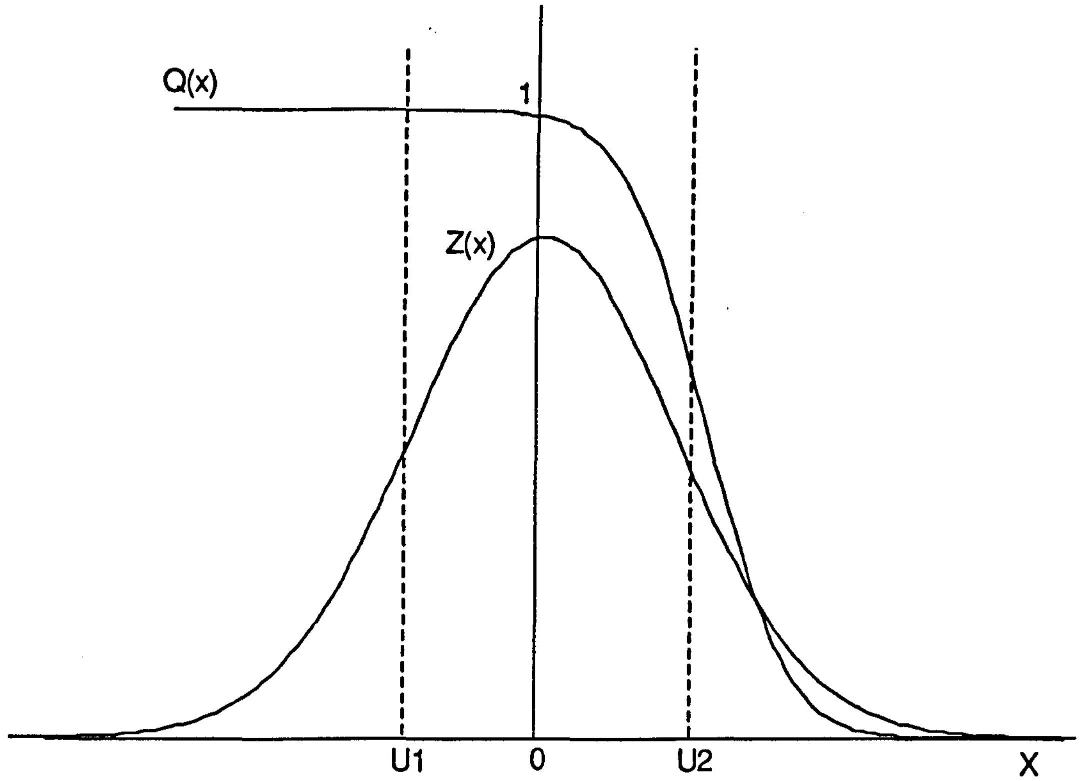
As is usual a Gaussian noise is assumed. By using the correlation matrix Eq(2.17), for a stationary Gaussian noise process can be written as [Abramowitz and Stegun 1965, 26.3.2]

$$f_2(x_1,x_2) = g(x_1,x_2,r) = \frac{1}{\sqrt{1-r^2}} Z(x_1) Z\left(\frac{x_2 - r x_1}{\sqrt{1-r^2}}\right) \qquad (3.9)$$

$$r = r(t_2 - t_1) = r(\tau)$$

where $r(t)$ is as given in Eq(2.18), and $Z(x)$ is the standard normal probability density function [Abramowitz and Stegun section 26.3.1]. Throughout this work the notation for normal probability functions is consistent with that of Abramowitz and Stegun. Much use is made of the formulae in Abramowitz and Stegun [1965], and future references will be denoted by A&S followed by the section number as appropriate. From Eq(3.7) and Eq(3.9) the probability of detecting a crossing in the interval t_1 t_2 is

$$P_x(t_1,t_2) = \int_{U_1}^{\infty} dx \; Z(x) Q\left(\frac{rx - U_2}{\sqrt{1-r^2}}\right) \qquad (3.10)$$



Fig[3.1] The high correlation approximation.

where symmetry of Z(x) has been used to change the limits, and Q(x) is the complementary cumulative normal probability function [A&S 26.2.3]. This simple form is convenient for numerical work as approximations are available for Q(x) [A&S 26.2.17].

Functions Z(x) and $Q\left(\frac{rx-U_2}{\sqrt{1-r^2}}\right)$ are depicted in Fig[3.1]. The cut off effect of function Q is centered on U_2, and has an effective width (at one standard deviation) of $\frac{\sqrt{1-r^2}}{r}$. As $r \longrightarrow 1$ this width reduces to 0, and the Q function becomes a reverse step function centered on U_2. This leads to the simple approximation for the crossing probability

$$P_x(t_1,t_2) \xrightarrow[r\to 1]{} P_x^C(t_1,t_2)$$

$$\boxed{P_xC = P_x^C(t_1,t_2) = \theta(U_2-U_1)\int_{U_1}^{U_2} dx Z(x) = \theta(U_2-U_1)\Big[P(U_2)-P(U_1)\Big]}$$

(3.11)

where P(U) is the standard normal cumulative distribution function [A&S 26.2.17]. The initial step function, $\theta(U)$, just expresses the requirement that the second signal value should be greater than the first for an upward threshold crossing. If signal values were known at the two sampling points, then Eq(3.11) could be evaluated easily. This limiting form should work as an approximation for $1-r<<1$.

The expressions in Eq(3.11) can be interpreted in terms of the deterministic signal by putting

$$\bar{U} = \frac{1}{2}(U_1+U_2) = \frac{1}{2\sigma}\Big[s(t_1)+s(t_2)-2\alpha\Big]$$

$$\text{and} \quad \Delta U = (U_2-U_1) = \frac{1}{\sigma}\Big[s(t_2)-s(t_1)\Big] \qquad (3.12)$$

Since the sampling frequency will be much greater than the bandwidth of the signal, $\bar{U}$ is the difference between the signal in the clocking interval and the threshold, normalised by the square root of the noise power; and ΔU is the normalised increment in the signal over the interval. Substituting these in Eq(3.11) and shifting the integration gives

$$P_x^C(t_1,t_2) = \theta(\Delta U)\int_{-\Delta U/2}^{+\Delta U/2} Z(x+\overline{U})\, dx \tag{3.13}$$

If the form of the signal is known, then the probability of detection in any clocking interval can be determined using this equation.

If the approximation holds, then it can be seen from (3.12) that the maximum probability of detection of a crossing occurs with the signal at the threshold value, and that the probability reduces in a roughly normal form, depending on the form of the signal.

Unlike the classical studies, which consider signal to be purely stochastic, it is possible, here, to take the noiseless limit. $P(U_1)$ and $P(U_2)$ in (3.11) are also cut off functions, with effective width σ, and become step functions in the noiseless case, so that (3.11) becomes

$$P_x^C(t_1,t_2) \xrightarrow[\sigma \to 0]{} \theta\Big[s(t_2)-s(t_1)\Big]\Big\{\theta\Big[s(t_2)-\alpha\Big]-\theta\Big[s(t_1)-\alpha\Big]\Big\} \tag{3.14}$$

This has a value 1 only if both $s(t_1)<\alpha$ and $s(t_2)>\alpha$, and is otherwise zero. This is the expected result for the threshold crossing of a noiseless deterministic signal.

3.5 Probability Of Detection Of a Single Crossing-General Case

From the results of section 3.2, for the general case, the probability of detecting a crossing in the interval t_1 to t_2 for the sum of a signal plus noise, taking the crossing in the positive sense (upcrossing) is given in Eq(3.7). In this equation the joint probability function $f_2(x_1,x_2)$ defined in Eq(2.11), represents a bivariate normal density function for the signal values U_1 and U_2 at times t_1 and t_2. For the general case, the joint probability density is factorised by changing the stochastic variables as determined by the eigenvectors of the correlation matrix. This shifts the problem of evaluation to that of determining the appropriate integration regions. This is discussed in some detail because the same techniques are used with the double crossing in Chapter Four.

The correlation matrix M of the time difference is defined in Eq(2.17). r cannot be approximated by unity at this stage, otherwise this matrix becomes singular. The eigenvalues of M are given by

$$\begin{vmatrix} 1-\lambda & r \\ r & 1-\lambda \end{vmatrix} = 0 \tag{3.15}$$

The roots are $\lambda_1 = 1-r$, $\lambda_2 = 1+r$, and axes corresponding to the new coordinate variables x_1', x_2' with their results derived as in Eq(3.A.1) in appendix 3.A.

$$x_1' = \frac{1}{\sqrt{2}}(x_1 - x_2) \qquad , \qquad x_2' = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

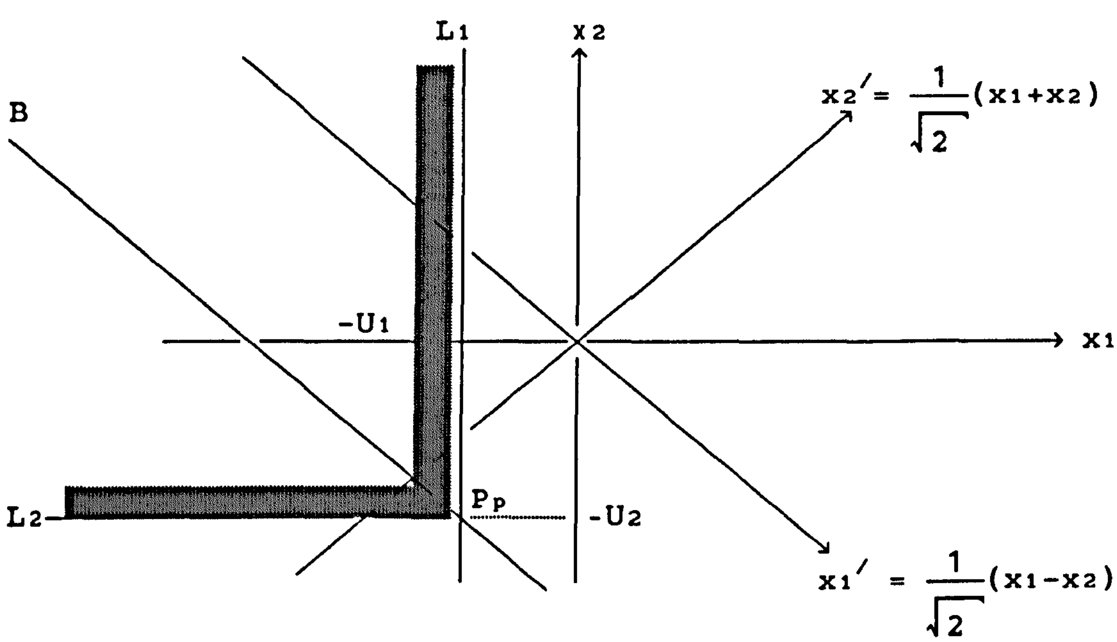
and from Eq(A.3.2) appendix 3.A

$$X^T.M^{-1}.X = \frac{x_1'^2}{1-r} + \frac{x_2'^2}{1+r}$$

Hence Eq(2.11) becomes

$$f_2(x_1,x_2) = \frac{1}{(2\Pi)(\det M)^{1/2}} \mathrm{EXP}\left[-\frac{1}{2}\left(\frac{x_1'^2}{1-r} + \frac{x_2'^2}{1+r}\right)\right] \qquad (3.16)$$

Having factorised the integrand it is now necessary to determine the integration region in the new variables. The region of integration for the first crossing variables is shown in Fig[3.2] and is the upper left portion bounded by the lines L_1, L_2, which intersect at point P_p.



Fig[3.2] The integration of the first crossing.

These limit lines must be expressed in terms of the coordinates x_1', x_2'

L_1: $x_1 = -U_1$ x_2 arbitrary

Hence

$$x_1' = \frac{1}{\sqrt{2}}(x_1 - x_2) = \frac{1}{\sqrt{2}}(-U_1 - x_2) \qquad (3.17)$$

$$x_2' = \frac{1}{\sqrt{2}}(x_1 + x_2) = \frac{1}{\sqrt{2}}(-U_1 + x_2) \qquad (3.18)$$

Substituting Eq(3.18) into Eq(3.17) yields

$$x_1' = \frac{1}{\sqrt{2}}(-2U_1 - \frac{1}{\sqrt{2}} x_2')$$

$$\therefore \quad L_1 \text{ is } \quad x_1' = -\sqrt{2}\, U_1 - x_2' \qquad (3.19)$$

L_2 is treated similarly

L_2: $x_2 = -U_2$ x_1 arbitrary

giving

$$L_2 \text{ is } \quad x_1' = \sqrt{2}\, U_2 + x_2' \qquad (3.20)$$

The new limits of integration are found by drawing line B which bisects the region of integration and runs parallel to the x_1', axis through point P_p. The expression for the lines and point in the x_1', x_2', variable are

P_p : $(-U_1, -U_2)$

L_1 : $x_1 = -U_1$ $\qquad L_1(x_2') = -\sqrt{2}\, U_1 - x_2'$

L_2 : $x_2 = -U_2$ $\qquad L_2(x_2') = \sqrt{2}\, U_2 + x_2'$

B : $x_2' = -U_1$

Define

$$U_1=\frac{1}{\sqrt{2}}(U_1''-U_1') \ , \ U_2=\frac{1}{\sqrt{2}}(U_1''+U_1')$$

so that (3.21)

$$U_1''=\frac{1}{\sqrt{2}}(U_1+U_2) \ , \ U_1'=\frac{1}{\sqrt{2}}(U_2-U_1)$$

The integral over x_1 therefore runs from $-\infty$ to either L_1 or L_2 depending on whether $x_2'>U_1''$ or $<U_1''$. Hence the integration limit only in Eq(3.8) becomes.

$$\int_{-\infty}^{-U_1} dx_1 \int_{-U_2}^{+\infty} dx_2 = \int_{-U_1''}^{+\infty} dx_2' \int_{-\infty}^{L_1(x_2')} dx_1' + \int_{-\infty}^{-U_1''} dx_2' \int_{-\infty}^{L_2(x_2')} dx_1'$$

With the change of variables completed, the probability of detecting a crossing in Eq(3.7) can be written as

$$P_x(t_1,t_2)=\left\{ \int_{-\frac{(U_1+U_2)}{\sqrt{2}}}^{+\infty} dx_2' \int_{-\infty}^{-\sqrt{2}\ U_1-x_2'} dx_1' + \int_{-\infty}^{-\frac{(U_1+U_2)}{\sqrt{2}}} dx_2' \int_{-\infty}^{\sqrt{2}\ U_1+x_2'} dx_1' \right\}$$

$$.f_2(x_1,x_2)$$

$$P_x(t_1,t_2) = \left[\int_{-U_1''}^{+\infty} dx_2' \int_{-\infty}^{L_1(x_2')} dx_1' + \int_{-\infty}^{-U_1''} dx_2' \int_{-\infty}^{L_2(x_2')} dx_1' \right]$$

$$\cdot \frac{1}{(2\pi)} \frac{1}{\sqrt{(1-r)(1+r)}} EXP\left(-\frac{1}{2}\frac{x_1'^2}{(1-r)}\right) EXP\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right)$$

$$P_x(t_1,t_2) = \left[\int_{-U_1''}^{+\infty} dx_2' \int_{-\infty}^{L_1(x_2')} dx_1' \frac{1}{\sqrt{2\pi(1-r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_1'^2}{(1-r)}\right)\right]$$

$$\cdot \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right) + \left[\int_{-\infty}^{-U_1''} dx_2' \int_{-\infty}^{L_2(x_2')} dx_1' \right.$$

$$\left. \cdot \frac{1}{\sqrt{2\pi(1-r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_1'^2}{(1-r)}\right)\right] \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_1'^2}{(1+r)}\right)$$

Using the standard normal cumulative distribution function P(x) [A&S 26.2.1 & 2] the integration over x_1' becomes

$$\frac{1}{\sqrt{2\pi(1-r)}} \int_{-\infty}^{L_1(x_2')} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_1'^2}{(1-r)}\right) dx_1' = P\left(\frac{-\sqrt{2}\ U_1 - x_2'}{\sqrt{1-r}}\right) \qquad (3.22)$$

Substituting these together with a similar expression for the integral over x_2' in the second term gives

$$P_x(t_1,t_2) = \left[\int_{-U_1''}^{+\infty} dx_2'\ P\left(\frac{-\sqrt{2}\ U_1 - x_2'}{\sqrt{1-r}}\right) \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right) \right.$$

$$\left. + \int_{-\infty}^{-U_1''} dx_2'\ P\left(\frac{\sqrt{2}\ U_2 + x_2'}{\sqrt{1-r}}\right) \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right)\right]$$

then by substituting the values of U_1, U_2 from Eq(3.21)

$$P_x(t_1,t_2) = \int_{-U_1''}^{+\infty} dx_2'\ P\left(\frac{-x_2' - U_1'' + U_1'}{\sqrt{1-r}}\right) \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right)$$

$$+ \int_{-\infty}^{-U_1''} dx_2'\ P\left(\frac{x_2' + U_1'' + U_1'}{\sqrt{1-r}}\right) \frac{1}{\sqrt{2\pi(1+r)}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right) \qquad (3.23)$$

Changing the first P function to a Q function [A&S 26.2.6], and rearranging the expression yields

$$P_x(t_1,t_2) = \left[\int_{-U_1''}^{+\infty} dx_2' \; Q\left(\frac{x_2'+U_1''-U_1'}{\sqrt{1-r}}\right) + \int_{-\infty}^{-U_1''} dx_2' \; P\left(\frac{x_2'+U_1''+U_1'}{\sqrt{1-r}}\right)\right]$$

$$\cdot \frac{1}{\sqrt{1+r}} \frac{1}{\sqrt{2\pi}} \mathrm{EXP}\left(-\frac{1}{2}\frac{x_2'^2}{(1+r)}\right) \qquad (3.24)$$

It is convenient to write the sum of the integrals of P and Q as a single integral by introducing the single function F(x,u,a) defined by

$$F(x,u,a) = [1-\theta(x)]\; P\left(\frac{x+u}{a}\right)+\theta(x)\; Q\left(\frac{x-u}{a}\right) \qquad (3.25)$$

so that, after putting

$$z = \frac{x_2'}{\sqrt{1+r}} \qquad (3.25a)$$

Eq(3.24) becomes

$$P_x(t_1,t_2) = \int_{-\infty}^{+\infty} dz \; F\left(z+\frac{U_1''}{\sqrt{1+r}}, \frac{U_1'}{\sqrt{1+r}}, \sqrt{\frac{1-r}{1+r}}\right) Z(z) \qquad (3.26)$$

Eqs(3.21) and (3.12) are used to express U_1'' and U_1' in terms of $\bar{U}$ and ΔU, the average and increment over the interval. Consequently Eq(3.26) becomes

$$P_x(t_1,t_2) = \int_{-\infty}^{+\infty} dz \; F\left[\left(z+\sqrt{\frac{2}{1+r}}\;\bar{U}\right), \frac{\Delta U}{\sqrt{2(1+r)}}, \sqrt{\frac{1-r}{1+r}}\right] Z(z) \qquad (3.27)$$

For convenience put

$$V = \sqrt{\frac{2}{1+r}}\ \bar{U} \qquad (3.28)$$

$$W = \frac{\Delta U}{\sqrt{2(1+r)}} \qquad (3.29)$$

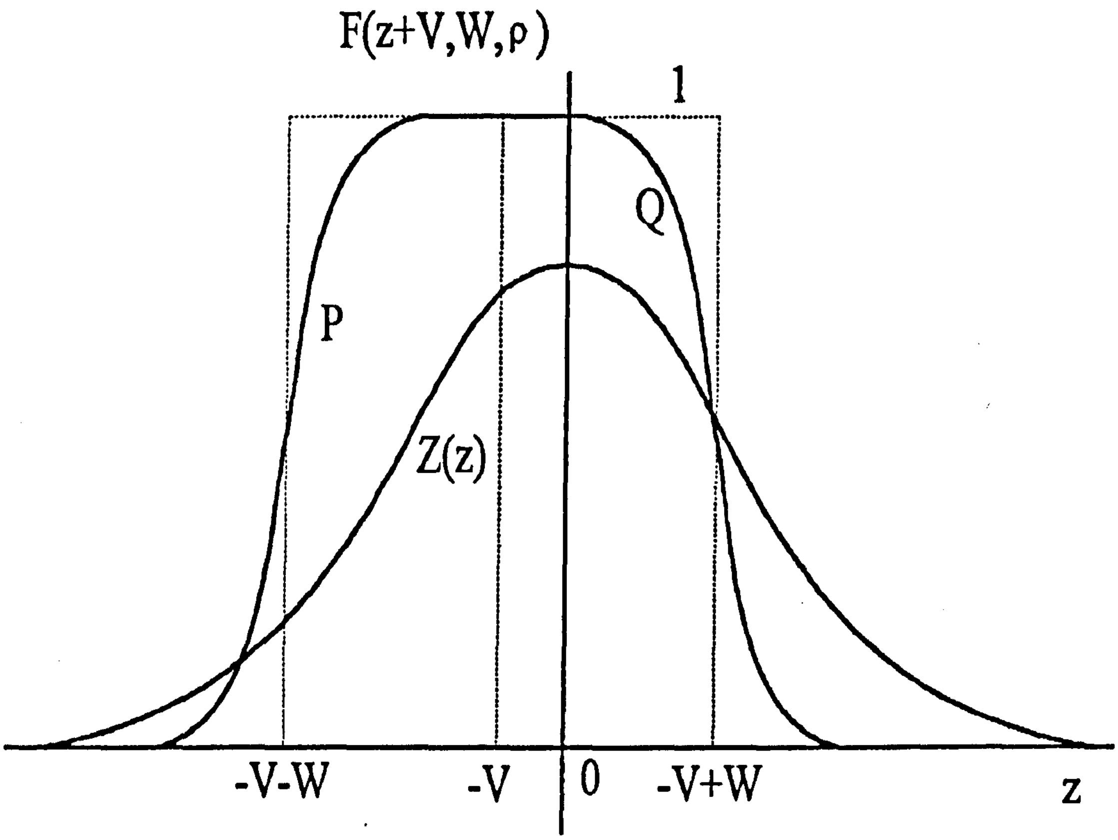
$$\rho = \sqrt{\frac{1-r}{1+r}} \qquad (3.30)$$

Then the probability for detection of a single crossing in a single interval t_1 to t_2 can be written quite generally, as

$$\boxed{P_x(t_1,t_2) = \int_{-\infty}^{+\infty} dz\ F(z+V,W,\rho)\ \ Z(z)} \qquad (3.31)$$

3.6 Approximations And Parameters For Sampling Crossing Detection-Incremental Signal to Noise Correlation Ratio

The functions of Eq(3.31) are shown in Fig[3.3]. The function $Z(z)$ is integrated over the window defined by $F(z+V,W,\rho)$. The width of $Z(z)$ is controlled by its standard deviation of 1. The width of the window to its half points is 2W, and the tails to the 2% points have an additional width of 4ρ. If $W<\rho$, then the window becomes peaked, and extends over a distance $2(W+2\rho)$. If in addition, this width is very much less than the width of Z, then Z can be approximated by its value at the centre of the window, -V. This will be called the high noise correlation approximation. On the other hand, if $W>>\rho$, then the P and Q cut offs are relatively steep, and can be approximated by step functions. This will be called the low noise correlation approximation.



Fig[3.3] Functions contributing to the single crossing probability integral.

The essential parameter controlling these approximations is

$$i_r = W/\rho = \frac{\Delta U}{\sqrt{2(1-r)}} = \frac{s(t_2)-s(t_1)}{\sigma\sqrt{2(1-r)}} \tag{3.32}$$

where Eqs(3.29, 3.30 and 3.12) have been used for the latter expressions. i_r may be called the incremental signal to noise correlation ratio (ISNCR).

The condition for the validity of the high noise correlation approximation is then

$$i_r < 1 \tag{3.33}$$

and high noise correlation means a high combination of noise power σ^2 and correlation r relative to the deterministic signal increment, giving a low ISNCR. Similarly the condition for validity of the low noise correlation approximation is

$$i_r > 1 \tag{3.34}$$

These conditions alone are not quite sufficient to determine the validity of the approximations. The variation in Z(z) and value of V will have some effects, but these effects are difficult to discern from the analytic expressions. In specific cases, validity of the approximations can be tested by calculation of the approximations, as in sections 3.7 and 3.8.

The correlation between the noise in successive samples is r, but the lack of correlation, 1-r, gives rise to a more convenient measure. Putting

$$1-r(t) = \frac{1}{2} e_r^2 \, \tau^2 \tag{3.35}$$

gives

$$e_r \tau = \sqrt{2(1-r)} \tag{3.36}$$

$$2\rho = e_r \tau \left(1 - \frac{1}{4} (e_r \tau)^2 \right)^{-1/2} \cong e_r \tau \tag{3.37}$$

e_r can simply be regarded as a parameter which specifies the correlation. However, with fast sampling and band limited noise, a good approximation is [Papoulis, section 11.2]

$$e_r^2 = -\frac{\frac{d^2R(0)}{dt^2}}{R(0)} = \frac{\ddot{R}(0)}{R(0)} = -\dot{r}'(0) \qquad (3.38)$$

i_r in Eq(3.32) becomes

$$i_r = \frac{\Delta U}{e_r \tau} \qquad (3.39)$$

With fast sampling a good approximation to the increment in the deterministic signal is [Newland 1980, section 8.4]

$$\Delta s = s(t_2) - s(t_1) = \dot{s}\tau$$

Hence

$$\Delta U = \dot{U}\tau = \dot{s}\tau/\sigma \qquad (3.40)$$

$$i_r = \frac{\dot{s}}{\sigma\, e_r} \qquad (3.41)$$

The following inequality for r, in terms of the bandwidth frequency f_B of the noise, and the sampling frequency f_C, may be obtained [Papoulis 1989, section 11.2]

$$1-r \le \frac{(2\pi f_B)^2}{2(f_C)^2}$$

For a sampling frequency of 60kHz, and bandwidth as noted, the following inequalities are found for 1-r, r, and $e_r\tau$:

f_B =1kHz gives $1-r \le 0.0054$, $r \ge 0.994$ $e_r\tau = 0.103$

f_B =2kHz gives $1-r \le 0.0219$, $r \ge 0.978$ $e_r\tau = 0.209$

From this it is clear that nearly total correlation is easily achieved.

3.7 High Noise Correlation

3.7.1 The High Noise Correlation Approximation To The Probability Of Crossing

The high noise correlation case corresponds to a low ISNCR, $i_r<1$. The integral for the probability of a single crossing, Eq(3.31), consists of the function Z(z) integrated over the window formed by function $F(z+V,W,\rho)$. The window is symmetric about -V, and when $i_r<1$ forms a narrow peak function. Z(z) is expanded in Taylor series about the point of symmetry, and for the approximation the zeroth order term, Z(-V) is used. This approximation takes the exact form of the F window, and approximates the Z(z) function. With the Z function fixed at Z(-V), the integral is symmetric about -V. Therefore it is only necessary to integrate over the right half, using the Q function from Eq(3.25), and then double the result. Thus the approximation, P_x^H to Eq(3.31) becomes

$$P_x^H(t_1,t_2) = 2\int_{-V}^{+\infty} dz \; Q\left(\frac{z+V-W}{\rho}\right) Z(-V) \tag{3.42}$$

Since Q is centered on z=(-V+W), it is convenient to change the variable of integration to give

$$P_x^H(t_1,t_2) = 2\rho \; Z(-V)\int_{-W/\rho}^{+\infty} dz \; Q(z) \tag{3.43}$$

A&S[26.2.41, 26.2.42 and 26.2.44] give the following relations for repeated integrals of the normal probability

$$I_1(x) + xI_0(x) - I_{-1}(x) = 0$$

$$I_1(x) = \int_x^{\infty} Q(t)\, dt$$

$$I_{-1}(x) = Z(x) \qquad\qquad I_0(x) = Q(x)$$

Substituting these in Eq(3.43), and using Eq(3.32) for W/ρ, yields

$$\boxed{PxH = P_x^H(t_1,t_2) = 2\rho\left\{Z(i_r) + i_r P(i_r)\right\} Z(-V)} \qquad (3.44)$$

Thus the probability of detection of a single crossing for a deterministic signal with additive Gaussian noise, in the low incremental signal to noise correlation approximation, is expressed in terms of ISNCR. This approximation is valid for small ISNCR, i.e. for ΔU small and r close to unity. The expressions are in terms of the ISNCR, and V, i.e. essentially in terms of $\bar{U}$, ΔU and r, and hence can be adapted to any deterministic signal irrespective of its function form.

The analogue crossing detection, e.g.by Schmitt trigger can be deduced from this form by taking the limit of an infinitesimal time between samples. If the sampling time is infinitesimal then $r \longrightarrow 1$ and ΔU is infinitesimal, $i_r \longrightarrow 0$ and the approximation becomes exact. Substituting Eqs(3.28, 37, 39, 40) into Eq(3.46)

$$P_x^H(t_1,t_2) \cong \tau\left(e_r Z(i_r)+\dot{U}\ P(i_r)\right)Z(-V) \tag{3.45}$$

and taking $\tau \longrightarrow dt$

$$\boxed{\frac{d}{dt}P_x(t_1)=\left(e_r Z(i_r)+\dot{U}\ P(i_r)\right)Z(-\bar{U})} \tag{3.46}$$

The probability density of crossing for a purely stochastic signal can also be obtained. With no deterministic signal $\bar{U}=-\alpha/\sigma$ (from Eq.3.12), $\dot{U}=0$ (from Eq.3.40) and $i_r=0$ (from eq.3.41). Because $i_r=0$ Eq(3.46) gives the exact result. Because there is no signal, and when the noise signal is stationary, the probability of detecting a signal crossing in the clocking interval τ is invariant, and Eq(3.46) becomes

$$P_x(\tau)=\frac{2\rho}{\sqrt{2\pi}}\ Z\left(\sqrt{\frac{2}{1+r}}\ \frac{\alpha}{\sigma}\right) \tag{3.47}$$

Taking $\tau \longrightarrow dt$, and $r \longrightarrow 1$, and using Eqs(3.30, 36, 38) for ρ, gives the the analogue result for the probability density of a crossing

$$P_x dt = \sqrt{\frac{-\ddot{r}(0)}{2\pi}}\ Z\left(\frac{\alpha}{\sigma}\right)dt \tag{3.48}$$

which is identical to the classical result, Eq(2.34) [Rice 1945, section 3; Newland 1980 section 8.3].

3.7.2 Corrections To The High Noise Correlation Approximation

Corrections to the high noise approximation, Eq(3.44), can be estimated by using higher order terms from the Taylor expansion of Z(x) in Eq(3.31). The Taylor expansion about the point of symmetry, -V, of function F is

$$Z(z)=Z(-V)+(z+V)\ Z'(-V)+\frac{1}{2}(z+V)^2\ Z''(-V) \qquad (3.49)$$

Because of the symmetry, odd order terms give zero on integration; and with the zeroth order term, Eq(3.42), half range integration can be used for the even order terms.

The second order correction, P^H_{cor} to the high noise approximation, Eq(3.42) for the probability of detecting a crossing is

$$P^H_{cor}(t_1,t_2)=\int_{-V}^{+\infty} dz\ Q\left(\frac{z+V-W}{\rho}\right)\ (z+V)^2\ Z''(-V) \qquad (3.50)$$

and with the same change of variable as for Eqs(3.42 & 3.43)

$$P^H_{cor}(t_1,t_2)=\rho^3\ Z''(-V)\int_{-W/\rho}^{+\infty} dz\ Q(z)\left(z+\frac{W}{\rho}\right)^2 \qquad (3.51)$$

Now

$$Z''(-V)=\ (V^2-1)Z(V)$$

and substituting this and Eq(3.32) in (3.51)

$$P^{H}_{cor}(t_1,t_2)= \rho^3(V^2-1)\ Z(V)\int_{-\iota_r}^{+\infty} dz\ Q(z)(z+\iota_r)^2 \tag{3.52}$$

The recurrence relations for repeated integrals of the normal probability integrals [A&S 26.2.44] can be extended to include products of the normal probability integral with powers of z. The method is discussed in appendix 3.B, where it is shown that Eq(3.B.14)

$$\int_{-a}^{\infty} Q(t)(t+a)^2\ dt = a(1+\frac{a^2}{3})P(a)+\frac{1}{3}(a^2+2)Z(-a)$$

Substituting this in Eq(3.52)

$$P^{H}_{cor}(t_1,t_2)= \rho^3(V^2-1)\ Z(V)\left\{\iota_r\left(1+\frac{\iota_r^{\,2}}{3}\right)P(\iota_r)+\frac{1}{3}\left(\iota_r^2+2\right)Z(\iota_r)\right\}$$

Hence

$$\boxed{PxHC=P^{H}_{xcor}(t_1,t_2)=\rho^3\ Z(V)(V^2-1)\left\{I(\iota_r)+\frac{\iota_r^{\,2}}{3}I(\iota_r)-\frac{Z(\iota_r)}{3}\right\}} \tag{3.53}$$

where

$$I(\iota_r) = Z(\iota_r)+\iota_r\ P(\iota_r)=\frac{PxH}{2\rho\ Z(V)} \tag{3.54}$$

From Eq(3.53) it is clear that the high noise correlation approximation depends on the ISNCR, ι_r. There is also a direct dependence on the correlation through the factor ρ, and dependence on the level of the deterministic signal $\bar{U}$ via V. However it may be noted that the combination $Z(V)(V^2-1)$ is a decreasing function of V, taking its maximum value at V=0, so that increasing the magnitude of V improves the approximation. The factor (V^2-1) arises in this second order correction because the second derivative of the normal distribution function is zero at its standard deviation points, i.e. $V=\pm1$ for the standard normal, and hence the second order correction is zero at these points.

In principle it is possible to continue with further higher order correction terms using the techniques of this section. However the numerical investigation of section 3.9 indicates that these will seldom be needed in practical usage.

3.8 Low Noise Correlation

3.8.1 The Low Noise Correlation Approximation To The Probability Of Crossing

As explained in section 3.6, when $i_r>1$, the width of the window, F, is very much greater than the widths of the cut off regions. The window can be approximated by a unity function with step cut offs about the mid points of the cut off regions, i.e. at $-(V+W)$ and $-(V-W)$. This approximation treats the $Z(z)$ function exactly, and approximates the F window, i.e. the opposite of the high noise correlation approximation. With the step cut off window, the probability of a threshold crossing, Eq(3.31), is approximated by

$$P_{xL} = P_x^L(t_1, t_2) = \int_{-V-W}^{-V+W} Z(z)dz = \theta(W)[Q(V-W)-Q(V+W)] \qquad (3.55)$$

$$= \theta(W)[P(V+W)-P(V-W)]$$

This approximation is very similar to the high correlation approximation, Eq(3.11), except that the condition $r \longrightarrow 1$ is replaced by the less stringent condition $\iota_r > 1$. The two become identical in the limit $r \longrightarrow 1$, and so this approximation also gives the correct result for a pure deterministic signal without noise.

3.8.2 Corrections To The Low Noise Correlation Approximation

Corrections to the low noise correlation approximation are found by taking the difference between the effect of the true window F and the step cut off window. Referring to Fig[3.3], it is seen that the correction at the lower cut off can be found by adding the effect of the lower tail of P, and subtracting the effect of the difference between unity and the upper tail of P. Using the P function from F Eq(3.25) in Eq(3.31), this correction becomes

$$\text{PxLC1}=\int_{-\infty}^{-V-W} dz\ P\left(\frac{z+V+W}{\rho}\right) Z(z)-\int_{-V-W}^{-V} dz\left\{1-P\left(\frac{z+V+W}{\rho}\right)\right\}Z(z) \qquad (3.56)$$

A change of variable to y=-(z+V+W) in the first integral, and to y=z+V+W in the second, and using reflection properties of P, Q and Z functions, gives

$$\text{PxCL1}=\int_{0}^{\infty}dy\ Q\left(\frac{y}{\rho}\right)Z\left(y+V+W\right)-\int_{0}^{W}dy\ Q\left(\frac{y}{\rho}\right)Z\left(-y+V+W\right) \qquad (3.57)$$

By $y=2.5\rho$, $Q(y/\rho)$ has decreased to 1.2% of its value at 0. Hence, since the width, ρ, of the cut off is much less than the width, W, of the window, the second integration can be extended to ∞, giving

$$\text{PxLC1}=\int_{0}^{\infty}dy\ Q\left(\frac{y}{\rho}\right)\left\{Z\left(y+V+W\right)-Z\left(-y+V+W\right)\right\} \qquad (3.58)$$

The extension of the second integral is certainly valid for $\ell_r \geq 2.5$.

The Z function is now expanded in a Taylor series similar to that of Eq(3.49), but about the point -V-W. Even orders of the expansion cancel in Eq(3.58), and odd orders double. Taking the first order term.

$$\text{PxLC1}=2\ Z'(V+W)\int_{0}^{\infty}Q\left(\frac{y}{\rho}\right)y\,dy = 2\rho^2\ Z'(V+W)\int_{0}^{\infty}Q(x)x\,dx \qquad (3.59)$$

where the change of variable makes the integral into the integral function $I_1^{(1)}(0)$of appendix 3.B, which takes the value 1/4, Eq(3.B.15). Hence

$$PxLC1=\frac{1}{2}\rho^2\ Z'(V+W) \tag{3.60}$$

Similar treatment at the upper cut off leads to the expression for the correction at the upper cut off

$$PxLC2=\int_0^\infty dy\ Q(\frac{y}{\rho})\left\{Z\left[y-V+W\right]-Z\left[-y-V+W\right]\right\} \tag{3.61}$$

which is analogous to Eq(3.58). Continuing with the Taylor expansion about -V+W eventually gives

$$PxLC2=\frac{1}{2}\rho^2\ Z'(V-W) \tag{3.62}$$

Differentiating the Z functions and adding the two contributions gives the correction to the low noise correlation approximation as

$$PxLC12=\frac{1}{2}\rho^2\left\{(V+W)Z(V+W)+(V-W)Z(V-W)\right\} \tag{3.63}$$

This correction term vanishes when V=0, i.e. at the most probable crossing point. This reduction of the correction to zero, when V=0, is independent of the Taylor series expansion. From Eq(3.58) and Eq(3.61) it is seen that PxLC1 and PxLC2 always cancel each other when V=0. The reduction to zero is a consequence of extending the integrals. To obtain some knowledge of the limits of the low noise correlation approximation, when V=0, it is necessary to investigate the effect of extending the integrals i.e between equations 3.57 and 3.55, the second integral has extended from 0 to W to 0 to ∞. The sum of the two extensions is

$$PxLCext=\int_{W}^{\infty}dy\ Q(\frac{y}{\rho})\left\{Z(-y+V+W)+Z(-y-V+W)\right\} \tag{3.64}$$

and since the extensions are in the negative contributions to PxLC eg. the negative term in Eq(3.57), this term should be added.

For $\rho<1$, the rate of variation of the Q function in Eq(3.64) is more rapid than that of the Z functions, and these can then be approximated by their values at lower limit and taken out of the integral, to give

$$PxLCext=\left\{Z(V)+Z(-V)\right\}\int_{W}^{\infty}dy\ Q(\frac{y}{\rho})=\ 2\rho\ Z(V)\int_{i_r}^{\infty}dx\ Q(x) \tag{3.65}$$

On changing the variable of integration, as above, the integral becomes the integral function $I_1(i_r)$ of appendix 3.B, which evaluates to $(I(i_r)-i_r)$, where $I(i_r)$ is the function defined in Eq(3.54). Thus, this extension to the correction becomes

$$PxLCext=PxH-2\rho i_r\ Z(V) \tag{3.66}$$

Finally, adding PxLC12 and PxLCext together gives the first order correction to the low noise correlation approximation as

$$\boxed{PxLC = PxH - 2\rho i_r \; Z(V) + \frac{1}{2}\rho^2\left\{(V+W)Z(V+W)+(V-W)Z(V-W)\right\}} \quad (3.67)$$

The appearance of the high noise correlation in the correction to the low noise correlation approximation indicates that the range of the corrected low noise correlation may have wide applicability. However this possibility must be investigated

3.9 Numerical Investigation Of Approximation To The Probability Of A Single Threshold Crossing

The approximations to the probability of detection of a single threshold crossing are useful because they involve only simple functions, normal probability and cumulative normal probability functions, and are therefore easy to evaluate numerically. They can be used not only for single threshold crossings, but also in nearly all practical cases of multiple crossings. It is therefore useful to investigate the range of applicability of these approximations. Originally, in this work, the investigation was based on pascal programs, but later transferred to MathCad documents. The numerical results reported in this section, Figs[3.4-13], have been calculated using MathCad. For convenience the approximations are denoted as:

PxC High correlation approximation Eq(3.11).

PxH High noise correlation approximation without corrections Eq(3.44).

PxH2 Corrected high noise correlation approximation to second order Eq(3.44) with the correction from Eq(3.53) and Eq(3.54).

PxL Low noise correlation approximation without corrections Eq(3.55).

PxL1 Corrected low noise correlation approximation Eq(3.55) with correction from Eq(3.67).

The first set of graphs, Figs[3.4-9], shows the various approximations as functions of the normalised increment in the deterministic signal, ΔU, over increasing values of the combination $e_r\tau$, i.e. decreasing correlation, for a mean deterministic signal $\bar{U}=0$ (the most probable crossing point).

Fig[3.4], with $e_r\tau=0.01$ ($r=0.99995$), is typical of the high correlation case. High correlation and low noise approximations, PxC, PxL and PxL1, coincide over the whole range, and give adequate approximation to the probability. The high noise correlation approximation, PxH, breaks at about $\Delta U=0.5$, and the corrected high noise correlation approximation, PxH2, at about $\Delta U=1.5$, corresponding to $i_r=50$ and 150 respectively, and showing a wider range of applicability than might be expected. In Fig[3.5], with $e_r\tau=0.1$ (0.995), PxC and PxL do not show the tail near $\Delta U=0$, and are no longer suitable approximations for low or negative ΔU. PxH and PxH2 breaks slightly lower than $\Delta U=0.5$ ($i_r=50$) and $\Delta U=1.5$ ($i_r=150$) respectively.

Fig[3.6], with $e_r\tau=0.2$ ($r=0.98$), shows that the high correlation approximation, PxC, is no longer valid over any range, and is not shown in any subsequent graphs. The corrected low noise correlation, PxL1, remains valid over the whole range, but this is the limit of its applicability. In Fig[3.7] $e_r\tau=0.4$ ($r=0.92$), PxL1 differs from the high noise correlation approximations, PxH and PxH2, at the low ΔU end, where the latter are expected to be valid; and from the uncorrected PxL at the upper end where this is expected to be valid. In Fig[3.8] $e_r\tau=0.6$ ($r=0.82$), the uncorrected high noise approximation, PxH, no longer meets the low noise correlation approximation, PxL, and therefore has limited applicability.

At this point it may be seen that the corrected low noise approximation, PxL1 is good over the whole range of ΔU up to a value of $e_r\tau=0.2$. However a wider range of $e_r\tau$ is covered by a combination of the corrected high noise approximation PxH2 for low and negative ΔU, together with the uncorrected low noise correlation approximation, PxL, for the upper range of ΔU. This combination of approximations is suitable for correlations $r>0.80$, which should easily be achievable with a high sampling rate with band limited noise. Over most of this range of r there is a substantial overlap in the ΔU range of validity of the two approximations.

Finally in this set, Fig[3.9] with $e_r\tau=1$ ($r=0.5$), shows PxH2 and PxL. The crossover between the two approximations occurs for a value of i_r of about 0.8, as might be expected. This crossover is not smooth, but the combination is still likely to be adequate for many purposes. If higher accuracy were required then this might be achieved by taking higher order corrections to the high correlation approximation.

It is expected that noise will perturb the total signal crossing from that expected from the deterministic signal alone. However the tail at negative ΔU in the above graphs shows that the noise can actually drive the crossing even when the deterministic signal is making a crossing in the opposite direction, provided the crossing rate is low. This effect is, however, reduced by high correlation in the noise, as shown by the decreasing tail for decreasing $e_r\tau$. Normalised signal increments $\Delta U\approx6$ cover the full width of the noise (to better than 99%) and give a probability of detection approaching 1, which is seen in the graphs.

Figs[3.10, 11], show the approximations for $e_r\tau=0.4$ ($r=0.92$), and $\bar{U}=0.5$ and (-0.5) respectively, and compared with Fig[3.7] it is seen that the effect of moving away from the expected threshold crossing point does little more than alter the skew of the functions.

Finally in the graphs of Figs[3.12, 13] approximations are plotted for fixed ΔU and varying $\bar{U}$ to simulate the passing of a deterministic signal (of ramp form) through a threshold. Fig[3.12] corresponds to validity of the corrected high noise correlation approximation, PxH2, and Fig[3.13] to validity of the low noise correlation approximation, PxL.

3.10 Application Of The Approximations For The Probability Of Detection Of A Single Crossing

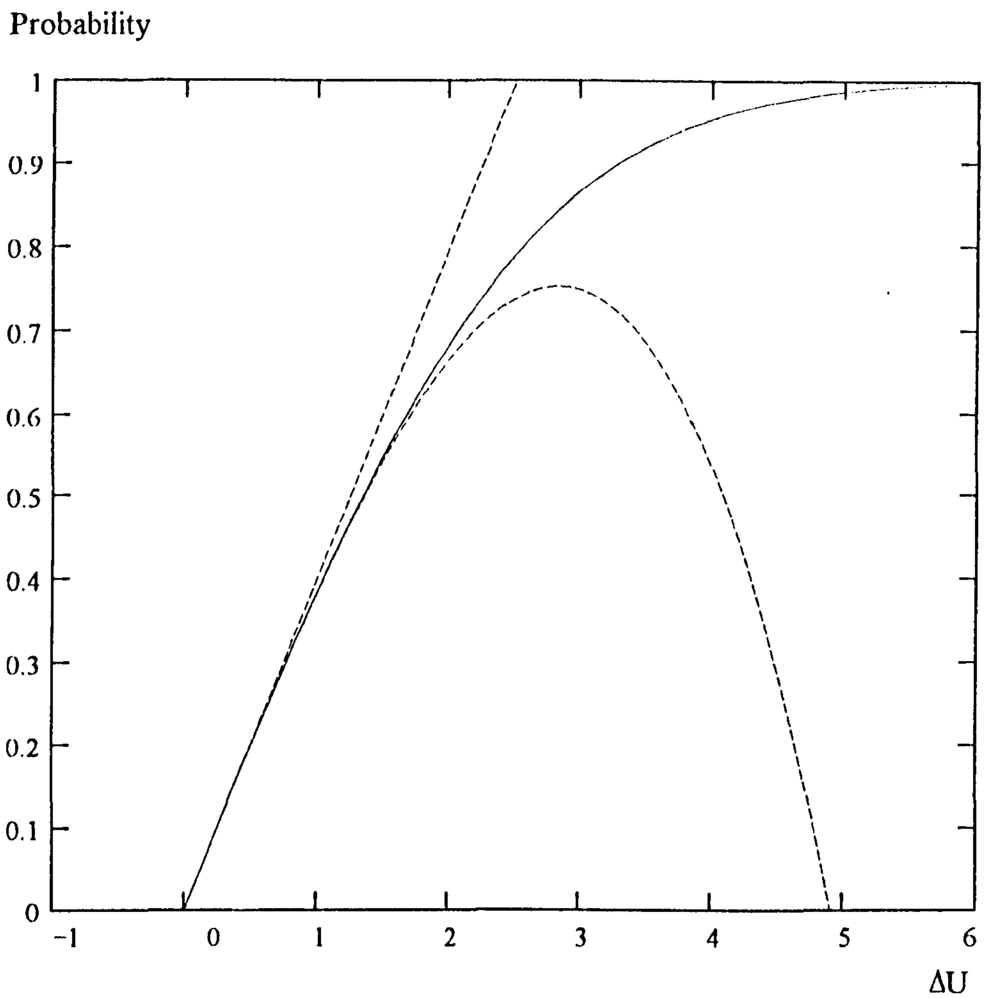
For most practical use of the approximations for the evaluation of the probability of detection of a single crossing by signal sampling, it is sufficient to limit attention to the high noise correlation approximation PxH2 (Eq.3.44 with Eq.3.53 and Eq.3.54) corrected to second order, and the corrected low noise correlation PxL1 (Eq.3.55 and Eq.3.67). Since the approximations and their corrections have closed mathematical forms the extra computation required for the corrections will be mostly of little significance. From examination of the grapgs of Figs[3.4-3.13], it may be concluded that

for the whole range of $e_r\tau$, and for $-0.5<\bar{U}<0.5$ (at least)

for $\Delta U<0.7$ PxH2 is valid (Eqs.3.44, 3.53, 3.54)

for $\Delta U>0.7$ PxL1 is valid (Eqs.3.55, 3.67)

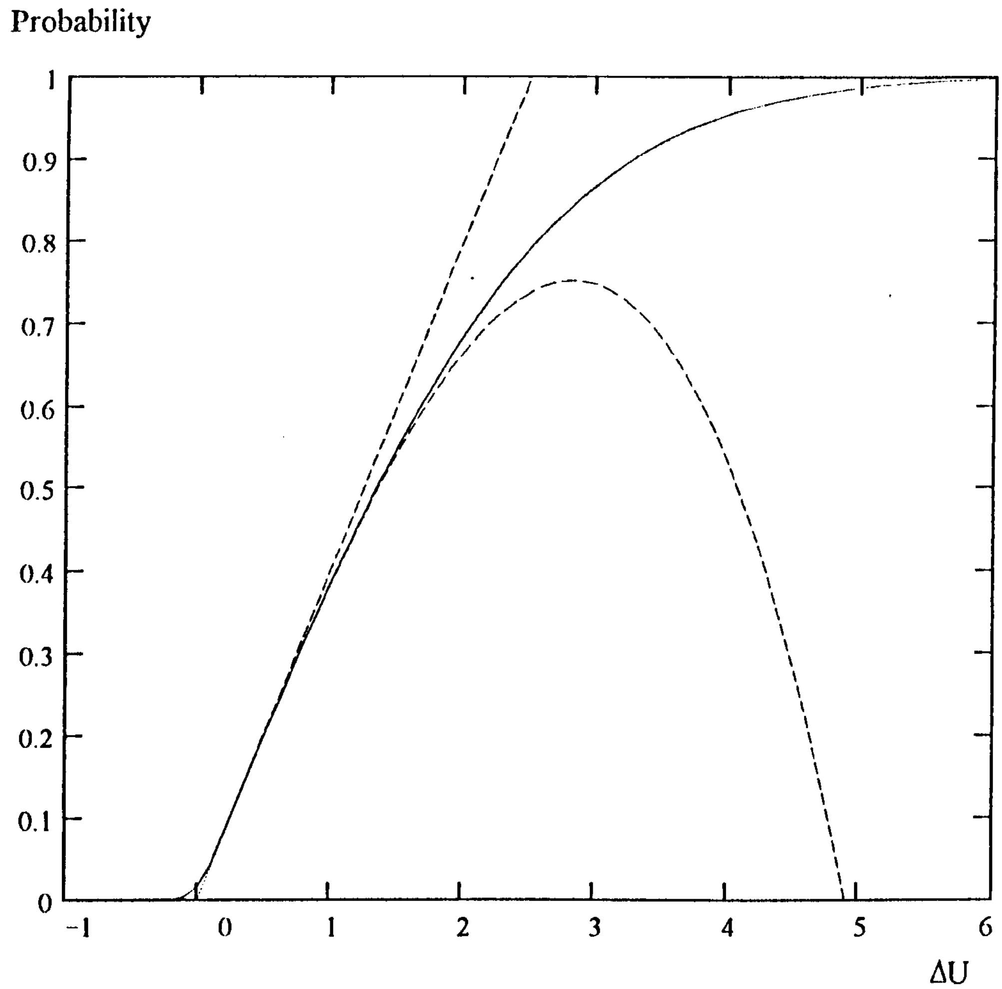
The maximum error in these approximations is not more than about 1% and occurs for $e_r\tau > 0.6$, and ΔU between 0.6 to 0.8.



Figure[3.4] Approximations to Probability of a Single Crossing

$e_r\tau = 0.01$ Ubar = 0

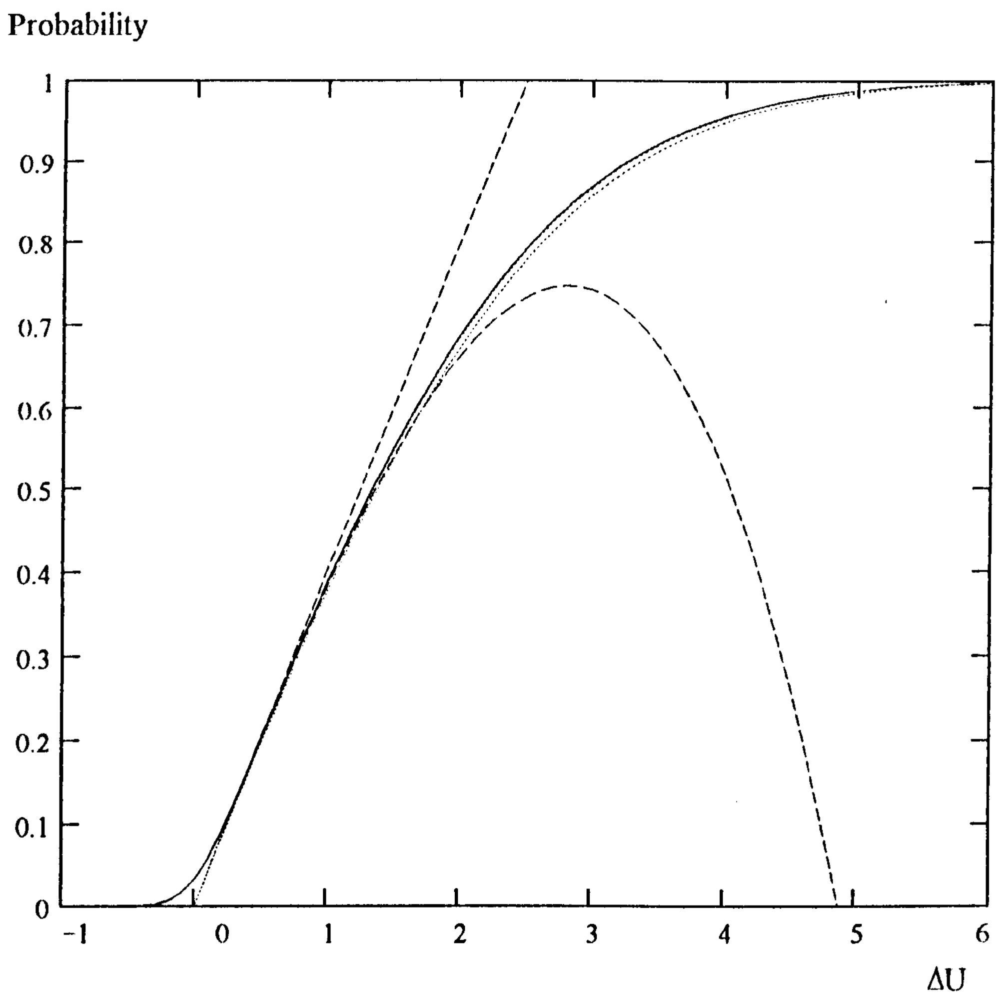
upper dashed - PxH high noise correlation
lower dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
solid ΔU>1 - also PxL low noise correlation
dotted ΔU>1 - PxC high correlation
dotted ΔU<1 - PxL low noise correlation



Figure[3.5] Approximations to Probability of a Single Crossing

$e_r\tau = 0.1$ Ubar = 0

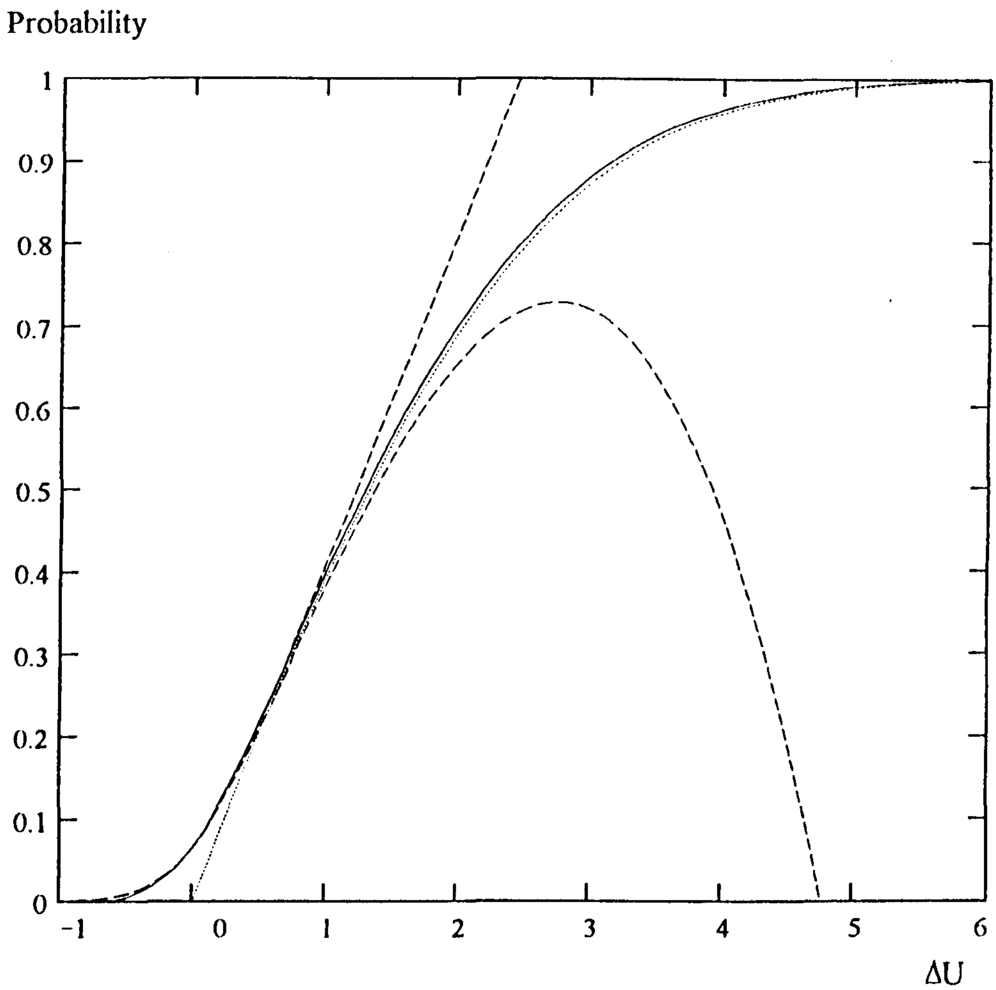
upper dashed - PxH high noise correlation
lower dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
solid ΔU>1 - also PxL low noise correlation
and PxC high correlation
dotted ΔU<1 - PxL low noise correlation
and PxC high correlation



Figure[3.6] Approximations to Probability of a Single Crossing

$e_r\tau = 0.2$ Ubar = 0

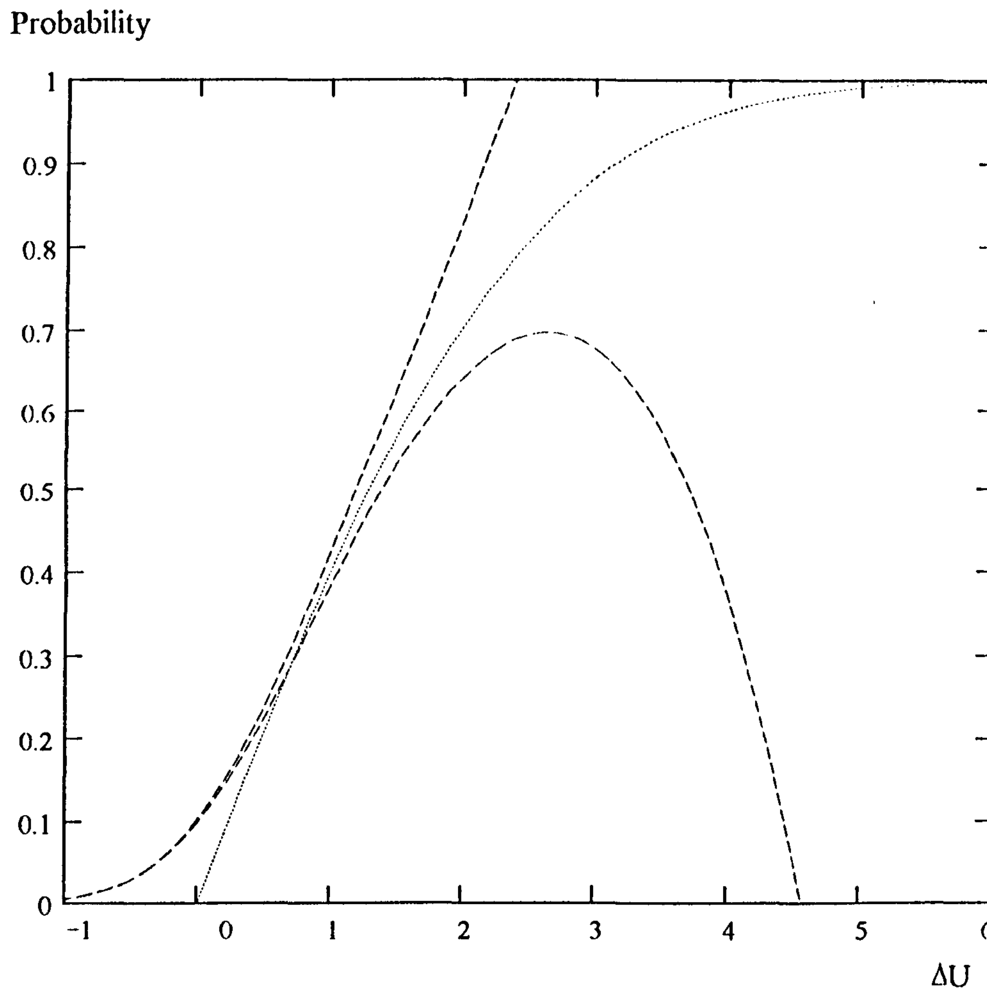
upper dashed - PxH high noise correlation
lower dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
solid ΔU>1 - also PxL low noise correlation
dotted ΔU>1 - PxC high correlation
dotted ΔU<1 - PxL low noise correlation



Figure[3.7] Approximations to Probability of a Single Crossing

$e_r\tau = 0.4$ Ubar = 0

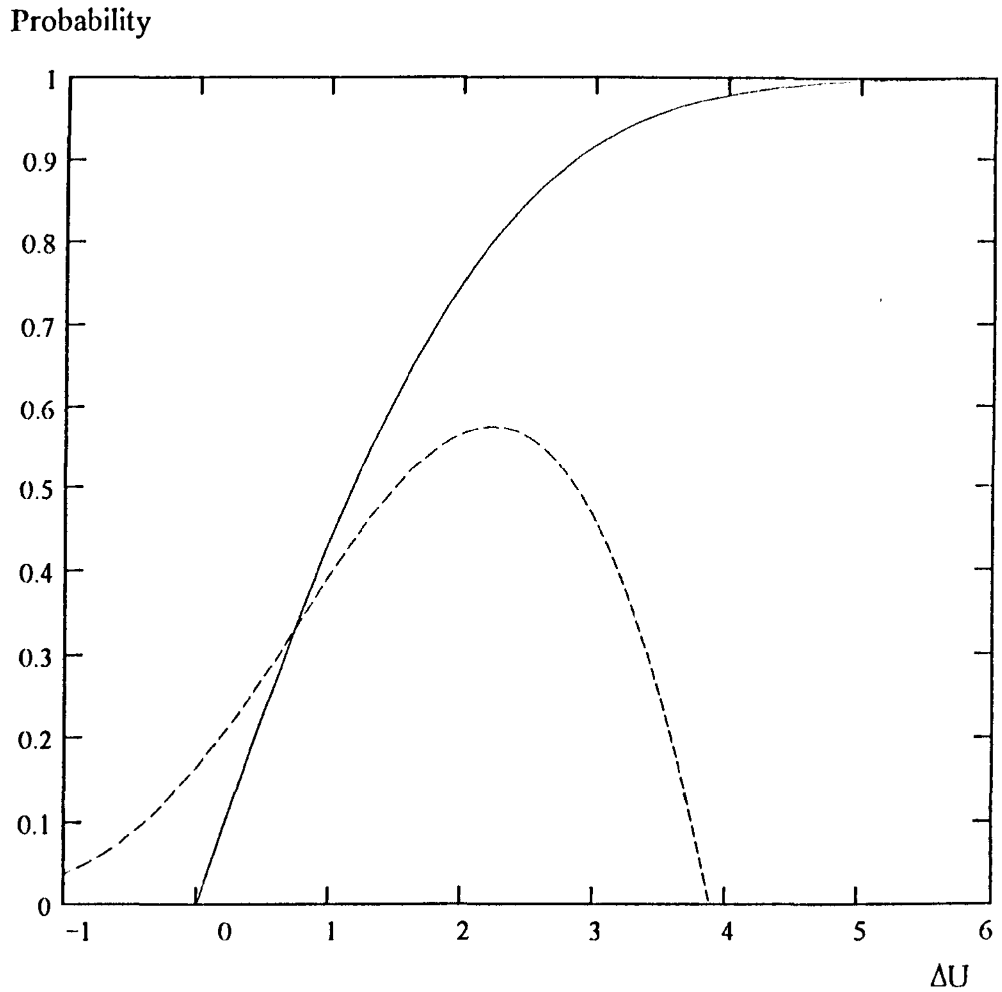
upper dashed - PxH high noise correlation
lower dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
dotted - PxL low noise correlation



Figure[3.8] Approximations to Probability of a Single Crossing

$e_r\tau = 0.6$ Ubar = 0

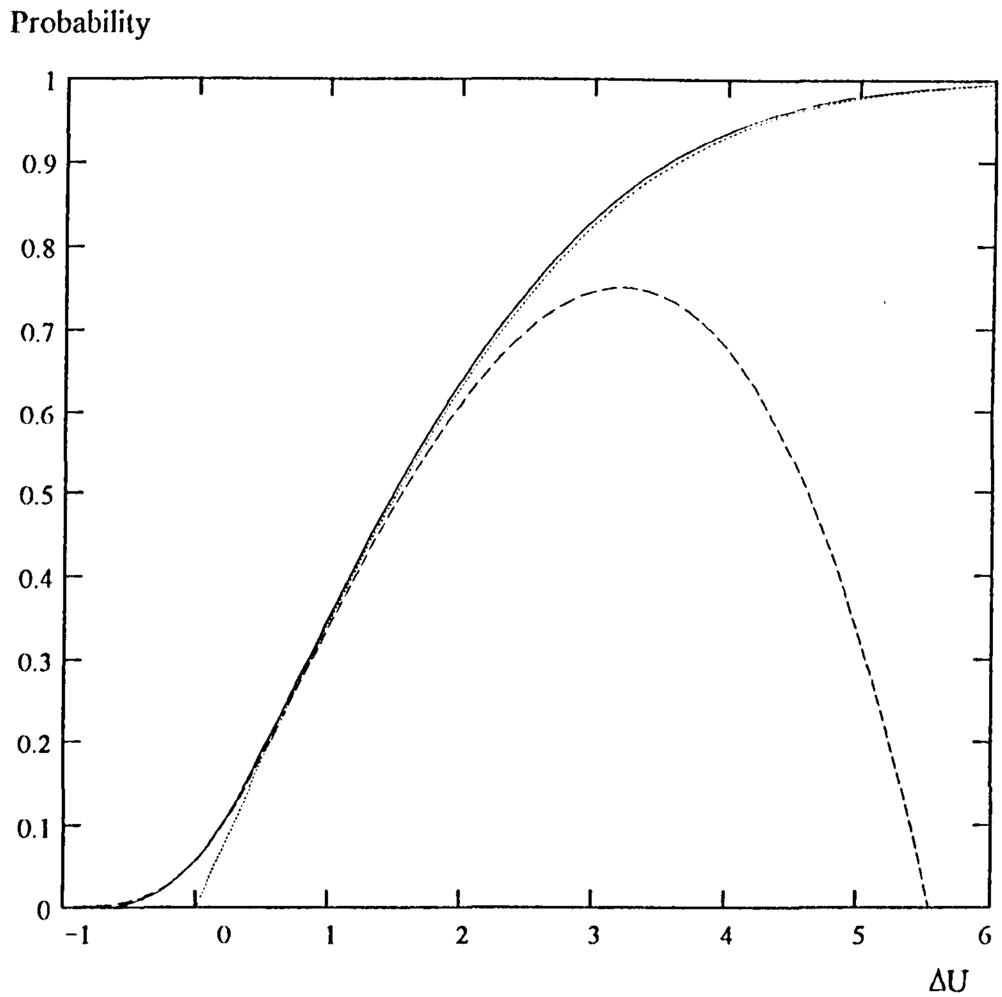
upper dashed - PxH high noise correlation
lower dashed - PxH2 corrected high noise correlation
dotted - PxL low noise correlation



Figure[3.9] Approximations to Probability of a Single Crossing

$e_r\tau = 1$ Ubar = 0

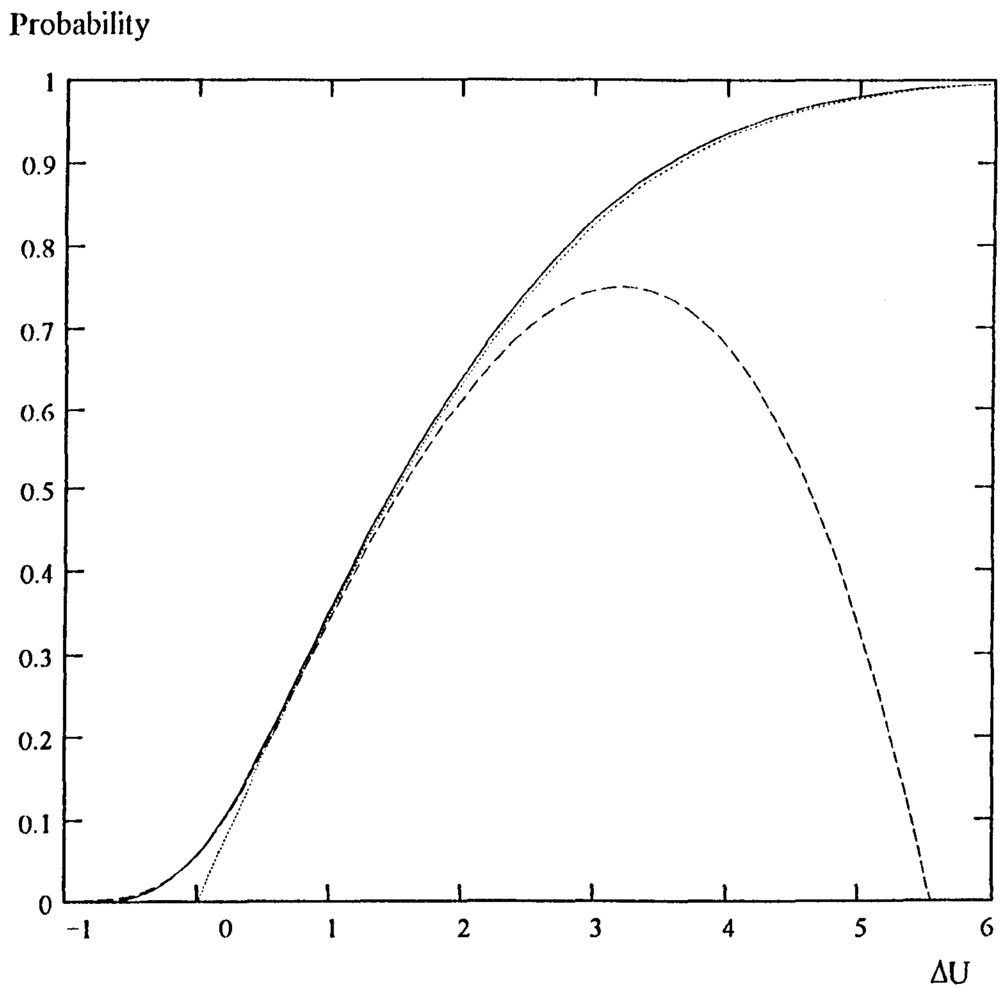
dashed - PxH2 corrected high noise correlation
solid - PxL low noise correlation



Figure[3.10] Approximations to Probability of a Single Crossing

$c_r\tau = 0.4$ Ubar = 0.5

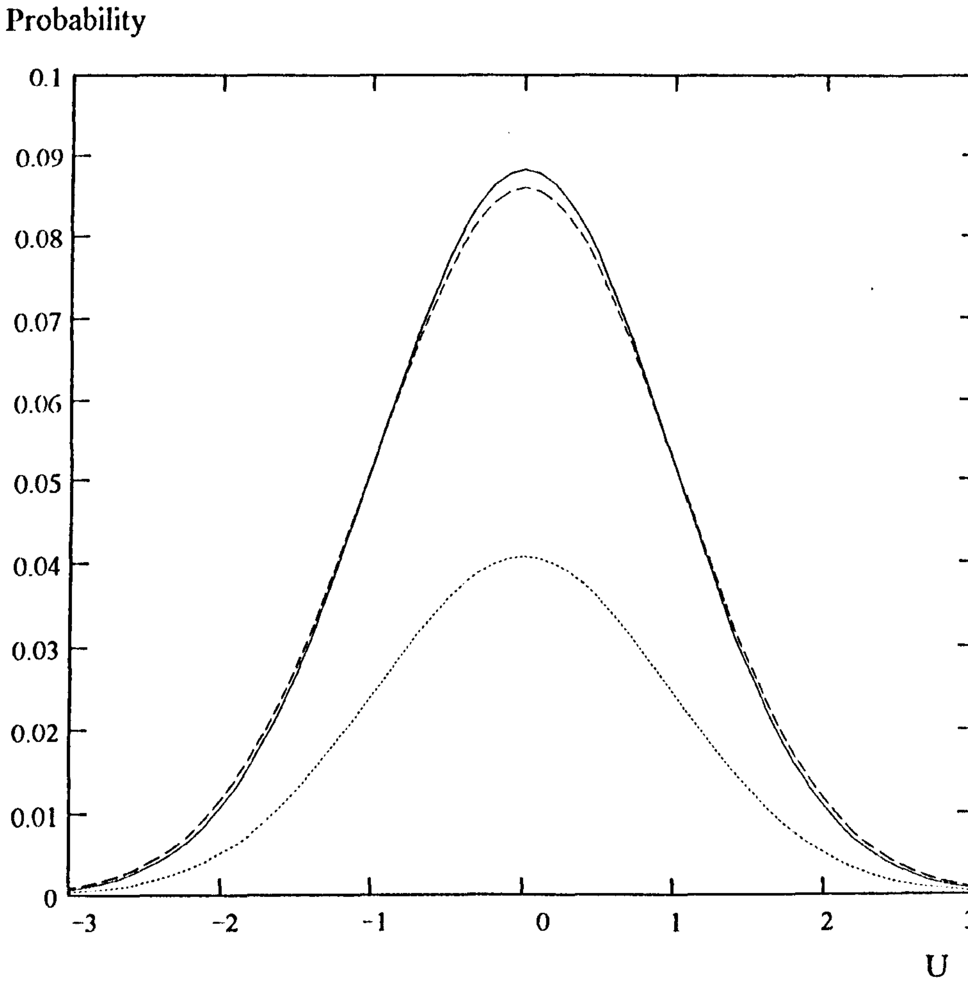
dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
dotted - PxL low noise correlation



Figure[3.11] Approximations to Probability of a Single Crossing

$e_r\tau = 0.4$ Ubar = -0.5

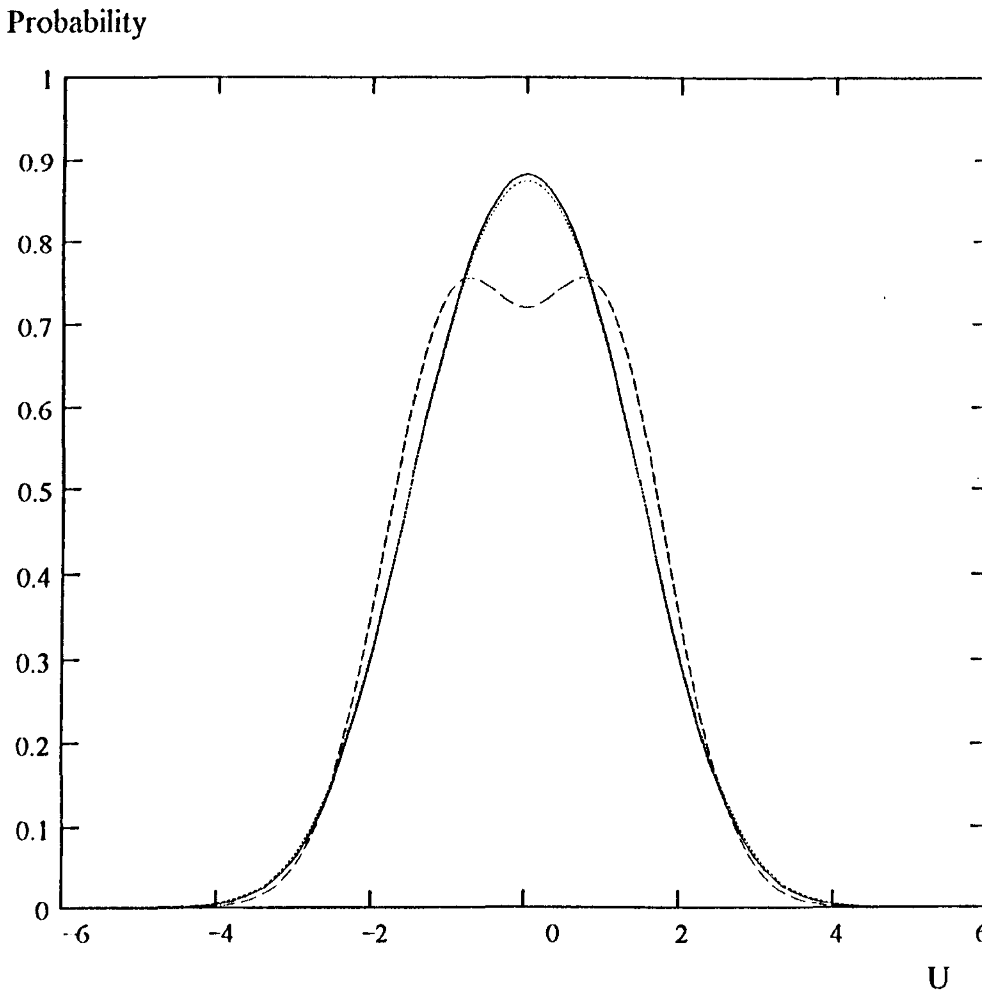
dashed	- PxH2 corrected high noise correlation
solid	- PxL1 corrected low noise correlation
dotted	- PxL low noise correlation



Figure[3.12] Approximations to Probability of a Single Crossing

$e_r\tau = 0.4$ $\Delta U = 0.1$

dashed	- PxH2	corrected high noise correlation
solid	- PxL1	corrected low noise correlation
dotted	- PxL	low noise correlation



Figure[3.13] Approximations to Probability of a Single Crossing

$e_r\tau = 0.4$ $\Delta U = 3$

dashed - PxH2 corrected high noise correlation
solid - PxL1 corrected low noise correlation
dotted - PxL low noise correlation

Chapter 4

The Joint Probability Of Two Crossings (Double Crossings Detection)

4.1 Introduction

In this chapter, use is made of the normalised autocorrelation function r(t) to characterise the dependence between the times of two successive detections of a threshold crossing of a signal with added noise, due to the effect of the time correlations in the noise.

In the presence of band limiting, noise is correlated over time, and this correlation should be taken into account when estimating characteristics of the deterministic signal from measurements of the signal with noise. In particular, the correlation should be included in determining the joint probability of two successive threshold crossings. The method used for single zero-crossing (in the previous chapter), where the correlation of crossing is high, is extended in order to find the probability of double crossing for an up-crossing (t_1,t_1+dt_1) followed by a down-crossing (t_2,t_2+dt_2) separated by time interval $t_x=t_2-t_1$.

For times which are short in comparison with the characteristic time of the band limiting, there will be strong correlation between the values of the noise. This can be expected to affect the time interval between detected threshold crossings, in comparison with the interval for the deterministic signal alone, or deterministic signal with uncorrelated noise. Significant effects might be anticipated, for example, in the determination of frequency of a signal with a frequency close to the band limit.

The tractability of the problem depends largely on the structure of the process. Again an ergodic zero mean stationary Gaussian random process is assumed.

A polynomial type approximation for the bivariate normal distribution is introduced; this allows a straight forward numerical integration to be carried out. This opens new channels for mathematical analysis and overcomes some of the difficulties of practical simulation.

A mathematical software aid (Derive) has been used where possible to confirm and aid evaluation of results.

4.2 Threshold Crossing Detection Using Sampling Signal For Double Crossing

As in Chapter Three, y(t) consists of a deterministic signal s(t), and additive zero mean stochastic noise n(t). It is initially assumed that threshold crossings are detected by signal sampling. The crossing time can only be detected within the sampling period τ. The condition for detection of a single crossing (section 3.2, Eq.3.3) is extended to give the condition of a joint detection of an upward crossing between times t_1 and t_2, and a down-crossing between t_3

and t_4 are

$$y(t_1) < \alpha_1 \quad \text{and} \quad y(t_2) > \alpha_1 \tag{4.1}$$

$$y(t_3) > \alpha_2 \quad \text{and} \quad y(t_4) < \alpha_2 \tag{4.2}$$

Where α_1, α_2 are the threshold values for first and second crossing respectively. n(t) and s(t) are assumed to be statistically independent, and for a Gaussian process it is convenient to renormalise the signals with respect to the standard deviation. The crossing criteria 4.1 and 4.2, then become

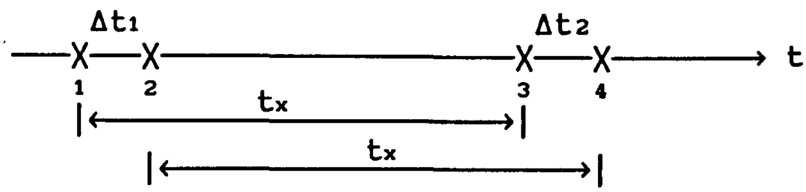
$$x(t_1) < \frac{\alpha_1 - s(t_1)}{\sigma_n} \;,\; x(t_2) > \frac{\alpha_1 - s(t_2)}{\sigma_n} \text{(Up crossing)} \tag{4.3}$$

$$x(t_3) > \frac{\alpha_2 - s(t_3)}{\sigma_n} \;,\; x(t_4) < \frac{\alpha_2 - s(t_4)}{\sigma_n} \text{(down crossing)} \tag{4.4}$$

Where σ_n is the noise standard error.

4.3 Probability Of Detection Of a Double Crossing

The up-crossing will be detected in the small interval Δt_1 from t_1 to t_2, if the noise has any value x_1 satisfying the condition from (4.3) at time t_1 together with any value x_2 satisfying the second condition at time t_2. Similarly, a down-crossing will be detected in the interval Δt_2 from t_3 to t_4, if the noise has any value x_3 satisfying the condition from (4.4) at time t_3 together with any value x_2 satisfying the second condition at time t_4. This may be depicted as in Fig[4.1].



Fig[4.1] The intervals of double crossing.

Hence, the detection probability of double crossing under these condition is

$$P_{2X} = \int_{-\infty}^{U1} dx_1 \int_{-U2}^{+\infty} dx_2 \int_{-U3}^{+\infty} dx_3 \int_{-\infty}^{U4} dx_4 \; f_4(x_1,x_2,x_3,x_4) \qquad (4.5)$$

where $f_4(x_1,x_2,x_3,x_4)$ is the fourth order joint probability density function for a Gaussian noise.

and

$$U_1 = \frac{s(t_1)-\alpha_1}{\sigma_n} \quad ; \quad U_2 = \frac{s(t_2)-\alpha_1}{\sigma_n}$$

$$U_3 = \frac{s(t_3)-\alpha_2}{\sigma_n} \quad ; \quad U_4 = \frac{s(t_4)-\alpha_2}{\sigma_n} \qquad (4.5a)$$

If deterministic signal values $s(t_1)\ldots s(t_4)$ are known, α_1, α_2 specified, and $f_4(x_1,x_2,x_3,x_4)$ known, then in principle (4.5) can be evaluated.

$U_1\ldots U_4$ can be called normalised deterministic signal values with a short clocking interval for signal sampling it is convenient to put

$$\bar{U}_1 = \frac{1}{2}(U_1+U_2)$$

$$\Delta U_1 = (U_2-U1)$$

$$\bar{U}_2 = \frac{1}{2}(U_3+U_4) \qquad (4.6)$$

$$\Delta U_2 = (U_4-U_3)$$

For short clocking intervals the approximations

$$\Delta U_1 = \dot{U}(t_1)\Delta t_1$$
$$\Delta U_2 = \dot{U}(t_2)\Delta t_2 \qquad (4.6a)$$

can be used. $\overline{U}_1$, $\overline{U}_2$ are mean normalised deterministic signal values and; ΔU_1, ΔU_2 are the normalised increment in the deterministic signal over the interval for each crossing.

4.4 Probability Density Function For Determining Joint Crossing

The joint probability function $f_4(x_1,x_2,x_3,x_4)$ in Eq(4.5), is modeled by a multivariate normal density function for the signal values U_1, U_2 at time t_1, t_2 and U_3, U_4 at time t_3, t_4; and has the form [Bendat 1958, section 3.4-4]

$$f_4(x_1,x_2,x_3,x_4) = \frac{1}{(2\pi)^2}\frac{1}{\sqrt{\det M}}\mathrm{Exp}\left[-\frac{1}{2}X^T.M^{-1}.X\right] \qquad (4.7)$$

$$X^T=(x_1,x_2,x_3,x_4)$$

The correlation matrix M is

$$M = \begin{bmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{12} & 1 & r_{23} & r_{24} \\ r_{13} & r_{23} & 1 & r_{34} \\ r_{14} & r_{24} & r_{34} & 1 \end{bmatrix}$$

Where $r_{ij} = r(t_j-t_i)$

By changing the stochastic variables as determined by the eigenvectors of the correlation matrix, using the same simple rules for the transformation of the coordinates which were effectively applied for single crossing in section 3.6, this will shift the problem of evaluation to that of determining the appropriate integration regions. So that the first crossing (up-crossing) is associated with an mean signal x_1' and rate of change (slope) y_1, and the second crossing (down-crossing) is similarly associated with an mean signal x_2' and rate of change (slope) y_2. The axes corresponding to the new coordinates variables are

$$x_1' = \frac{1}{2}(x_1 + x_2) \;, \; x_2' = \frac{1}{2}(x_3 + x_4)$$
$$y_1 = \frac{1}{\Delta t_1}(x_2 - x_1) \;, \; y_2 = \frac{1}{\Delta t_2}(x_4 - x_3) \qquad (4.8)$$

and for simplification $\Delta t_1 = \Delta t_2 = \Delta t$

with $\quad Z^T = (x_1', x_2', y_1, y_2) = (X'^T, y^T)$

Then $\quad Z = T.X$

where

$$T = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ -1/\Delta t & 1/\Delta t & 0 & 0 \\ 0 & 0 & -1/\Delta t & 1/\Delta t \end{bmatrix}$$

From which the quadratic form becomes

$$X^T.M^{-1}.X = X^T.T^T.(T^T)^{-1}.M^{-1}.T^{-1}.T.X$$
$$= Z^T.M_z^{-1}.Z$$

where $\quad M_z = T.M.T^T$

$$
M_z=\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ -1/\Delta t & 1/\Delta t & 0 & 0 \\ 0 & 0 & -1/\Delta t & 1/\Delta t \end{bmatrix}\begin{bmatrix} 1 & r_{12} & r_{13} & r_{14} \\ r_{12} & 1 & r_{23} & r_{24} \\ r_{13} & r_{23} & 1 & r_{34} \\ r_{14} & r_{24} & r_{34} & 1 \end{bmatrix}
$$

$$
\cdot\begin{bmatrix} 1/2 & 0 & -1/\Delta t & 0 \\ 1/2 & 0 & 1/\Delta t & 0 \\ 0 & 1/2 & 0 & -1/\Delta t \\ 0 & 1/2 & 0 & 1/\Delta t \end{bmatrix}
$$

As required then M_z becomes

$$
M_z=\begin{bmatrix} 1/2\ (1+r_{12}) & 1/4\ (r_{13}+r_{14}+r_{23}+r_{24}) & 0 & 1/2\ \Delta t(r_{14}-r_{13}+r_{24}-r_{23}) \\ 1/4\ (r_{13}+r_{14}+r_{23}+r_{24}) & 1/2\ (1+r_{34}) & 1/2\ \Delta t(r_{23}-r_{13}+r_{24}-r_{14}) & 0 \\ 0 & 1/2\ \Delta t(r_{23}-r_{13}+r_{24}-r_{14}) & (1/\Delta t)^2.2(1-r_{12}) & (1/\Delta t)^2(r_{24}+r_{13}-r_{23}-r_{14}) \\ 1/2\ \Delta t(r_{14}-r_{13}+r_{24}-r_{23}) & 0 & (1/\Delta t)^2(r_{24}+r_{13}-r_{23}-r_{14}) & (1/\Delta t)^2.2(1-r_{34}) \end{bmatrix}
$$

Then by writing the correlation function as explicit function of time and using Taylor's theorem.

$$r_{12}=r(\Delta t)=1+\dot r_o\Delta t+\frac{1}{2}\ddot r_o(\Delta t)^2$$

$$r_{13}=r(t_x)=r_x$$

$$r_{14}=r(t_x+\Delta t)=r_x+\dot r_x\Delta t+\frac{1}{2}\ddot r_x(\Delta t)^2$$

$$r_{23}=r(t_x-\Delta t)=r_x-\dot r_x\Delta t+\frac{1}{2}\ddot r_x(\Delta t)^2$$

$$r_{24}=r(t_x)=r_x$$

$$r_{34}=r(\Delta t)=1+r_o\Delta t+\frac{1}{2}\ddot r_o(\Delta t)^2$$

Now evaluating the elements of M_z in the limit $\Delta t\longrightarrow 0$ gives

$$M_z = \begin{bmatrix} 1 & r_x & 0 & \dot{r}_x \\ r_x & 1 & -\dot{r}_x & 0 \\ 0 & -\dot{r}_x & -\ddot{r}_0 & -\ddot{r}_x \\ \dot{r}_x & 0 & -\ddot{r}_x & -\ddot{r}_0 \end{bmatrix} \tag{4.9}$$

where

$r_x = r_{13} = r_{24}$, The normalized correlation between x_1', x_2' Of the noise alone.

$\dot{r}_x = dr_{13}/dt$

$\ddot{r}_0 = d^2r(0)/dt^2$,In practice this will be negative value

$\ddot{r}_x = d^2r_{13}/dt^2$

Making use of the property of a Gaussian process that the random variables produced by a linear transformation of (x_1,x_2,x_3,x_4) will be also Gaussian [Peebles 1987, 5.5].

$$f_4(x_1,x_2,x_3,x_4) = f_z(x_1', x_2', y_1, y_2) \tag{4.10}$$

Now noting the expression for $f_4(x)$ and substituting for Z in the exponential Eq(4.7).

$$f_4 = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{\det M}} \mathrm{Exp}\left[-\frac{1}{2} Z^T.M_z^{-1}.Z\right]$$

$$= \frac{\sqrt{\det M_z}}{\sqrt{\det M}} \frac{1}{(2\pi)^2} \frac{1}{\sqrt{\det M_z}} \mathrm{Exp}\left[-\frac{1}{2} Z^T.M_z^{-1}.Z\right]$$

Now $\det M_z = \det(T.M.T^T) = \det(T).\det(M).\det(T)$

and $\det T = \det T^T$

so that $$\frac{\sqrt{\det M_z}}{\sqrt{\det M}} = \det(T)$$

Now the relation between elements of area in the two planes is determined by the well-know Jacobian transformation of coordinates where

$$f_4(x) = \frac{1}{|J|} f_z(x)$$

Hence, the transformed density function is

$$|J| = \det T$$

In this formula, the variables (x_1, x_2, x_3, x_4) on the left hand side of 4.10 must be replaced by their appropriate (x_1', x_2', y_1, y_2). Also the absolute value of J should be used since the probabilities $f_4(x)$ and $f_z(x)$ are positive quantities between zero and one.

$$f_4(x_1, x_2, x_3, x_4)\, dx_1\, dx_2\, dx_3\, dx_4 =$$

$$\frac{\det T}{\sqrt{\det M_z}} \frac{1}{(2\pi)^2} \mathrm{Exp}\left[-\frac{1}{2} Z^T . M_z^{-1} . Z\right] \frac{1}{\det T} dx_1'\, dx_2'\, dy_1\, dy_2$$

Noting the cancellation of the detT

$$\boxed{f_z(x_1', x_2', y_1, y_2) = \frac{1}{\sqrt{\det M_z}} \frac{1}{(2\pi)^2} \mathrm{Exp}\left[-\frac{1}{2} Z^T . M_z^{-1} . Z\right]} \quad (4.11)$$

This is the correct probability density function to be used in conjunction with the transformed coordinates, x_1', x_2', y_1, y_2.

4.5 Reduction Of The Quadratic Form

The main difficulties of double crossing are that of computation complexity and that of geometrical visulisation. To simplify these, the block matrix structure of M_z is utilised, with blocks related to first and second crossings respectively. [Useful theorems and results are summarised in an appendix of Patel and Munro, 1982].

$$M_z = \begin{bmatrix} A & D \\ C & E \end{bmatrix}$$

With

$$A = \begin{bmatrix} 1 & r_x \\ r_x & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & \dot{r}_x \\ -\dot{r}_x & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -\dot{r}_x \\ \dot{r}_x & 0 \end{bmatrix} \qquad E = \begin{bmatrix} -\ddot{r}_o & -\ddot{r}_x \\ -\ddot{r}_x & -\ddot{r}_o \end{bmatrix}$$

Then

$$M_z^{-1} = \begin{bmatrix} A^{-1} + A^{-1}DS^{-1}CA^{-1} & -A^{-1}DS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

with $S = E - C.A^{-1}.D$

and

$\det M_z = \det A \det S$

and from Eq(4.9), $D = C^T$

The associated block structure of Z is

and $Z = (X'^T, y^T)$

where $X'^T = (x_1', x_2')$

$y^T = (y_1, y_2)$

So that the quadratic form in (4.11) becomes.

$$Z^T.M_z^{-1}.Z = X'^T.A^{-1}.X' + (y-B)^T.S^{-1}.(y-B)$$

where $B = C.A^{-1}.X'$

such that $B^T = (b_1, b_2)$

Now

$$A^{-1} = \frac{1}{1-r_x^2}\begin{bmatrix} 1 & -r_x \\ -r_x & 1 \end{bmatrix}$$

$$\det A = 1-r_x^2$$

$$C.A^{-1} = \frac{\dot{r}_x}{1-r_x^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -r_x \\ -r_x & 1 \end{bmatrix} = \frac{\dot{r}_x}{1-r_x^2}\begin{bmatrix} r_x & -1 \\ 1 & -r_x \end{bmatrix}$$

and

$$b_1 = \frac{\dot{r}_x}{1-r_x^2}(r_x\, x_1' - x_2')$$

$$b_2 = \frac{\dot{r}_x}{1-r_x^2}(x_1' - x_2'\, r_x)$$

$$C.A^{-1}.D = \frac{\dot{r}_x^2}{1-r_x^2}\begin{bmatrix} r_x & -1 \\ 1 & -r_x \end{bmatrix}\begin{bmatrix} 0 & \dot{r}_x \\ -\dot{r}_x & 0 \end{bmatrix}$$

$$= \frac{\dot{r}_x^2}{1-r_x^2}\begin{bmatrix} 1 & r_x \\ r_x & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} -\dot{r}_{\dot{0}} & -\dot{r}_{\dot{x}} \\ -\dot{r}_{\dot{x}} & -\dot{r}_{\dot{0}} \end{bmatrix} - \frac{\dot{r}_x^2}{1-r_x^2}\begin{bmatrix} 1 & r_x \\ r_x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\dot{r}_{\dot{0}} - \dfrac{\dot{r}_x^2}{1-r_x^2} & -\dot{r}_{\dot{x}} - \dfrac{\dot{r}_x^2\, r_x}{1-r_x^2} \\ -\dot{r}_{\dot{x}} - \dfrac{\dot{r}_x^2\, r_x}{1-r_x^2} & -\dot{r}_{\dot{0}} - \dfrac{\dot{r}_x^2}{1-r_x^2} \end{bmatrix}$$

Now put

$$S = \sigma_y^2\begin{bmatrix} 1 & \rho_y \\ \rho_y & 1 \end{bmatrix}$$

$$\det S = \sigma_y^4\ (1-\rho_y^2)$$

Elements of S, called the covariance matrix of the random variables. The density Eq(4.11) is multivariate Gaussian density function. For our special case, the fourth-order joint probability density function, the inverse of covariance matrix becomes [Peebles 1987];

$$S^{-1} = \frac{1}{\sigma_y^2(1-\rho_y^2)}\begin{bmatrix} 1 & -\rho_y \\ -\rho_y & 1 \end{bmatrix}$$

ρ_y is the dimensionless correlation coefficient and its value is not affected by a change of scale. Values of ρ_y near +1 or -1 indicate high degree of linearity(correlation), while ρ_y near 0 indicates a lack of linearity(correlation), between the random variables of noise, and also that any two uncorrelated Gaussian random variables are statistically independent. Hence the normalized covariance is

$$\sigma_y^2 = -\ddot{r}_0 - \frac{\dot{r}_x^2}{1-r_x^2}$$

and the correlation coefficient is

$$\rho_y = \frac{1}{\sigma_y^2}\left(-\ddot{r}_x - \frac{\dot{r}_x^2\, r_x}{1-r_x^2}\right)$$

$$\rho_y = r_x + \frac{1}{\sigma_y^2}\left(r_x\, \ddot{r}_0 - \ddot{r}_x\right)$$

Finally, in the last step substitute the equivalent of $Z^T.M_z^{-1}.Z$ and $\det M_z$ in Eq(4.11). Then after some reorganization of terms

$$
f_z(x_1',x_2',y_1,y_2)dx_1'\,dx_2'\,dy_1\,dy_2 =
$$

$$
\frac{1}{\sqrt{1-r_x^2}}\,\frac{1}{(2\pi)}\mathrm{Exp}\left[-\frac{1}{2}\,X'^T.A^{-1}.X'\right]\frac{1}{\sqrt{1-\rho_y^2}}\,\frac{1}{\sigma_y^2}\,\frac{1}{(2\pi)} \tag{4.12}
$$

$$
.\mathrm{Exp}\left[-\frac{1}{2}(y-B)^T.S^{-1}.(y-B)\right] dx_1'\,dx_2'\,dy_1\,dy_2
$$

The above equation has the form of a product of two bivariate normal distributions, with B as function of x_1',x_2'. Now to establish the correspondence between the second function of Eq(4.12) and σ_y

$$
(y-B)^T.S^{-1}.(y-B) = \frac{1}{\sigma_y}(y-B)^T\,\frac{1}{(1-\rho_y^2)}\begin{bmatrix} 1 & -\rho_y \\ -\rho_y & 1 \end{bmatrix}\frac{1}{\sigma_y}(y-B)
$$

by putting

$$
\begin{aligned}
y_1' &= \frac{1}{\sigma_y}y_1 \quad , \quad y_2' = \frac{1}{\sigma_y}y_2 \\
dy_1' &= \frac{1}{\sigma_y}dy_1 \quad , \quad dy_2' = \frac{1}{\sigma_y}y_2
\end{aligned} \tag{4.13}
$$

For computational purposes, the following form may be more convenient.

$$
\begin{aligned}
x_1' &= \frac{1}{2}(x_1+x_2) \quad , \quad x_2' = \frac{1}{2}(x_3+x_4) \\
y_1' &= \frac{1}{\Delta t\ \sigma_y}(x_2-x_1) \quad , \quad y_2' = \frac{1}{\Delta t\ \sigma_y}(x_4-x_3)
\end{aligned} \tag{4.14}
$$

From which it follows that

$$
\begin{aligned}
x_1 &= x_1' - \frac{1}{2}\Delta t\sigma_y y_1' \\
x_2 &= x_1' + \frac{1}{2}\Delta t\sigma_y y_1' \\
x_3 &= x_2' - \frac{1}{2}\Delta t\sigma_y y_2' \\
x_4 &= x_2' + \frac{1}{2}\Delta t\sigma_y y_2'
\end{aligned} \tag{4.15}
$$

Hence Eq(4.12) becomes

$$\boxed{\begin{aligned}&f_z(x_1',x_2',y_1,y_2)dx_1'dx_2'dy_1dy_2=\\&\frac{1}{\sqrt{1-r_x^2}}\frac{1}{(2\pi)}\mathrm{Exp}\left[-\frac{1}{2}\,X'^T.A^{-1}.X'\right]\\&\cdot\frac{1}{\sqrt{1-\rho_y^2}}\frac{1}{(2\pi)}\mathrm{Exp}\left[-\frac{1}{2}\,(y'-B')^T.S'^{-1}.(y'-B')\right]dx_1'\ dx_2'\ dy_1'\ dy_2'\end{aligned}}$$

(4.16)

where

$$b_1' = \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(r_x\ x_1'-x_2') \tag{4.17}$$

$$b_2' = \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(x_1'-x_2'\ r_x) \tag{4.18}$$

$$A^{-1} = \frac{1}{1-r_x^2}\begin{bmatrix}1 & -r_x\\ -r_x & 1\end{bmatrix} \tag{4.19}$$

$$\det A = 1-r_x^2 \tag{4.20}$$

$$S'^{-1} = \frac{1}{(1-\rho_y^2)}\begin{bmatrix}1 & -\rho_y\\ -\rho_y & 1\end{bmatrix} \tag{4.21}$$

$$\rho_y = r_x+\frac{1}{\sigma_y^2}\left(r_x\ \ddot{r}_o-\ddot{r}_x\right) \tag{4.22}$$

$$\sigma_y^2 = -\ddot{r}_o - \frac{\dot{r}_x^2}{1-r_x^2} \tag{4.23}$$

<u>4.6 Double Crossing and Limits Of Integration</u>

In order to determine the probability of double crossing Eq(4.5), the attempt is made to divide the integration into two parts using the slope of each crossing.

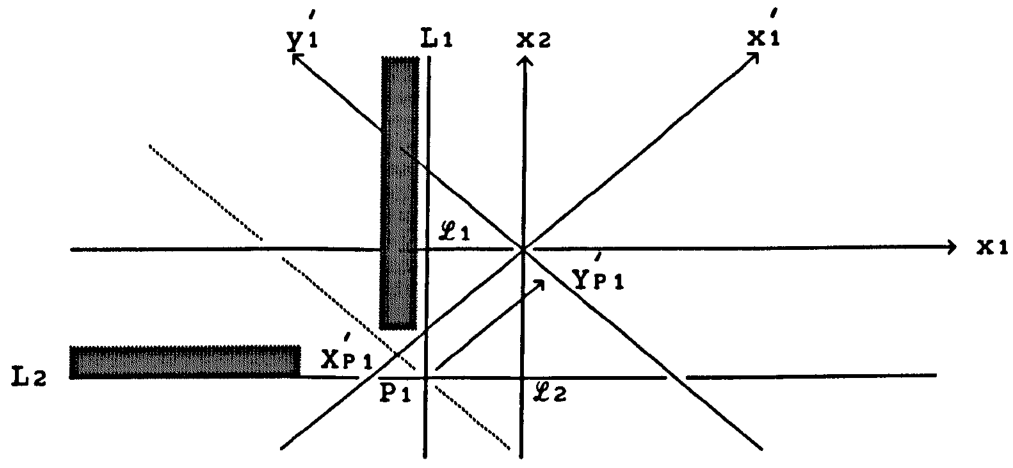
The integration limits for an up-crossing slope, for the first zero-crossing, lies within the shaded wedge area in the x_1, x_2 plane, as shown in Fig[4.2]. The succeeding algebraic operations can be simplified, by using the substitutions shown below, where

$\mathscr{L}_1 = -U_1$; $\mathscr{L}_2 = -U_2$

and the limit lines are

$L_1 \quad : \quad x_1 = \mathscr{L}_1$

$L_2 \quad : \quad x_2 = \mathscr{L}_2$



Fig[4.2] Condition for a positive slope (up-crossing) of level zero in time interval Δt_1.

Converting these to x_1', y_1' coordinates, as discussed in chapter 3 appendix[3.A] and using the conditions which were introduced in Eqs(4.6), (4.15) yields

$$L_1 \quad : \quad \mathcal{L}_1 = -(\bar{U}_1 - \frac{1}{2}\Delta U_1)$$

where $$x_1' = \frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_1$$

$$L_2 \quad : \quad \mathcal{L}_2 = -(\bar{U}_1 + \frac{1}{2}\Delta U_1)$$

where $$x_1' = -\frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_2$$

and the point P_1, the first crossing is given by the coordinates

$$X_{P1}' = \frac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2) = -\bar{U}_1 \qquad (4.24)$$

$$Y_{P1}' = \frac{(\mathcal{L}_2 - \mathcal{L}_1)}{\Delta t_1 \sigma_y} = -\frac{\Delta U_1}{\Delta t_1 \ \sigma_y} = -\frac{\dot{U}_1}{\sigma_y}$$

where $\dot{U}_1$ is now $\dot{U}(t_1)$. X_{P1}', Y_{P1}' are the normalised deterministic signal and its slope at the mid-interval. The integral over the shaded area becomes

$$\int_{-\infty}^{\mathcal{L}_1} dx_1 \int_{\mathcal{L}_2}^{+\infty} dx_2 = \int_{Y_{P1}'}^{+\infty} dy_1' \int_{-\frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_2}^{+\frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_1} dx_1' \qquad (4.24a)$$

Now as Δt_1 is small the interval of integration over x_1' is small and so the integral can be approximated by replacing x_1' by its mid-interval value X_P' in the function and multiplying by the width of the interval. Therefore $L_1 = L_2$, and

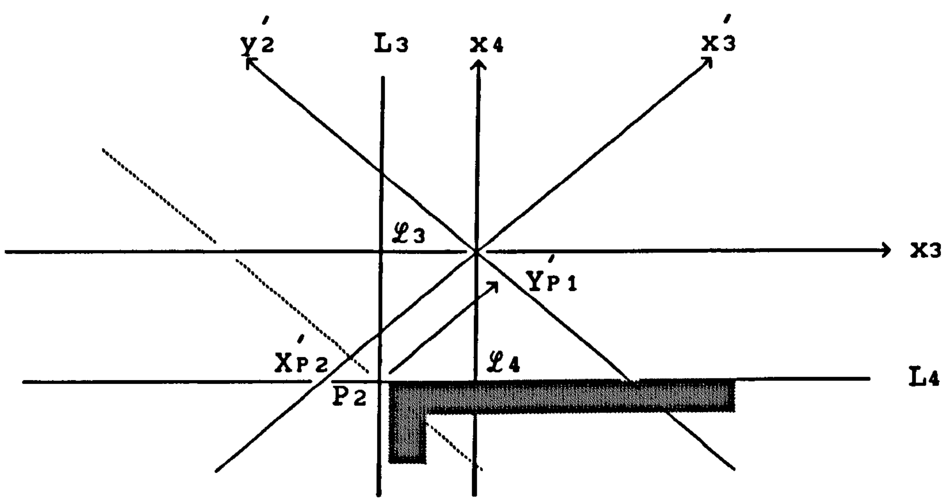
$$\int_{-\frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_2}^{+\frac{1}{2}\Delta t_1 \sigma_y y_1' + \mathcal{L}_1} dx_1' \ f(x_1') \quad = (\Delta t_1 \sigma_y y_1' + \ (\mathcal{L}_1 - \mathcal{L}_2)) \ f(X_{p1}')$$

where $(\mathcal{L}_1 - \mathcal{L}_2) = \Delta U_1$. Hence Eq(4.24a) becomes

$$\int_{-\infty}^{\mathscr{L}_1} dx_1 \int_{\mathscr{L}_2}^{+\infty} dx_2 \ f(x_1') = \int_{Y_{P1}'}^{+\infty} dy_1' \left\{ \Delta t_1 \sigma_y (y_1' - Y_{P1}') \right\} f \Big|_{x_1' = X_{P1}'} \quad (4.25)$$

The x_1' interval only becomes wide for large y_1' where the probability density becomes very small, and as $\Delta t_2 \longrightarrow 0$, the approximation is exact.

The above manipulation can be extended for the second crossing (down-crossing), where the integral limits lie within the shaded wedge area in the plane x_3, x_4, and x_2', y_2', as shown in Fig[4.3].



Fig[4.3] Condition for down-crossing of level zero in time interval Δt_2.

Similarly as above, where

$\mathscr{L}_3 = -U_3$; $\mathscr{L}_4 = -U_4$

and the limit lines are

$L_3 \quad : \quad x_3 = \mathscr{L}_3$

$L_4 \quad : \quad x_4 = \mathscr{L}_4$

so that, x_2' integration runs from L_3 to L_4. Using similar manipulation to convert to x_2', y_2' coordinates, and the condition which was introduced in Eq(4.6) and (4.15)

$$L_3 \quad : \quad \mathcal{L}_3 = -(\bar{U}_2 - \frac{1}{2}\Delta U_2)$$

$$\text{where} \quad x_2' = \frac{1}{2}\Delta t_2 \sigma_y y_2' + \mathcal{L}_3$$

$$L_4 \quad : \quad \mathcal{L}_4 = -(\bar{U}_2 + \frac{1}{2}\Delta U_2)$$

$$\text{where} \quad x_2' = \frac{1}{2}\Delta t_2 \sigma_y y_2' + \mathcal{L}_4$$

and the point p_2, coordinate for second crossing is

$$\begin{aligned} X_{P2}' &= \frac{1}{2}(\mathcal{L}_3 + \mathcal{L}_4) = -\bar{U}_2 \\ Y_{P2}' &= \frac{(\mathcal{L}_4 - \mathcal{L}_3)}{\Delta t_2 \sigma_y} = -\frac{\Delta U_2}{\Delta t_2 \ \sigma_y} = -\frac{\dot{U}_2}{\sigma_y} \end{aligned} \tag{4.26}$$

where $\dot{U}_2$ is now $\dot{U}(t_2)$. In a similar way, carrying out the simple integration for the second part of Eq(4.5) for down-crossing,

$$\int_{\mathcal{L}_3}^{+\infty} dx_3 \int_{-\infty}^{\mathcal{L}_4} dx_4 \ f(x_2') = \int_{-\infty}^{Y_{P2}'} dy_2' \left\{ -\Delta t_2 \sigma_y (y_2' - Y_{P2}') \right\} f \bigg|_{x_2' = X_{P2}'} \tag{4.26a}$$

The probability of double crossing in the two intervals Δt_1, Δt_2 can be calculated from the joint probability density function Eq(4.16) and the integration evaluation in (4.25), and (4.26a).

Hence the probability of double crossing in (4.5) becomes

$$P_{2X} = -\Delta t_1 \Delta t_2 \sigma_y^2 \int_{Y'_{P1}}^{+\infty} dy'_1 \int_{-\infty}^{Y'_{P2}} dy'_2 \, (y'_1 - Y'_{P1})(y'_2 - Y'_{P2})$$

$$\cdot \frac{1}{\sqrt{1-r_x^2}} \frac{1}{(2\pi)} \mathrm{Exp}\left[-\frac{1}{2} X_P'^T . A^{-1} . X'_P\right]$$

$$\cdot \frac{1}{\sqrt{1-\rho_y^2}} \frac{1}{(2\pi)} \mathrm{Exp}\left[-\frac{1}{2}(y' - B'_P)^T . S^{-1} . (y' - B'_P)\right]$$

where

$$X_{P1}'^{T} = (X'_{P1}, X'_{P2}) = (-\bar{U}_1, -\bar{U}_2)$$

$$B'_P = (b'_{P1}, b'_{P2})^T$$

$$b'_{P1} = \frac{\dot{r}_x}{\sigma_y (1-r_x^2)} (r_x X'_{P1} - X'_{P2})$$

$$b'_{P2} = \frac{\dot{r}_x}{\sigma_y (1-r_x^2)} (X'_{P1} - X'_{P2} r_x)$$

Finally, a further change of variable for simplicity, let

$$Z_1 = (y'_1 - b'_{P1}) \quad , \quad Z_2 = (y'_2 - b'_{P2})$$

$$dz_1 = dy'_1 \quad , \quad dz_2 = dy'_2$$

After some rearrangement of terms, the second exponential can be written as

$$P_{2X} = -\Delta t_1 \Delta t_2 \, \sigma_y^2 \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \, (Z_1 - a_1)(Z_2 - a_2)$$

$$\cdot \frac{1}{\sqrt{1-r_x^2}} \frac{1}{(2\pi)} \mathrm{Exp}\left[-\frac{1}{2} X_P'^T . A^{-1} . X'_P\right]$$

$$\cdot \frac{1}{\sqrt{1-\rho_y^2}} \frac{1}{(2\pi)} \mathrm{Exp}\left[-\frac{1}{2} Z^T . S'^{-1} . Z\right]$$

where

$$a_1 = -Y'_{P1} + b'_{P1} = -\left(\frac{\dot{U}_1}{\sigma_y} + \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(r_x X'_{P1} - X'_{P2})\right)$$

$$\tag{4.27}$$

$$a_2 = -Y'_{P2} + b'_{P2} = -\left(\frac{\dot{U}_2}{\sigma_y} + \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(X'_{P1} - X'_{P2}\, r_x)\right)$$

and

$Z^T = (Z_1, Z_2)$

In this form, the exponentials and associated factors are standard bivariate normal distributions, and the complexity is hidden in the limits a_1 a_2, which can in principle be evaluated.

It is now convenient to take the subscripts 1,2 as referring to these instantaneous crossing times, and to drop the (′) from the notation with(y) from ($\rho_y = \rho$). Consequently, the probability of detection of double crossings of a signal consisting of a deterministic signal and band limited Gaussian noise which has an up-crossing in the infinitesimal interval dt_1 and a down-crossing in the infinitesimal interval dt_2 can be written as

$$\boxed{\begin{aligned} P_{2X} &= -dt_1 dt_2\, \sigma_y^2\, g(X_{p1}, X_{p2}, r_x)\, \mathcal{J}(a_1, a_2, \rho) \\ \mathcal{J}(a_1, a_2, \rho) &= \int_{a_1}^{\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\, (Z_1 - a_1)(Z_2 - a_2)\, g(Z_1, Z_2, \rho) \end{aligned}} \tag{4.28}$$

where g function is bivariate normal function [A&S ,26.3.1,26.3.2]. The probability of detection of a double crossing can be specified in terms of the probability density

$$\boxed{\begin{aligned} PD_{2x}(t_1,t_2) &= \frac{P_{2x}}{dt_1 dt_2} = -\sigma_y^2\, g(X_{p1},X_{p2},r_x)\,\mathcal{J}(a_1,a_2,\rho) \\ \mathcal{J}(a_1,a_2,\rho) &= \int_{a_1}^{\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ (Z_1-a_1)(Z_2-a_2)\ g(Z_1,Z_2,\rho) \end{aligned}} \qquad (4.29)$$

where times t_1, t_2 are the times of the signals X_{p1}, X_{p2}.

The double integral is simplified by employing the mathematical techniques discussed in appendix 4.A Eq(4.A.14), so that the above expression becomes

$$\boxed{\begin{aligned} PD_{2x}(t_1,t_2) = {} & \sigma_y^2\ g(X_{p1},X_{p2},r_x) \\ & \cdot \left\{ (1-\rho^2)\ g(a_1,a_2,\rho) - a_1\ Q\left(\frac{a_1-\rho\ a_2}{\sqrt{1-\rho^2}}\right)\ Z(a_2) \right. \\ & \left. + a_2\ Z(a_1)\ P\left(\frac{a_2-\rho\ a_1}{\sqrt{1-\rho^2}}\right) - (a_1 a_2+\rho)\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g(Z_1,Z_2,\rho) \right\} \end{aligned}} \qquad (4.30)$$

From (4.20), (4.22; 4.23) and (4.27)

$$\det A = 1-r_x^2 \qquad (4.30a)$$

$$\sigma_y^2 = -\ddot{r}_0 - \frac{\dot{r}_x^2}{1-r_x^2} \qquad (4.30b)$$

$$\rho = r_x + \frac{1}{\sigma_y^2}\left(r_x\ \ddot{r}_0 - \ddot{r}_x\right) \qquad (4.30c)$$

$$a_1 = -Y_{P1} + b_{P1} = -\left(\frac{\dot{U}_1}{\sigma_y} + \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(r_x\ X_{P1} - X_{P2})\right) \qquad (4.30d)$$

$$a_2 = -Y_{P2} + b_{P2} = -\left(\frac{\dot{U}_2}{\sigma_y} + \frac{\dot{r}_x}{\sigma_y(1-r_x^2)}(X_{P1} - X_{P2}\ r_x)\right) \qquad (4.30e)$$

X_{P1}, X_{P2} are the instantaneous normalised deterministic signals in the infinitesimal interval for the double zero-crossing.

Eq(4.30) can be written as

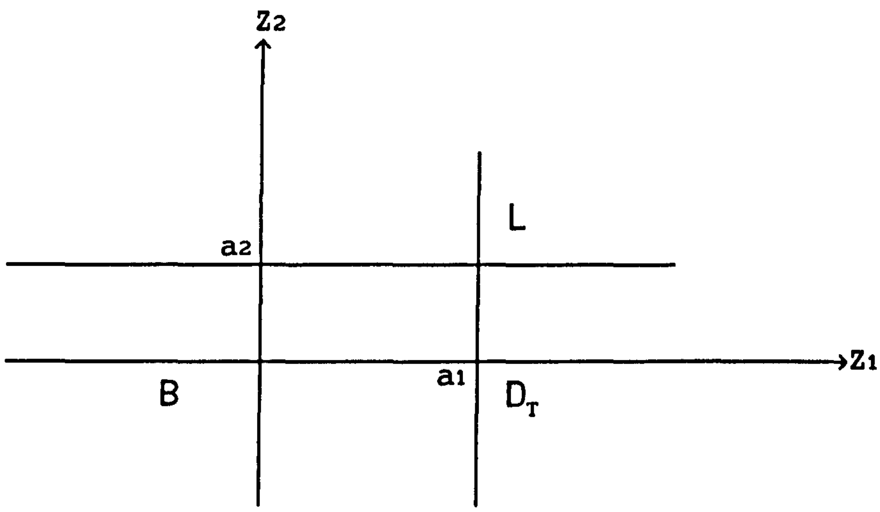
$$\boxed{\begin{aligned} PD_{2x}(t_1,t_2) &= \sigma_y^2\, g(X_{P1},X_{P2},r_x) \\ &\cdot \left\{(1-\rho^2)\, g(a_1,a_2,\rho) - a_1\, Q\left(\frac{a_1-\rho\, a_2}{\sqrt{1-\rho^2}}\right) Z(a_2) \right. \\ &\left. + a_2\, Z(a_1)\, P\left(\frac{a_2-\rho\, a_1}{\sqrt{1-\rho^2}}\right) - (a_1 . a_2+\rho)\, D_T(a_1,a_2,\rho)\right\} \end{aligned}} \qquad (4.31)$$

where the integral over the bivariate normal distribution has been written as

$$D_T(a_1,a_2,\rho) = \int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\; g(Z_1,Z_2,\rho) \qquad (4.32)$$

4.7 Evaluation Of Integrals Of The Bivariate Normal Distribution

Owen[1956] considered the cumulative probability of the bivariate normal distribution with zero means and unit variances. This corresponds to the area B of Fig[4.4](lower left quadrant with upper limits a_1 a_2). Values of B were tabulated to sixth decimal place over grids of the two upper limits. Gidon & Gurland[1978](denoted by G&G for brevity) used this tabulation to produce a polynomial approximation for integrals of the bivariate normal distribution (b.n.d), effectively over the region L in Fig[4.4] in the upper right quadrant.



Fig[4.4] The quadrant of b.n.d.

The function D_T of Eq(4.31 & 4.32) must be put into a form consistent with the polynomial approximation of G&G, and in particular values of the limits must be used which ensure that the integral lies entirely in the upper right quadrant. A particular difficulty arises from the signs of a_1 & a_2 in Eq(4.31 & 4.32) which may take positive or negative values in any combination. Using the relations of A&S 26.3.7.....26.3.10, the following can be used for D_T

a_1	a_2	$D_T(a_1,a_2,\rho)$ =
+	+	$Q(\|a_1\|)-L(\|a_1\|,\|a_2\|,\rho)$
+	-	$L(\|a_1\|,\|a_2\|,-\rho)$
-	+	$1-Q(\|a_1\|)-Q(\|a_2\|)+L(\|a_1\|,\|a_2\|,-\rho)$
-	-	$Q(\|a_2\|)-L(\|a_1\|,\|a_2\|,\rho)$ (4.33)

In these expressions $L(|a_1|,|a_2|,\pm\rho)$ are always in the upper right quadrant.

[G&G 1978] give the integral over B as

$$B(a_1,a_2,\rho) = P(a_1)+P(a_2)-1+D(r,\theta_h)+D(r,\theta_k)-b \tag{4.34}$$

Where b=0 if $a_1a_2>0$ or if $a_1a_2=0$, and b=1/2, otherwise. However in the present case $|a_1|,|a_2|>0$ and so b can be ignored.

Using the same relation of A&S it is found that

$$\boxed{\begin{aligned} L(a_1,a_2,\rho) &= D(r,\theta_h)+D(r,\theta_k) \\ L(a_1,a_2,\rho) &= D_1+D_2 \end{aligned}} \tag{4.35}$$

G&G give the polynomial expression for $D(r,\theta)$; r, θ_h, θ_k are determined from a_1 a_2 by a transformation to polar coordinates with a circular b.n.d appendix 4.B. This gives

$$r^2 = (|a_1^2| + 2\rho|a_1||a_2| + |a_2^2|)/(1-\rho^2) \tag{4.36}$$

$$\tan\theta_h = |a_1|(1-\rho^2)^{0.5}/(|a_2| - \rho|a_1|) \tag{4.37}$$

$$\tan\theta_k = |a_2|(1-\rho^2)^{0.5}/(|a_1| - \rho|a_2|) \tag{4.38}$$

$$D(r,\theta_h) = D_1 = \frac{1}{2}Q(r)d(r,\theta_h) \tag{4.39}$$

$$D(r,\theta_k) = D_2 = \frac{1}{2}Q(r)d(r,\theta_k) \tag{4.40}$$

where $d(r,\theta)$ is given by

$$d(r,\theta) = (b_0 + b_1 r + b_2 r^2)\theta + (b_3 r + b_4 r^2)\theta^3 + (b_5 r + b_6 r^2)\theta^5 \tag{4.41}$$

where the numerical coefficients are given by G&G[1978, table 1]. Because of the previous quadrant transformation Eq(4.34) θ_h and θ_k are never negative.

The maximum absolute error resulting from this approximation to $d(r,\theta)$ is between (2.E-05) in the first interval to (14.E-05) in the last interval.

4.8 Evaluation Of The Joint Probability Of A Detection Of A Double Crossing

The procedure for the evaluation of the joint probability of detection of a double crossing can now be summarised. The final expression for the joint probability of double crossing detection is given by Eq((4.31)

$$PD_{2x}(t_1,t_2)= \sigma_y^2\; g(X_{p1},X_{p2},r_x) \cdot \left\{(1-\rho^2)\; g(a_1,a_2,\rho) - a_1\, Q\left(\frac{a_1-\rho\, a_2}{\sqrt{1-\rho^2}}\right) Z(a_2) + a_2\; Z(a_1)\; P\left(\frac{a_2-\rho\, a_1}{\sqrt{1-\rho^2}}\right) - (a_1 . a_2+\rho)\; D_T(a_1,a_2,\rho)\right\} \quad \begin{Bmatrix}4.31\\ \text{again}\end{Bmatrix}$$

where

$$D_T(a_1,a_2,\rho) = \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2\; g(Z_1,Z_2,\rho) \qquad (4.32\ \text{again})$$

All the expressions in Eq(4.31) are standard probability functions [A&S 26.3.1 & 2] with variables as defined by Eqs(4.30a to e), except for $D_T(a_1,a_2,\rho)$ the integral of the bivariate normal distribution. Evaluation of $D_T(a_1,a_2,\rho)$ depends on the signs of a_1 and a_2 vis:

a_1	a_2	$D_T(a_1,a_2,\rho) =$
+	+	$Q(\lvert a_1\rvert)-L(\lvert a_1\rvert,\lvert a_2\rvert,\rho)$
+	-	$L(\lvert a_1\rvert,\lvert a_2\rvert,-\rho)$
-	+	$1-Q(\lvert a_1\rvert)-Q(\lvert a_2\rvert)+L(\lvert a_1\rvert,\lvert a_2\rvert,-\rho)$
-	-	$Q(\lvert a_2\rvert)-L(\lvert a_1\rvert,\lvert a_2\rvert,\rho)$

$$\begin{Bmatrix}4.33\\ \text{again}\end{Bmatrix}$$

and $L(|a_1|,|a_2|,\rho)$ is evaluated using

$$L(a_1,a_2,\rho)=D(r,\theta_h)+D(r,\theta_k) \qquad (4.35\ \text{again})$$

where

$$r^2 =(|a_1{}^2|+2\rho|a_1||a_2|+|a_2{}^2|)/(1-\rho^2) \quad \begin{Bmatrix} 4.36 \\ \text{again} \end{Bmatrix}$$

$$\tan\theta_h =|a_1|(1-\rho^2)^{0.5}/(|a_2|-\rho|a_1|) \quad \begin{Bmatrix} 4.37 \\ \text{again} \end{Bmatrix}$$

$$\tan\theta_k =|a_2|(1-\rho^2)^{0.5}/(|a_1|-\rho|a_2|) \quad \begin{Bmatrix} 4.38 \\ \text{again} \end{Bmatrix}$$

$$D(r,\theta_h)=D_1=\frac{1}{2}Q(r)d(r,\theta_h) \quad \begin{Bmatrix} 4.39 \\ \text{again} \end{Bmatrix}$$

$$D(r,\theta_k)=D_2=\frac{1}{2}Q(r)d(r,\theta_k) \quad \begin{Bmatrix} 4.40 \\ \text{again} \end{Bmatrix}$$

Finally $d(r,\theta)$ is evaluated numerically using

$$d(r,\theta)=(b_0+b_1r+b_2r^2)\theta+(b_3r+b_4r^2)\theta^3+(b_5r+b_6r^2)\theta^5 \quad \begin{Bmatrix} 4.41 \\ \text{again} \end{Bmatrix}$$

Different numerical coefficients b_1 to b_6 are used for different ranges of r and values can be found in Gidon & Gurland[1978, table 1]; they are also given in the program of Appendix 5.A pages 243, 244.

Chapter 5

Numerical Evaluation Of The Probability Of Crossing For Sinusoid Signals With Band Limited Noise

5.1 Introduction

This chapter uses the theoretical results of Chapter Four for the general case for the probability of detection of a double crossing, to investigate the detection of a sinusoidal signal with added Gaussian noise. Numerical integration of these theoretical expressions is necessary.

Eq(4.30) gives the joint probability density of crossings at specific times t_1 and t_2. In investigating a sine signal the phase is not usually known, and because of the noise it is not possible to predict, within any cycle, precisely when the first crossing will occur. A more practical probability is therefore the joint probability density per cycle that a first crossing will occur followed by a second crossing within a small interval dt_x after a lag of time t_x; i.e. the relevant probability is

$$PDC(t_x)dt_x = \int_{cycle} dt_1 PD_{2x}(t_1,t_2)dt_2 \tag{5.1}$$

$$t_2 = t_1 + t_x \tag{5.2}$$

This would be relevant to a detector which was triggered by the first crossing and measured the time to the second crossing.

Detection of a crossing with a sinusoidal deterministic signal will be optimum where the signal itself passes through zero, and so the thresholds are taken as $\alpha_1 = \alpha_2 = 0$ throughout.

In practice any signal passes through a band-limited channel, which produces correlations in the noise. Noise correlation can produce structure in the probability with respect to the time between crossings, which competes with the time expected for the signal alone. For high SNR the crossing is dominated by the signal and the peak is nearly a delta function. For very low SNR, the structure due to correlation is significant. A range of SNRs is investigated to determine the range of the correlation effects. The band-limiting effects are modeled by two different LP filters: a Brickwall filter, and a second order Butterworth; these being extreme cases of ideal sharp frequency cut off and slow cut-off.

Sinusoidal signals at band edge, and half band edge frequency are investigated. The rate at which the band edge frequency sinusoid crosses zero is the fastest allowed by the band, and so this frequency gives a strong indication of the behaviour for general deterministic signals.

The numerical calculations were carried out by a pascal program, which can be used for any SNR and detection window, and can easily be adapted to other filters and nonzero thresholds. This program is presented in appendix 5.A. The probability density function was evaluated over a range of delay times, and the results are displayed graphically in Figs[5.4 to 5.7], (at the end of this chapter).

Finally, the work is extended to calculate the probability of crossing, within a window of finite interval. This is included within the program, and the window can be changed for optimum detection design.

5.2 Normalisation Of The Sinusoid

To obtain numerical estimates based on the theoretical analysis of Chapter Four, it is necessary to specify both the deterministic signal and the noise correlation.

The sinusoid is a useful basic signal, and in this work the deterministic signal is taken to be the sinusoid with fixed phase

$$s(t) = A \sin W_s t \qquad (5.3)$$

where A and W_s are signal amplitude and angular frequency respectively. The calculations can be made more general by first normalising these with respect to noise power σ_n^2 and noise bandwidth W_B, to give

$$\frac{s(t)}{\sigma_n} = SNR \sin W_B R_s t$$

where

$$R_s = \frac{W_s}{W_B} \tag{5.4}$$

$$SNR = \frac{A}{\sigma_n} \tag{5.5}$$

SNR is the signal amplitude to noise standard deviation ratio, and R_s is the signal to bandwidth frequency ratio. W_B is being used in Eq(5.4) as a frequency standard so that time can also be normalised to

$$W_B t = \tau \tag{5.6}$$

giving

$$\frac{s(t)}{\sigma_n} = SNR \sin R_s \tau \tag{5.7}$$

An alternative specification for time is

$$k = \frac{1}{2\pi} W_s t = \frac{1}{2\pi} R_s \tau \tag{5.8}$$

where $0 \le k \le 1$ covers a complete cycle values, τ is normalised with respect to the bandwidth of the noise and k is normalised with respect to the period of the signal.

The first detected crossing is taken to be at time t_1 and the second detection at time $t_2 = t_1 + t_x$ which are specified respectively by τ_1 ; $\tau_2 = \tau_1 + \tau_x$ or by k_1; $k_2 = k_1 + k_x$.

Using these conventions the normalised signals and slopes in 4.24 & 4.26 become

$$X_{P1} = SNR \sin 2\pi k_1 \tag{5.9}$$

$$Y_{P1} = \frac{1}{\sigma_{By}} SNR\ R_s \cos 2\pi k_1 \tag{5.10}$$

$$X_{P2} = SNR \sin 2\pi (k_1 + k_x) \tag{5.11}$$

$$Y_{P2} = \frac{1}{\sigma_{By}} SNR\ R_s \cos 2\pi (k_1 + k_x) \tag{5.12}$$

where $\sigma_{By} = \sigma_y W_B$ (5.13)

5.3 Normalisation Of Noise Characteristics

Noise characteristics are scaled with respect to the noise bandwidth using Eq(5.6), which gives the scaling $d\tau = W_B dt$ on differentiation. Then

$$r_x = r_x$$

$$\dot{r}_x = W_B\, \dot{r}_{Bx} \quad , \quad \dot{r}_{Bx} = \frac{d}{d\tau} r(t) \Big|_{\tau=\tau_x}$$

$$\ddot{r}_x = W_B^2 \ddot{r}_{Bx} \quad , \quad \ddot{r}_{Bx} = \frac{d^2}{d\tau^2} r(t) \Big|_{\tau=\tau_x}$$

$$\ddot{r}_o = W_B^2 \ddot{r}_{Bo} \quad , \quad \ddot{r}_{Bo} = \frac{d^2}{d\tau^2} r(t) \Big|_{\tau=0}$$

The noise correlation parameters of Eqs(4.30a,b,c,d,e) then scale to

$$\det A = 1 - r_x^2 \tag{5.14}$$

$$\sigma_y^2 = -\ddot{r}_{Bo} - \frac{\dot{r}_{Bx}^2}{1 - r_x^2} \tag{5.15}$$

and

$$\rho = r_x + \frac{1}{\sigma_{By}^2} (r_x\, \ddot{r}_{Bo} - \ddot{r}_{Bx}) \tag{5.16}$$

$$a_1 = -\left(Y_{P1} + \frac{\dot{r}_{Bx}}{\sigma_{By} (1 - r_x^2)} (r_x\, X_{p1} - X_{p2}) \right) \tag{5.17}$$

$$a_2 = -\left(Y_{P2} + \frac{\dot{r}_{Bx}}{\sigma_{By} (1 - r_x^2)} (X_{p1} - X_{p2}\, r_x) \right) \tag{5.18}$$

5.4 Correlation Parameters For Low Pass Filters

5.4.1 The Second Order Butterworth Filter

For the second Butterworth filter the noise autocorrelation function is given by [Papoulis 1984, ex.10-14]

$$r_x = r(t) = \mathrm{Exp}\left(-\frac{W_B\ t}{\sqrt{2}}\right)\left[\cos\left(\frac{W_B\ t}{\sqrt{2}}\right)+\sin\left(\frac{W_B\ t}{\sqrt{2}}\right)\right]$$

Differentiating r_x with respect to t, gives

$$\dot{r}_x = -\sqrt{2}\ W_B \sin\left(\frac{W_B\ t}{\sqrt{2}}\right)\ \mathrm{Exp}\left(-\frac{W_B\ t}{\sqrt{2}}\right)$$

$$\ddot{r}_x = W_B^2\ \mathrm{Exp}\left(-\frac{W_B\ t}{\sqrt{2}}\right)\left[\sin\left(\frac{W_B\ t}{\sqrt{2}}\right)-\cos\left(\frac{W_B\ t}{\sqrt{2}}\right)\right]$$

Hence, the scaled parameters are

$$r_x = \mathrm{Exp}\left(-\frac{\tau}{\sqrt{2}}\right)\left[\cos\left(\frac{\tau}{\sqrt{2}}\right)+\sin\left(\frac{\tau}{\sqrt{2}}\right)\right] \tag{5.19}$$

$$\dot{r}_{Bx} = -\sqrt{2}\ \sin\left(\frac{\tau}{\sqrt{2}}\right)\ \mathrm{Exp}\left(-\frac{\tau}{\sqrt{2}}\right) \tag{5.20}$$

$$\ddot{r}_{Bx} = \mathrm{Exp}\left(-\frac{\tau}{\sqrt{2}}\right)\left[\sin\left(\frac{\tau}{\sqrt{2}}\right)-\cos\left(\frac{\tau}{\sqrt{2}}\right)\right] \tag{5.21}$$

and for stationary process, where the ensemble average are independent of time t

$$\ddot{r}_{Bo} = -1 \tag{5.22}$$

Fig[5.1], shows the variation of r_x, $\dot{r}_{Bx}$, $\ddot{r}_{Bx}$, detA, σ_{By}, ρ and $(1-\rho^2)$, with respect to τ.

5.4.2 The Brickwall Filter

For the Brickwall filter (ideal) the noise autocorrelation function is [Papoulis 1984, ex.10-26)

$$r_x = r(t) = \frac{\sin W_B t}{W_B t}$$

$$\dot{r}_x = \frac{1}{t}(\cos W_B t - r_x)$$

$$\ddot{r}_x = -\frac{2}{t}\dot{r}_x - W_B^2 r_x$$

$$\ddot{r}_o = -\frac{1}{3} W_B^2$$

As before, the scaled parameters are

$$r_x = \frac{\sin \tau}{\tau} \tag{5.23}$$

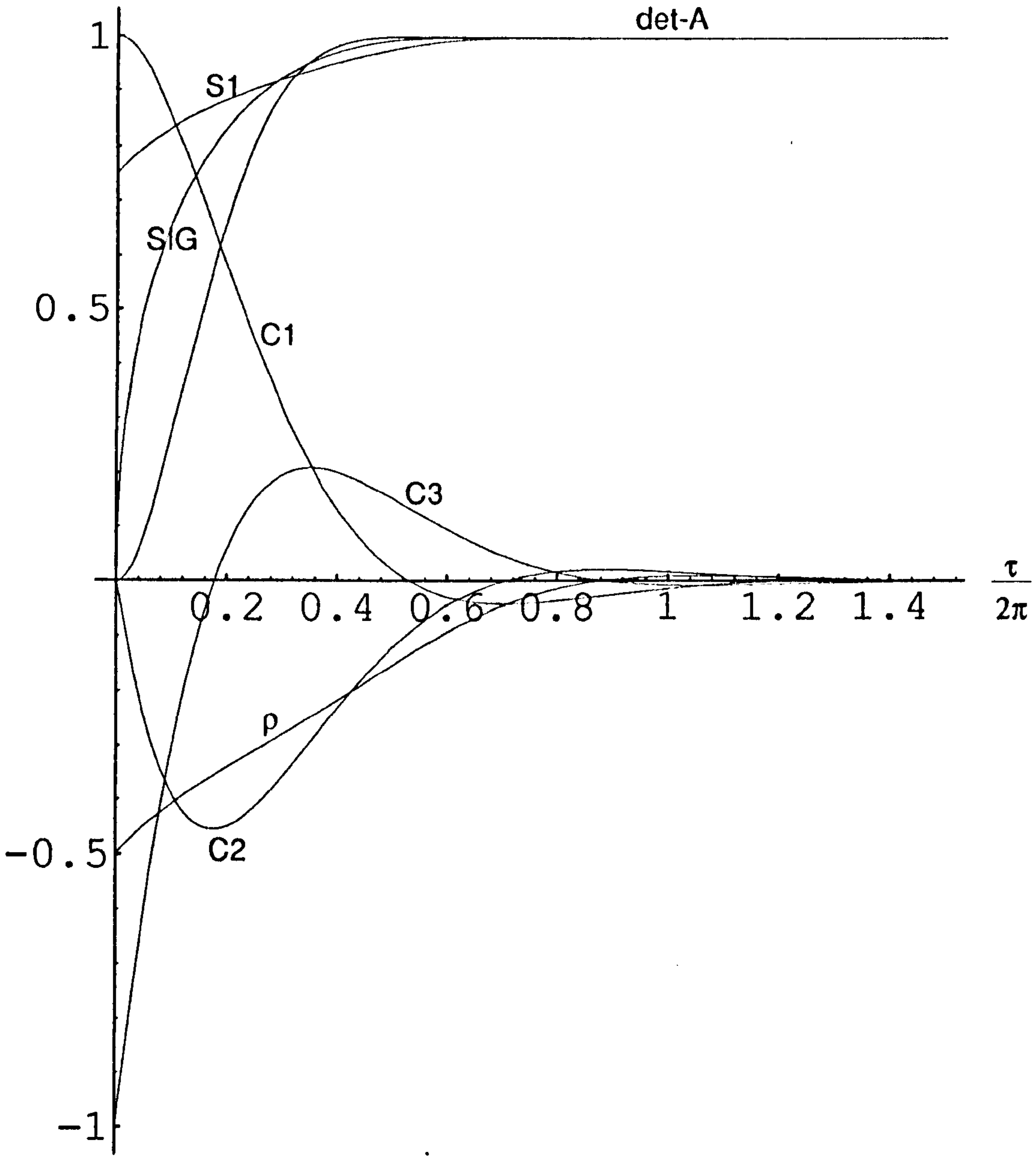
$$\dot{r}_{Bx} = \frac{1}{\tau}(\cos\tau - r_x) \tag{5.24}$$

$$\dot{r}^{\bullet}_{Bx} = -\frac{2}{\tau}\dot{r}_{Bx} - r_x \tag{5.25}$$

and for stationary process, where the ensemble average are independent of time t

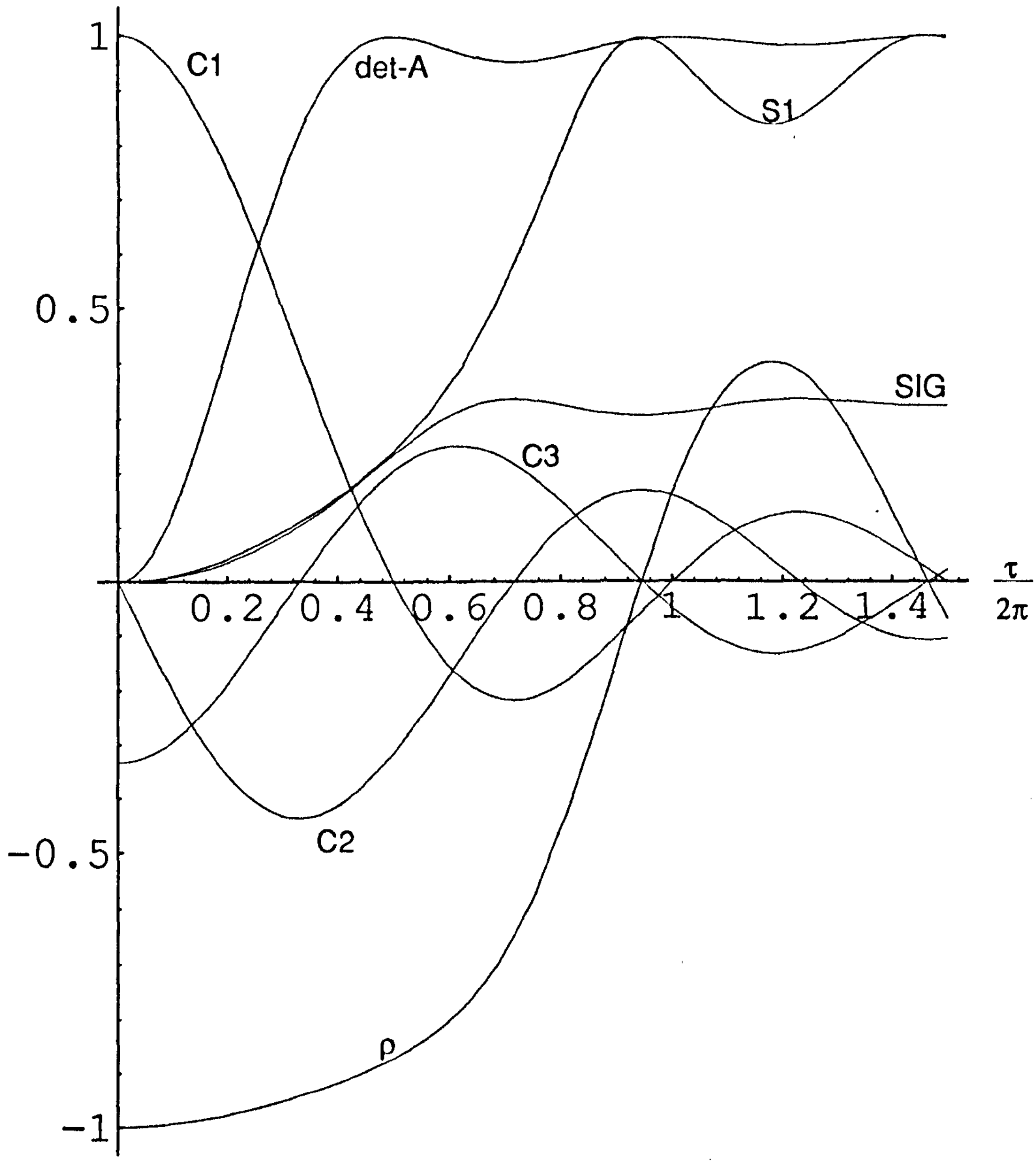
$$\ddot{r}_{Bo} = -\frac{1}{3} \tag{5.26}$$

Figs[5.2 & 5.3], show the variation of r_x, $\dot{r}_{Bx}$, $\dot{r}^{\bullet}_{Bx}$, detA, σ_{By} , ρ and $(1-\rho^2)$, over a short and large range and long range of τ respectively.



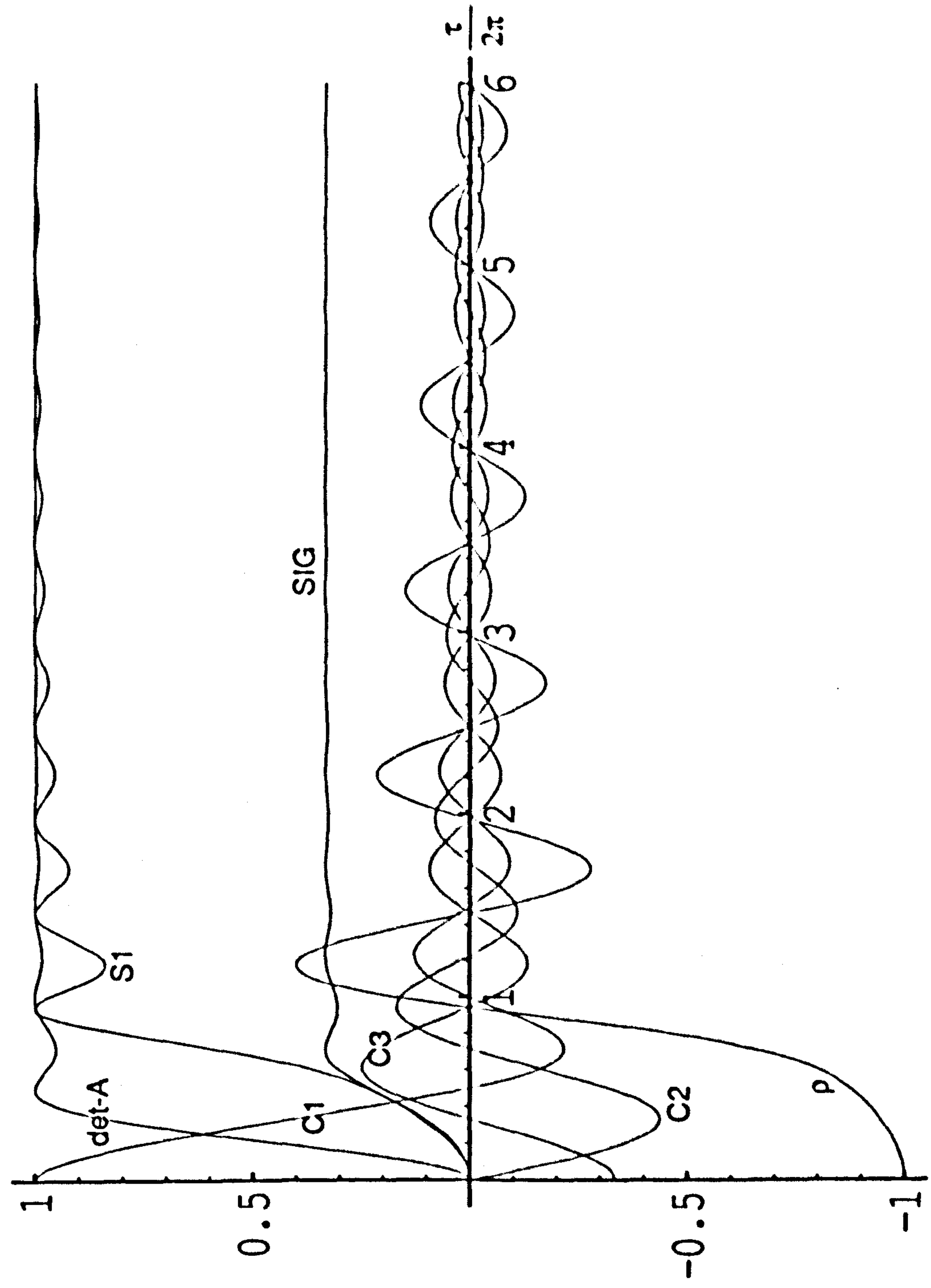
Fig[5.1] Normalised coefficient for Butterworth filter.

C1=r_x, C2=$\dot{r}_{Bx}$, C3=$\cdot\dot{r}_{Bx}$,S1=$(1-\rho^2)$, SIG=σ_{By}, det-A=detA



Fig[5.2] Normalised coefficient for Brickwall filter.

$C1=r_x$, $C2=\dot{r}_{Bx}$, $C3={}^{\cdot}\dot{r}_{Bx}$, $S1=(1-\rho^2)$, $SIG=\sigma_{By}$, det-A=detA



Fig[5.3] Normalised coefficient for Brickwall filter over long range of τ.

$C1=r_x$, $C2=\dot{r}_{Bx}$, $C3=\ddot{r}_{Bx}$, $S1=(1-\rho^2)$, $SIG=\sigma_{By}$, det-A=detA

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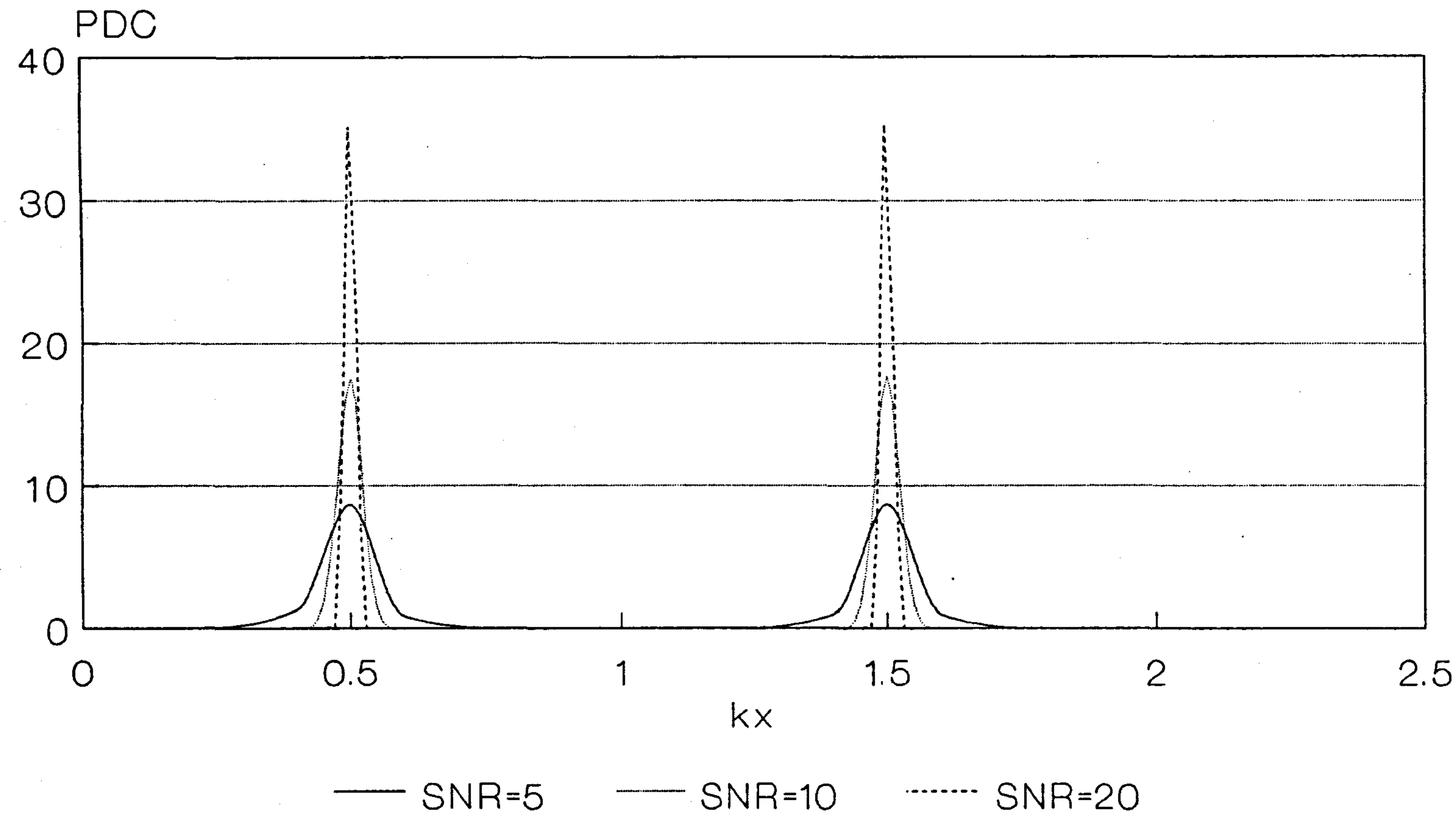
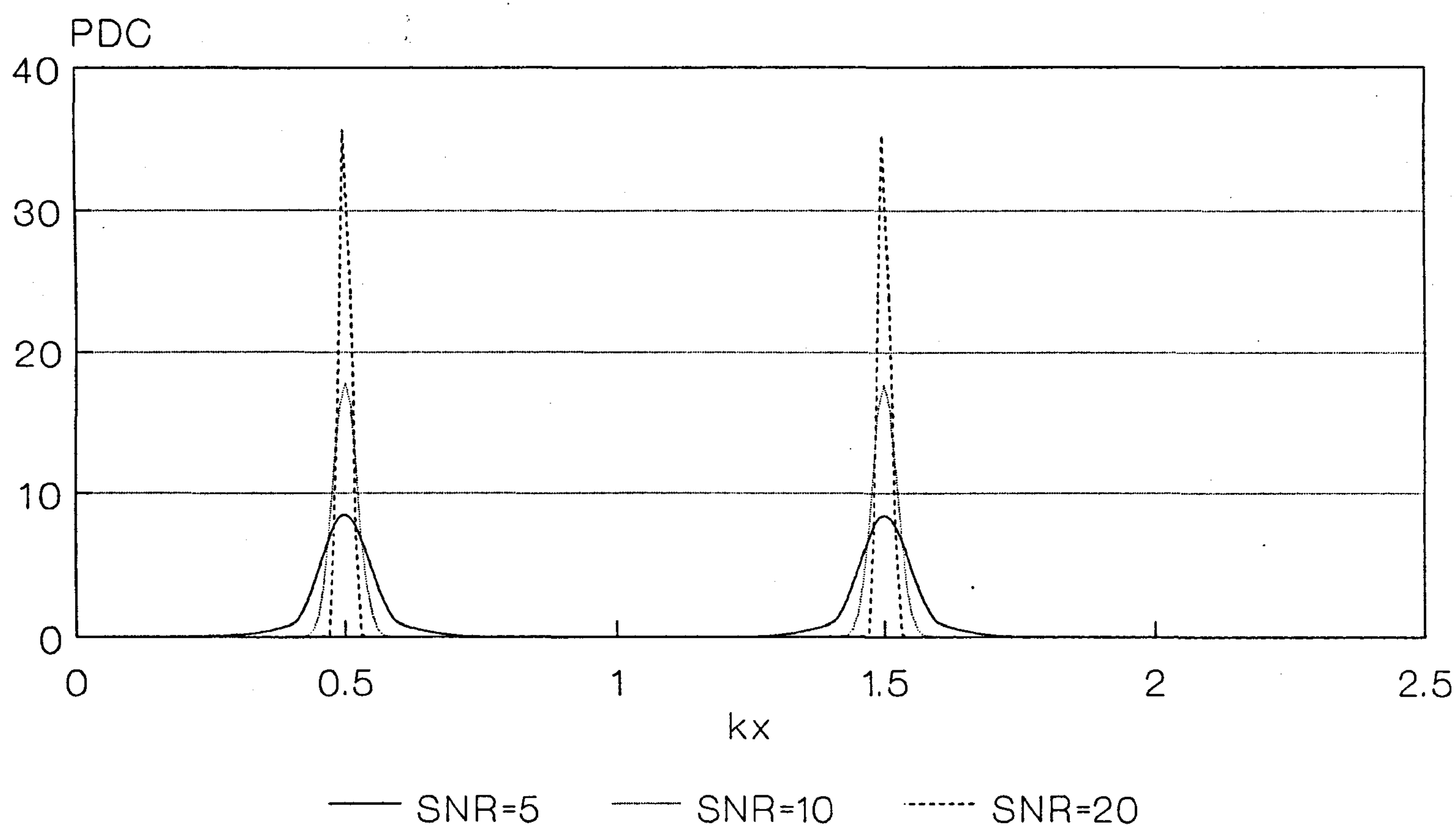


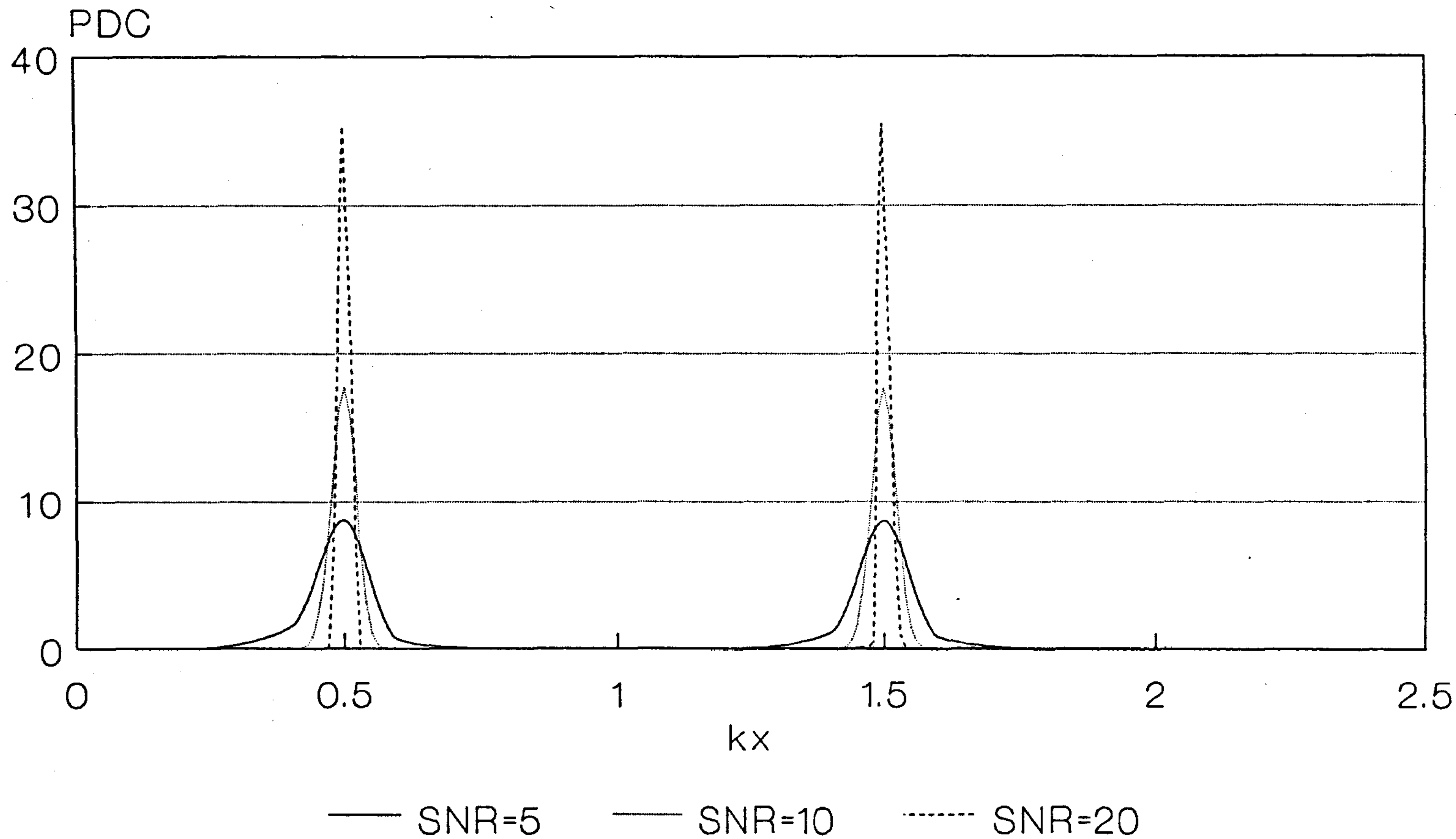
Rs=WS/WB= 1

Fig[5.4a] PDC for Butterwoth filter.



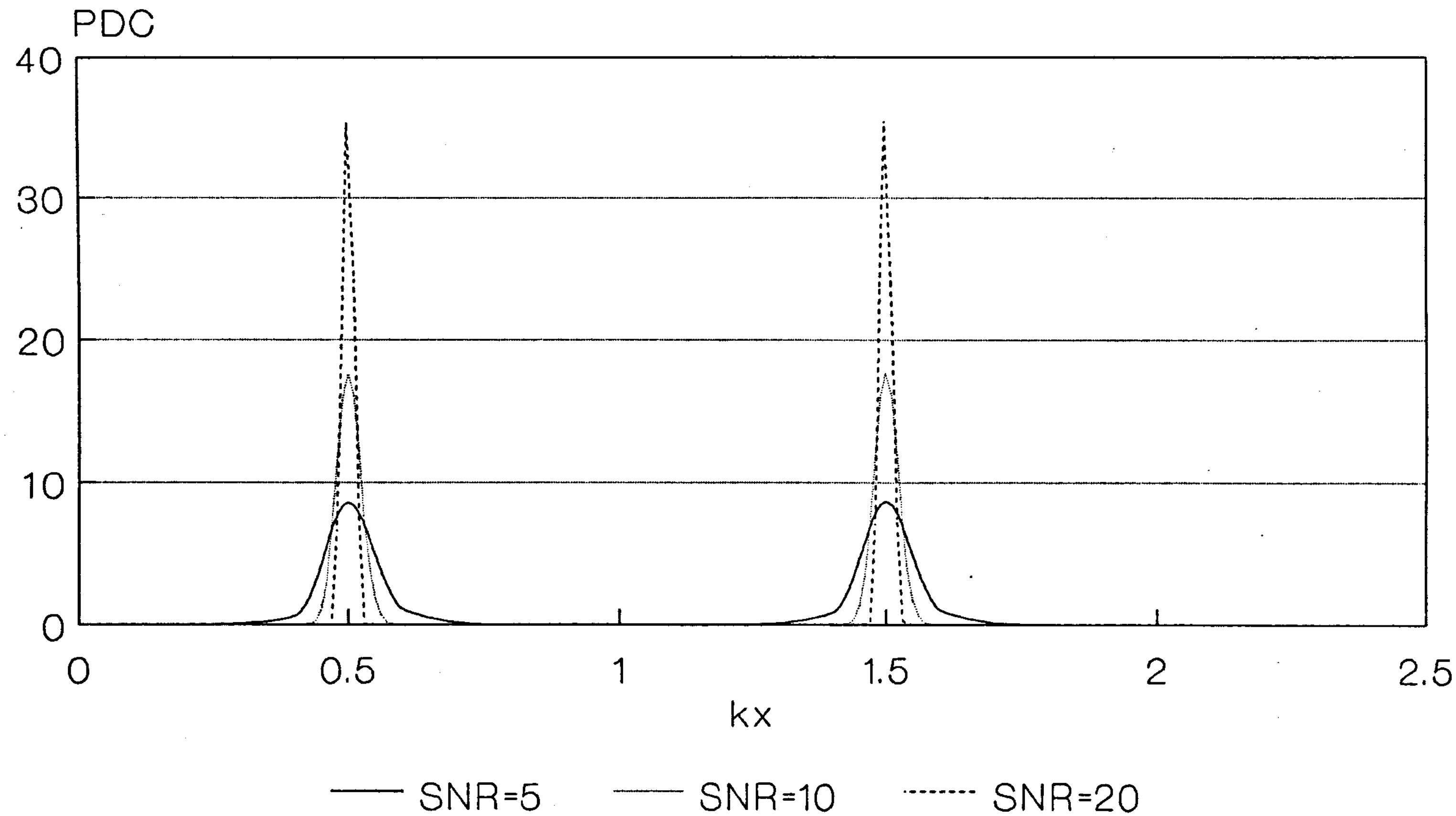
Rs=WS/WB= 0.5

Fig[5.4b] PDC for Butterworth filter



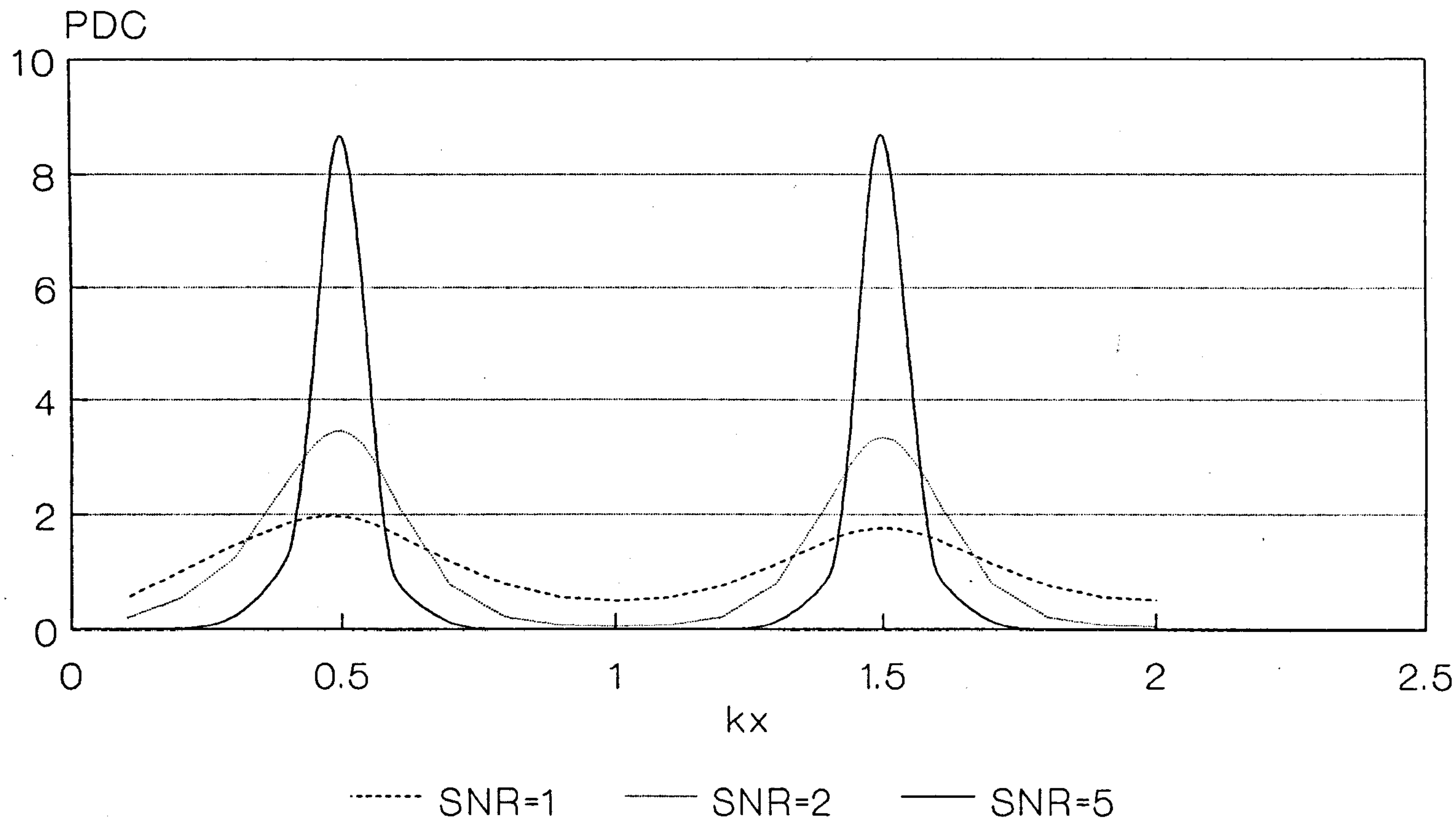
Rs=Ws/WB= 1

Fig[5.4c] PDC for Brickwall filter.



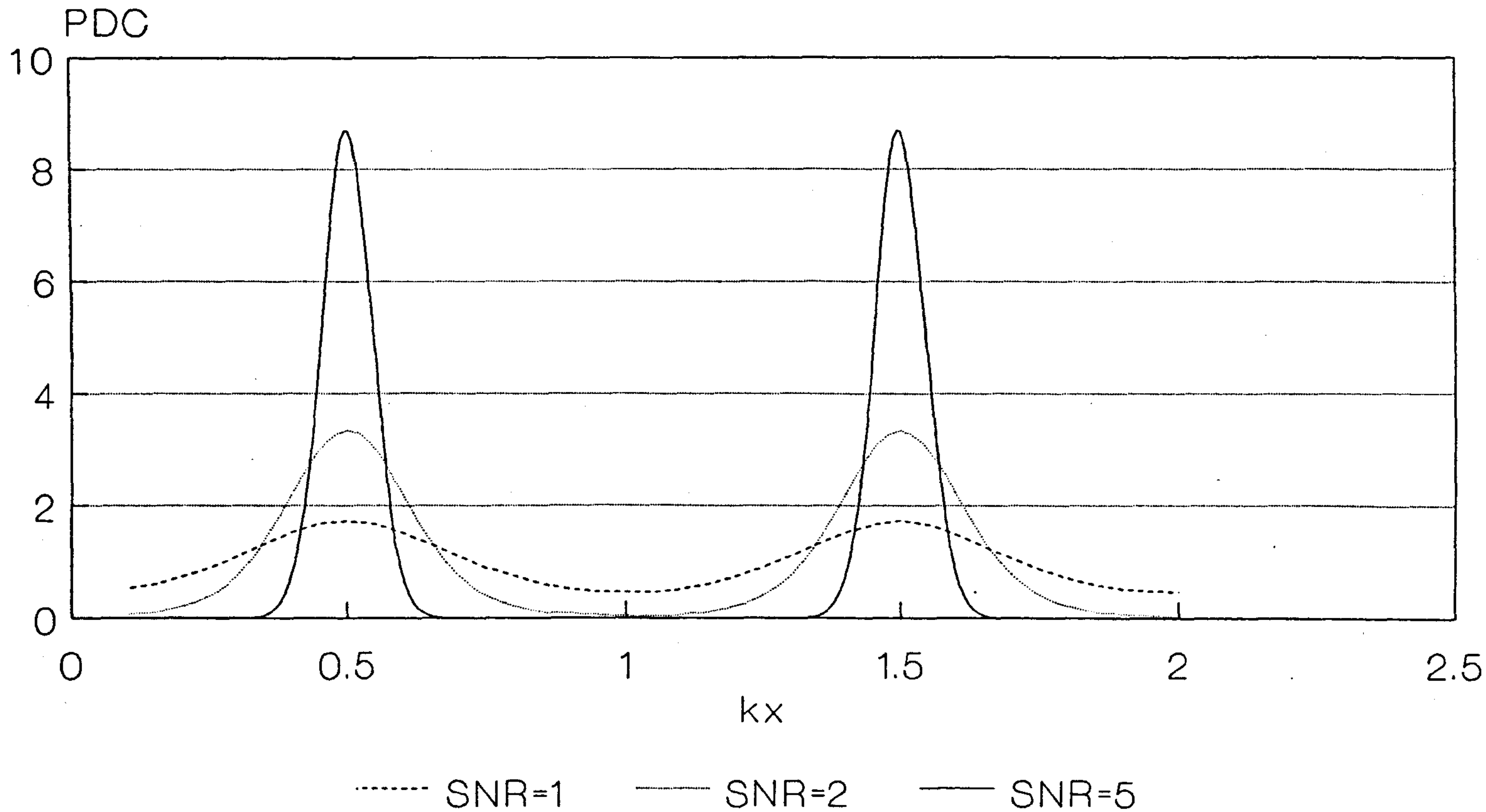
Rs=Ws/WB=0.5

Fig[5.4d] PDC for Brickwall Filter



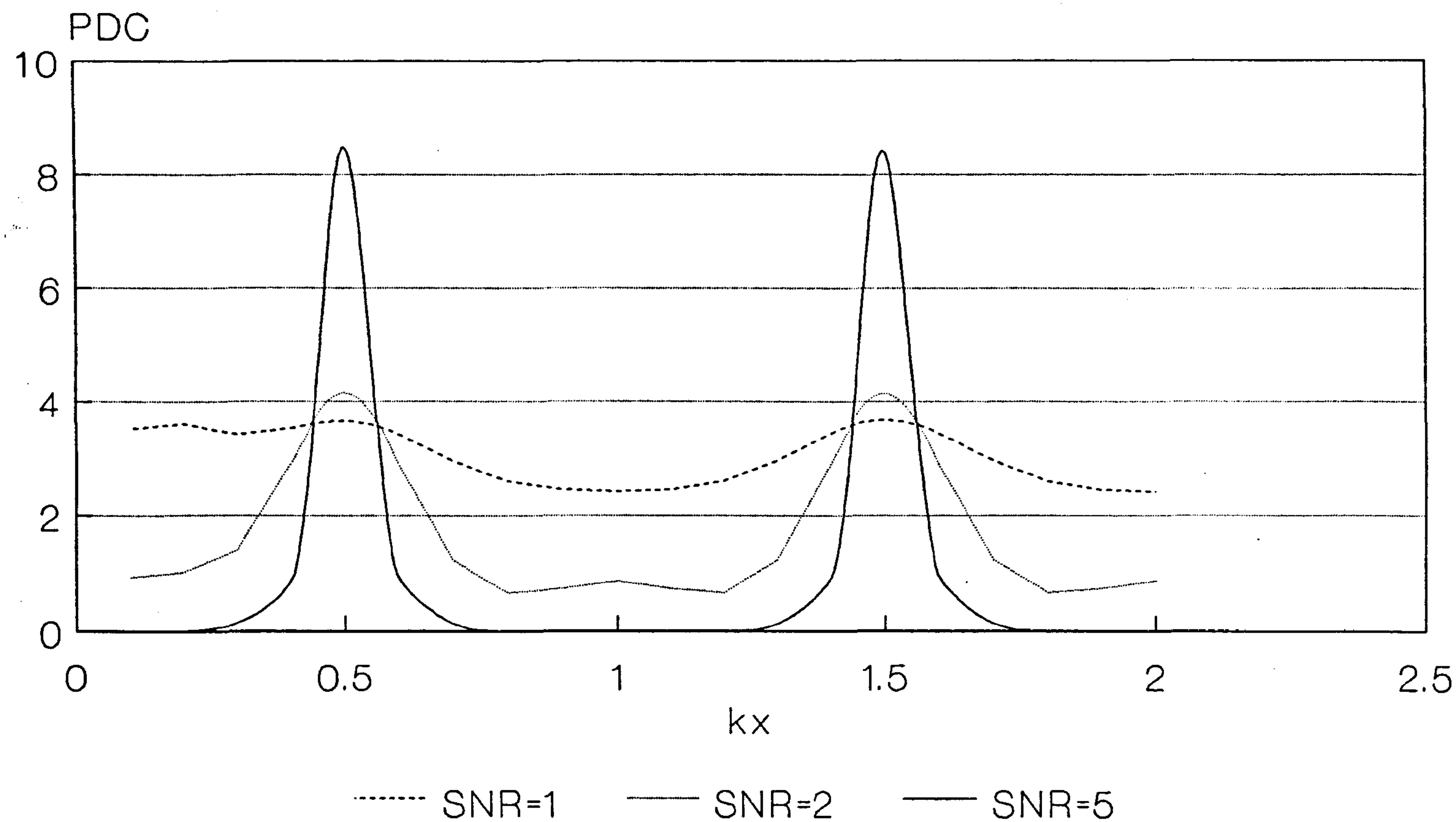
Rs=WS/WB= 1

Fig[5.5a] PDC for Butterwoth filter.



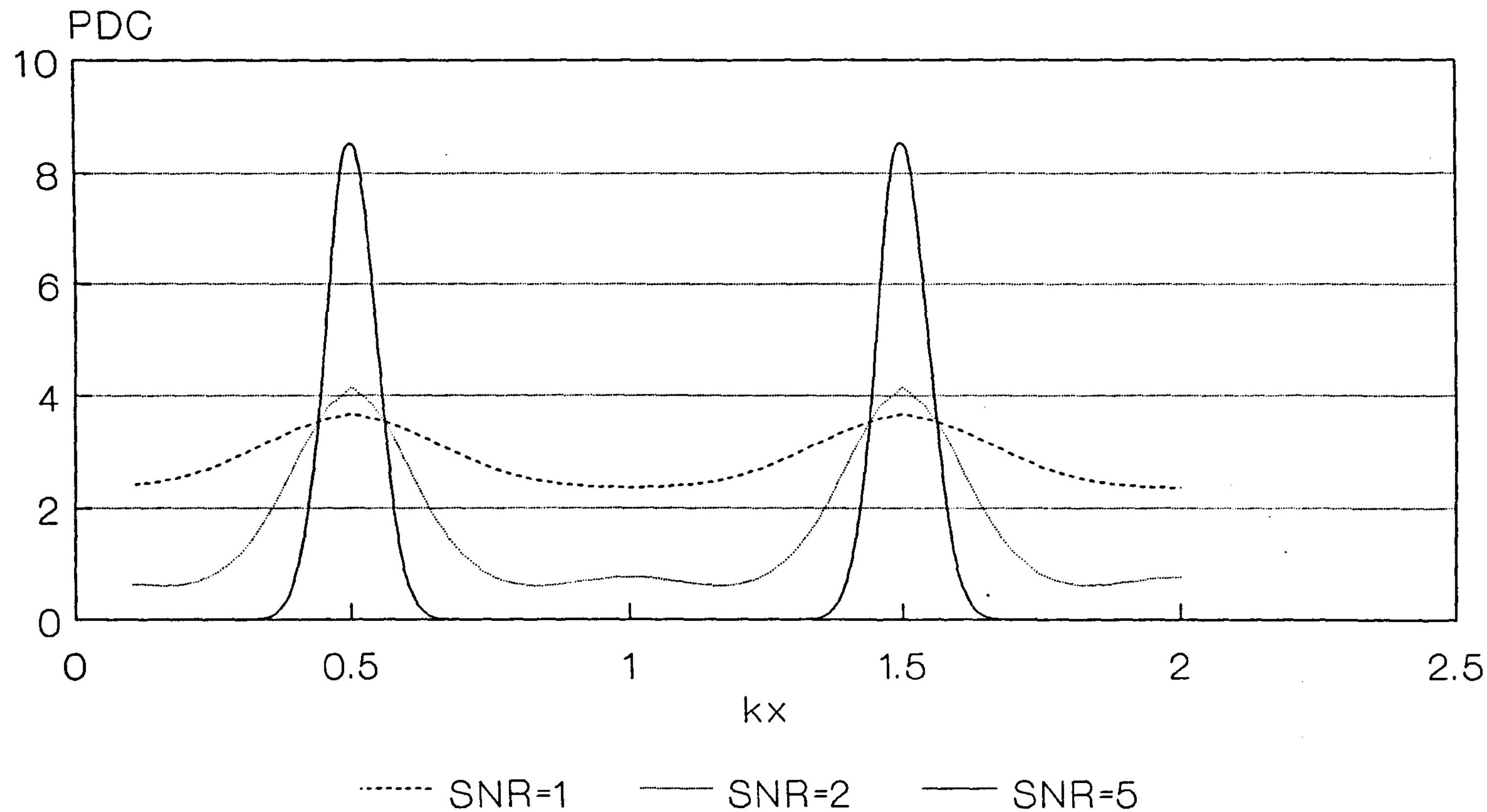
Rs=WS/WB= 1

**Fig[5.5b] PDC for ButterWorth filter
the case of no correlation**



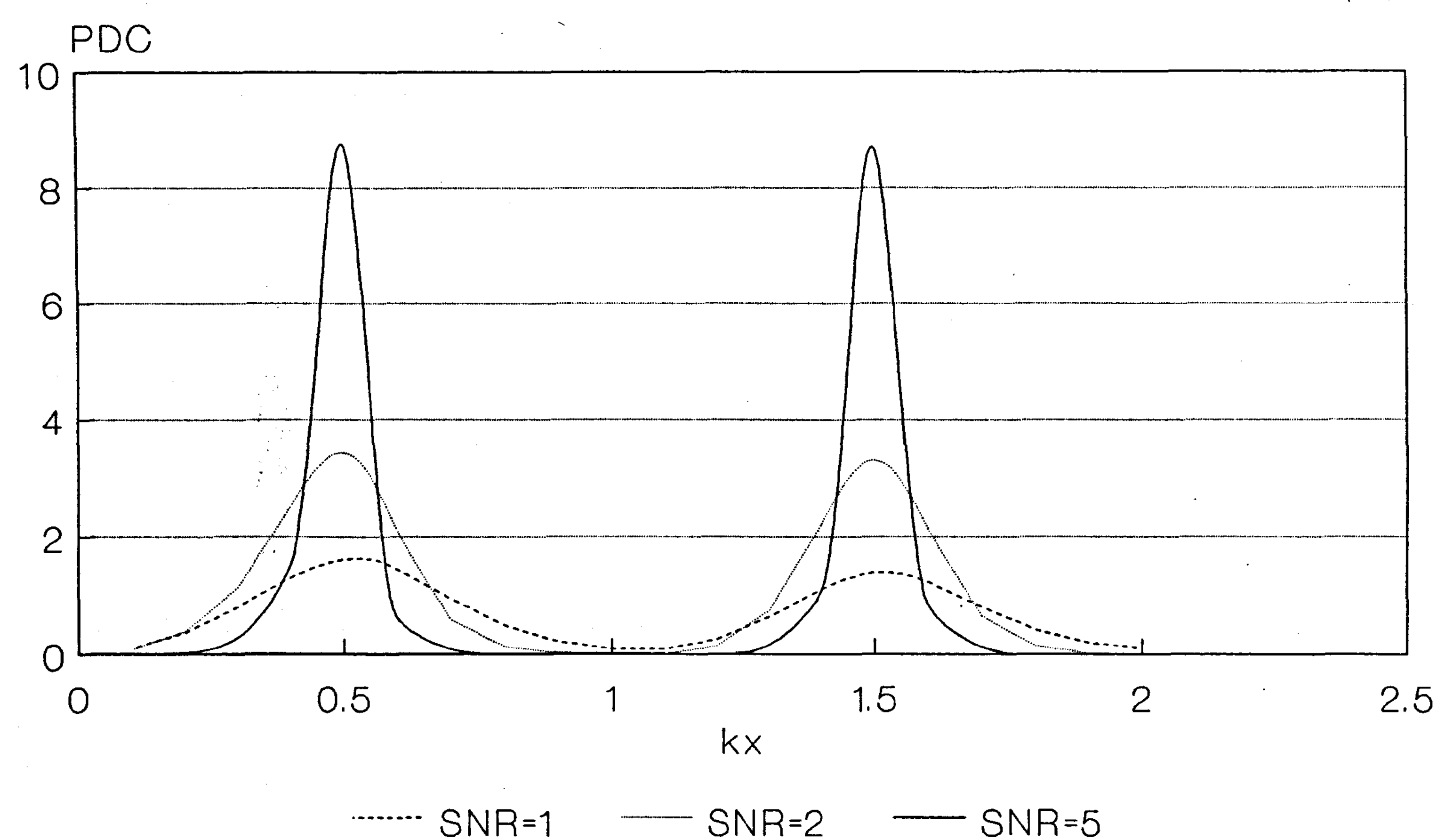
Rs=WS/WB= 0.5

Fig[5.5c] PDC for Butterworth filter



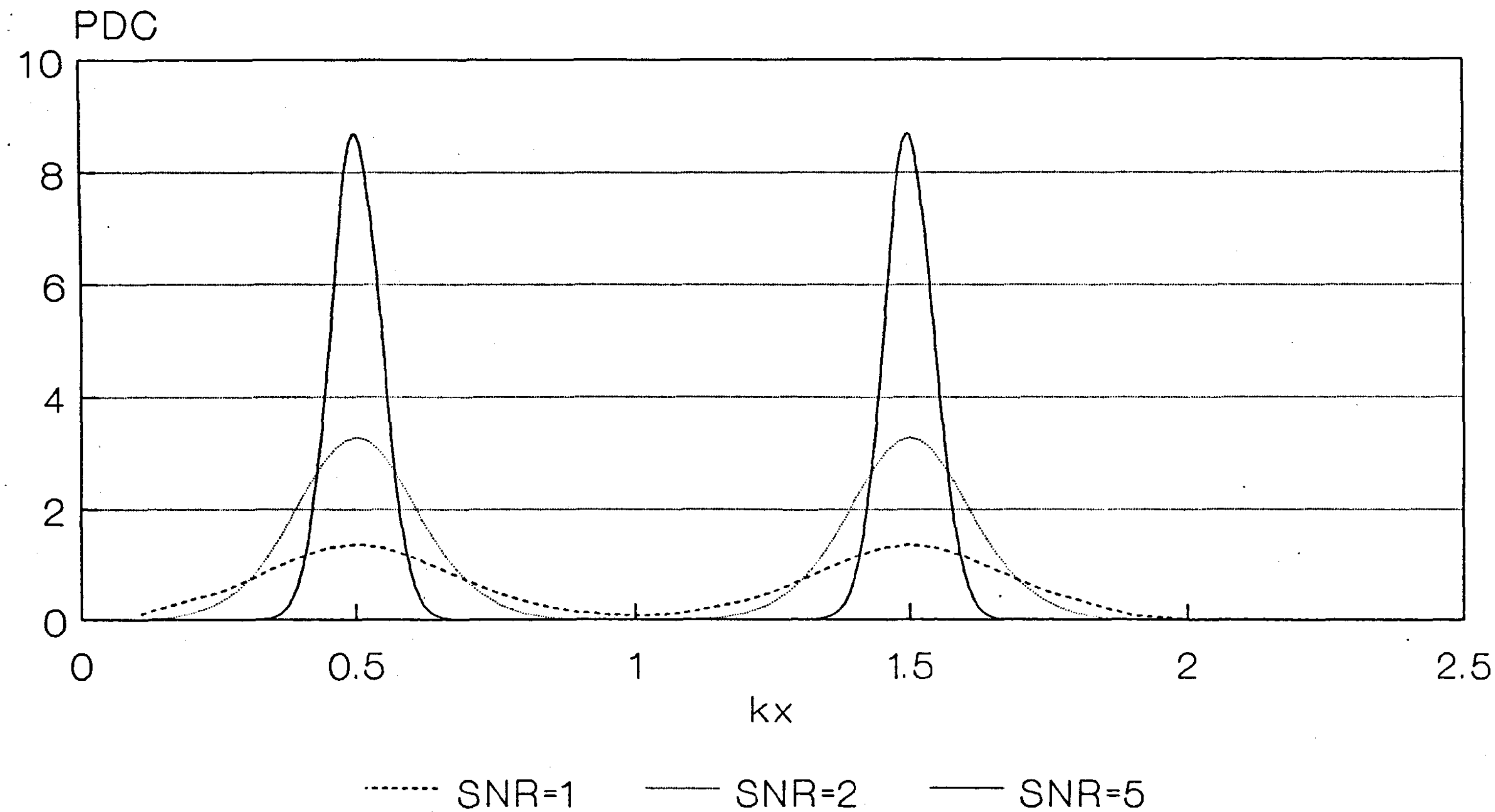
Rs=WS/WB= 0.5

Fig[5.5d] PDC for ButterWorth filter the case of no correlation



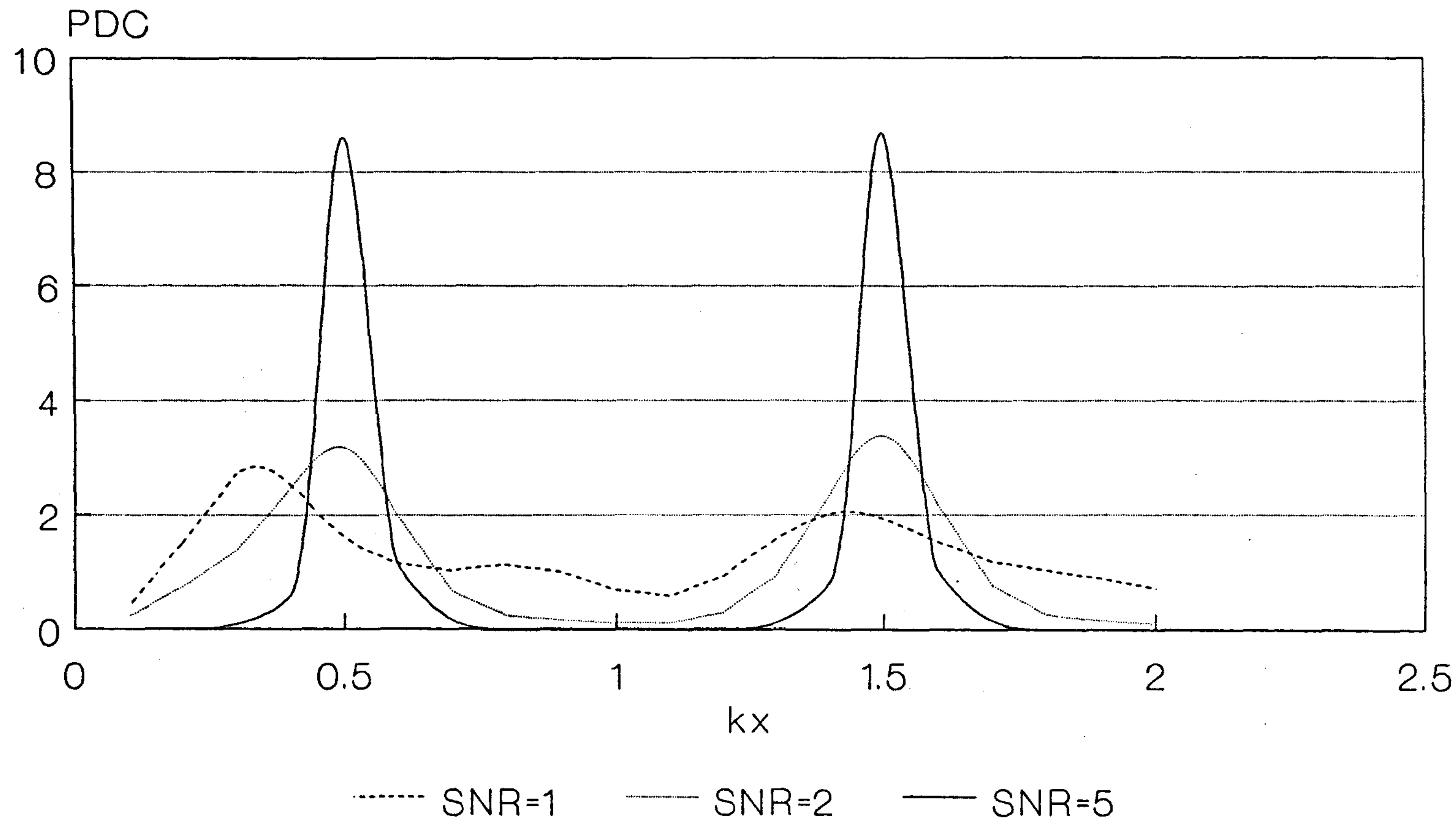
Rs=Ws/WB= 1

Fig[5.5e] PDC for Brickwall filter.



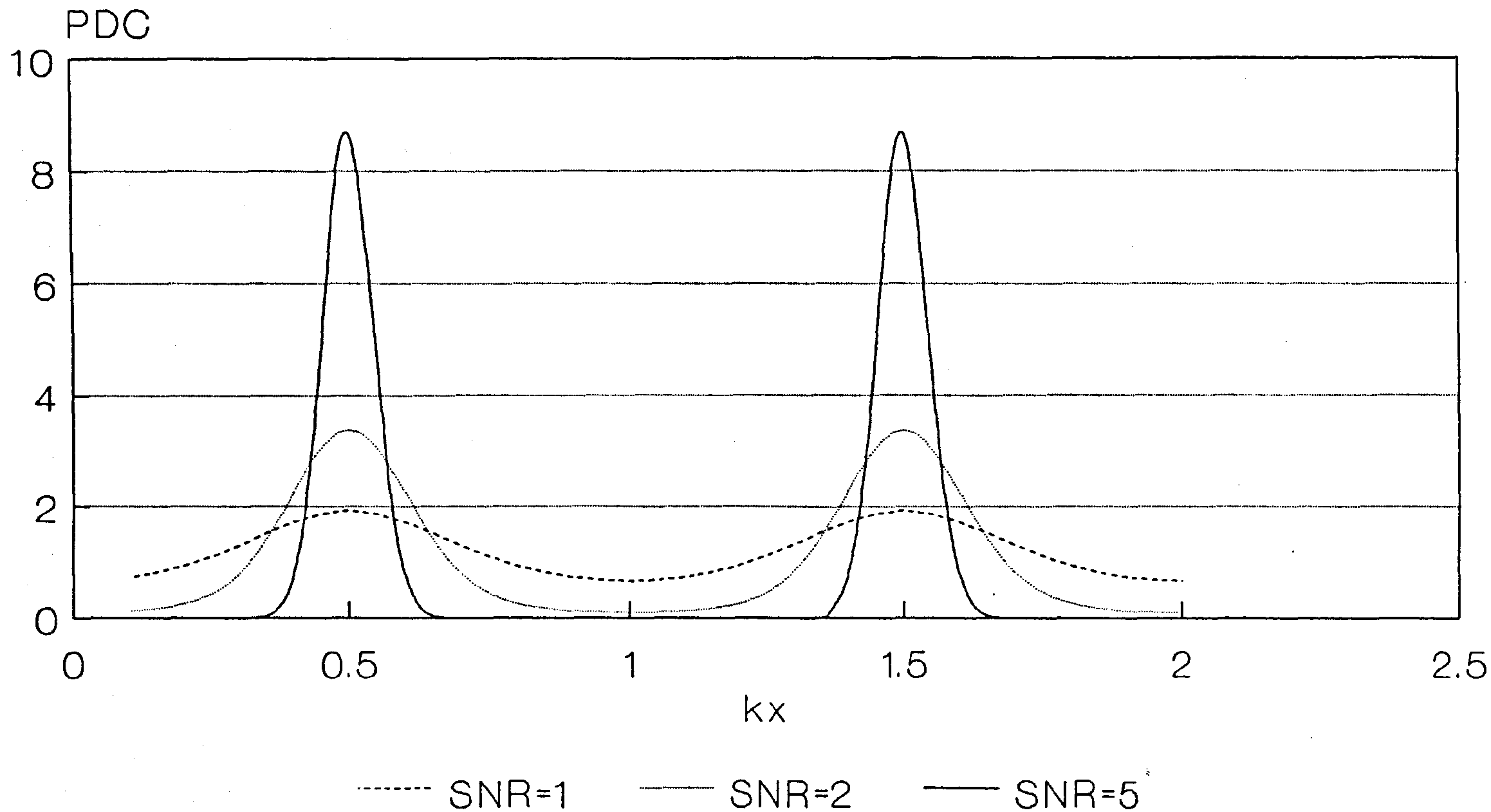
Rs=WS/WB=1

Fig[5.5f] PDC for Brickwall filter
the case of no correlation



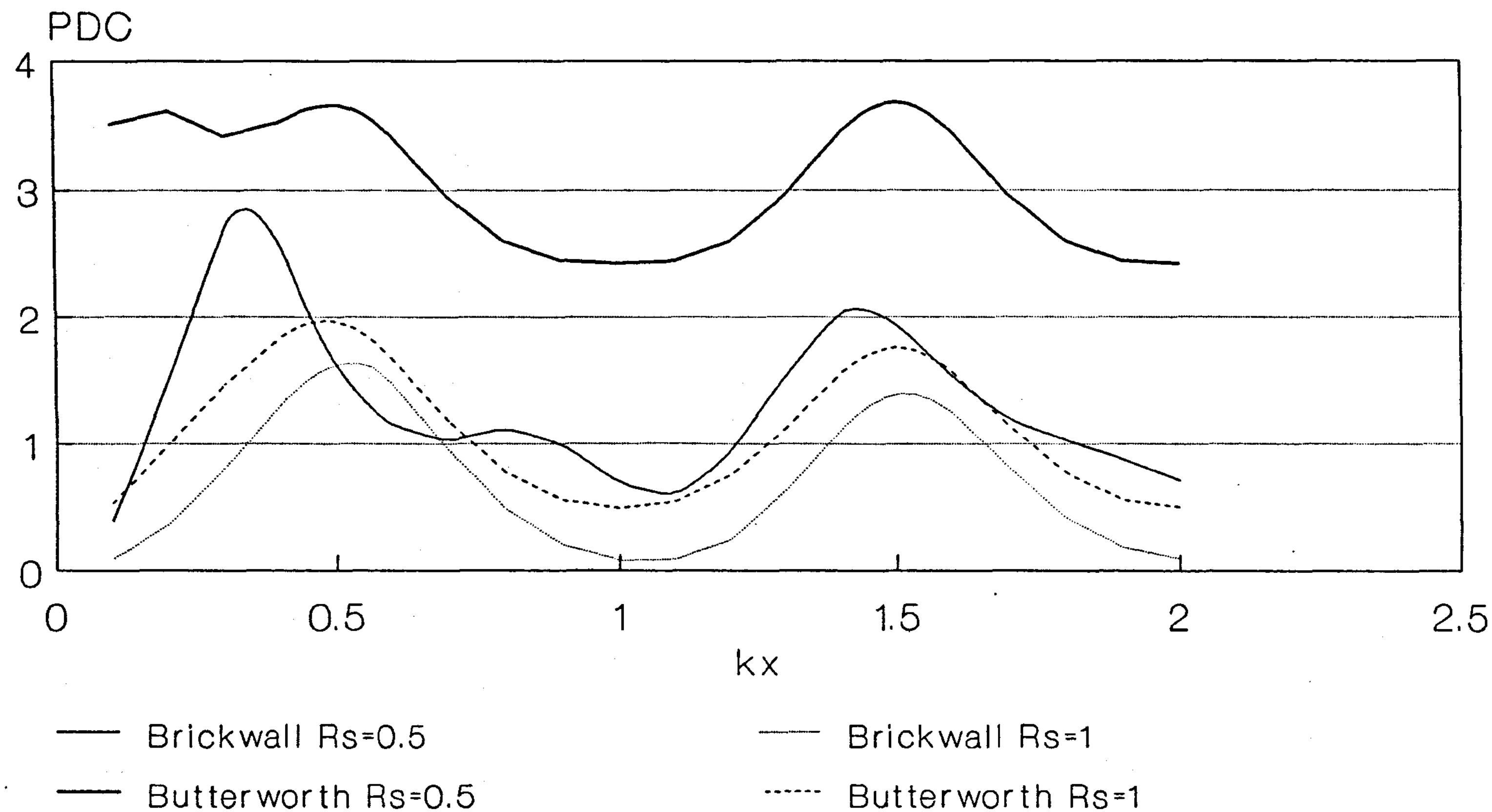
Rs=Ws/WB=0.5

Fig[5.5g] PDC for Brickwall Filter

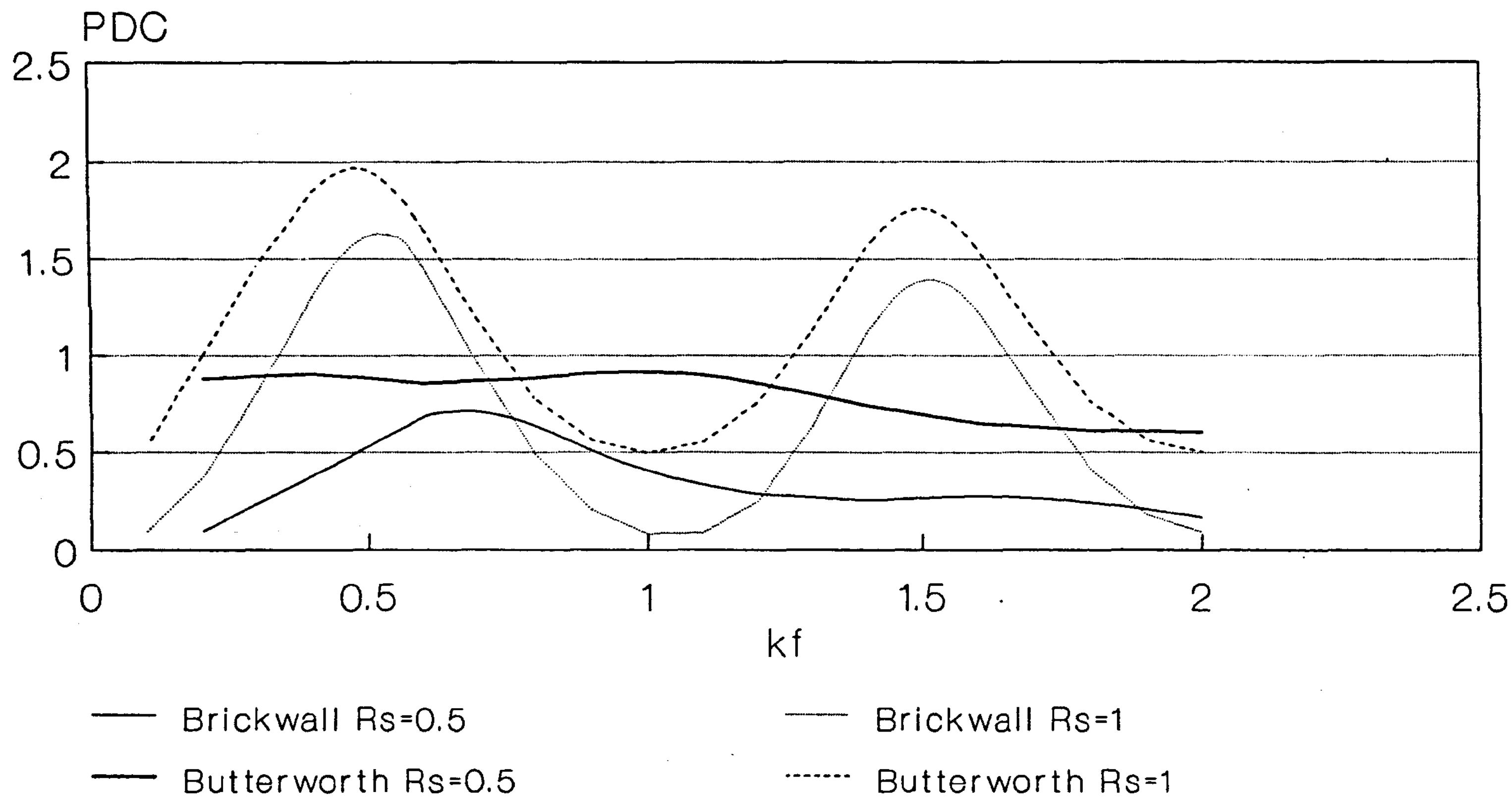


Rs=WS/WB= 0.5

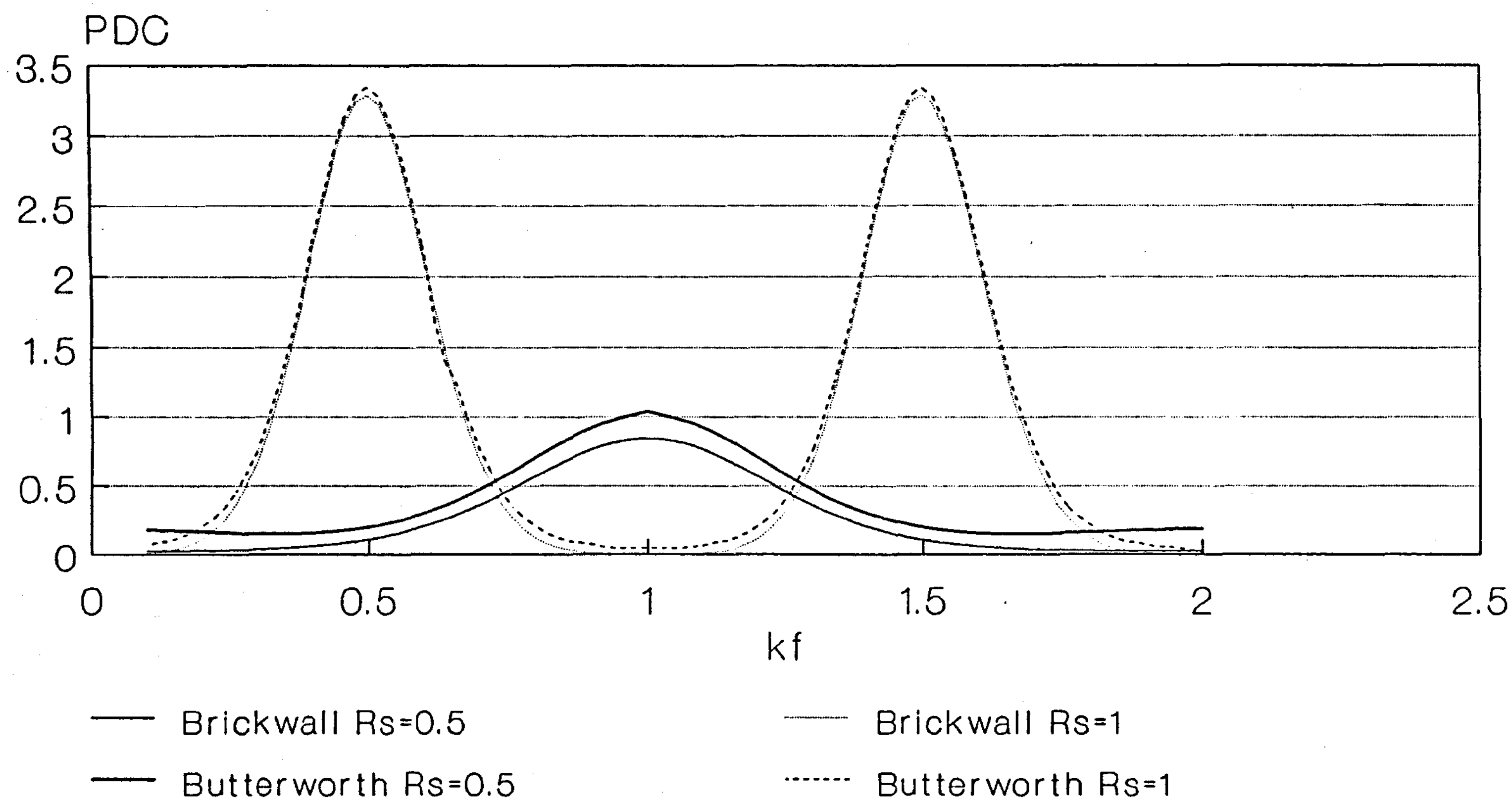
Fig[5.5h] PDC for Brickwall filter
the case of no correlation



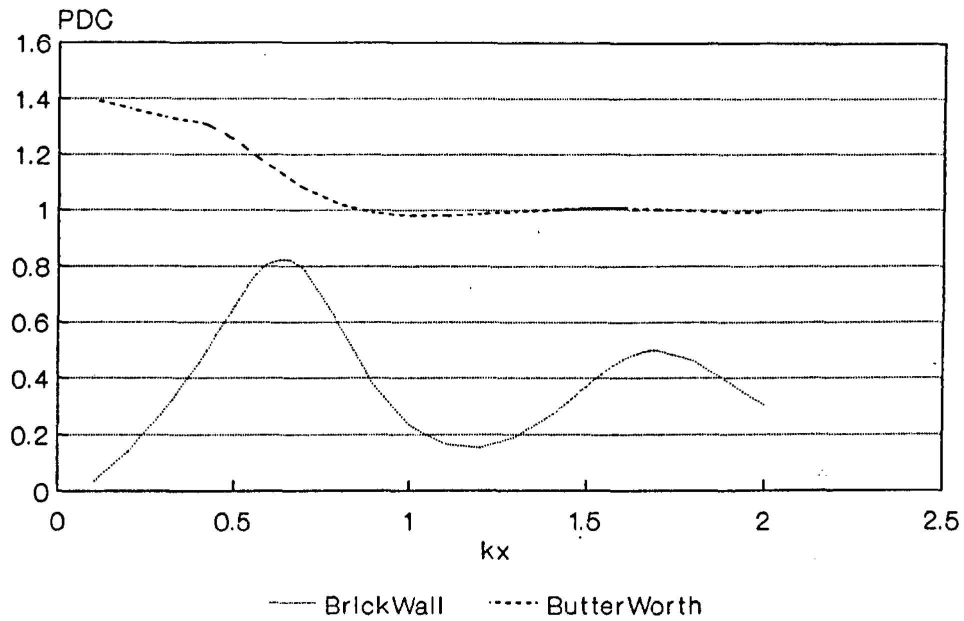
Fig[5.6a]PDC for SNR=1, for both filters



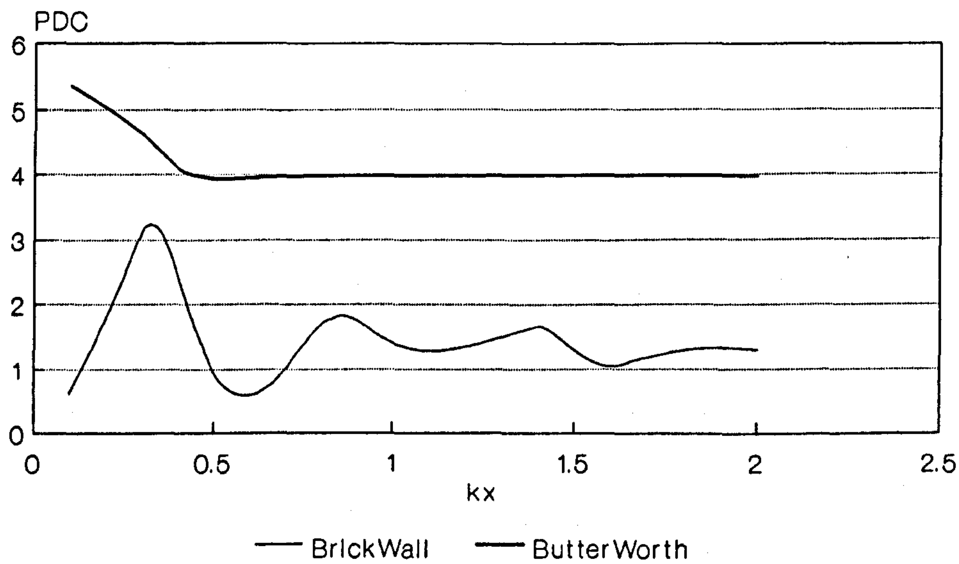
Fig[5.6b] PDC for SNR=1, for both filters with same time scaling.



Fig[5.6c] PDC for SNR=1, for the case of no correlation with same time scaling.

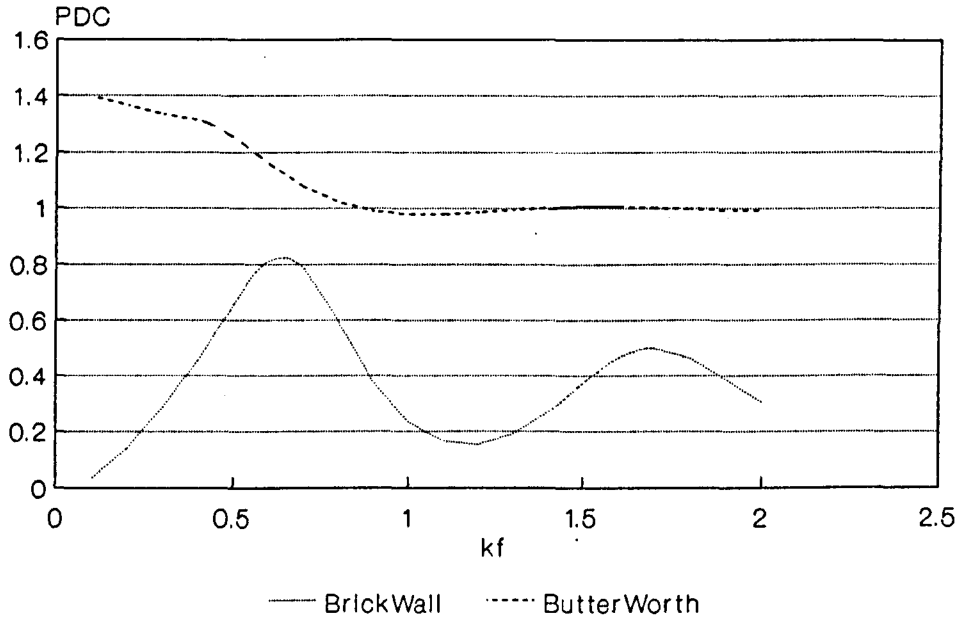


SNR=0.1, Rs=Ws/WB=1

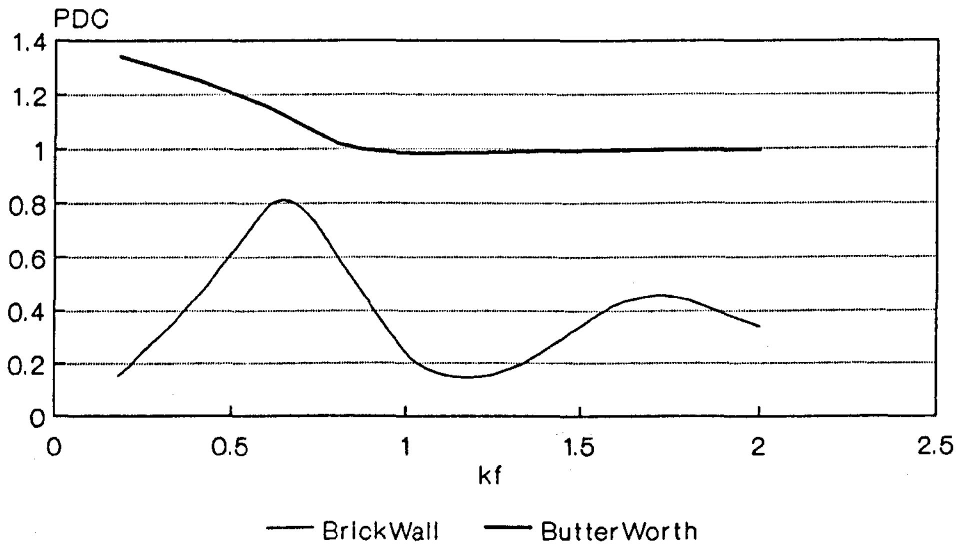


SNR=0.1, Rs=Ws/WB=0.5

Fig[5.7a] PDC for very low SNR.

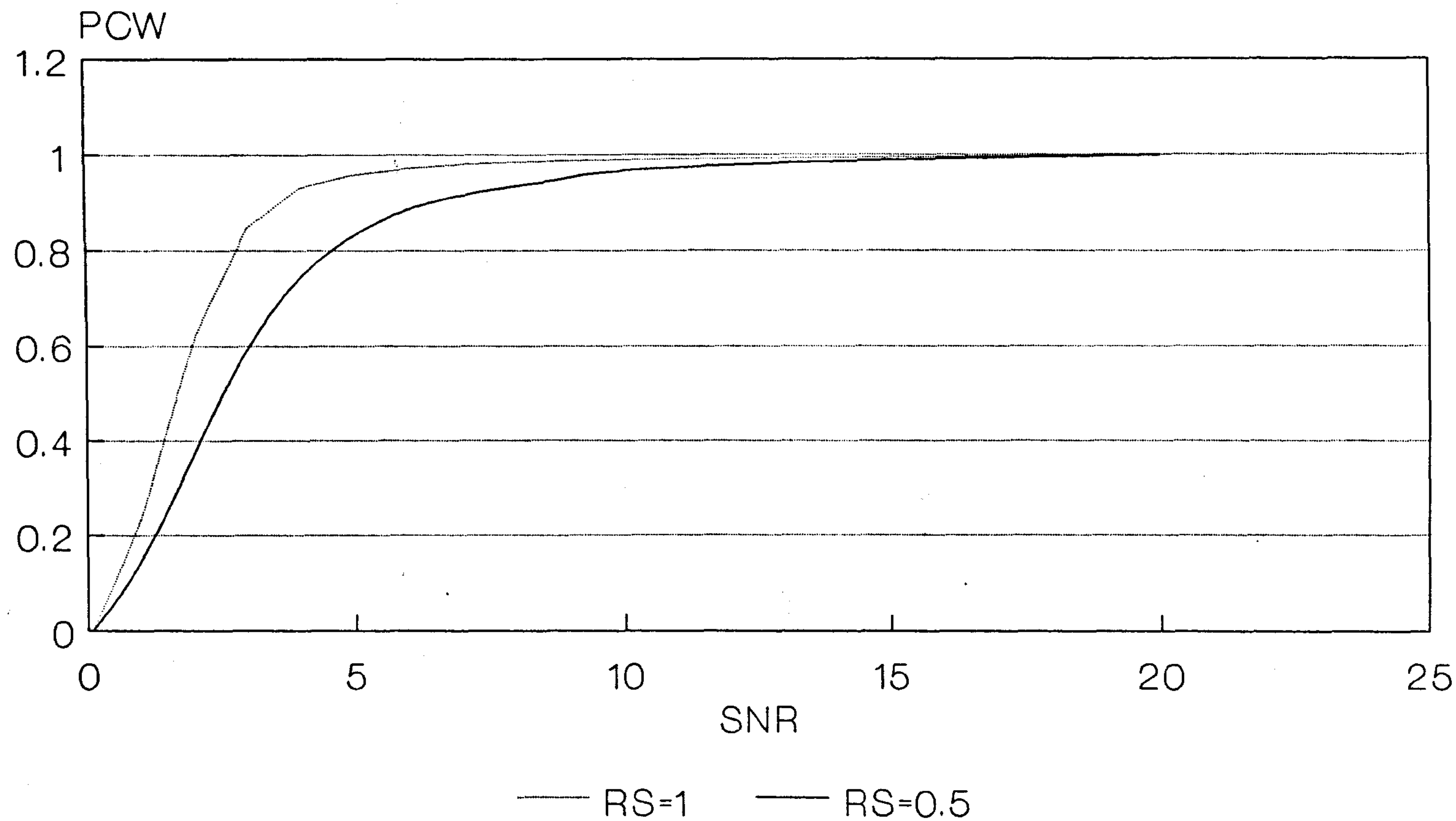


SNR=0.1, Rs=Ws/WB=1

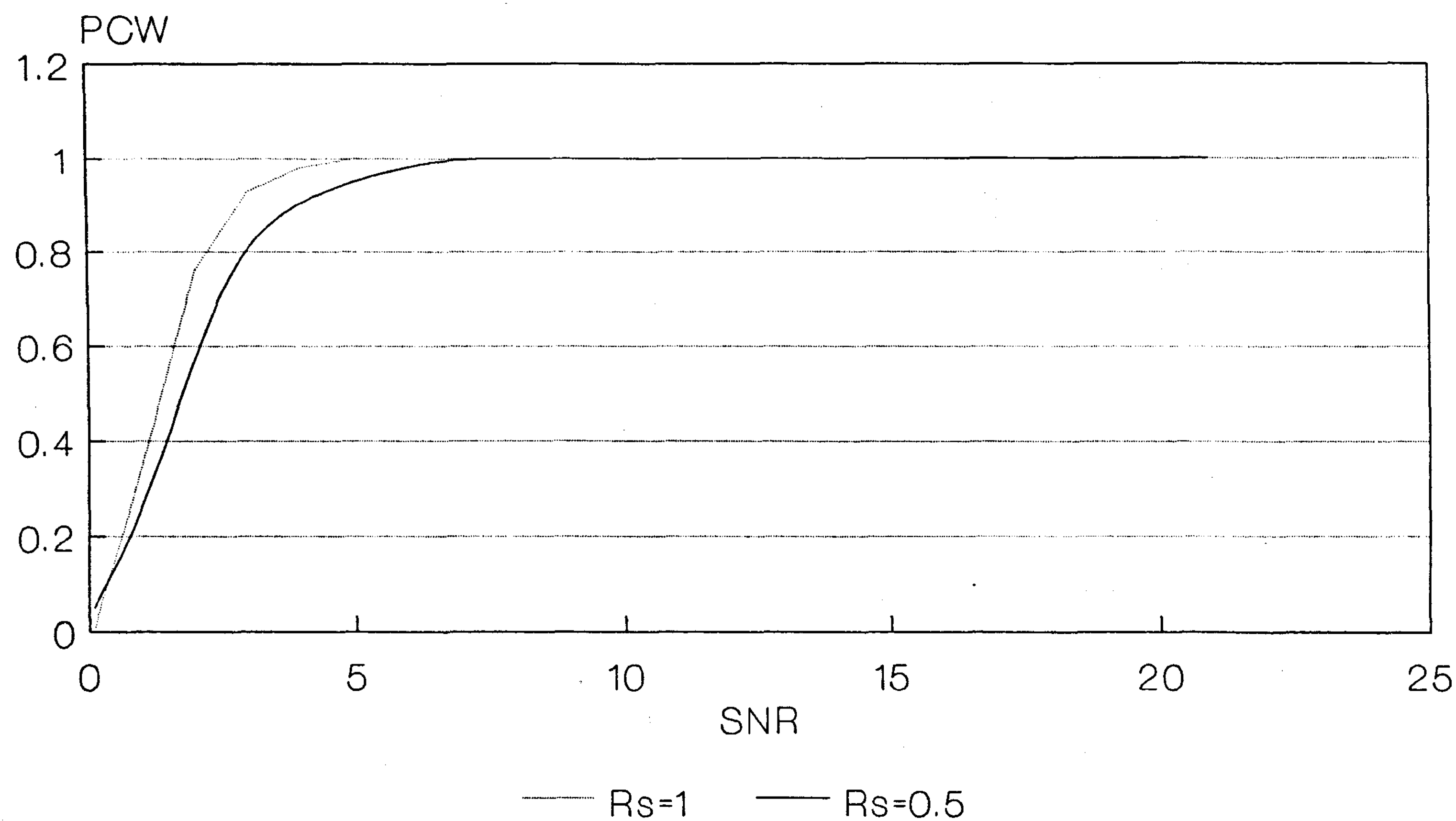


SNR=0.1, Rs=Ws/WB=0.5

Fig[5.7b] PDC for very low SNR, with same time scaling.



Fig[5.8a] PCW for Butterworth filter.



Fig[5.8b] PCW for brickwall filter.

Chapter 6

Effect Of Noise Correlation On The Probability Of Double Crossing: Discussion Of Numerical Results

6.1 Introduction

For specific signals and detailed design of systems it will be necessary to carry out calculations similar to those of Chapter Five, and compare in detail the results of such calculations. However the results for pure sinusoids show a number of effects which can be expected to be more general. These are discussed in this chapter.

6.2 Correlation Parameters

The graph (Figure 5.1) of the variation of correlation parameters with respect to τ for the 2nd order Butterworth filtered noise shows that all correlation parameters have decayed to the no correlation situation by about one period of the filter frequency($\tau/2\pi$=1). At this delay time, the probability functions will not be significantly different from the zero correlation case (this is confirmed by comparison of calculations of the joint probability density of double crossings with and without correlation, discussed below). Thus probability function results for the Butterworth filter for delay times corresponding to about one period of the filter frequency or greater can be taken as a standard for comparison. One period of the filter frequency corresponds to k_x=1 for R_s=1 and k_x=0.5 for

$R_s=0.5$.

Graphs (Figures 5.2 & 5.3) of the correlation parameters for the Brickwall filter show that Correlation effects are greatly extended compared with the Butterworth filter, and have certainly not decayed by one period of the filter frequency. However, the most significant parameters σ_{By} and detA have almost decayed to no correlation values by one period. When the noise does not dominate, it may be anticipated that again the correlation effects will not be significant (This is also confirmed below).

The Brickwall filter has the sharpest possible cut off, and is the worst possible case for correlation effects. In general, practical filters will not show correlation effects as strong as those for Brickwall filter. Filters with a sharp cut-off will be closer to the Brickwall case, and closer to the Butterworth case with a slow cut off. Plotting correlation curves, similar to those of Figs[5.1 to 5.3], for a particular filter will give an indication of the extent of correlation effects.

6.3 Overall Effects Of noise and Its Correlations

For high SNRs, Fig[5.4a to d], the signal crossings are driven by the deterministic signal. The exact time between crossings is randomly perturbed by the noise. As the SNR decreases, Figs[5.5a to h], the effect of the noise is to reduce the value of the probability of crossing at the times determined by the deterministic signal, and to spread the probability into the window of the next deterministic signal crossing. These effects tend to a uniform double crossing probability density, seen for longer time scales in Figs[5.7a to b].

For very high noise (amplitude SNR=0.1, Fig 5.7b) the comparison between the crossing probability with different frequencies plotted on the same time scale ($\tau/2\pi$) shows that the signal has very little effect. The deterministic signal crossings are at $\tau/2\pi=0.5$ for Rs=1, and at $\tau/2\pi=1$ for Rs=0.5, but there is very little evidence of these except with the Butterworth filter at Rs=1. Crossings are therefore noise driven, and structure is caused by correlation in the noise. Structure effects due to the correlation die away to a uniform background over a time scale which is consistent with the time scale of the correlation parameters. Thus, for very high noise and lower time scales the correlation structure in the noise produces probability density maxima at times related to the band correlation effects, and not to the deterministic signal crossing. These can be regarded as false maxima.

Whilst there are maxima in the probability related to the crossings of the deterministic signal, then in principle information about the deterministic signal can be extracted. (This may take a very long time when the maxima are very low). However, as the noise increases the false maxima due to correlations compete with, and eventually dominate, the reducing maxima corresponding to the deterministic signal. When correlation maxima dominate, information about the deterministic signal cannot be extracted. Limits of usable SNRs are therefore more likely to be caused by the correlation effects than by white noise effects.

6.4 PDC With High SNR

Figures 5.4a to 5.4d show the joint PDC for Butterworth and Brickwall filters. Scaled delay times k_x are extended to $k_x=2$ to show the periodic repeat of the probability maximum. In the case of the Butterworth filter this also extends the delay sufficiently to show the probability where correlation effects have become negligible.

It can be seen, as expected, that the probability of detection of a crossing peaks at the position of the crossing of the deterministic signal. The peak tends towards a delta function form for very high SNR.

For Butterworth filtered noise with amplitude SNRs 5 to 20 graph 5.4a (Rs=1) probability peaks at $k_x=1.5$ and graph 5.4b, (Rs=0.5) peaks at 0.5 and 1.5 are at delays for which correlation effects are expected to be insignificant. To within the precision of the graphical display, these peaks are identical, which confirms the absence of correlation effects (this is verified for SNR 5 as discussed below). On the graphs, a sensitive indicator of the presence of correlation is the vertical placement of data points. In the no correlation case there are corresponding pairs of the points at exactly the same level on either side of the centre of the peak.

The peak at 0.5 on graph 5.4a (Butterworth R_s=1) does show slight displacement of corresponding points, indicating some correlation effects, but the differences from the other peaks are otherwise not discernible. For almost all practical purposes the correlation effects can be ignored. On Figures 5.4c and 5.4d for the Brickwall filter (R_s=1 & 0.5 respectively) all peaks are similar to the peak at 0.5 with R_s=1 for the Butterworth, and similar comments apply.

It can therefore be concluded that for amplitude SNRs of 5 or greater the effects of noise correlations between the two crossings can be ignored. This is irrespective of the bandwidth of the noise. Although this is a negative result, nevertheless it is a result of some importance, because it justifies the common practice of ignoring correlation effects.

When correlation effects between the two crossings are negligible, the joint probability of a double crossing is simply the product of the probabilities of the two single crossings (evaluated at the correct times with respect to the deterministic signal). It is not necessary to use the full theory of Chapter Four & Five. Instead, the methods of Chapter Three can be used for the probability densities of each individual crossing, and these combined for the double crossing. The results of Chapter Three, in terms of signal increments, can be adapted to any functional form of deterministic signal, and can also be used with sampled signal processing.

Similarly, with amplitude SNRs of 5 or greater, probability of multiple crossings can be analysed by combining probability densities for the individual single crossings.

6.5 PDC With Low SNR

Figures 5.5a to 5.5h, for amplitude SNRs 5, 2 and 1, show increasing effects of correlation as the SNR is lowered. Figures 5.5a, c, e, g, show results of the full calculation with correlated noise based on Eq(5.28). For comparison Figs[5.5b, d, f and h], show results of use of the simplified formula Eq(5.29) for noise without correlation.

For amplitude SNR 5, comparison of the double crossing probability densities for correlated noise with those for uncorrelated noise, confirm the assertion, made above, that correlation can be ignored for most practical cases; and also that the second peak (kx=1.5) for the Butterworth filter can be taken as a no correlation comparison.

For Rs=1 (Signal with band cut off frequency) and amplitude SNR 2, Fig[5.5a and e] show a barely noticeable increase in probability over the first maximum. The increase in probability becomes more significant for $kx(\tau/2\pi) < 0.4$. For Rs=0.5 (Signal frequency at half band cut off, Figs 5.5c and g) the probability shows a noticeable increase for $kx < 0.25$ ($\tau/2\pi < 0.5$), which is consistent with that for Rs=1. It may be concluded that for band effects with a slow roll-off, correlation effects can be ignored for times greater than about $\tau/2\pi = 0.4$. The increase in probability for lower $kx(\tau/2\pi)$ are false crossing effects due to the correlation structure. In any practical application it would be best to remove these by a time window with a lower limit and, above this lower limit, again the no correlation approximation could be used.

With the Butterworth filter for SNR=2 and Rs=0.5, Fig[5.5c], there is a small maximum at kx=1. This occurs where the deterministic signal has a crossing in the wrong direction, but because the signal is close to zero the noise is able to produce a crossing in the opposite direction. This effect is not related to correlation as is shown by the same maximum as the no correlation case Fig[5.5d]. The effect only occurs with bands having a slow roll-off, which allow noise with higher frequencies and shorter time scales, and is not shown by the Brickwall case. The effect is also absent at higher frequencies, relative to the bandwidth, Rs=1 Fig[5.5a, b], because, in this case the real rate of change of the deterministic signal is too fast.

For amplitude SNR=1, Figs[5.5a to h] correlation effects produce significant differences from the no correlation cases, up to and including the first maximum for the Butterworth filter, and beyond the second maxima for the Brickwall filter. The correlation effects becomes more apparent for deterministic signals away from the band edge (Rs=0.5). Comparison of Figs[5.6a and b], (SNR=1) with Figs[5.7a and b], (SNR=0.1) indicate that amplitude SNR=1 is transitional between ratios for which deterministic signal effects are dominant, and the very high noise case dominated by noise and its correlation effects. Details of the structure with SNR=1, Fig[5.6b] and its differences from the no correlation case Fig[5.6c] can be explained by reference to the very high noise case Fig[5.7b]. Fig[5.6b], showing the probability densities on the same time scale ($\tau/2\pi$) also indicate that it would be difficult to use SNR=1 for finding characteristics of the deterministic signal, or for communication such as demodulation of FSK signals.

6.6 PDC With Very High Noise

The case of very high noise has been discussed in section 6.3. Structure in the probability is related to the correlation effects. From comparison of the structure of the probability densities shown in Fig[5.7b] with the correlation parameters, Figs[5.1 to 5.2], it is not possible to discern any simple connection between the probability structure and the correlation parameters other than the decay time scale.

6.7 Probability Of Double Crossing Within a Time Window

In the calculations the time window used for the second crossing is the half period about the expected time of the second crossing of the deterministic signal. As can be seen from Figs[5.4a to h] this window is suitable for high SNRs, although a narrower window could be used in most cases. For practical use with low SNRs the window would need to be adjusted for the specific application in order to avoid the correlational effects, as discussed in section 6.5. These calculations are intended to be illustrate the technique. The results are displayed in Figs[5.8a to b].

6.8 Practical Limits On The Use Of Low Signal To Noise Ratios

As discussed above, the calculations of probability densities of double crossings show competition between structure in the probability due to the deterministic signal and structure due to the correlation in the noise. For usability the deterministic signal structure should be distinguishable. From the examples above it has been found that, when the deterministic signal structure is distinguishable, the probability density of double crossing can be approximated by the no correlation case. Correlational calculations are only needed to determine the limits. Determination of performance parameters can be accomplished using the no correlation methods of Chapter Three. The correlations affecting the double crossing are essentially those over the time scales between the crossings. The methods of Chapter Three can be used for the very short time correlation effects, which arise when sampled signal methods are used for detecting threshold crossings.

The band limiting effects, which cause the correlations in the noise, have been modeled, by a second order Butterworth filter and Brickwall filter. These can be taken as extreme cases. For the sinusoidal signal used in the above calculations the results indicate that an amplitude SNR of about 2 is the limit for use of the opposed crossings separated by half a signal period. If the pairs of crossings were to be taken over larger separations, to avoid correlation effects, then the limit would in principle be lower, but at the expense of very long times to obtain the number of crossings for reasonable confidence in the deduced values. The results for the sinusoid can be taken as a very strong indicator of the limits for other signals because no signal will have a faster zero-crossing than the band edge sinusoid ($R_s=1$). The methods of Chapter Three are independent of the signal function, and depend only on the value and incremental change (or rate of change) at the threshold crossing, and can therefore be adapted for use with any form of signal.

Chapter 7

A Zero-Crossing Detector Simulator And Demonstrator

7.1 Introduction

Theoretical studies based on the methods of the previous chapters can be used for preliminary design of practical zero-crossing detectors. However in real situations the noise may not be Gaussian, and the precise channel filtering effects (prior to deliberate filtering to reduce noise) will not be known. It is therefore useful to have a simulation of the detection system, which can be used with real signals to adjust the theoretical design. The simulation of a DSP based sampled signal zero-crossing detector is discussed in this chapter.

For demonstration purposes, a sinusoidal signal with added wide-band Gaussian noise is used for input. This allows comparison with the theoretical studies of the previous chapters.

7.2 Overview Of Simulator-Demonstrator

Fig[7.1], shows the scheme of the detector-simulator. This consists of two main sections: a detector simulator preceded by a signal source for demonstration purposes.

The detector is implemented using a Loughborough Sound Images (LSI) DSP development board incorporating a TMS32020 Texas Instrument digital signal processor, and installed in a PC. The incoming signal is sampled and digitised, and then passes through a low-pass filter (LPF). The LPF is followed by a zero-crossing detector, which implements the double crossing detection, and measures the times between crossings. The times between crossings are passed to the accumulator of the TMS processor, from where they are accessed by the PC for analysis of probabilities. The LPF can be used as a noise-limiting filter for simulation and design purposes. In the demonstration and comparison with theory it is used to model band limiting effects.

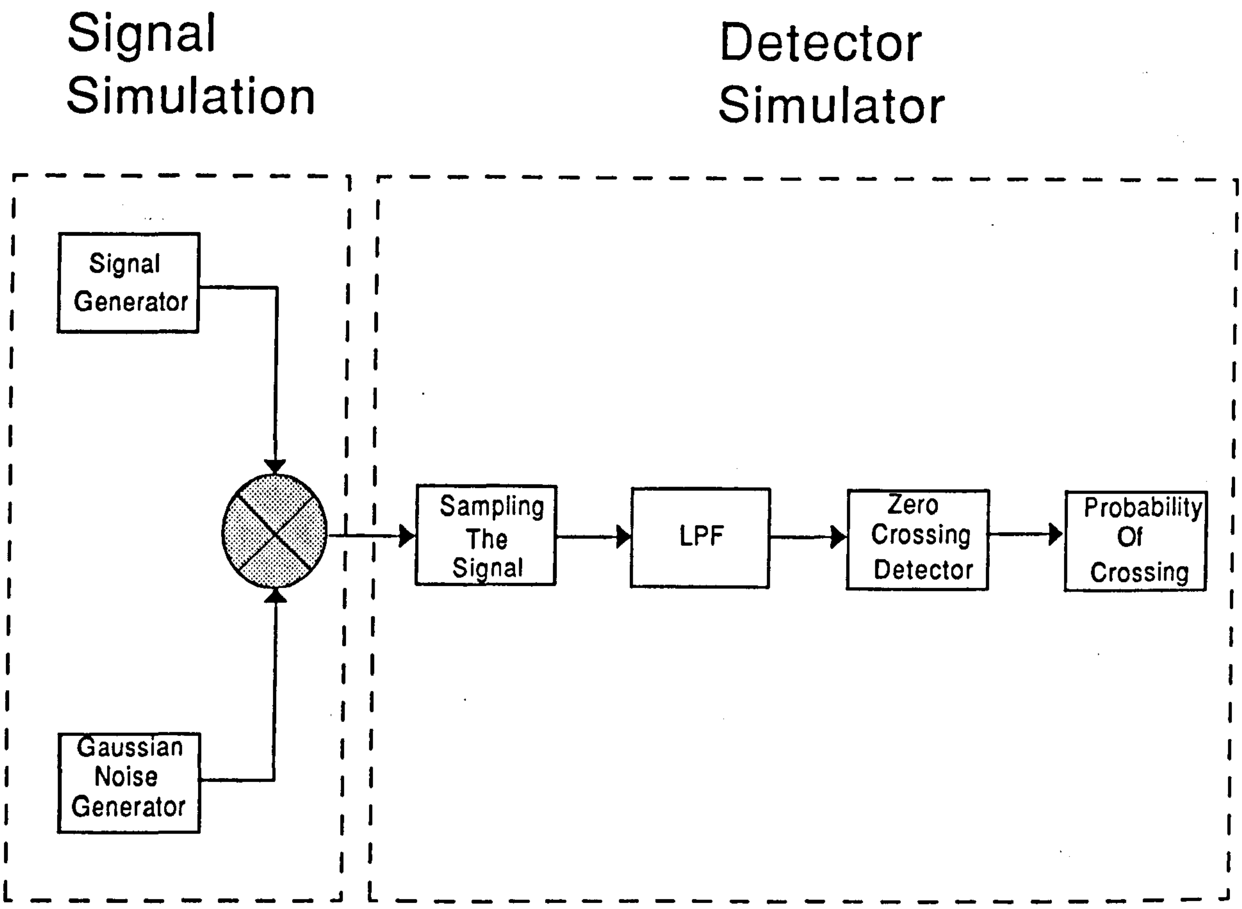
To simulate an incoming signal for demonstration and comparison purposes, a signal generator is used to provide a sinusoidal signal, which is mixed with Gaussian noise before being fed to the detector. The Gaussian noise source is one which is used routinely by the communications group of Hull University for Gaussian noise simulation.

For this study the sampling rate of the TMS system is set at 60kHz using the internal clock. Bandwidth effects are modeled for this study by using the LPF with the cut-off at 2kHz, and sinusoidal signals of 1KHz and 2kHz, respectively are used. These give relative frequency values corresponding to the theoretical studies.

Since a Brickwall filter cannot be implemented in practice, 8th order Chebyshev and Butterworth filters are used for the sharp cut-off. Both these 8th order filters have impulse responses close to the ideal [Poularkas

1985]. The Chebyshev filter has a sharper frequency cut-off than the Butterworth filter but has ripples in the frequency response, whereas the Butterworth response is smooth.

For the slow cut-off a 2nd order Butterworth, as in the theory, was used. In addition a 2nd order Chebyshev was implemented. The software can be easily adapted to other filters.



Fig[7.1] Block diagram of experimental set up.

The process software consists of TMS code for the operation and control of the DSP board, embedded in a C program for overall control and data analysis. The software is written to analyse times between crossings and also to determine the probability of a second crossing within a specifiable time window.

The probability density distribution of zero-crossing (PDC) was measured over a range of SNR corresponding to the theoretical study, and the results are displayed in Fig[7.4a,b,c,d,e,f,g,h], Fig[7.5a,b,c,d,e,f,g,h] and Fig[7.6a,b].

Finally, the work is extended to the probability of crossing within a window of finite interval, corresponding to Chapter Five.

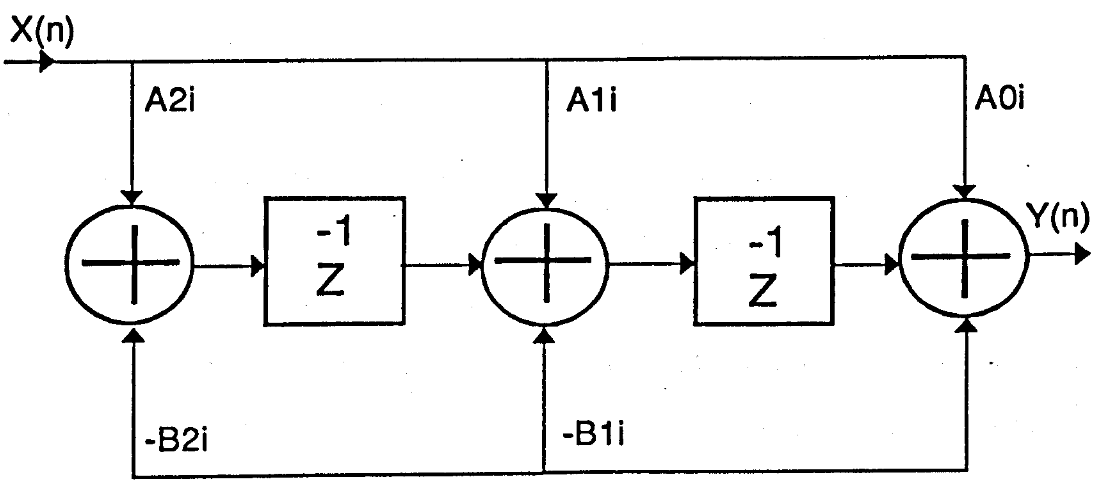
7.3 Implementation Of Filters

The DSP filters are implemented in IIR form, rather than FIR form, in order to reduce memory requirements and execution times. The Hypersignal software package (HSP) was used to determine the coefficients. This uses Steiglitz's method [Hypersignal users manual 1990] to give the coefficients Aoi, A1i, A2i, B1i, B2i for a series of biquad elements as shown in Fig[7.2]. The coefficients used in this work are given in Appendix 7.A.

The coefficients are initially stored in program memory, and then moved to data memory. In this case the algorithm can be structured so that the 32-bit accumulator of the TMS32020 acts as a storage register and carries the output of one of the second-order subsections to the input of the next second-order subsection. The implementation of the cascade of any order IIR filter can be summarized as follows

1. Load the new input value X(n)
2. Operate on the first section
3. Leave the output of the first section in the accumulator.
4. Operate on the second section in the same way as the first section, remembering that the accumulator already contains the output of the previous section.
5. Continue the same procedure for nth section. The output of the last section is the filter output Y(n).

This procedure can be applied to the IIR filter implementation of higher orders.



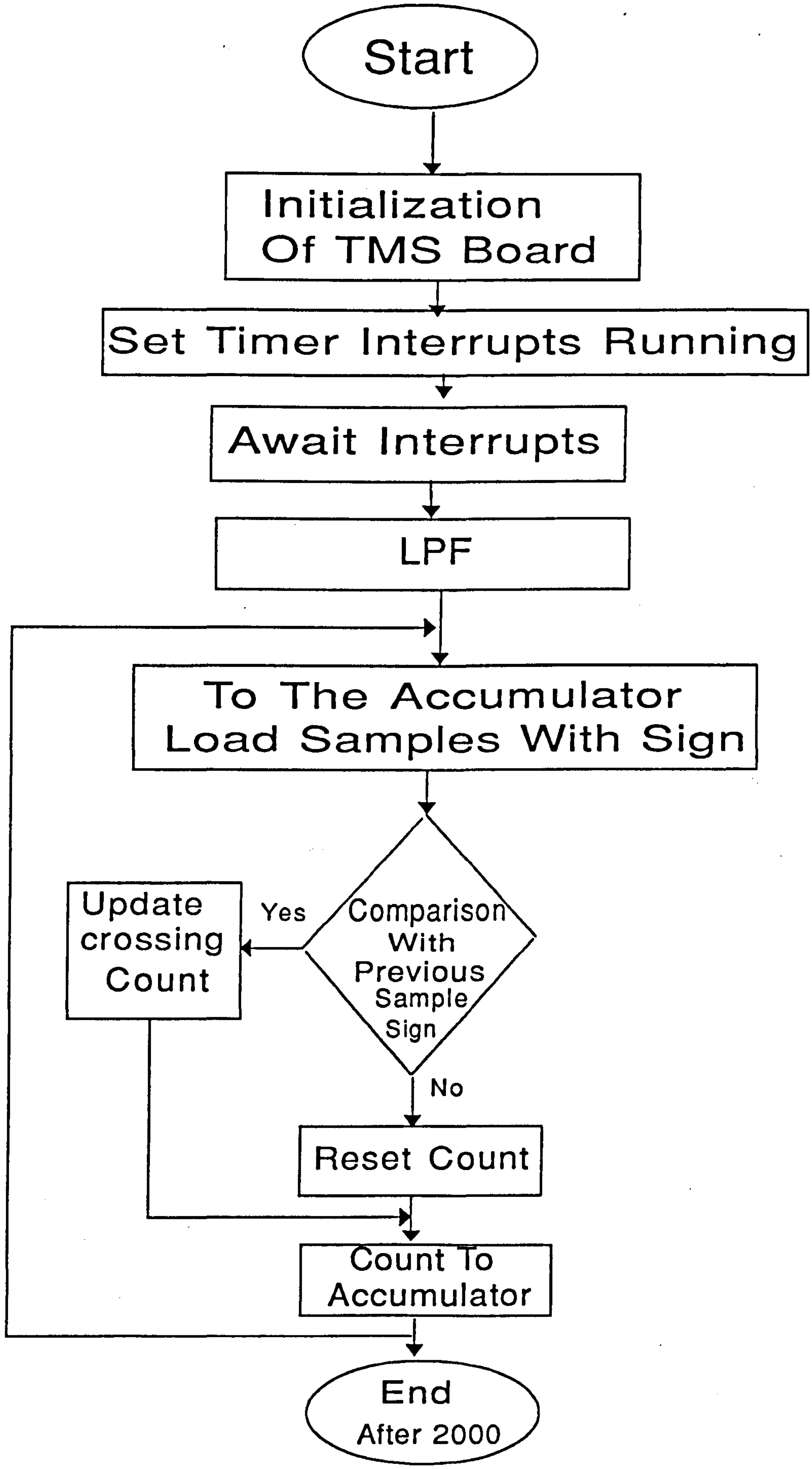
Fig[7.2] Structure of a biquad elements for use in implementing IIR filter.

7.4 Implementation Of The Zero-Crossing Detector And Data Analysis

Using the TMS board, the first upward zero-crossing is detected by a change in sample value from negative to positive. This event starts a counter which counts the clock pulses until a downward crossing is encountered, detected by a change in sample values from positive to negative. The count is then output from the TMS board to the PC, the counter reset, and the detection process repeated. A flowchart of this process is shown in Fig[7.3].

As a count value is grabbed by the PC from the TMS accumulator, the entry of the corresponding element of an accumulator array is incremented. The accumulator array has 60 elements corresponding to the clock count 0 to 59, so that finally the accumulator array contains the number frequency distribution of the clock counts between successive crossings. From the clock frequency and count, the time between crossings can be found. In This work 2000 double crossings were timed (run times of the order 2 minutes) in each case. This should give a good estimation for probabilities.

For the probability of detection of a second crossing within a time window (comparable with section 5.6), the first crossing starts the clock, and the counter counts for the number of pulses corresponding to the required delay. The second crossing detection routine



Fig[7.3] Zero Crossing detector flow diagram.

is reactivated for the number of clock pulses required for the time window, and if a crossing is detected an accumulator (not an array in this case) is incremented. The counter is then reset and the process repeated. In this work 500 runs were taken for each time window, again this should give a good estimate of probabilities.

The above method for probability of a crossing for a given delay measures, in a strict sense, the probability of a crossing within a clock period, conditional upon both the occurrence of the first crossing, and upon there being no second crossing prior to the detected second crossing. The theory and calculations of Chapter Four, Five, and Six are strictly for a joint crossing, which is unconditional in both these respects. The condition upon the first crossing is unlikely to have an effect with the signals and noise used in this work, as a first crossing will occur in any cycle, and more than one single crossing per cycle is unlikely. However, if the noise bandwidth were wide compared to the signal frequency, then a correction for the single crossing probability would be needed.

The effect of previous second crossings on the conditional probability is to reduce this with respect to the unconditional joint probability at longer time delays. This effect will not be noticeable for delays within the first period in low noise cases, but will be increasingly important as noise is increased. The probability of a double crossing irrespective of intervening crossings can be estimated using the window method with a very narrow window. However, this will be at the expense of greatly increased run times, and has not been investigated in this work. However, the simulator and demonstrator system can be used for such investigation.

7.5 Input Signal Demonstrator

The DSP filter and zero-crossing detector system can in principle be adapted to any baseband signal, and so can be used for investigation and simulation for design purposes. To demonstrate its use, and to compare with theory, an input signal was simulated. The input signal in this work consisted of a deterministic sinusoid with added Gaussian noise.

The sinusoidal signal was obtained from a common type of laboratory signal generator. The signal from this was fed into a noise generator where Gaussian noise was added to it.

The Gaussian noise generator is used routinely by the communications research group at Hull University. It is based on a 32 bit shift register, which produces a maximal length pseudo random sequence. For each shift, the levels of the first 31 bits are weighted by resistor and summed by an operational amplifier. The resulting noise signal passes through circuitry giving an adjustable level shift, low pass filtering, noise width (amplitude); and adds an external signal (in the current work: the sinusoid). The values of the weighting resistors are chosen to produce noise from a Gaussian distribution [George, Brian and Gordon 1967] and [Kramer 1965].

Investigations by others indicate that the distribution from this type of noise generator falls away noticeably from a true Gaussian distribution beyond about 1.5 standard deviations, and has fewer large deviates. This will affect the width of the crossing probability distributions for low noise.

7.6 Simulation Results

The results of the simulation are shown in Figs[7.4a to 7.8b], whose labelling corresponds as closely as possible to the results of the theoretical numerical calculations of Figs[5.4a to 5.8b]. The probability density estimates of Figs[7.4 to 6], in all cases, have an absolute normalisation obtained from the class width.

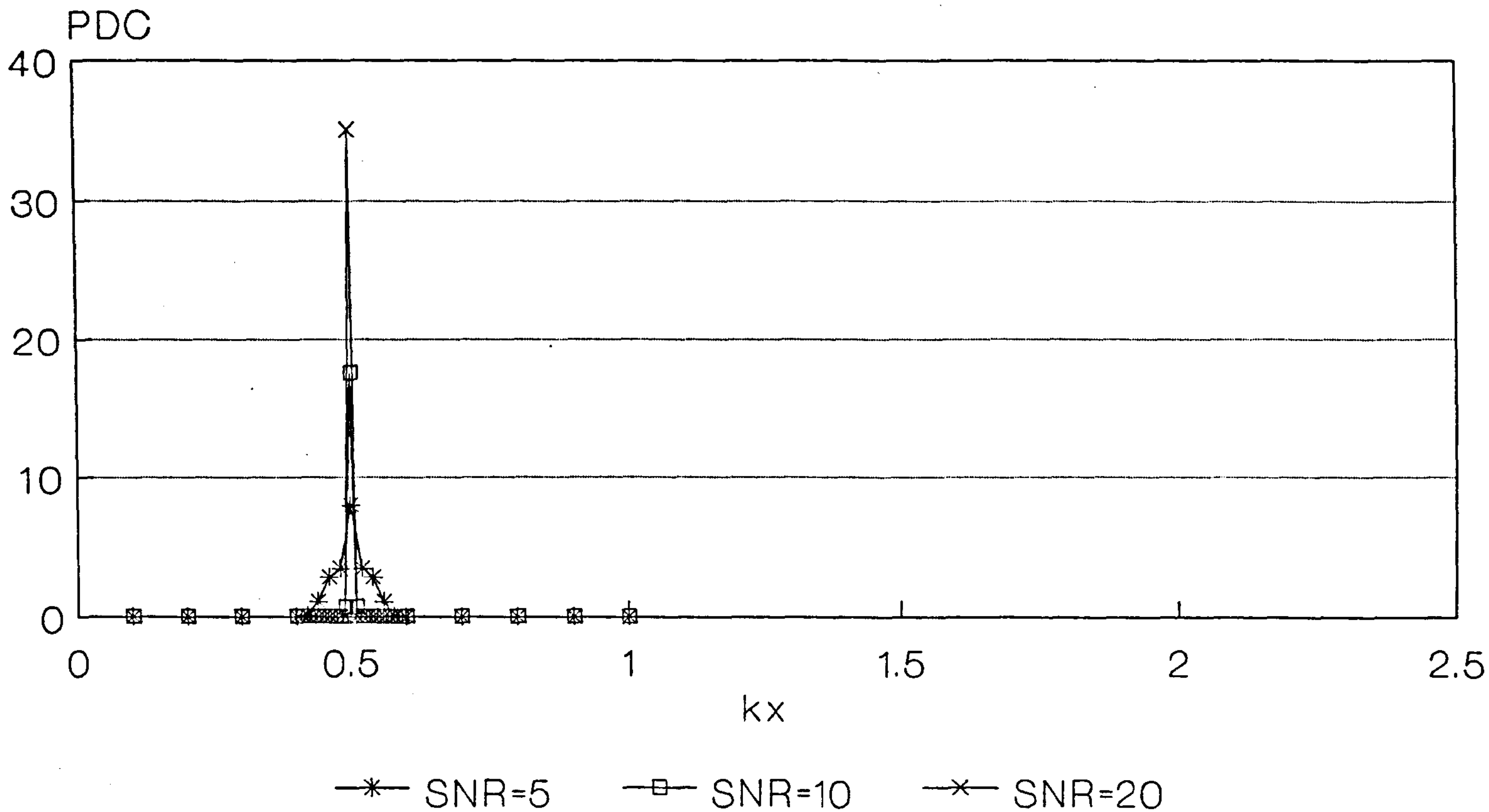
The high SNR cases (as discussed in section 6.4) are shown in Figs[7.4a to 7.4h]. Figs[7.4a and b] are for the second order Butterworth filter, corresponding to Figs[5.4a and b]; Figs[7.4c, d and g, h] are for 8th order Butterworth and Chebyshev filters respectively, and can be compared with the Brickwall results of Figs[5.4c, d]. Second order Chebyshev filter simulations are shown in Figs[7.4e, f]. In all cases the maxima of the probabilities have values corresponding very closely to the theoretical calculations. The widths are slightly narrower than the theoretical calculations, but this may be attributable to the fall off of the noise simulator from true normal (as discussed in section 7.5).

The results for the low SNR compare with the theoretically based calculations down to a SNR 2 (Figures 7.5a, b with Figures 5.5a, c; Figures 7.5c, d, g, h with Figures 5.5e, g) in a similar manner to the high SNR cases.

For SNRs of 1 and less (Figures 7.5 as above, and Figures 7.6a and b compared with 5.7a and b) the effects of the condition of no intervening crossings (as discussed in section 7.4) are evident, especially in the sharp cut-off of probability in Fig[7.6a and b].

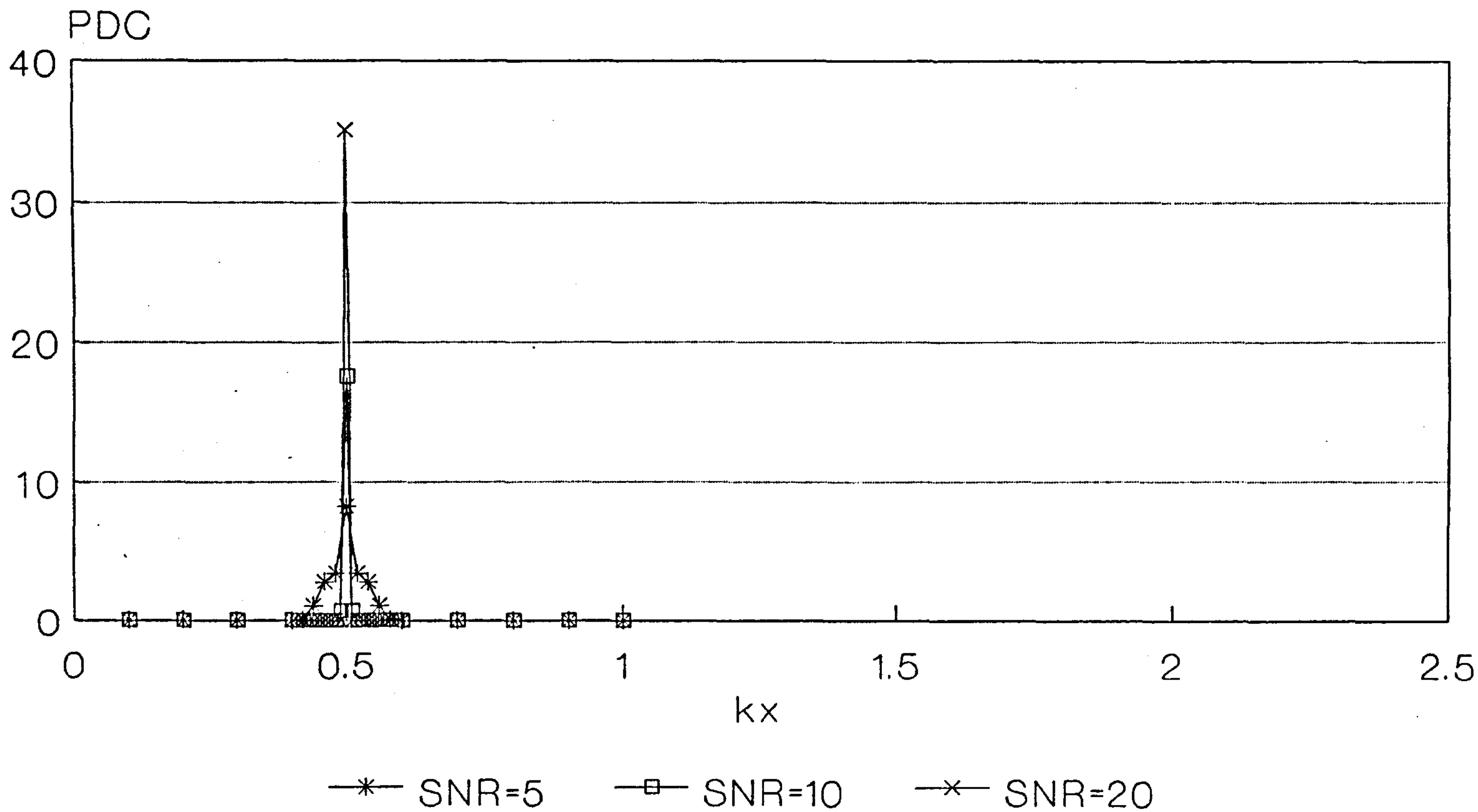
Simulation results for the probabilities of crossing within the window are shown in Fig[7.7a and b]; and comparison with the theoretically based calculations are shown in Figs[7.8a, b, c, d]. These correspond well. Moving from the second order in Figs[7.8a and b] to the 8th order and Brickwall filters in Fig[7.8c and d], it can be seen that the higher order filters tend towards the Brickwall case as the upper limit of probability, as is expected.

These simulation results indicate that theoretical studies followed by investigation of real signals using a simulated signal processing system will be a useful, if not essential, procedure as part of the design process for time domain analysis of signals with noise. This will be the case especially with low signal to noise ratios, where correlation effects can be important. For low noise, both theory and simulator action should evaluate the true conditional probability of the system simulated (the current simulator of this work needs slight modification to evaluate the probability of a double crossing irrespective of intervening crossings).



Rs=WS/WB= 1

Fig[7.4a] PDC for 2nd order Butterworth



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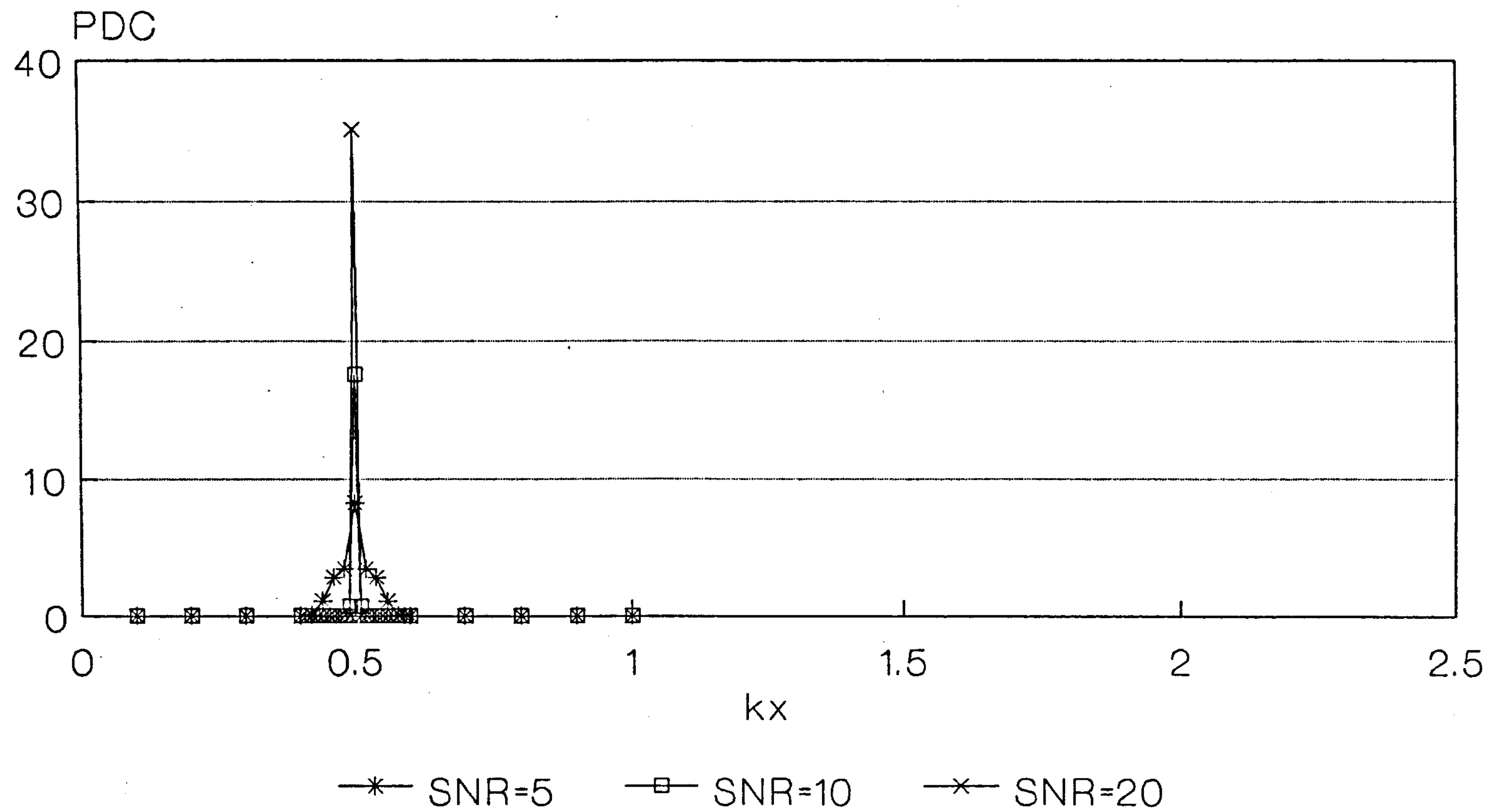
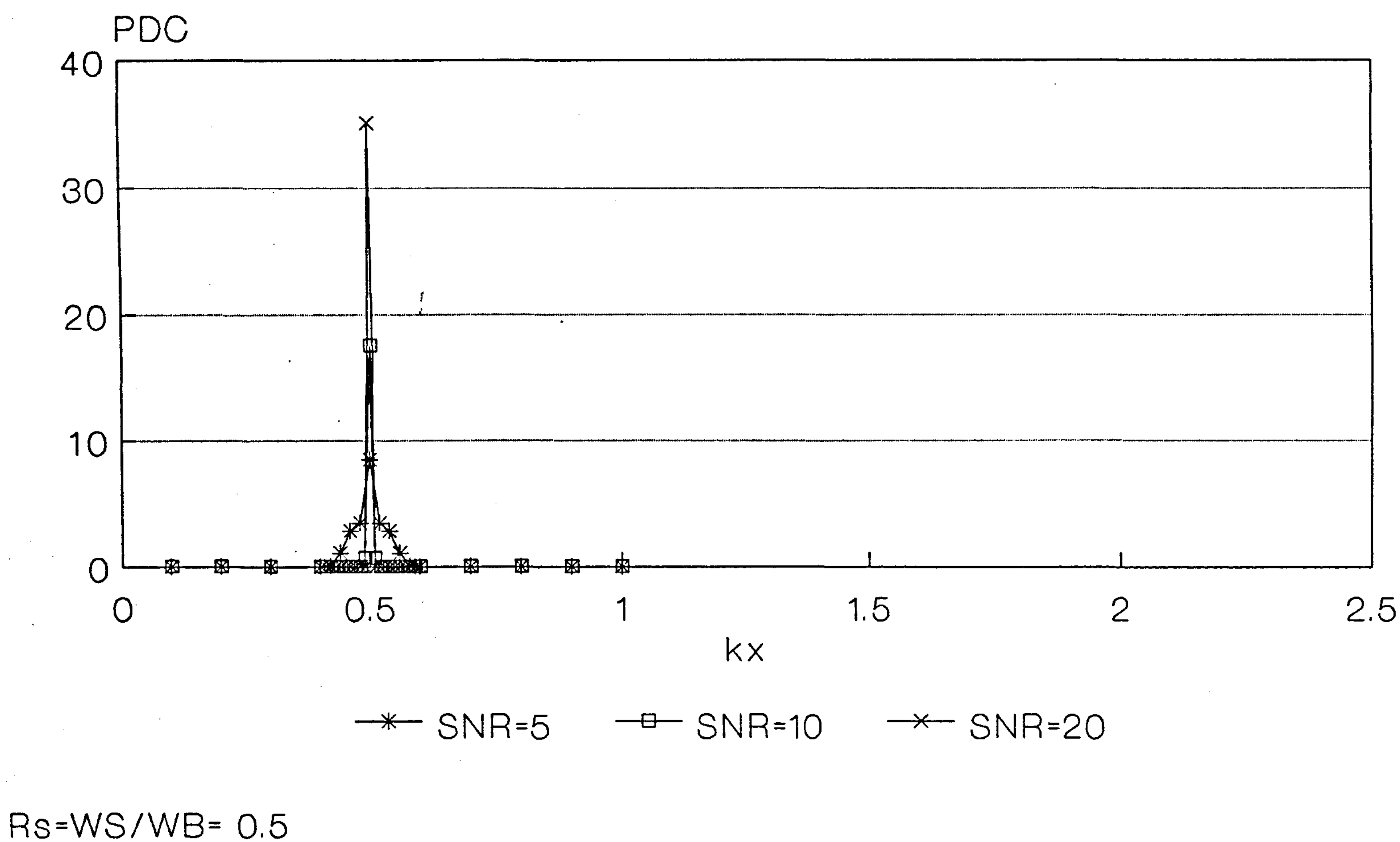
Fig[7.4b] PDC for 2nd order Butterworth

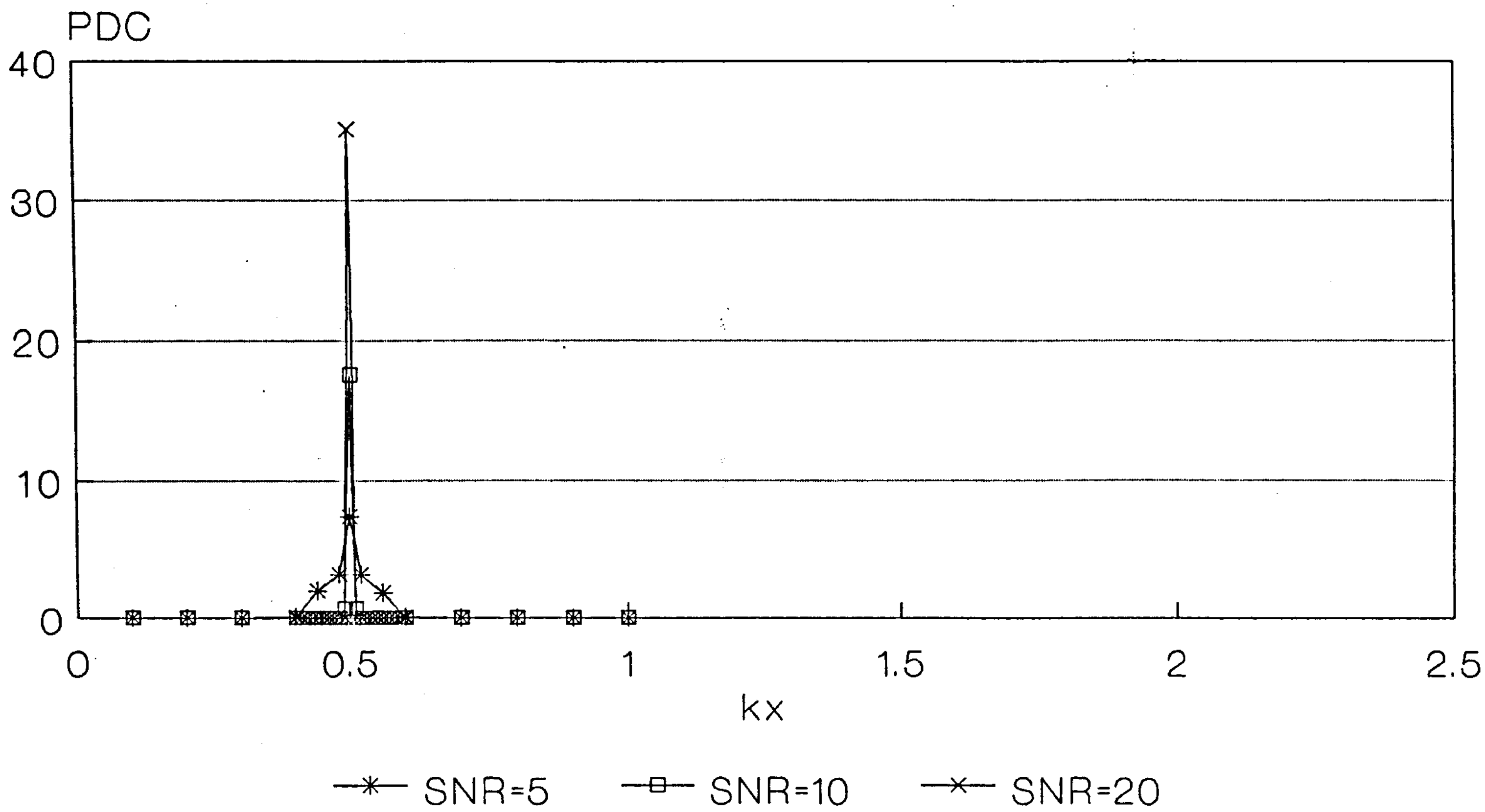


Rs=WS/WB= 1

Fig[7.4c] PDC for 8th order Butterworth

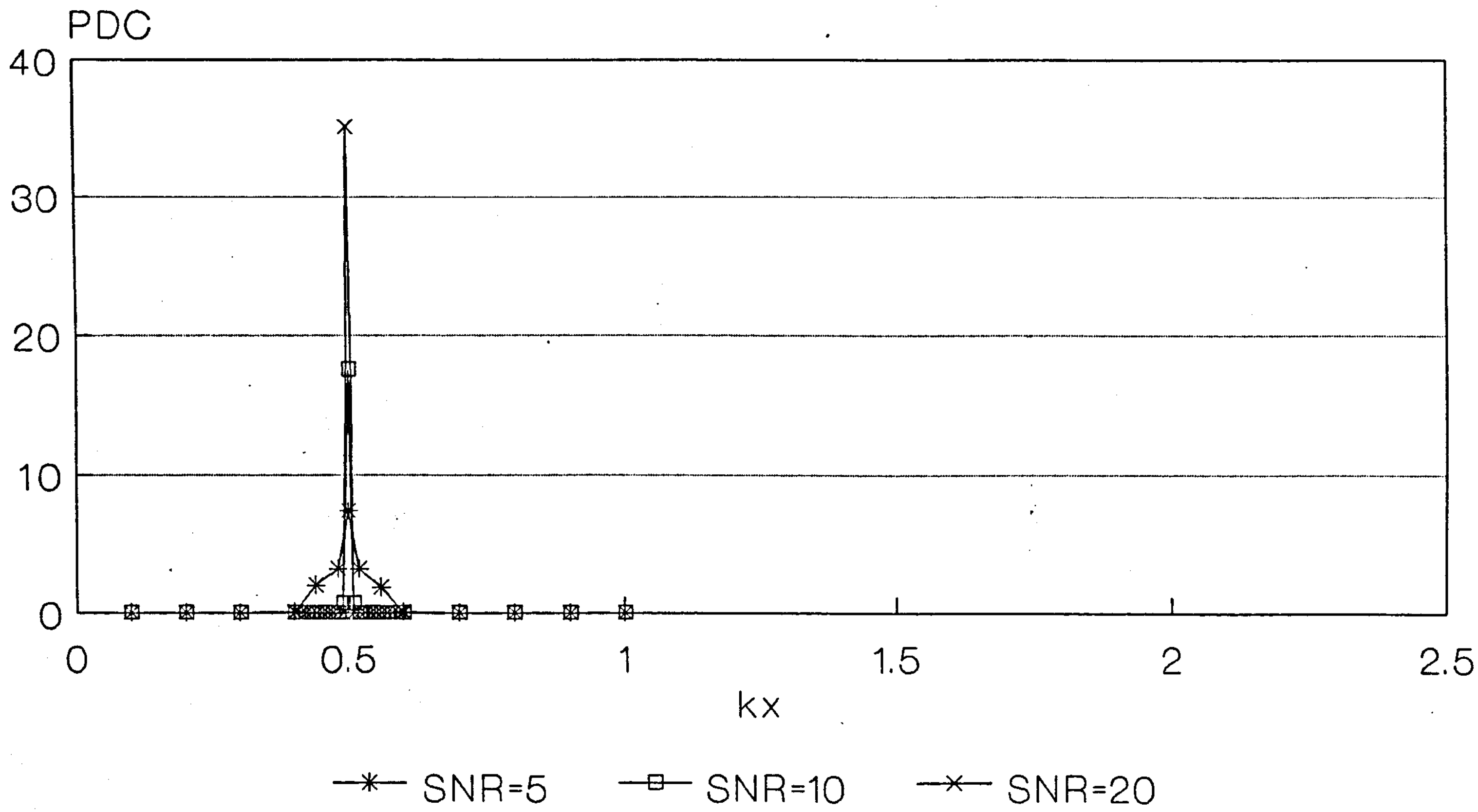


Fig[7.4d] PDC for 8th order Butterworth



Rs=WS/WB= 1

Fig[7.4e] PDC for 2nd order Chebyshev



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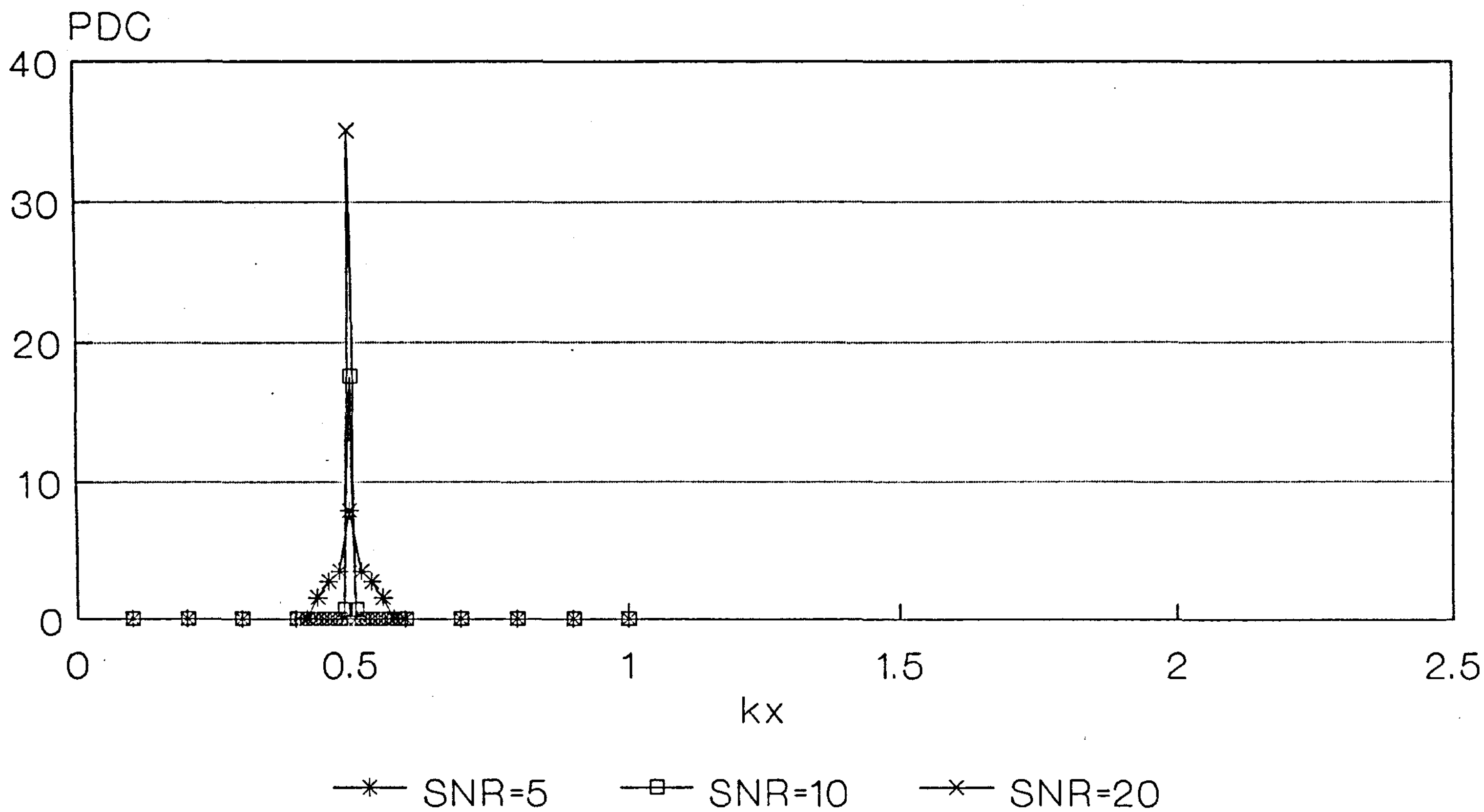
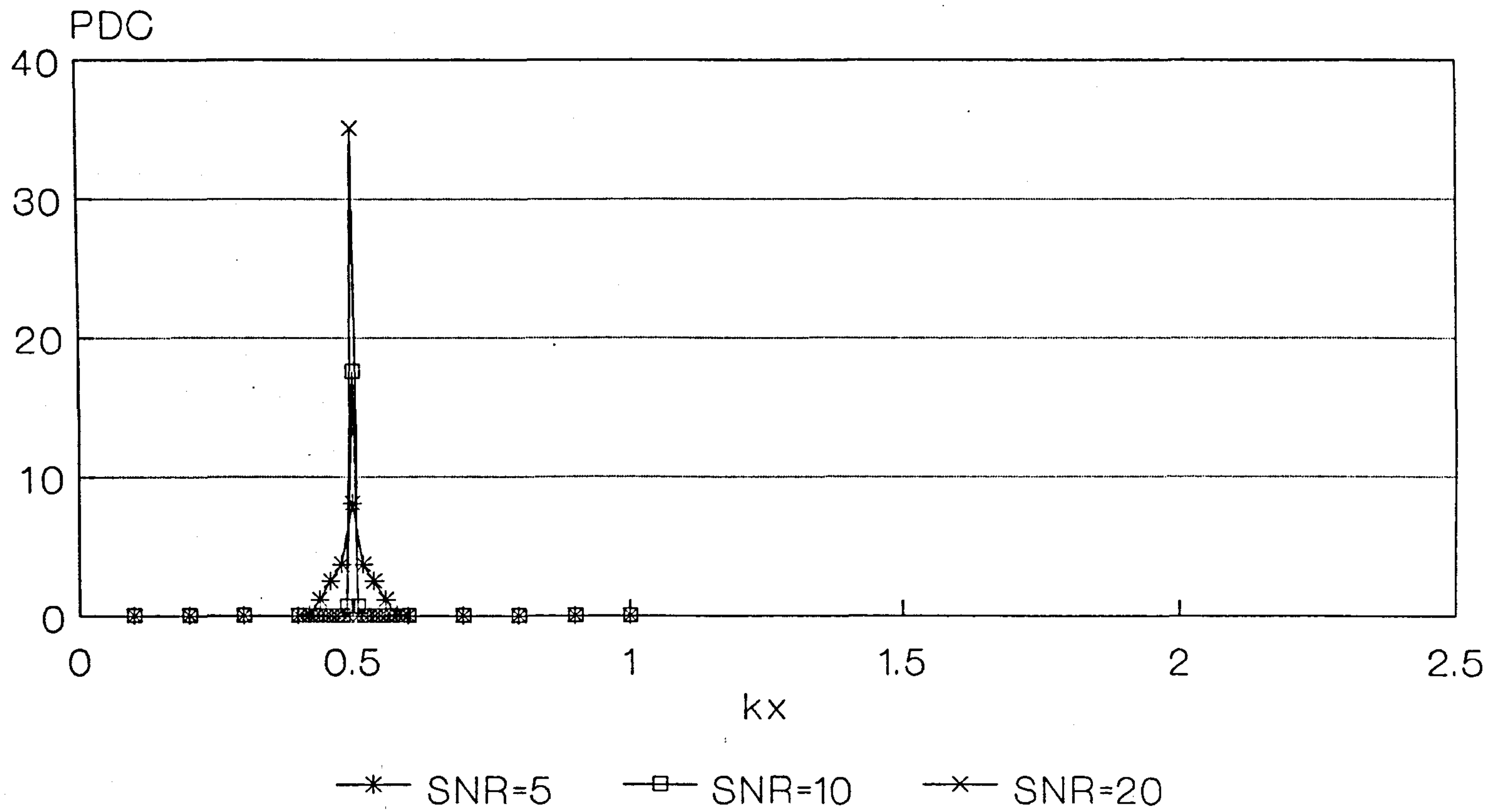
Fig[7.4f] PDC for 2nd order Chebyshev



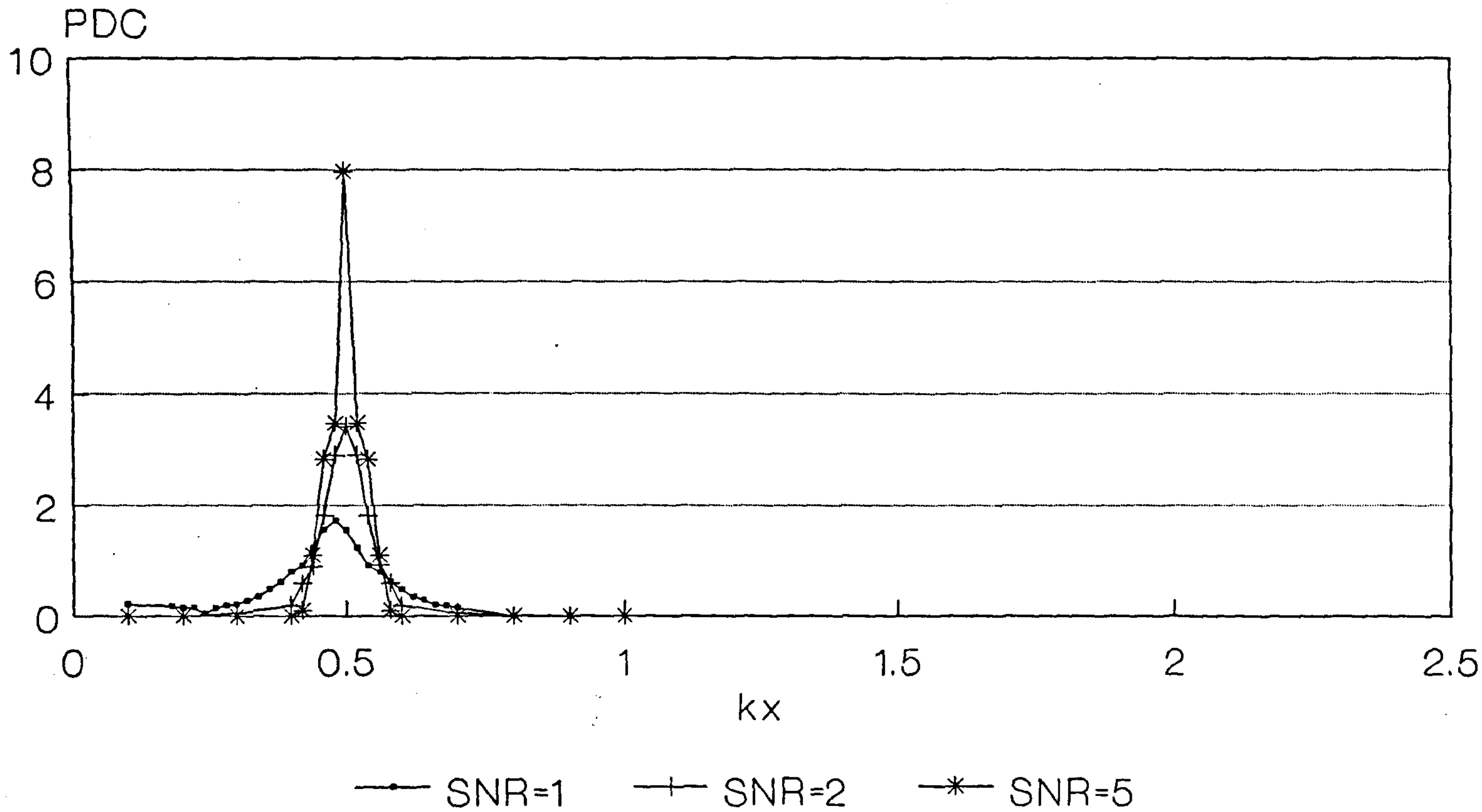
Rs=WS/WB= 1

Fig[7.4g] PDC for 8th order Chebyshev



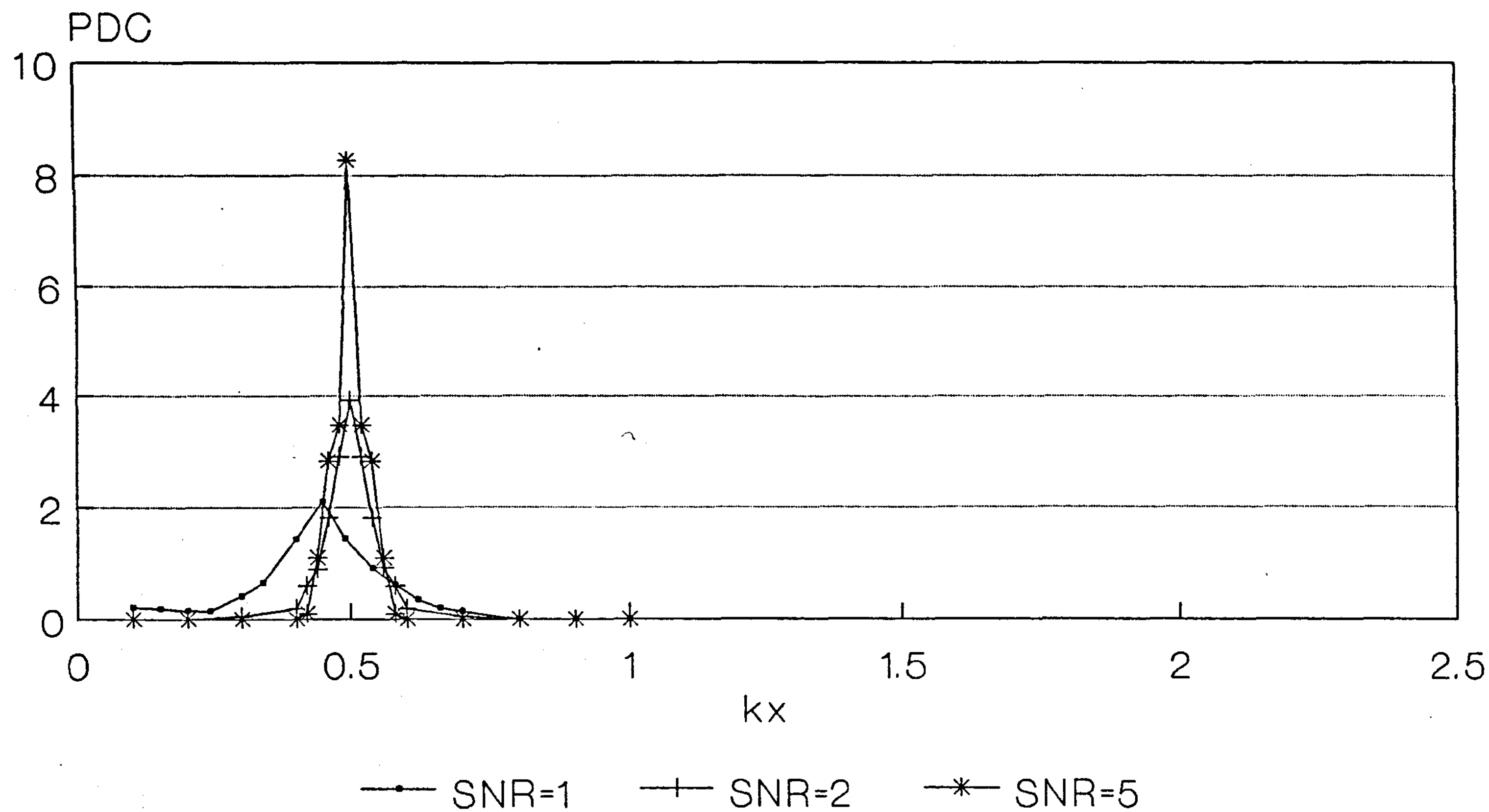
Rs=WS/WB= 0.5

Fig[7.4h] PDC for 8th order Chebyshev



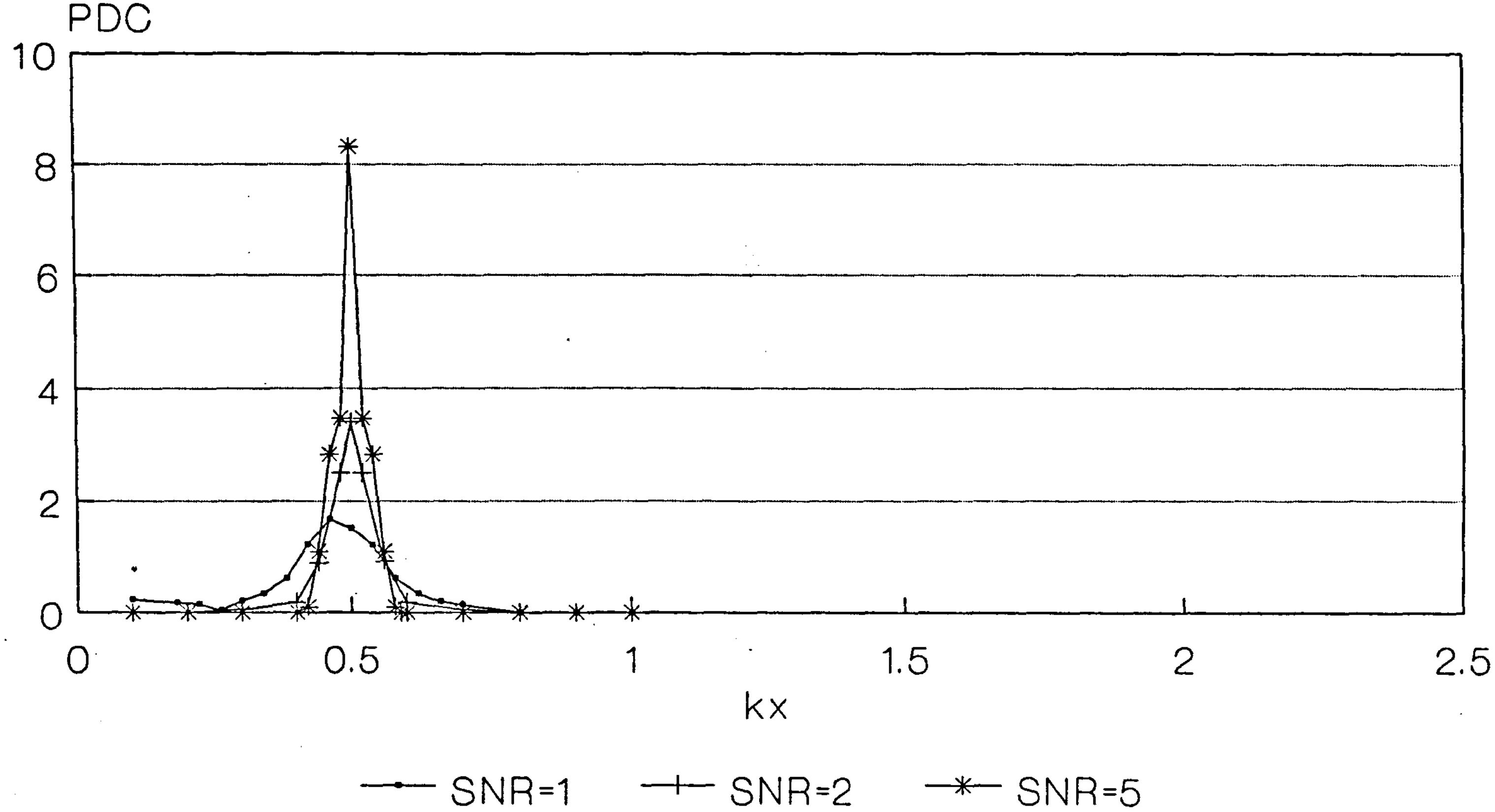
Rs=WS/WB= 1

Fig[7.5a] PDC for 2nd order Butterworth



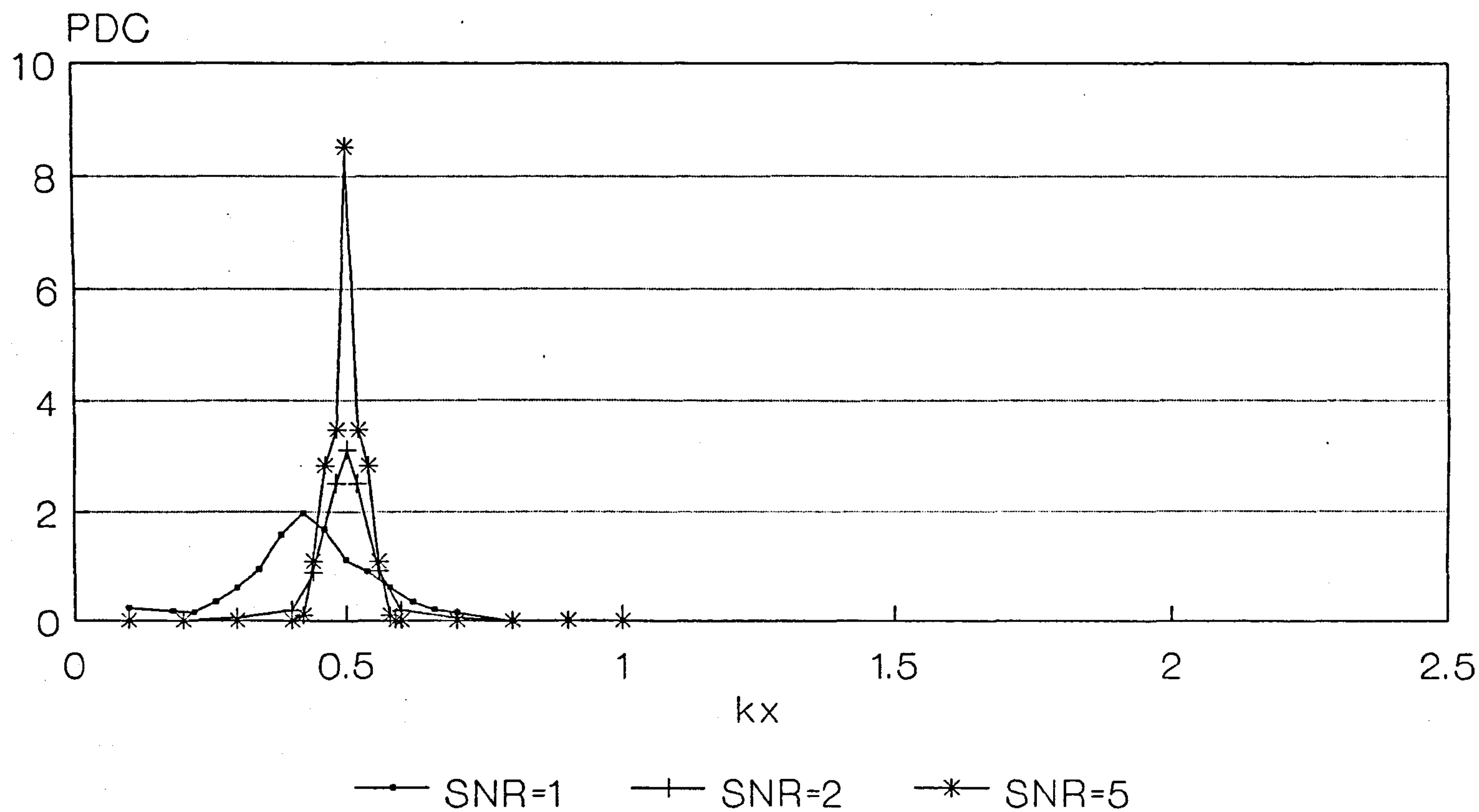
Rs=WS/WB= 0.5

Fig[7.5b] PDC for 2nd order Butterworth



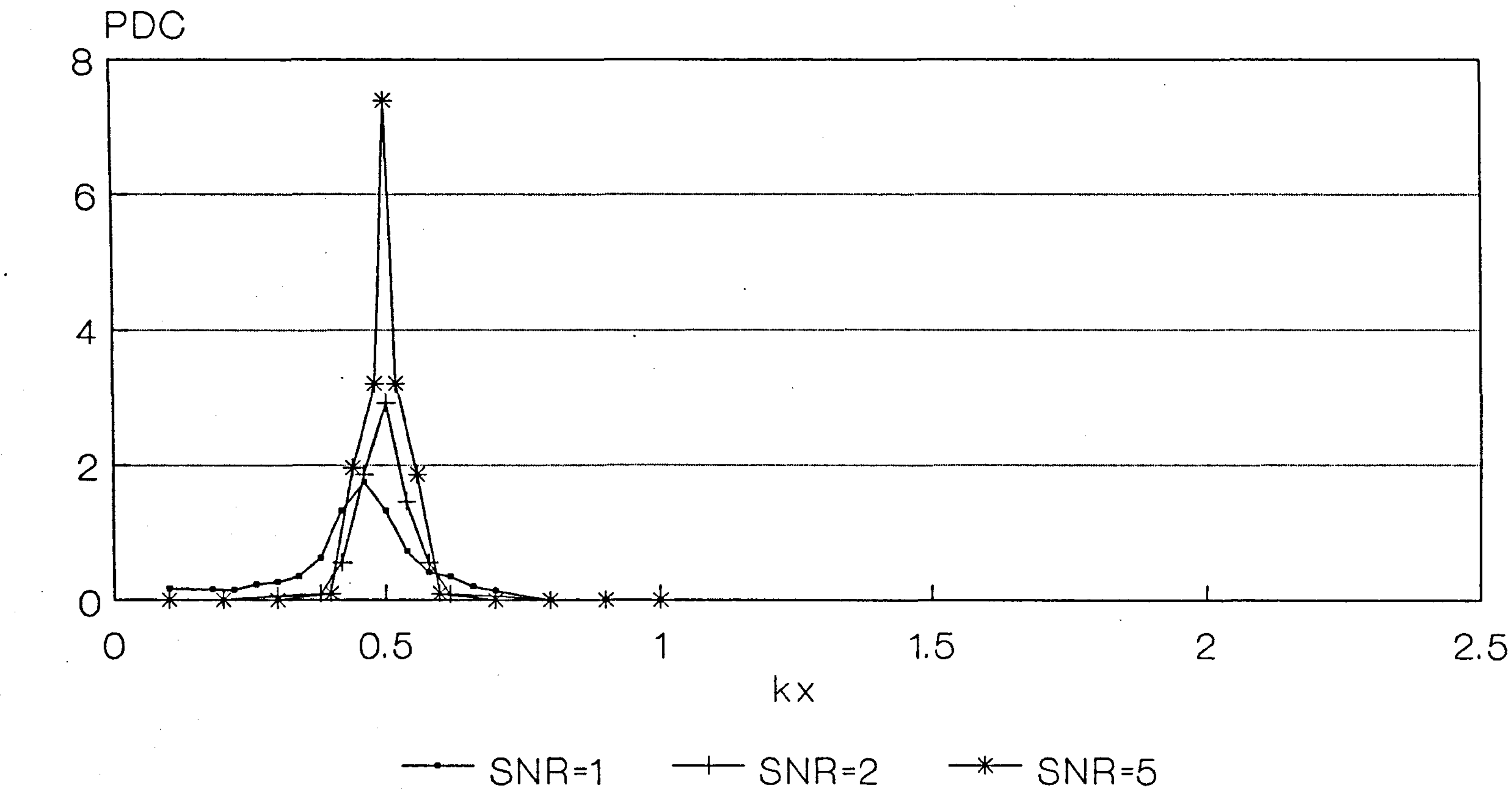
Rs=WS/WB= 1

Fig[7.5c] PDC for 8th order Butterworth



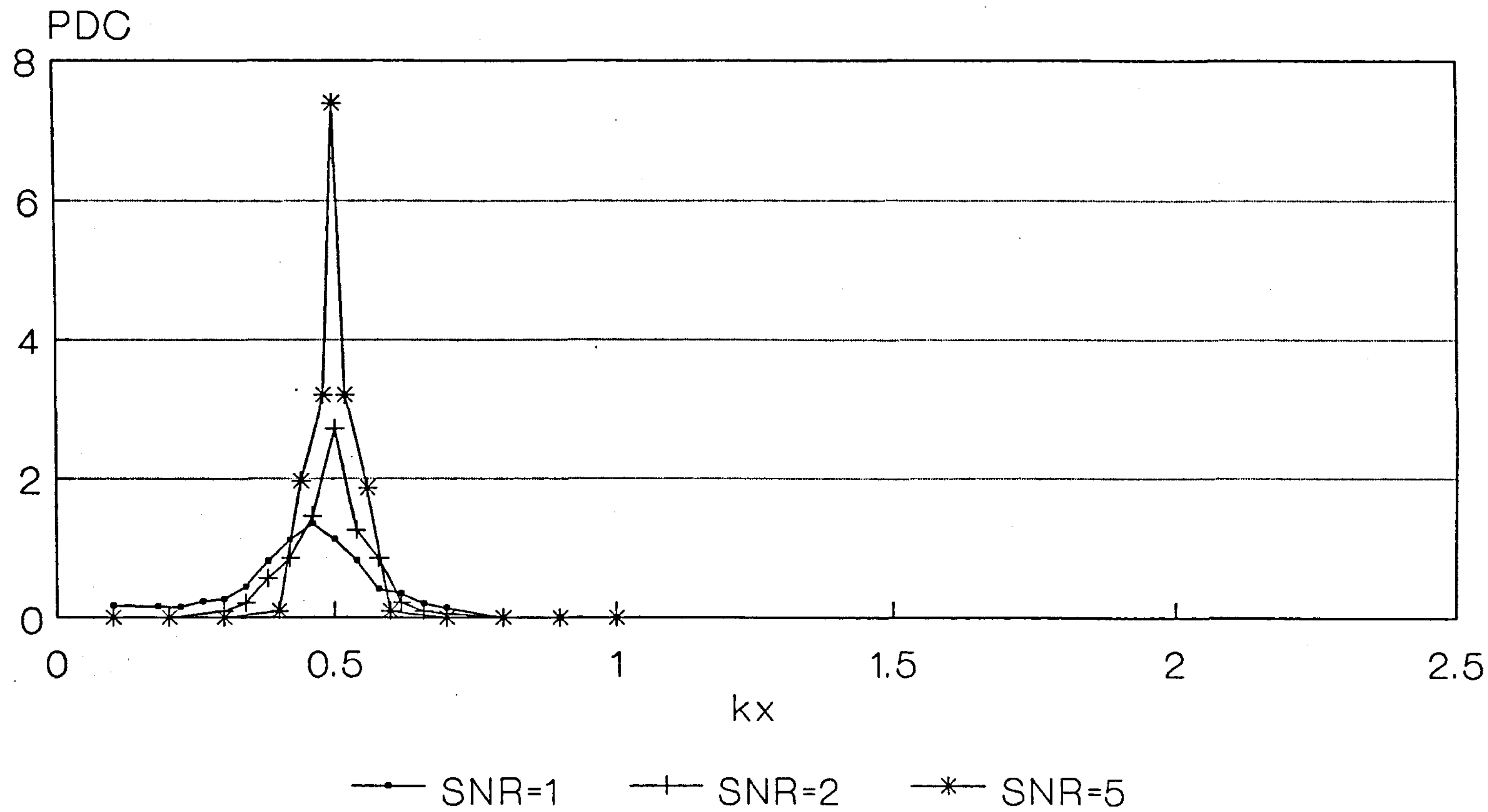
Rs=WS/WB= 0.5

Fig[7.5d] PDC for 8th order Butterworth



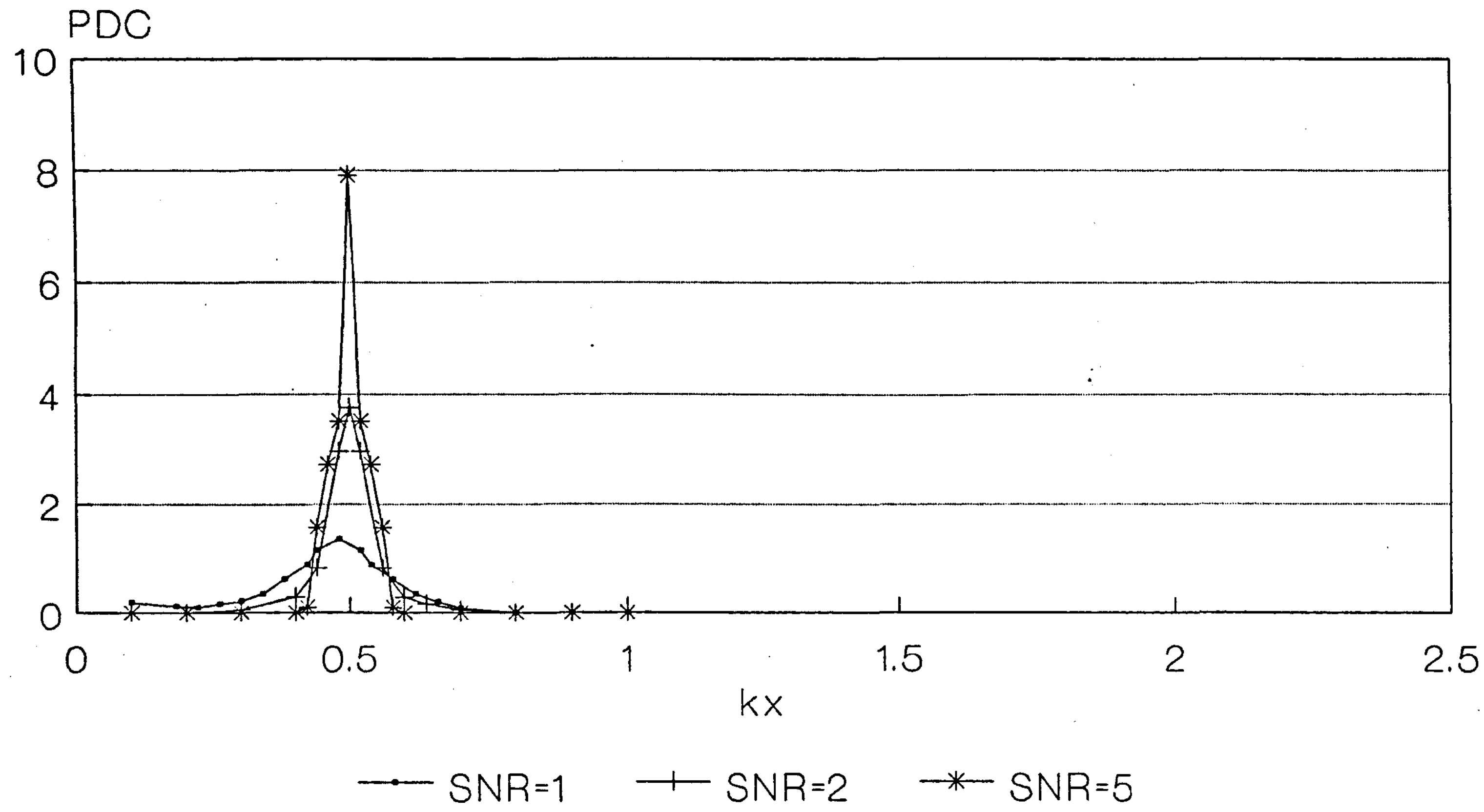
Rs=WS/WB= 1

Fig[7.5e] PDC for 2nd order Chebyshev



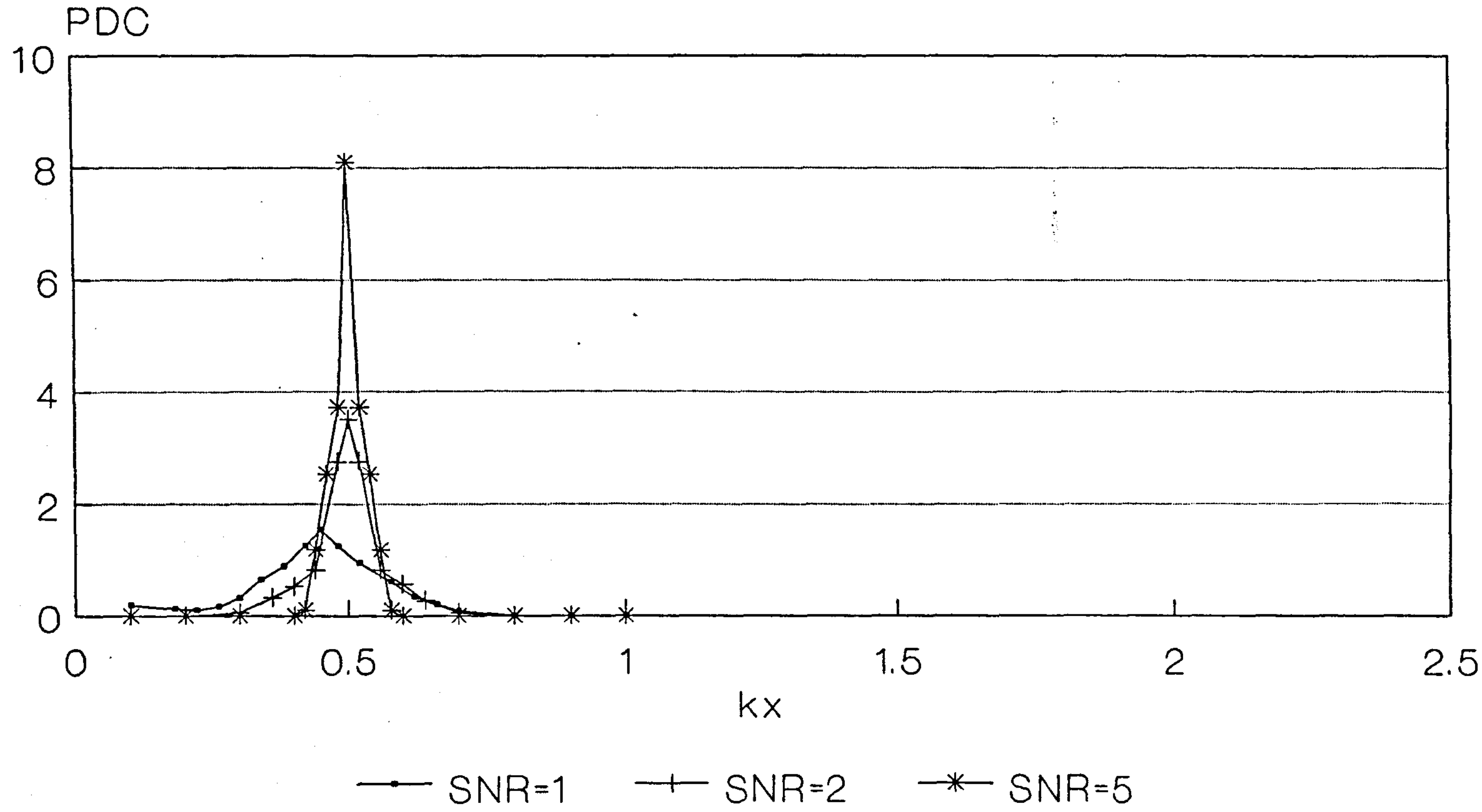
Rs=WS/WB= 0.5

Fig[7.5f] PDC for 2nd order Chebyshev



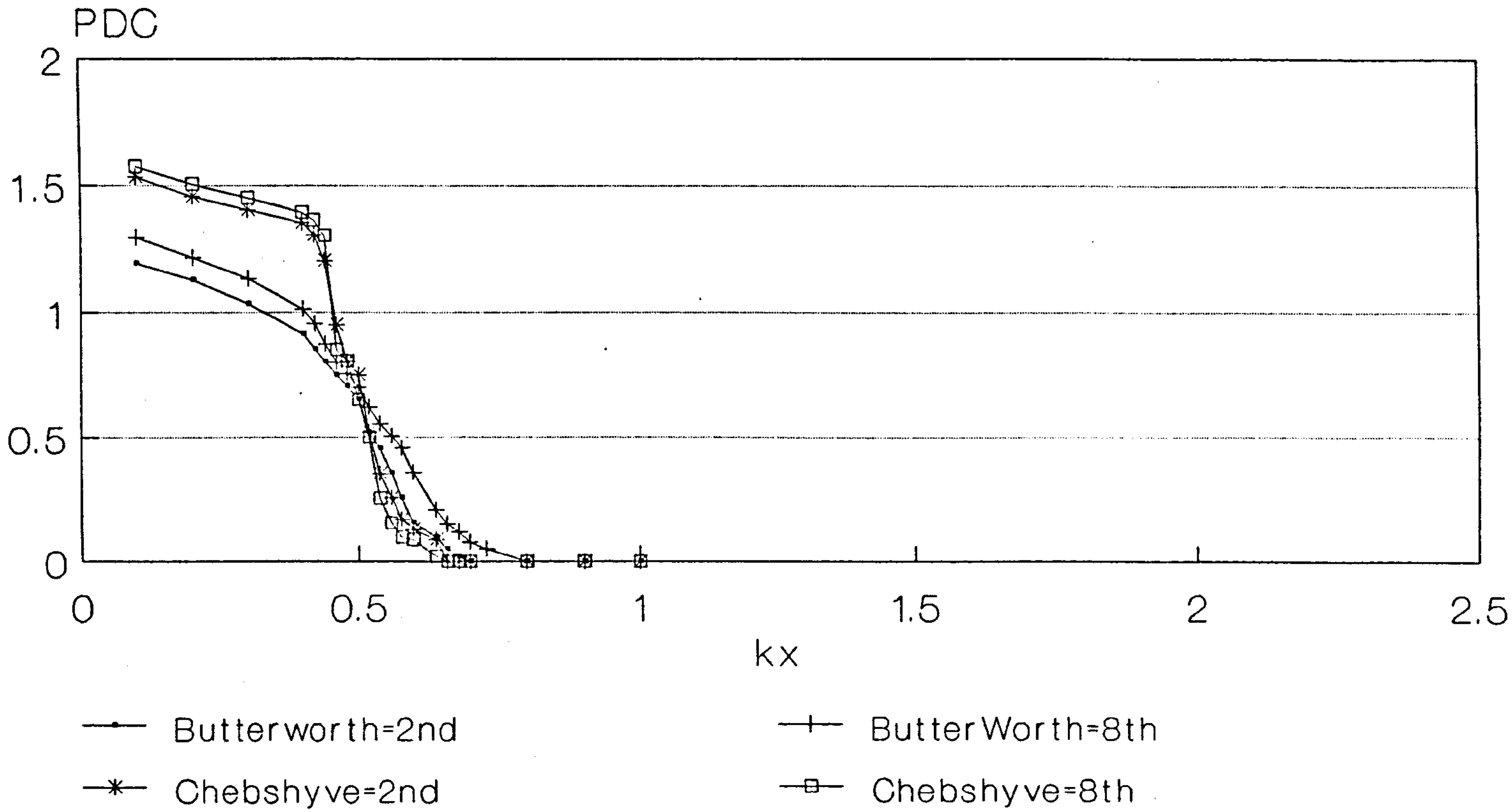
Rs=WS/WB= 1

Fig[7.5g] PDC for 8th order Chebyshev



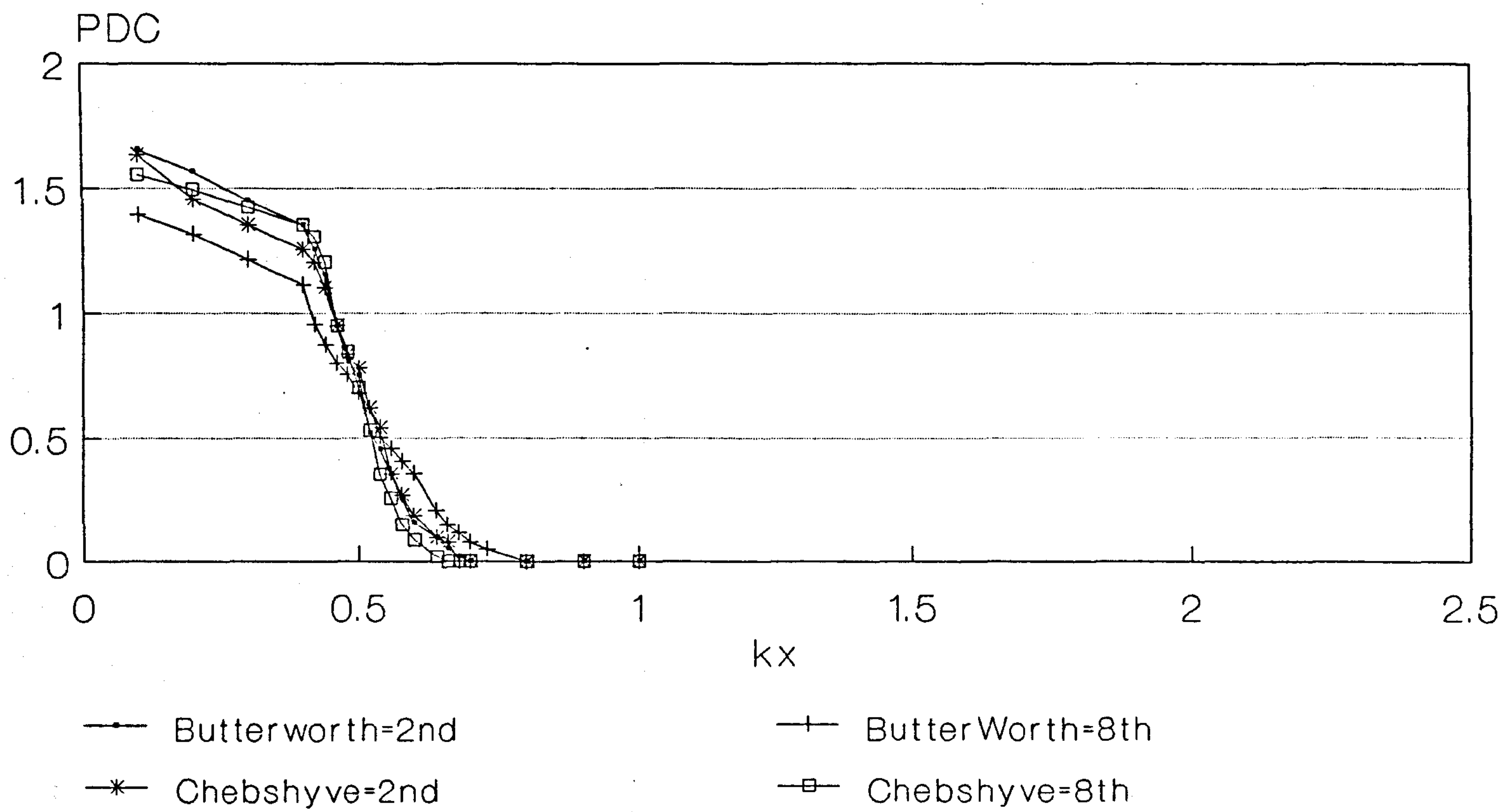
Rs=WS/WB= 0.5

Fig[7.5h] PDC for 8th order Chebyshev



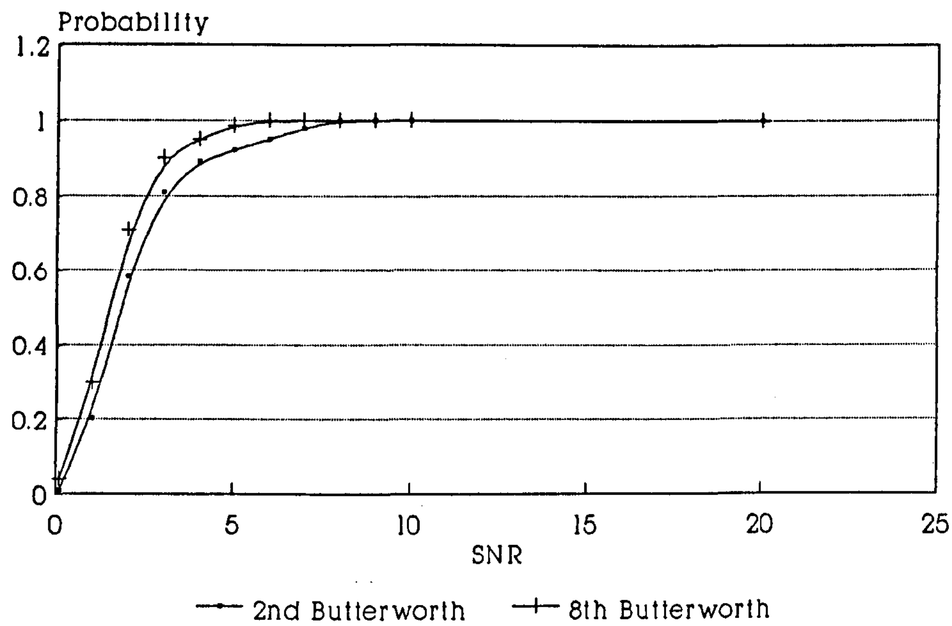
SNR=0.1, Rs=Ws/WB=1

Fig[7.6a] PDC for 2nd & 8th order Butterworth and Chebshev filters.

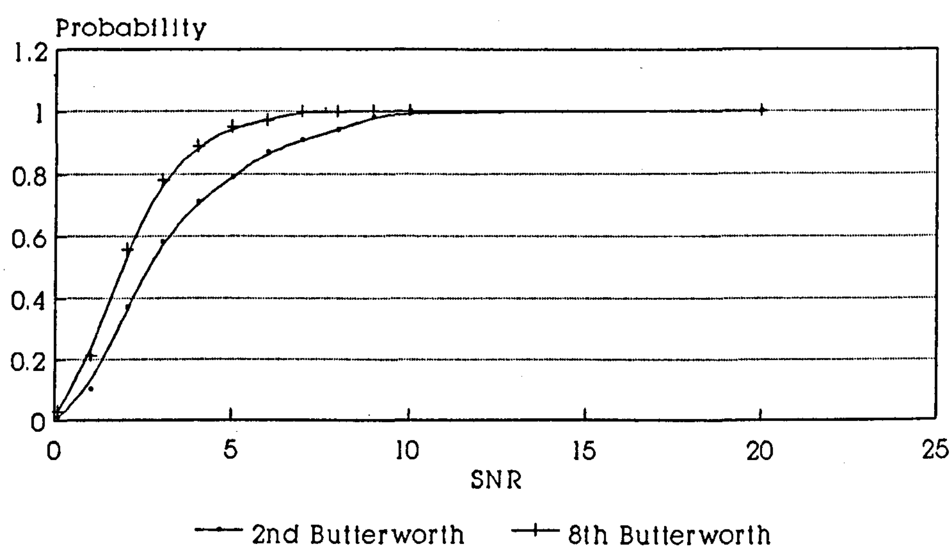


SNR=0.1, Rs=Ws/WB=0.5

Fig[7.6b] PDC for 2nd & 8th order Butterworth and Chebshev filters.

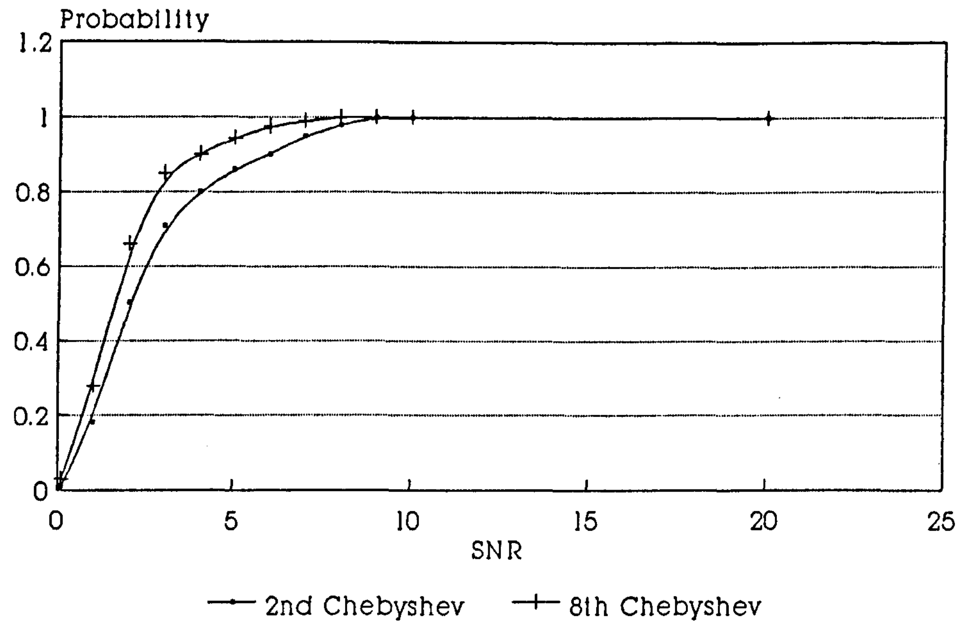


Rs=WS/WB=1

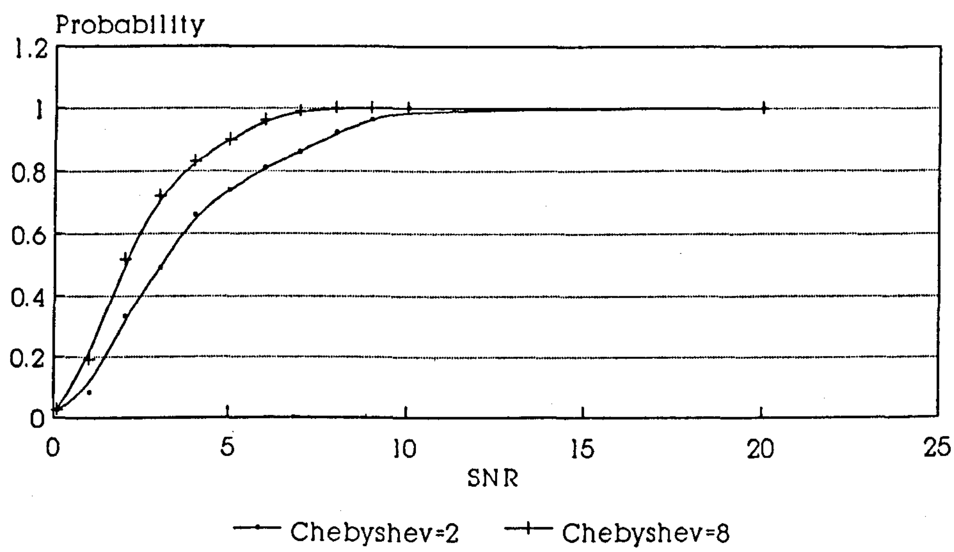


Rs=WS/WB=0.5

Fig(7.7a) Probability of crossing for Butterworth filter.

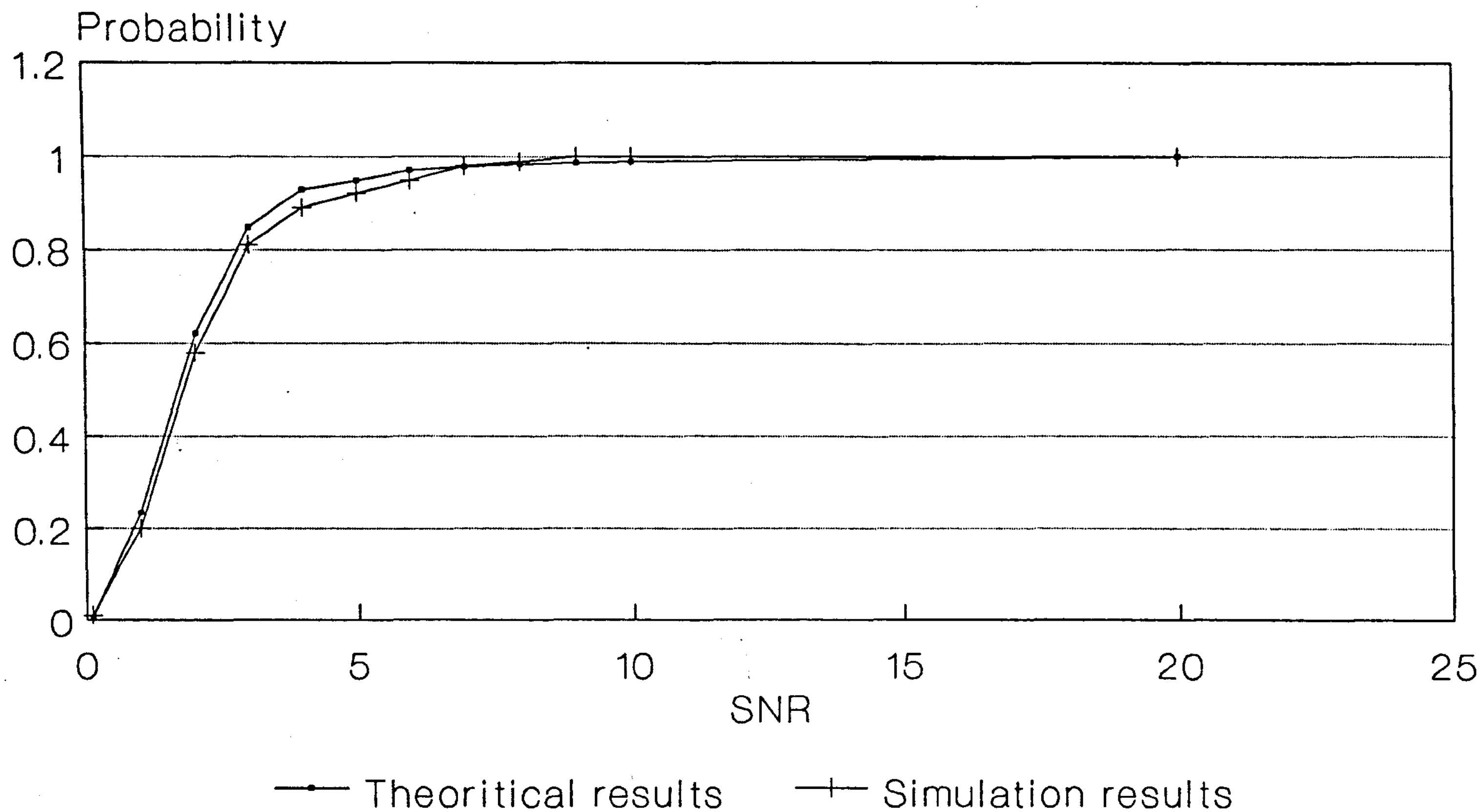


Rs=WS/WB=1



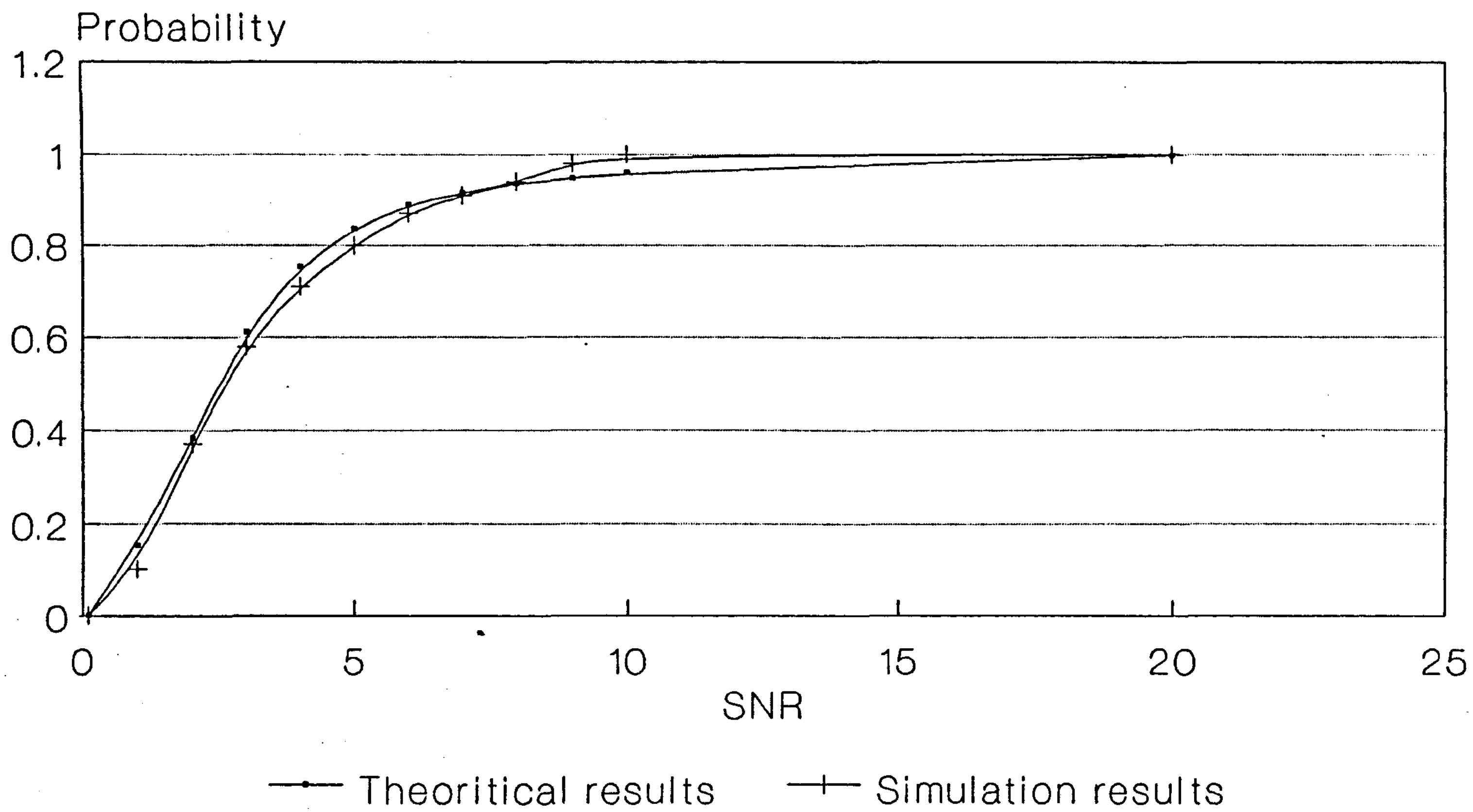
Rs=WS/WB=0.5

Fig(7.7b) Probability of crossing for Chebyshev filter.



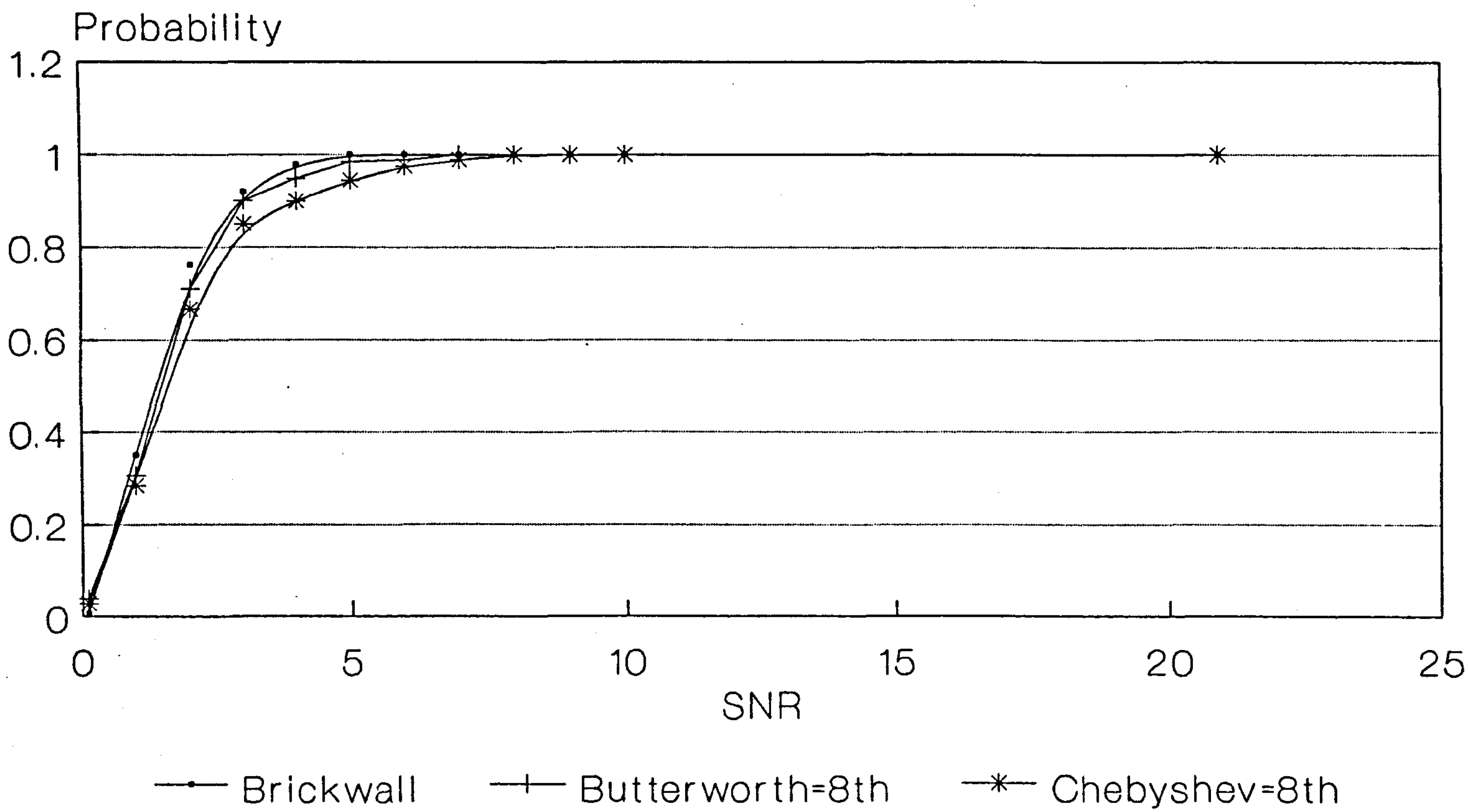
RS=WS/WB=1

Fig[7.8a] Comparison results for 2nd order Butterworth filter.



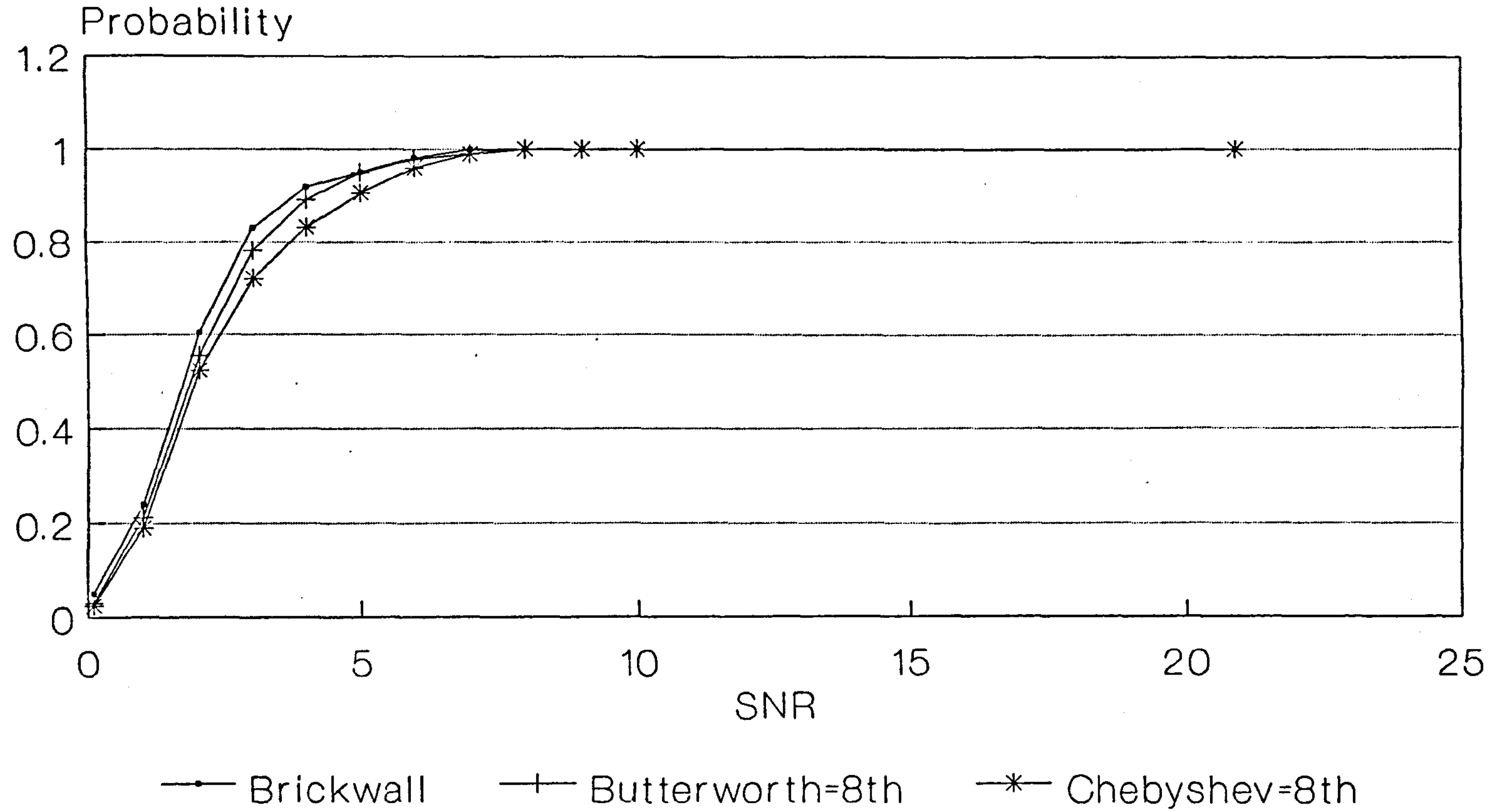
RS=WS/WB=O.5

Fig[7.8b] Comparison results for 2nd order Butterworth filter.



Rs=WS/WB= 1

Fig[7.8c] Comparison results between theoritical & simulation for Brickwall.



Rs=WS/WB= 0.5

Fig[7.8d] Comparison results between theoritical & simulation for Brickwall.

Chapter 8

Conclusions And Further Work

8.1 Conclusions

This work has presented theoretical methods for studying the probability of detecting threshold crossings of a signal. Single crossings, and pairs of crossings separated by defined intervals, have been discussed in detail. The threshold crossing problem is a classic problem of signal processing. However, this study differs from previous work in that the signal is taken as a deterministic signal with stochastic noise, for which the aim is to determine parameters or characteristics of the deterministic signal. Older work treated the signal as stochastic and determined stochastic or statistical properties. The treatment in this work is suitable for use with modern methods of signal processing including DSP. A DSP simulator and demonstrator for double crossings has been included in this work.

The methods are based on study of the distribution with time of the probability of detecting a crossing, and is possible because the total signal can be expressed in terms of the definite characteristics of the deterministic signal and the statistical characteristics of the noise. This enables the probability of detecting a crossing to be related to the expected crossings of the deterministic signal.

The treatment for the probability of a single threshold crossing makes no assumption about the form of the deterministic signal. Sampled signal processing is assumed, and the effect of correlations, which will be present in the noise for fast sampling, is taken into account. The infinitesimal interval limit is taken in order to cover standard continuous trigger detection, and results are identical to those of previous work, in cases treated by the latter.

The work is extended to the joint probability of detection of two crossings, where the second crossing may be separated by any time interval from the first. In practice it is not necessary to extend the interval beyond about two periods of the band limit of the signal, because correlation effects become negligible before this. A sinusoidal signal with added Gaussian noise is used. The method can be easily extended to other forms of deterministic signal, especially in numerical work, because the major complexities arise from the noise, and particularly its correlation effects. The band limiting effects which cause the correlations in the noise have been modeled by a second order Butterworth filter and Brickwall filter. These can be taken as extreme cases.

The analysis of the probability is based on a quadrivariate normal distribution, requiring four integrations to evaluate probability time densities, a time integration for probability of detection of a second crossing after a first crossing, and a further integration with respect to time for the probability of a second crossing within a time window. These are reduced, after much theoretical manipulation, to expressions involving integrals of the bivariate distribution, and only the time integrations. To reduce the problem of the integrals over the bivariate normal distribution, whilst maintaining sufficient accuracy, an adaptation of a polynomial approximation to these integrals is introduced in this work. This leaves only the time integrals, and makes numerical evaluation viable on a modern PC.

A pascal program is used for numerical evaluation, which is structured by the use of procedures. The probability of double crossing for any filter can be evaluated by incorporating a procedure for the normalised correlation function [see Papoulis 1989, chapter 10] for the required filter.

The DSP simulator was used to verify the theoretical analysis. The results of these simulations are comparable over a wide range of SNR, the filters used.

This work shows that the effects of noise correlation between the two crossings can be ignored for amplitude SNR of about 2 and above. For these SNRs the deterministic signal structure is distinguishable, and the probability density of double crossing can be approximated by the no correlation case. This is irrespective of the bandwidth of the noise. This is an important result, because it justifies the common practice of ignoring correlation effects in engineering applications. Thus, the joint probability of a double crossing can be expressed as a product of the probabilities of two single crossings, evaluated at the correct times with respect to the deterministic signal. This clearly extends to joint probabilities for multiple crossings, which can be obtained simply by taking the product of the probabilities for the respective single crossings. Therefore, the single crossing probability is important for time domain methods of signal analysis.

At low signal amplitude to noise ratios, less than about 1, the correlation effects become significant for intervals up to about the period of the band frequency, and produce a structure in the probability which masks the crossing of the deterministic signal. This will have the effect of giving a lower limit to the SNR from which information about the deterministic signal can be obtained, which is dependent not only on the SNR but also upon the correlation effects. Thus, at low signal to noise ratios, errors in demodulation will depend on both SNR and noise correlation effects, and channel capacities will not depend solely on SNRs. Noise bandwidth correlation effects are likely to be more severe in frequency domain signal processing because these rely on a continuous time interval, especially the short intervals. Time domain demodulation could still remain viable by selecting an interval between the crossings beyond the correlation effects, where the joint probability was the product of the two single crossings, e.g. at the second or third expected crossing. Errors would then arise simply from statistical sampling, and would depend only on the SNR.

8.2 Further Work

8.2.1 Zero-Crossing Detector For (FSK)

The FSK receiver consists of two channels, one tuned to frequency f_1, the other to frequency f_2. The output of the two channels are compared to determine whether one binary symbol or the other was transmitted.

This Frequency demodulation FM process can be carried out by using the zero-crossings detector (as channel) described in this work. As an example let f_1 be on period frequency corresponding to kx=1, for Rs=1, and f_2 be the second frequency corresponding to kx=0.5 for Rs=0.5.

This can be achieved practically by implementing the theoretical results in Chapter Four & Five, where the first crossing would trigger the detector, which after a delay time t_x would then detect any second crossing within a time window, which can be measured mathematically for any time interval (Chapter Five). The detector will come to a halt after that and repeat the same procedure for the next information signal.

An advantage of time domain detection as a demodulation method is its adaptability. Using DSP methods the delay time and window can, in principle, be adjusted to various and varying signals, by altering the respective counts, or possibly by adjusting the clocking rate. Although management and control of such adjustment would be non-trivial. Since zero crossing detection relies only on the relatively short times between zero crossings it is not sensitive to the synchronisation of the deterministic signal, and would be applicable to synchronous, asynchronous and non-synchronous channels, and also synchronous channels with accidentally varying synchronisation.

It would be useful to investigate the bit error rate and channel capacity of time domain demodulation and compare this with more standard approaches. The methods of this work are particularly applicable to low SNR cases. The window of detection for each frequency could be optimised to produce minimum error. Having selected the intervals and windows for minimum error, and having estimated the errors, it is possible to determine the number of periods over which each frequency should run to produce an acceptable overall error, and to estimate channel capacities.

8.2.2 Multiple Threshold Crossing Detection

It might also be noted that the use of multiple thresholds, including non-zero thresholds, might improve the noise rejection, because this allows for use of more details of the deterministic signal. A number of thresholds could be set, with windows and delay times selected according to the expected deterministic signal, essentially following the shape of the signal. Such multiple thresholds would be separated by intervals of less than the half period used for successive zero crossings, and so noise correlation effects would be even more important. A full analysis would require high dimensional integrals over high dimensional multivariate probability distributions. A study of two successive thresholds would provide the minimum necessary information about correlation effects and would be a prerequisite for further study. In Chapter Four of this work the theory of the probability of a double crossing is given for arbitrary thresholds and times, and the methods of Chapter Five and Six, can be easily adapted to thresholds other than zero and intervals other than half periods. The basic methods are also independent of the deterministic signal, provided some functional form is given. The work of this thesis can therefore be extended to give a preliminary study of the theory of multiple thresholding.

8.2.3 Zero-Crossing Detection of Radar Signals In Noise

The two basic operations performed by radar are firstly, detection of the presence of reflecting objects, and secondly, extraction of information from the received waveform to obtain such target data as position, velocity and, perhaps, size. The operations of detection and extraction may be performed separately and in either order although a radar that is a good detection device is usually a good radar for extracting information, and vice versa.

The portion of the radar receiver which extracts the modulation from the carrier is called the detector. Almost all radar detection decisions are based upon comparing the output of a receiver with some threshold level Lynn[1987]. If the envelope of the receiver output exceeds a pre-established threshold, a signal is said to be present. The purpose of the threshold is to divide the output into a region of no detection, and region of detection; or in other words, the threshold detector allows a choice between one of two hypotheses. One hypothesis is that the receiver output is due to noise alone; the other is that the output is due to signal-plus-noise.

The information contained in the zero-crossings of the received waveform can be used for detecting the presence of signal in noise [Skolnik 1981]. The zero-crossing detector destroys amplitude information. If the exact phase of the echo carrier were known, it would be possible to design a detector which makes optimum use of both the phase information and the amplitude information contained in the echo signal. It would perform more efficiently than a detector which used either amplitude information only or phase information only.

The particular parameter of interest is the distance between the crossings of the waveform along the zero axis. This distance is related to the instantaneous period of the waveform. The variations in this distance will depend on whether signal-plus-noise is present or noise alone is present. Using the analysis for probability of double crossing (PDC) in a unit interval of time, a target signal is said to be present if this PDC is less than a predetermined value and absent if this is exceeded. The greater SNR, the higher is PDC.

There is the unavoidable question of how this method would work on real signals. This is left for further research.

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APPENDIX 3.A

The evaluation Of Eigenvector And The Transformation Of Quadratic Forms

In section 3.5 the initial coordinate system is rotated to a new system, which decouples the bivariate normal distribution. The decoupling is based on the eigenvectors of the correlation matrix M of Eq(3.15) the characteristic equation of M is

$$\begin{vmatrix} 1-\lambda & r \\ r & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - r^2 = 0$$

whose roots

$$\lambda_1 = 1-r \qquad , \qquad \lambda_2 = 1+r$$

are the eigenvalues of M. Standard methods then give the corresponding eigenvectors as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad\qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

from which the transformation matrix B is formed with

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad\qquad B^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = B^T$$

Now the quadratic form in the bivariate normal distribution function, Eq(2.20), becomes

$$X^T.M^{-1}.X = (BX)^T.(B^T MB)^{-1}.(BX) = X'^T.M'^{-1}.X'$$

where

$$X' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = BX = \frac{1}{\sqrt{2}}\begin{pmatrix} x_1+x_2 \\ x_1-x_2 \end{pmatrix}$$

$$M'^{-1} = B^T M B = \begin{pmatrix} \frac{1}{1-r} & 0 \\ 0 & \frac{1}{1+r} \end{pmatrix}$$

Hence the coordinate transformation

$$x_1' = \frac{x_1 - x_2}{\sqrt{2}} \quad , \quad x_2' = \frac{x_1 + x_2}{\sqrt{2}} \tag{A.3.1}$$

makes the quadratic form

$$X^T . M^{-1} . X = \frac{x_1'^2}{1-r} + \frac{x_2'^2}{1+r} \tag{A.3.2}$$

which is used in Eq(3.7) to give Eq(3.16).

APPENDIX 3.B

Reducing The Probability Of High Noise Approximation By Repeated Integrals Of The Normal Probability

The integrals in the approximations in section 3.7 can be reduced by use of repeated normal probability integrals. Some basic rules for these can be found in A&S[26.2.41 to 26.2.46], but some extensions are needed for this work. The basic relations are [A&S 26.2.41 to 26.2.46]

$$I_n(x) = \int_x^{+\infty} I_{n-1}(t)dt$$

where $I_{-1}(x) = Z(x)$ (3.B.1)

$$I_0(x) = \int_x^{\infty} Z(t)dt = Q(x) = 1-P(x) \qquad (3.B.2)$$

and

$$I_1(x) = \int_x^{\infty} Q(t) \quad dt \qquad (3.B.3)$$

However for this work it is necessary to find expressions for the integral

$$\int_{-a}^{\infty} Q(t)(t+a)^2 \, dt = \int_{-a}^{\infty} Q(t) \; t^2 dt + 2a\int_{-a}^{\infty} Q(t)tdt + a^2\int_{-a}^{\infty} Q(t)dt$$

$$= I^{(2)}(-a) + 2aI_0^{(1)}(-a) + a^2 I_0(-a) \qquad (3.B.4)$$

and integrals of the form

$$I_1^{(n)}(x) = \int_x^{\infty} I_m(t) \; t^n \, dt \qquad (3.B.5)$$

for n up to 2, and m=0 are needed

For n=0 Eq(3.B.1) becomes

$$I_1(x) + xI_0 - I_{-1}(x) = 0 \qquad (3.B.6)$$

which on integration gives

$$I_2(x) + I_0^{(1)}(x) - I_0(x) = 0 \qquad (3.B.7)$$

For n=1 Eq(3.B.1) gives

$$2I_2(x) + xI_1(x) - I_0(x) = 0$$

$$I_2(x) = \frac{1}{2} I_0(x) - \frac{1}{2} xI_1(x) \qquad (3.B.8)$$

Substituting this expression for $I_2(x)$ in Eq(3.B.7), and rearranging gives

$$I_0^{(1)}(x) = \frac{1}{2}\left[I_0(x) + xI_1(x)\right] \qquad (3.B.9)$$

which is required for the integral in the central term on the R.H.S. of Eq(3.B.4).

If Eq(3.B.6) is multiplied by x and integrated, then this gives

$$I_1^{(1)}(x) + I_0^{(2)}(x) - I_{-1}^{(1)}(x) = 0 \qquad (3.B.10)$$

Integration of Eq(3.B.8) gives

$$2I_3(x) + I_1^{(1)}(x) - I_1(x) = 0 \qquad (3.B.11)$$

Substituting for $I_1^{(1)}(x)$ in Eq(3.B.11) and rearranging gives

$$I_0^{(2)}(x) = 2I_3(x) - I_1(x) - I_{-1}^{(1)}(x)$$

The last term on the right is

$$I_{-1}^{(1)}(x) = \int_x^\infty Z(t)\ t\ dt = Z(x) = I_{-1}(x)$$

Hence

$$I_0^{(2)}(x) = 2I_3(x) - I_1(x) + I_{-1}(x) \qquad (3.B.12)$$

$I_3(x)$ is expressed in terms of lower orders using Eq(3.B.1) for n=2 and n=1, to give

$$I_3(x)=\frac{1}{3}\left(1+\frac{1}{2}x^2\right)I_1(x)-\frac{1}{6}xI_0(x)$$

Substituting this in Eq(3.B.13), and collecting terms

$$I_0^{(2)}(x)= \frac{1}{3}\left(x^2-1\right)I_1(x)-\frac{1}{3}xI_0(x)+I_{-1}(x) \qquad (3.B.13)$$

as required by the first integral of Eq(3.B.4).

Substituting Eqs(3.B.9 and 13) into Eq(3.B.4), collecting terms, and expressing $I_0(-a)$ and $I_{-1}(-a)$ in terms of Q(-a) or P(a) and Z(-a) respectively

$$\boxed{\int_{-a}^{\infty}Q(t)(t+a)^2\,dt =a(1+\frac{a^2}{3})P(a)+\frac{1}{3}(a^2+2)Z(-a)} \qquad (3.B.14)$$

which is the expression used in Eq(3.52).

It may be seen from Eq(3.B.9) that

$$\int_0^{\infty}Q(t)tdt= I_0^{(1)}(0)= \frac{1}{2}\,Q(o) = \frac{1}{4} \qquad (3.B.15)$$

which is needed for Eq(3.59).

APPENDIX 4.A

Evalution Of Double Integrals For Probability Of Double Crossing

In this appendix details are given of the evaluation of double integrals needed in Eq(4.29), section 4.6. The double integral is

$$\mathcal{J}(a_1,a_2,\rho) = \int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2 \; (Z_1-a_1)(Z_2-a_2)\; g(Z_1,Z_2,\rho) \qquad (4.A.1)$$

g is always positive see [A&S, 26.3.1,26.3.2]; (Z_1-a_1) is always positive and (Z_2-a_2) is always negative, because of the limits on the integration. Therefore $\mathcal{J}(a_1,a_2,\rho)$ is always negative. But this cancels the overall negative from Eq(4.29), so that PD_{2X} is always positive.

In order to compute the integral it is reduced using partial differentiation and partial integration to terms involving integrals over the standard normal and bivariate normal distribution without the Z_1, Z_2 factors. The integral can be expanded

$$(Z_1-a_1)(Z_2-a_2)\; g(Z_1,Z_2,\rho) = Z_1Z_2g - a_1Z_2g - a_2Z_1g + a_1a_2g \quad (4.A.2)$$

and the factors with Z_1, Z_2 written as partial derivatives of g, form [A&S , 26.3.1], then

$$g = g(Z_1,Z_2,\rho) = \frac{1}{2\pi}\,\frac{1}{\sqrt{1-\rho^2}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{Z_1^2-2\rho Z_1Z_2+Z_2^2}{1-\rho^2}\right\}$$

$$\frac{\partial g}{\partial Z_1} = \frac{1}{2\pi}\,\frac{1}{\sqrt{1-\rho^2}}\,\frac{\partial}{\partial Z}\mathrm{Exp}\left\{-\,\frac{1}{2}\,\frac{Z_1^2-2\rho Z_1Z_2+Z_2^2}{1-\rho^2}\right\}$$

$$\frac{\partial g}{\partial Z_1} = g\frac{\partial}{\partial Z_1}\left\{-\frac{1}{2}\;\frac{Z_1^2-2\rho Z_1 Z_2+Z_2^2}{1-\rho^2}\right\}$$

$$\frac{\partial g}{\partial Z_1} = -\frac{1}{1-\rho^2}(Z_1-\rho Z_2)\; g(Z_1,Z_2,\rho)$$

similarly

$$\frac{\partial g}{\partial Z_2} = -\frac{1}{1-\rho^2}(Z_2-\rho Z_1)\; g(Z_1,Z_2,\rho)$$

now

$$-(1-\rho^2)\frac{\partial g}{\partial Z_1} = Z_1\; g-\rho Z_2\; g \qquad (4.A.3)$$

$$-(1-\rho^2)\frac{\partial g}{\partial Z_2} = \rho Z_1\; g-Z_2\; g \qquad (4.A.4)$$

Eq(4.A.3) is multiplied by ρ, to obtain

$$-\rho(1-\rho^2)\frac{\partial g}{\partial Z_1} = \rho Z_1\; g-\rho^2\; Z_2\; g$$

Now subtracting from Eq(4.A.4)

$$(1-\rho^2)\left(-\frac{\partial g}{\partial Z_2}-\rho\frac{\partial g}{\partial Z_1}\right) = 0+(1-\rho^2)\; Z_2\; g$$

Hence

$$Z_2\; g = -\frac{\partial g}{\partial Z_2}-\rho\frac{\partial g}{\partial Z_1} \qquad (4.A.5)$$

and

$$Z_1\; g = -\frac{\partial g}{\partial Z_1}-\rho\frac{\partial g}{\partial Z_2} \qquad (4.A.6)$$

Using Eq(4.A.5)

$$\frac{\partial g}{\partial Z_2} = -Z_2\; g-\rho\frac{\partial g}{\partial Z_1}$$

which, differentiating becomes

$$\frac{\partial^2 g}{\partial Z_2^2} = -\frac{\partial}{\partial Z_2}(Z_2\; g)-\rho\frac{\partial^2 g}{\partial Z_1\;\partial Z_2}$$

$$\frac{\partial^2 g}{\partial Z_2^2} = -g-Z_2\frac{\partial g}{\partial Z_2}-\rho\frac{\partial^2 g}{\partial Z_1\;\partial Z_2} \qquad (4.A.7)$$

Now from Eq(4.A.2) by substituting Eq(4.A.5,6,7) in the first part the following formula is derived

$$Z_1Z_2\ g = Z_1(Z_2\ g)$$

$$= -Z_1\left(\frac{\partial g}{\partial Z_2}+\rho\frac{\partial g}{\partial Z_1}\right)$$

$$= -\frac{\partial}{\partial Z_2}(Z_1\ g)-\rho Z_1\frac{\partial g}{\partial Z_1}$$

$$= \frac{\partial}{\partial Z_2}\left(\frac{\partial g}{\partial Z_1}+\rho\frac{\partial g}{\partial Z_2}\right)-\rho Z_1\frac{\partial g}{\partial Z_1}$$

$$= \frac{\partial^2 g}{\partial Z_1\ \partial Z_2}+\rho\frac{\partial^2 g}{\partial Z_2{}^2}-\rho Z_1\frac{\partial g}{\partial Z_1}$$

$$= \frac{\partial^2 g}{\partial Z_1\ \partial Z_2}-\rho\left(g+Z_2\frac{\partial g}{\partial Z_2}+\rho\frac{\partial^2 g}{\partial Z_1\ \partial Z_2}\right)-\rho Z_1\frac{\partial g}{\partial Z_1}$$

$$Z_1Z_2\ g = (1-\rho^2)\frac{\partial^2 g}{\partial Z_1\ \partial Z_2}-\rho Z_1\frac{\partial g}{\partial Z_1}-\rho Z_2\frac{\partial g}{\partial Z_2}-\rho g \qquad (4.A.8)$$

Eq(4.A.2) is now evaluated by making the substitution using Eqs(4.A.5,6,8)

$$(Z_1-a_1)(Z_2-a_2)\ g(Z_1,Z_2,\rho) = (1-\rho^2)\frac{\partial^2 g}{\partial Z_1\ \partial Z_2}-\rho Z_1\frac{\partial g}{\partial Z_1}-\rho Z_2\frac{\partial g}{\partial Z_2}$$

$$-\rho g + a_1\frac{\partial g}{\partial Z_2}+\rho a_1\frac{\partial g}{\partial Z_1}+a_2\frac{\partial g}{\partial Z_1}+\rho a_2\frac{\partial g}{\partial Z_2}+a_1a_2 g$$

after some straight forward analysis, becomes

$$= (1-\rho^2)\frac{\partial^2 g}{\partial Z_1\partial Z_2}+(a_2-\rho(Z_1-a_1))\frac{\partial g}{\partial Z_1}+(a_1-\rho(Z_2-a_2))\frac{\partial g}{\partial Z_2}+(a_1a_2-\rho)g \qquad (4.A.9)$$

The desired Eq(4.A.1) is now given as

$$\mathcal{J}(a_1,a_2,\rho) = \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \quad .\text{Eq}(4.A.9)$$

To carry out the four parts of the integration, each part is considered separately. The first integral becomes

$$(1-\rho^2)\int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \frac{\partial^2 g}{\partial Z_1\, \partial Z_2} = (1-\rho^2)\int_{a1}^{+\infty} dZ_1 \left(\int_{-\infty}^{a2} dZ_2 \frac{\partial^2 g}{\partial Z_1\, \partial Z_2}\right)$$

$$= (1-\rho^2)\int_{a1}^{+\infty} dZ_1 \left(\frac{\partial g}{\partial Z_2}\right)_{-\infty}^{a2} = (1-\rho^2)\int_{a1}^{+\infty} dZ_1 \frac{\partial}{\partial Z_1} g(a_1,a_2,\rho)$$

$$= (1-\rho^2) g(a_1,a_2,\rho)\Big|_{a1}^{+\infty} = -(1-\rho^2) g(a_1,a_2,\rho) \qquad (4.A.10)$$

To integrate the second part of Eq(4.A.9), the g function is taken as [A&S, 26.3.2].

$$g(Z_1,Z_2,\rho) = \frac{1}{\sqrt{1-\rho^2}} Z\left(\frac{Z_1-\rho Z_2}{\sqrt{1-\rho^2}}\right) \quad Z(Z_2)$$

Now integrate the second part of (4.A.9)

$$= \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \; (a_2-\rho(Z_1-a_1)) \frac{\partial g}{\partial Z_1}$$

$$= \int_{-\infty}^{a2} dZ_2 \; \left\{(a_2-\rho(Z_1-a_1)) \; g\Big|_{a1}^{+\infty} + \int_{a1}^{+\infty} dZ_1 \; \rho g\right\}$$

$$= -\int_{-\infty}^{a2} dZ_2 a_2 g(Z_2,a_1,\rho) + \rho \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \; g$$

$$= -a_2 Z(a_1) \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{a2} dZ_2 \; Z\left(\frac{Z_2-\rho\; a_1}{\sqrt{1-\rho^2}}\right) + \rho \int_{a1}^{+\infty} dZ_1 \int_{-\infty}^{a2} dZ_2 \; g$$

$$=-a_2 P\left(\frac{a_2-\rho\ a_1}{\sqrt{1-\rho^2}}\right)\ Z(a_1)+\rho\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g \tag{4.A.11}$$

Similarly integrate the third part of Eq(4.A.9), to give

$$\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ (a_1-\rho(Z_2-a_2))\frac{\partial g}{\partial Z_2}=\int_{a_1}^{+\infty} dZ_1\left\{(a_1-\rho(Z_2-a_2))g\Big|_{-\infty}^{a_2}+\int_{-\infty}^{a_2} dZ_2\ \rho g\right\}$$

$$=\int_{a_1}^{+\infty} dZ_1\ a_1 g(Z_1,a_2,\rho)+\rho\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g$$

$$=a_1 Z(a_2)\frac{1}{\sqrt{1-\rho^2}}\int_{a_1}^{+\infty} dZ_1\ Z\left(\frac{Z_1-\rho\ a_2}{\sqrt{1-\rho^2}}\right)+\rho\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g$$

$$=a_1 Q\left(\frac{a_1-\rho\ a_2}{\sqrt{1-\rho^2}}\right)Z(a_2)+\rho\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g \tag{4.A.12}$$

Finally the last term in Eq(4.A.9) to be integrated is

$$=(a_1 a_2-\rho)\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g \tag{4.A.13}$$

Now, it follows quite simply, by collecting all the above terms Eq(4.A.10,11,12,13) and substituting in Eq(4.A.1), that

$$\boxed{\begin{aligned}\mathcal{J}(a_1,a_2,\rho) &=-(1-\rho^2)\ g(a_1,a_2,\rho)\\ &+a_1\ Q\left(\frac{a_1-\rho\ a_2}{\sqrt{1-\rho^2}}\right)\ Z(a_2)-a_2\ P\left(\frac{a_2-\rho\ a_1}{\sqrt{1-\rho^2}}\right)\ Z(a_1)\\ &+(a_1 a_2-\rho)\int_{a_1}^{+\infty} dZ_1 \int_{-\infty}^{a_2} dZ_2\ g(Z_1,Z_2,\rho)\end{aligned}} \tag{4.A.14}$$

APPENDIX 4.B

Transformation To Circular b.n.d

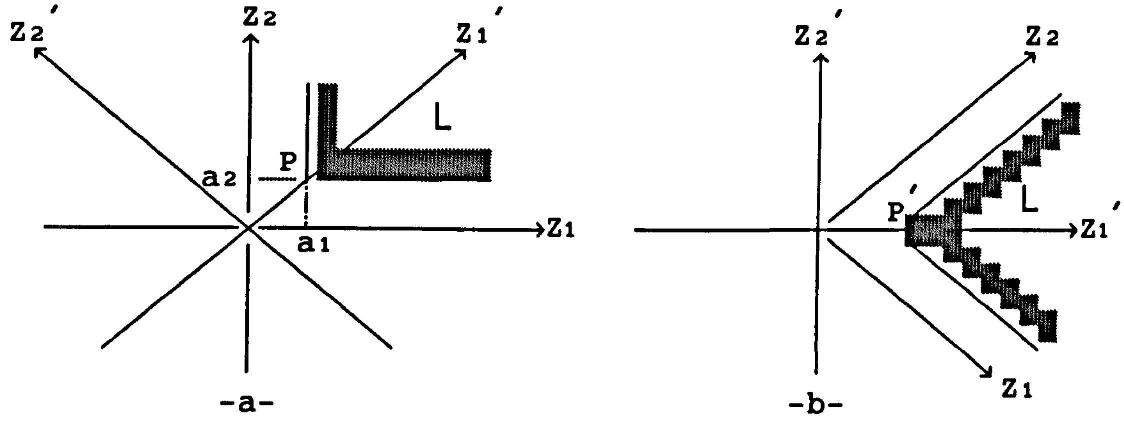
G&G[1978] derived various relations for bivariate normal probability $B(a_1,a_2,\rho)$. These must be adapted to our function $D(a_1,a_2,\rho)$ so that existing $g(Z_1,Z_2,\rho)$ in Eq(4.31, 32) may be evaluated. The first step for finding integrals of bivariate normal distribution is to make a rotation of the axes to reduce the function under integral. Let Z_1' and Z_2' be the transformation of coordinates of Z_1, Z_2 defined in [A&S, 26.3.22], as

$$Z_1' = \frac{1}{\sqrt{2(1+\rho)}}(Z_1+Z_2) \quad ; \quad Z_2' = \frac{-1}{\sqrt{2(1-\rho)}}(Z_1-Z_2) \qquad (4.B.1)$$

i.e.

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{2}}\left(\sqrt{1+\rho}\; Z_1' - \sqrt{1-\rho}\; Z_2'\right) \\ Z_2 &= \frac{1}{\sqrt{2}}\left(\sqrt{1+\rho}\; Z_1' + \sqrt{1-\rho}\; Z_2'\right) \end{aligned} \qquad (4.B.2)$$

For $\rho^2<1$, let P be the point in the locus of constant values of (Z_1,Z_2), will be an ellipse centred on coordinates (a_1,a_2) [Peebles 1987], as shown in Fig[4.B.1a]. The integral of the circular bivariate normal variate over the right labeled L, triangle shown in Fig[4.B.1a]. The coordinate transformation takes the original triangle into another in Z_1',Z_2' plane, as shown if Fig[4.B.1b]



Fig[4.B.1] Transformation to circular b.n.d.

The coordinate of the point P' in Z_1' Z_2' are (a_1', a_2') and are given by [A&S, 26.3.22]

$$a_1' = \frac{1}{\sqrt{2(1+\rho)}}(a_1+a_2) \qquad a_2' = \frac{-1}{\sqrt{2(1-\rho)}}(a_1-a_2) \qquad (4.B.3)$$

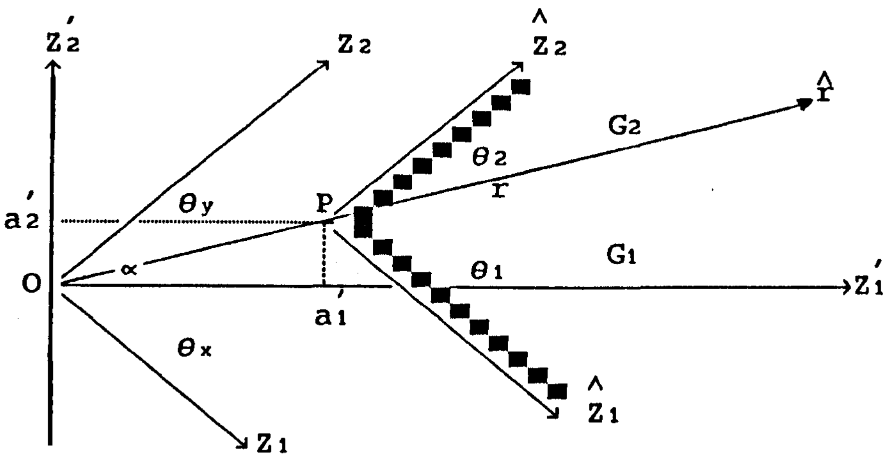
The distance of P' from the origin is

$$r = |op'| = \sqrt{a_1'^2 + a_2'^2}$$

G&G show that

$$r = \frac{a_1^2 - 2\rho\, a_1 a_2 + a_2^2}{1-\rho^2} \qquad (4.B.4)$$

Fig[4.B.2],shows the geometrical relationships for all these further transformation rotates, all vectors about O through the same angle α in the opposite direction. The unit vector $\hat{r}$ is in the direction of OP' and i,j two mutually perpendicular vectors each of length one.



Fig[4.B.2] Definition of $B(a_1,a_2,\rho)$ function.

The unit vector $\hat{r}$ in the direction of op' is

$$\hat{r} = \cos\alpha\ \hat{i}' + \sin\alpha\ \hat{j}'$$

where $\cos\alpha = \frac{a_1'}{r}$, $\sin\alpha = \frac{a_2'}{r}$ (4.B.5)

similarly unit vectors $\hat{Z}_1$, $\hat{Z}_2$ in the direction of the old Z_1 and Z_2 axes are, where θ_x, θ_y are the polar coordinates of the tip of the vector r, their angles are taken from the Z_1' to Z_1 and Z_2 axes respectively. This conversion will ensure that θ_x is a negative angle as shown in Fig[4.B.2].

$$\hat{Z}_1 = \cos\theta_x\ \hat{i}' + \sin\theta_x\ \hat{j}'$$
$$\hat{Z}_2 = \cos\theta_y\ \hat{i}' + \sin\theta_y\ \hat{j}' \qquad (4.B.6)$$

This is the coordinate representation of the rotation relative to the basis i,j. Now θ_1 is the angle from $\hat{Z}_1$ to

$\hat{Z}_2$ hence the coordinate rotation is Peebles[1987]

$$\begin{aligned} \cos\theta_1 &= \cos\alpha \cos\theta_x + \sin\alpha \sin\theta_x \\ \sin\theta_1 &= \sin\alpha \cos\theta_x - \cos\alpha \sin\theta_x \end{aligned} \qquad (4.B.7)$$

similarly with θ_2 taken from $\hat{Z}_2$ to $\hat{Z}_1$

$$\begin{aligned} \cos\theta_2 &= \cos\alpha \cos\theta_y + \sin\alpha \sin\theta_y \\ \sin\theta_2 &= \sin\alpha \cos\theta_y - \cos\alpha \sin\theta_y \end{aligned} \qquad (4.B.8)$$

From Eq(4.B.1)

$$Z_2' = -\sqrt{\frac{1+\rho}{1-\rho}}\, Z_1'$$

Hence

$$\tan\theta_x = -\sqrt{\frac{1+\rho}{1-\rho}} \quad ,\cos\theta_x = \sqrt{\frac{1-\rho}{2}} \quad ,\sin\theta_x = -\sqrt{\frac{1+\rho}{2}} \qquad (4.B.9)$$

Similarly, by substituting in (4.B.1) gives the equation for the Z_2-axis

$$Z_2' = \sqrt{\frac{1+\rho}{1-\rho}}\, Z_1' \quad , \quad \cos\theta_y = \sqrt{\frac{1-\rho}{2}} \quad , \quad \sin\theta_y = -\sqrt{\frac{1+\rho}{2}} \qquad (4.B.10)$$

substituting Eq(4.B.5), (4.B.9), into Eq(4.B.7)

$$\begin{aligned} \cos\theta_1 &= \frac{1}{2r}\left[(a_1+a_2)\sqrt{\frac{1-\rho}{1+\rho}} + (a_1-a_2)\sqrt{\frac{1+\rho}{1-\rho}}\right] \\ &= \frac{1}{2r}\,\frac{1}{\sqrt{1-\rho^2}}\left[(a_1+a_2)(1-\rho) + (a_1-a_2)(1+\rho)\right] \end{aligned}$$

$$\cos\theta_1 = \frac{1}{r}\,\frac{1}{\sqrt{1-\rho^2}}(a_1-\rho\, a_2) \qquad (4.B.11)$$

and also

$$\sin\theta_1 = \frac{1}{2r}\left[-(a_1-a_2)\sqrt{\frac{1-\rho}{1-\rho}} + (a_1+a_2)\sqrt{\frac{1+\rho}{1+\rho}}\right]$$

$$\therefore \sin\theta_1 = \frac{a_2}{r} \qquad (4.B.12)$$

Hence

$$\tan\theta_1 = \frac{a_2\sqrt{1-\rho^2}}{a_1-\rho\ a_2} = \tan\theta_k \qquad (4.B.13)$$

where θ_k is angle used by G&G[1987, Eq(5)], similarly from Eq(4.B.5), (4.B.10) substituted into Eq(4.B.8) gives

$$\cos\theta_2 = \frac{1}{2r}\left[(a_1+a_2)\sqrt{\frac{1-\rho}{1+\rho}} - (a_1-a_2)\sqrt{\frac{1+\rho}{1-\rho}}\right]$$

$$\therefore \cos\theta_2 = \frac{1}{r}\frac{1}{\sqrt{1-\rho^2}}(a_2-\rho a_1) \qquad (4.B.14)$$

and

$$\sin\theta_2 = \frac{1}{2r}\left[-(a_1-a_2)\sqrt{\frac{1-\rho}{1-\rho}} - (a_1+a_2)\sqrt{\frac{1+\rho}{1+\rho}}\right]$$

$$\sin\theta_2 = -\frac{a_1}{r} \qquad (4.B.15)$$

$$\tan\theta_2 = -\frac{a_1\sqrt{1-\rho^2}}{(a_2-\rho\ a_1)} = \tan(-\theta_h) \qquad (4.B.15)$$

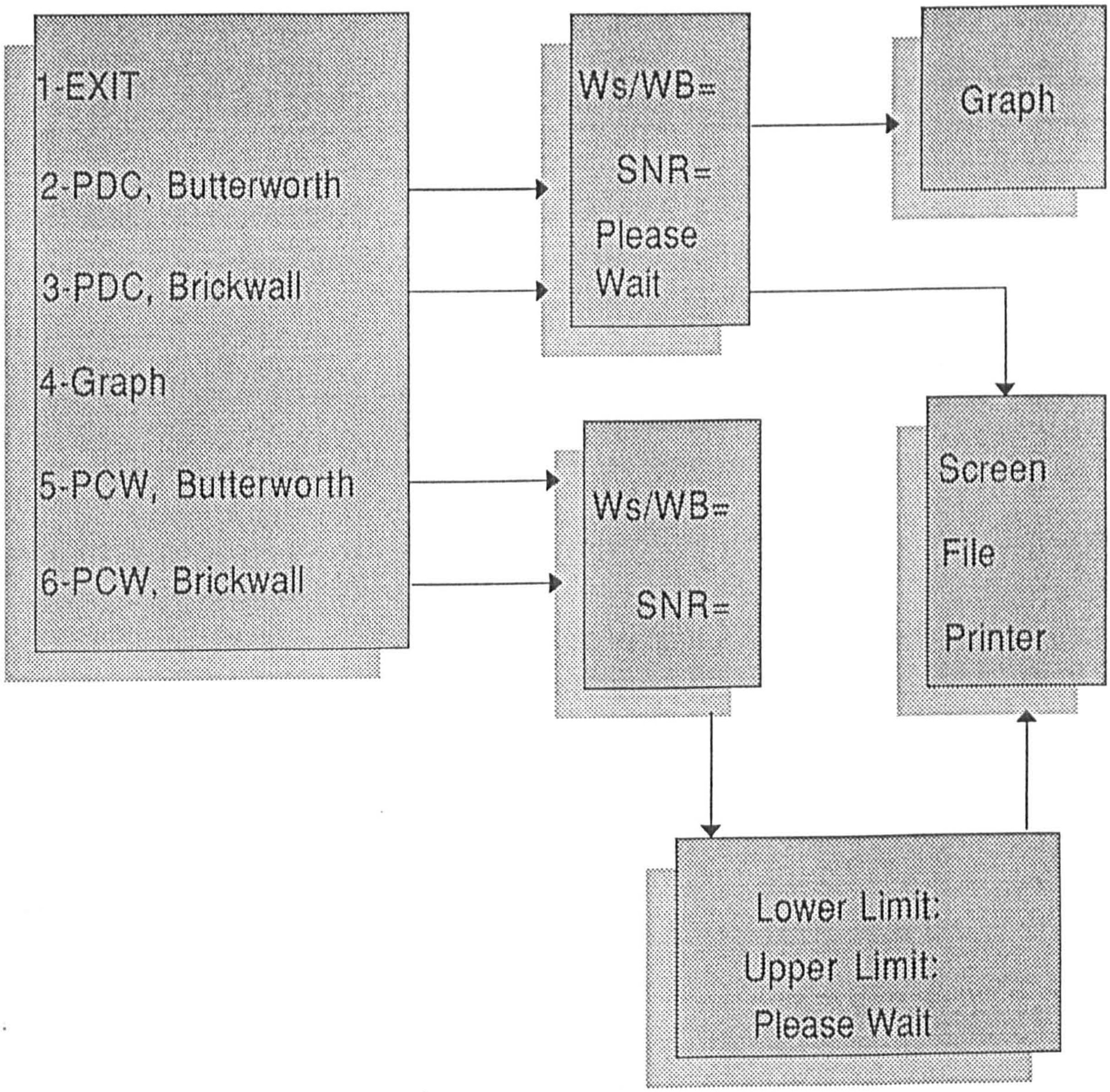
θ_h, θ_k are the angles used by G&G[1987, Eq(5)], and

$$\boxed{\begin{array}{l} D(r,\theta_1) = D(r,\theta_k) \\ D(r,-\theta_2) = D(r,\theta_h) \end{array}} \qquad (4.B.17)$$

The above geometrical transformation assume a_1 a_2 positive as is always the case in this work (Eq 4.33).

Appendix 5.A

A Program Demonstration Used In Chapter 4 & 5



Fig[5.A.1] The program block diagram.

The program is written in Turbo Pascal version 4. It is menu driven, as shown in the block diagram Fig[5.A.1]. To calculate at fixed delay time tx between crossings, the joint probability density with respect to the second crossing, per cycle over the first crossing, (PDC): choose the filter (no. 2 or 3) from the menu, specify (W_S/W_B) value and SNR; then wait for the program to be executed; the result is saved to disk. This result can be seen visually on screen as a graph by choosing no. 4 or printed out.

To find the probability of detection of double crossing within a time interval (PCW): choose the filter (no. 5 or 6) from the menu, choose the values for (W_S/W_B), and SNR. Specify the window interval t_{W1} (lower limit) and t_{W2} (upper limit); wait for the program to be executed; and eventually choose whether to save to disk, display or print.

The program can be used for any SNR and detection window interval, and can easily be adapted to other filters and nonzero thresholds.

```
program Probability_Of_Double_Crossing;

{--------------------------------------------------------------------------}
{-                                                          -}
{-      purpose:this program demonstrates integration with Adaptive      -}
{-              Quadrature methods and Gaussian Quadrature.          -}
{-                                                          -}
{-                                                          -}
{-      Unit   :Integrat procedure Adaptive_Gauss_Quadrature          -}
{-                                                          -}
{--------------------------------------------------------------------------}
{$I Float.inc}  { Determines what type Float means. }
{$I-}      { Disable I/O error trapping }
{$R+}      { Enable range checking }

uses
 Dos, Crt, GDriver, GKernel, GWindow, GShell,LeastSqr ,Integrat,common,turbo3;

const
     pi=3.141592653;

   IOerr : boolean = false;     { Flags an I/O error  }
 FirstFit : boolean = true;     { Needed for precedure results.  }
 WriteToFile : boolean = false; { Set true for disk output.  }

var
 XData, YData : TNColumnVector; { Data points (X,Y)  }
 NumPoints,M : integer;          { # of points  }
 Solution : TNRowVector;        { Coefficients of the l.s. fit  }
 YFit : TNColumnVector;         { Least squares solution  }
                      { at XData points  }
 Residual : TNColumnVector;     { YFit - YData  }
 StandardDeviation : Float;
                      { Indicates goodness of fit  }
 TNError : byte;             { Flags if something went wrong  }
 OutFile : text;            { Output file  }
 Multiplier : Float;          { Used in some modules  }
 Constant : Float;           { Used in some modules  }
```

```
  WData : TNColumnVector;       { Transformed X-values  }
  ZData : TNColumnVector;       { Transformed Y-values  }
  Basis : TNmatrix;             { Matrix of basis functions  }
  OutfileName : TNString40;     { File name for data and coefficients  }
  A, B : PlotArray;
  Choice:integer;
LowerLimitfir,UpperLimitfir:Real;
LowerLimitSec,UpperLimitSec:Real;
LowerLimit, UpperLimit : Real;  { Limits of integration }
Count,Nlo,Nup : integer;        { Maximum number of subintervals used }
                          { to approximate the integral }
    ING,Integral: Real;         { Value of the integral }
T,C1,C2,C3,C4,det_A,SIG,Py:Real;
K2,a1,a2,X1,X2,X3,X4,F1,F2,Rs,SNR,S1,S2,TA,Num:Real;
Xmin,Xmax,glx,Result:Real;

(*******************************************************************
)
(*                  Demonstration program                 *)
(*******************************************************************
)

procedure Delay(N : Float);
var
 I : Float;
 J : integer;
 Ch : char;
 Quit : boolean;

begin
 I := 0;
 Ch := ' ';
 repeat
  I := I + 1;
  Quit := false;
  if KeyPressed then
  begin
   Ch := ReadKey;
```

```
      Quit := (Ch = ^C);
      if (Ch = '') and KeyPressed then
      begin
        Ch := ReadKey;
        Quit := (Ch = 'O');
        Ch := ' ';
      end;
    end;
    if Quit then
    begin
      LeaveGraphic;
      Halt;
    end;
  until (Ch = ^M) or (I >= N);
end; { Delay }


procedure ClearEol(I : integer);
begin
  GotoXY(1, I);
  Write('                                                                              ');
end; { ClearEol }

procedure Msg(S : WrkString);
begin
  ClearEol(25);
  GotoXY(1, 25);
  Write(S);
  Delay(750);
end; { Msg }

procedure SelectIBM;
begin
  SelectWorld(2);
  SelectWindow(10);
end; { SelectIBM }

procedure DefineWindowIBM(I, X1, Y1, X2, Y2 : integer);
```

```
begin
 DefineWindow(I, Trunc(X1 / 79 * XMaxGlb + 0.001), Trunc(Y1 / 199 * YMaxGlb
+ 0.001),
            Trunc(X2 / 79 * XMaxGlb + 0.5), Trunc(Y2 / 199 * YMaxGlb + 0.5));
end; { DefineWindow }


{--------------------------------------------------------------------------}

procedure Intro;
var
 I : integer;
begin
 SetHeaderOff;
 DefineWindowIBM(1, 5, 40, 32, 80);
 DefineWindowIBM(2, 16, 55, 43, 95);
 DefineWindowIBM(4, 15, 60, 43, 135);
 SelectWindow(1);
 DrawBorder;
 SelectIBM;
 DrawTextW(52, 144, 5, 'HWCRG');
 StoreWindow(1);
 Delay(500);
 SelectWindow(2);
 SetBackground(0);
 DrawBorder;
 SelectIBM;
 DrawTextW(140, 129, 5, 'HWCRG');
 StoreWindow(2);
 Delay(500);
 for I := 1 to 4 do
 begin
  RestoreWindow(1, 0, 0);
  Delay(800);
  RestoreWindow(2, 0, 0);
  Delay(800);
 end;
 for I := 1 to 8 do
```

```
begin
  RestoreWindow(1, Trunc(3 * I * XMaxGlb / 79), Trunc(9 * I * YMaxGlb / 199));
  Delay(500);
  RestoreWindow(2, Trunc(3 * I * XMaxGlb / 79), Trunc(9 * I * YMaxGlb / 199));
  Delay(500);
end;
Delay(500);
ClearScreen;
InvertScreen;
SetColorBlack;
SelectIBM;
DrawTextW(50, 180, 3, 'UNIVERSITY OF HULL');
DrawTextW(40, 160, 3, 'DEPARTEMENT OF ELECTRONIC ENG.');
DrawTextW(40, 140, 3, 'Ph.D PROJECT,ZEROCROSSING');
DrawTextW(20, 120, 4, 'BY');
DrawTextW(40, 60, 3,  'DR.A.G.HALL');
DrawTextW(40, 50, 3,  'MR.SAM.N.AL-JAJJOKA');
CopyScreen;
SelectScreen(1);
Delay(5000);
DefineWindowIBM(1, 1, 125, 35, 168);
StoreWindow(1);
DefineWindowIBM(2, 36, 125, 78, 168);
StoreWindow(2);
for I := 1 to Trunc(34 * XMaxGlb / 79) do
begin
  SelectWindow(1);
  MoveHor(1, false);
  ReDefineWindow(1, X1RefGlb, Y1RefGlb, X2RefGlb - 1, Y2RefGlb);
  SelectWindow(2);
  MoveHor(-1, false);
  ReDefineWindow(2, X1RefGlb + 1, Y1RefGlb, X2RefGlb, Y2RefGlb);
end;
SetColorWhite;
DrawTextW(X1RefGlb - 56, Y1RefGlb + 96, 4, '_');
SetColorBlack;
Delay(1000);
RestoreWindow(1, 0, 0);
```

```
  RestoreWindow(2, 0, 0);
  GotoXY(1, 25);
  ClrEOL;
  GotoXY(1, 25);
 Write('*************************************************************
*****************');
  Delay(3000);
  SetColorWhite;
  SetHeaderOn;
 end; { Intro }

 {************************The Menu
 Demonstration****************************}

 Function MenuChoice(OldChoice : integer) : integer;
 const
  Null   = #0;
  Cr     = #13;
  Bs     = #8;
  CtrlC  = #3;
  Esc    = #27;
  Prompt : string[6] = '==>   ';
  N = 5;
  MenuItems : array [0..5] of string[35] = ('Exit this demonstration',
                          'Second Order ButterWorth Filter',
                          'BrickWall Filter',
                          'Draw  The Graph',
                          'Zerocrossing,ButterWorth Filter',
                          'Zerocrossing,BrickWall Fliter');
var
 X, Y, I : integer;
 J : word;
 Ch : char;
 Quit : boolean;

begin
 OldChoice := OldChoice + 1;
 if OldChoice > N - 1 then
```

```
  OldChoice := 1;
 GotoXY(8, 24);
 Write('Select item with SPACE / BACKSPACE , then hit RETURN');
 Y := 6 - N div 2;
X := 25;
 for I := 0 to N do
 begin
  GotoXY(X - 4, Y + I * 3);
  Write(I:1, ' -- ', MenuItems[I]);
 end;
 I := OldChoice;
 GotoXY(X - 10, Y + I * 3);
 Write(Prompt, I:1, ' -- ', MenuItems[I]);
 GotoXY(62, Y + I * 3);
 Write('<==');

 J := 0;
 Ch := '#';
 Quit := FALSE;
 while (Ch <> Cr) do
 begin
  Ch := '#';
  repeat until keypressed;

  if KeyPressed then
  begin            { Read a keystroke! }
   Ch := ReadKey;
   J := 0;
  end;

  case Ch of
   Bs       : begin
               GotoXY(X - 10, Y + I * 3);
               Write('     ');
               GotoXY(62, Y + I * 3);
               Write('   ');
               I := I - 1;
               if I = -1 then
```

```
      I := N;
     GotoXY(X - 10, Y + I * 3);
     Write(Prompt, I:1, ' -- ', MenuItems[I]);
     GotoXY(62, Y + I * 3);
     Write('<==');
    end;

' '    : begin
     GotoXY(X - 10, Y + I * 3);
     Write('     ');
     GotoXY(62, Y + I * 3);
     Write('   ');
     I := I + 1;
     if I > N then
      I := 0;
     GotoXY(X - 10, Y + I * 3);
     Write(Prompt, I:1, ' -- ', MenuItems[I]);
     GotoXY(62, Y + I * 3);
     Write('<==');
    end;

'0'..'2'  : begin
     if (Ord(Ch) - 48) <> I then
     begin
      GotoXY(X - 10, Y + I * 3);
      Write('     ');
      GotoXY(62, Y + I * 3);
      Write('   ');
      I := Ord(Ch) - 48;
      GotoXY(X - 10, Y + I * 3);
      Write(Prompt, I:1, ' -- ', MenuItems[I]);
      GotoXY(62, Y + I * 3);
      Write('<==');
     end;
    end;

Esc, CtrlC : begin
     LeaveGraphic;
```

```
            Halt;
          end;

    Null     : begin
              if KeyPressed then
              begin
                Ch := ReadKey;
                if Ch = 'O' then    { The End key was hit! }
                begin
                  LeaveGraphic;
                  Halt;
                end;
              end;
            end;
  end; { case }
  J := J + 1;
  if J > 32767 then
    Ch := Cr;
 end; { while Ch <> Cr do }

 MenuChoice := I;
end; { MenuChoice }

{-----------------------------------------------------------------------}
{-                                                                     -}
{-   Purpose: This program provides routines from the Turbo Graphix    -}
{-                                                                     -}
{-   Unit used from the Turbo Numeric Toolbox: LeastSqr.TPU            -}
{-                                                                     -}
{-   Units used from the Turbo Graphix Toolbox:                        -}
{-     GDriver.TPU, GKernel.TPU, GWindow.TPU, GShell.TPU               -}
{-                                                                     -}
{-  The data file FILTER.TXT  must also be in the current directory.   -}
{-                                                                     -}
{-----------------------------------------------------------------------}

procedure IOCheck;
```

```
{-------------------------------------------------}
{- Check for I/O error; print message if needed.  -}
{-------------------------------------------------}

type
  String80 = string[80];

var
  IOcode : integer;

procedure Error(Msg : String80);
begin
  Writeln;
  Write(^G); { Beep! }
  Writeln(Msg);
  Writeln;
end; { procedure Error }

begin { procedure IOCheck }
  IOcode := IOresult;
  IOerr := IOcode <> 0;
  if IOerr then
    case IOcode of
      2   : Error('File not found.');
      3   : Error('Path not found.');
      4   : Error('Too many open files.');
      5   : Error('File access denied.');
      6   : Error('Invalid file handle.');
      12  : Error('Invalid file access code.');
      15  : Error('Invalid drive number.');
      16  : Error('Cannot remove current directory.');
      17  : Error('Cannot rename across drives.');
      100 : Error('Disk read error.');
      101 : Error('Disk write error.');
      102 : Error('File not assigned.');
      103 : Error('File not open.');
      104 : Error('File not open for input.');
      105 : Error('File not open for output.');
```

```
    106 : Error('Invalid numeric format.');
   else
    begin
     Writeln;
     Writeln(^G);
     Writeln('Unidentified error message = ', IOcode, '. See manual.');
     Writeln;
    end;
   end; { case }
end; { procedure IOCheck }

function FileExists(Fname : TNString40) : boolean;
var
 CheckFile : file;
begin
 Assign(CheckFile, Fname);
 {$I-} Reset(CheckFile); {$I+}
  if IOresult = 0 then
  begin
   FileExists := true;
   Close(CheckFile)
  end
 else
  FileExists := false;
end; { function FileExists }


procedure Initialize(var XData, YData, YFit, Residual : TNColumnVector;
                     var Solution                     : TNRowVector;
                     var TNError                      : byte);

{-------------------------------------------------------}
{- Output: XData, YData, Solution, YFit,               -}
{-        Residual, TNError                            -}
{-                                                     -}
{- This procedure initializes the above variables to zero -}
{-------------------------------------------------------}
```

```
begin
  FillChar(XData, SizeOf(XData), 0);
  FillChar(YData, SizeOf(XData), 0);
  FillChar(Solution, SizeOf(Solution), 0);
  FillChar(YFit, SizeOf(XData), 0);
  FillChar(Residual, SizeOf(XData), 0);
  TNError := 0;
end; { procedure Initialize }




procedure ReportError(PrintString : TNString40; TNError : byte);
var
  Answer : char;

begin
  Writeln(PrintString, ' Error = ', TNError);
  Writeln('Strike any key to continue.');
  Answer := ReadKey;
end; { procedure ReportError }

procedure HelpScreen;
var
  Answer : char;

begin
  if XScreenMaxGlb = 719 then
    { Hercules }
    Write('14X9')
  else
    { CGA or others }
    Write('8X8');
  Writeln('.FON, and ERROR.MSG');
  Writeln('  must also be on the current directory at run time.');
  Writeln('To run your own data files, use: InFileName');
  Halt;
```

```
end; { procedure HelpScreen }


procedure GetData(var NumPoints   : integer;
            var XData, YData : TNColumnVector;
            var OutFileName  : TNString40);

{------------------------------------------------------------}
{- Output: NumPoints, XData, YData, OutFileName.           -}
{-                                                         -}
{- This procedure reads the number of data points (NumPoints),-}
{- gets XData, YData and the OutFileName.                  -}
{------------------------------------------------------------}

var
 InFileName : TNString40;

procedure GetTwoVectorsFromFile(InFileName   : TNString40;
                  var NumPoints    : integer;
                  var XData, YData : TNColumnVector);

{------------------------------------------------------------}
{- Output: NumPoints, XData, YData                         -}
{-                                                         -}
{- This procedure reads in the data points from a data file. -}
{------------------------------------------------------------}

var
 InFile : text;

begin
 if not FileExists(InFileName) then
 begin
  Writeln('Input data file ', InFileName,' not found.');
  Halt;
 end;
 Assign(InFile, InFileName);
 Reset(InFile);
```

```
  NumPoints := 0;
  while not EOF(InFile) do
  begin
   NumPoints := Succ(NumPoints);
   {$I-} Readln(InFile, XData[NumPoints], YData[NumPoints]);{$I+}
   IOCheck;
  end;
  Close(InFile);
end; { procedure GetTwoVectorsFromFile }


begin { procedure GetData }
  InFileName := 'A:FILTER.txt';
  if ParamCount > 0 then
  begin
   if (ParamStr(1) = '?') or (ParamCount <> 2) then
     HelpScreen;
   InFileName := ParamStr(1);
  end;
  GetTwoVectorsFromFile(InFileName, NumPoints, XData, YData);
end; { procedure GetData }

{ ######## GRAPHICS PROCEDURES STARTING HERE ############ }

type
  Strg20 = string[20];
  ColorArray = array[1..4] of byte;

const
  WindowColor : ColorArray = (Green, Magenta, LightCyan, White);
  BackColor   : ColorArray = (Black, Black, Black, Black);
  NotEnoughMemory : boolean = false;

var

  Ans : char;
  WindowNUM : byte;
```

```
procedure SetColors(Fore, Back : integer);
begin
 if MaxForeground = 15 then
 begin
  SetForeGroundColor(Fore);
  SetBackGroundColor(Back);
 end;
end; { procedure SetColors }



procedure GetGraphData(var X, Y      : TNColumnVector;
                var A : PlotArray);
var
 Index : byte;

begin
 for Index := 1 to NumPoints do
 begin
       A[Index, 1] := X[Index];
       A[Index, 2] := Y[Index];
 end;
end; { procedure GetGraphData }


procedure StartGraphics(var Xdata, Ydata : TnColumnVector;
                var       A   : PlotArray);
begin
 GetGraphData(Xdata, Ydata, A);
 RAMScreenGlb := FALSE;
 InitGraphic;
 SetColors(WindowColor[1], BackColor[1]);

 DefineWindow(1, 0, 0, XMaxGlb, YMaxGlb);
 FindWorld(1, A,Numpoints, 1, 1.0);
 DefineHeader(1,
    '  ZEROCROSSING PROBABILITY GRAPH');
```

```
  SetHeaderOn;
  DrawBorder;
  DrawAxis(8, 7, 0, 0, 0, 0, -1, -1, true);
  DrawPolygon(A, 1,NumPoints - 1 , 3, 3, -1);
end; { procedure StartGraphics }

Procedure Plot_The_Graph;    { program Plot The Graph }
begin
  ClearScreen;
  Initialize(XData, YData, YFit, Residual, Solution, TNError);
  GetData(NumPoints, XData, YData, OutFileName);
  StartGraphics(Xdata, YData, A);
repeat until keypressed;

end;


procedure GetLimits(var Xmin : Real;
                   var Xmax : Real);

{--------------------------------------------------------}
{- Output: LowerLimit, UpperLimit,Of Second Integration    -}
{-                                          -}
{- This procedure assigns values to the limits of          -}
{- integration from keyboard input                       -}
{--------------------------------------------------------}

begin
 repeat
  repeat
   Write('Lower limit of Second Integration? ');
   Readln(Xmin);
   IOCheck;
  until not IOerr;
  Writeln;
  repeat
   Write('Upper limit of Second Integration? ');
   Readln(Xmax);
```

```
    IOCheck;
   until not IOerr;
   if Xmin = Xmax Then
   begin
    Writeln;
    Writeln(' The limits of integration must be different.');
    Writeln;
   end;
  until Xmin <> Xmax;
  GetOutputFile(OutFile);
 end; { procedure GetLimits }

 procedure Results_Area(Result : Real);

 {--------------------------------------------------------}
 {- This procedure outputs the results to the device OutFile -}
 {--------------------------------------------------------}

 begin
  Writeln(OutFile, 'Result  :' : 15, Result: 10:10);
 end; { procedure Results }



 { ############## GRAPHIX PROCEDURES END HERE ############# }

 {------------------------------------------------------------------------}
 {-                                                  -}
 {-   Turbo Pascal Numerical Methods Toolbox                        -}
 {-                                                  -}
 {-       Purpose: This program demonstrates integration with          -}
 {-               Gaussian Quadrature Methods Using 32 points.        -}
 {-                                                  -}
 {------------------------------------------------------------------------}

Procedure BrickWall_Signal_Calculate(VAR C1,C2,C3,C4,det_A,SIG,Py:Real);

Begin
```

```
    IF ABS(T) < (1E-08)  Then
    C1:=1 - (1/6)*sqr(T)
    ELSE
    C1:=sin(T)/(T);

    C2:=(1.0/T)*(cos(T)-C1);
    C3:=-C1-((2.0/T)*C2);
    C4:=-(1.0/3.0);
  det_A:=1-sqr(C1);
   SIG:=SQRT(-C4-(SQR(C2)/(det_A)));
    Py:=C1 + (1/SQR(SIG))*((C1*C4)-C3);
End;


Procedure ButterWorth_Signal_Calculate(VAR C1,C2,C3,C4,det_A,SIG,Py:Real);

Begin

    C1:=EXP(-(T/SQRT(2.0)))*(cos(T/SQRT(2.0))+sin(T/SQRT(2.0)));
    C2:=EXP(-(T/SQRT(2.0)))*-(SQRT(2.0))*sin(T/SQRT(2.0));
    C3:=EXP(-(T/SQRT(2.0)))*(sin(T/SQRT(2.0))-cos(T/SQRT(2.0)));
    C4:=-1.0;
  det_A:=1-sqr(C1);
   SIG:=SQRT(-C4-(SQR(C2)/(det_A)));
    Py:=C1 + (1/SQR(SIG))*((C1*C4)-C3);
End;


Function Z(x:Real):Real;

 begin
    if(x<=4.5) and (x>=-4.5) then
    Z:=exp(-(x*x)/2.0)/sqrt(2.0*pi)
    else
    Z:=0.0
    end;
```

```
Function NQ(x:real):real;
const
    J=0.2316419;
    b1=0.319381530;
    b2=-0.356563782;
    b3=1.781477937;
    b4=-1.821255978;
    b5=1.330274429;
var
  N,N2,N3,N4,N5:real;
begin
   IF (x<0)  Then
   Writeln('Error');
   IF (x > 4.1)  Then
   NQ:=0.0

   Else

  begin
    N:=1/(1+J*x);
   N2:=N*N;
   N3:=N*N2;
   N4:=N*N3;
   N5:=N*N4;
   NQ:=Z(x)*(b1*N+b2*N2+b3*N3+b4*N4+b5*N5);
end;
end;

Function Q(x:real):real;
begin
IF X >=0 THEN
Q:=NQ(x)
ELSE
Q:=1-NQ(abs(x));
END;

Function P(x:real):real;
```

```
begin
IF X >=0 THEN
P:=1 - NQ(x)
ELSE
P:=NQ(abs(x));
END;



Function g(x,y,u:Real):real;
begin
g:=(1.0/(SQRT(1-sqr(u))))*Z(x)*Z((y-(u*x))/(SQRT(1-sqr(u))));
end;

Function d(V,x:Real):Real;
var
b0,b1,b2,b3,b4,b5,b6:real;
begin
If (V >=0.0) AND (V<=0.5)  Then
b0:=0.63662; b1:=-0.28829; b2:=0.076413; b3:=0.12887;  b4:=-0.077647; b5:=-
0.0044233;  b6:=0.018735;
If (V >0.5)  AND (V<=0.75) Then
b0:=0.63558; b1:=-0.28084; b2:=0.065589; b3:=0.12014;  b4:=-0.060133; b5:=-
0.00030492; b6:=0.010472;
If (V >0.75) AND (V<=1.0)  Then
b0:=0.63335; b1:=-0.27177; b2:=0.057479; b3:=0.11144;  b4:=-0.048636;
b5:=0.0029908;  b6:=0.0061419;
If (V >1.0)  AND (V<=1.25) Then
b0:=0.62937; b1:=-0.26033; b2:=0.050024; b3:=0.10217;  b4:=-0.039384;
b5:=0.0060024;  b6:=0.0031245;
If (V >1.25) AND (V<=1.50) Then
b0:=0.62336; b1:=-0.24710; b2:=0.043290; b3:=0.092844; b4:=-0.031926;
b5:=0.0086426;  b6:=0.0010043;
If (V >1.50) AND (V<=1.75) Then
b0:=0.61527; b1:=-0.23284; b2:=0.037374; b3:=0.084331; b4:=-0.026231;
b5:=0.010246;  b6:=-0.000094248;
If (V >1.75) AND (V<=2.0)  Then
```

```
b0:=0.60562; b1:=-0.21839; b2:=0.032265; b3:=0.075772; b4:=-0.021329;
b5:=0.012186;  b6:=-0.0012182;
If (V >2.0)  AND (V<=2.25) Then
b0:=0.59428; b1:=-0.20388; b2:=0.027847; b3:=0.068301; b4:=-0.017595;
b5:=0.013178;  b6:=-0.0017135;
If (V >2.25) AND (V<=2.5)  Then
b0:=0.58176; b1:=-0.18985; b2:=0.024082; b3:=0.061673; b4:=-0.014647;
b5:=0.013651;  b6:=-0.0019261;
If (V >2.5)  AND (V<=2.75) Then
b0:=0.56868; b1:=-0.17668; b2:=0.020906; b3:=0.055531; b4:=-0.012189;
b5:=0.014086;  b6:=-0.0021016;
If (V >2.75) AND (V<=3.0)  Then
b0:=0.55500; b1:=-0.16430; b2:=0.018215; b3:=0.050453; b4:=-0.010343;
b5:=0.013727;  b6:=-0.0019700;
If (V >3.0)  AND (V<=3.25) Then
b0:=0.53820; b1:=-0.15095; b2:=0.015631; b3:=0.046000; b4:=-0.0088589;
b5:=0.013298;  b6:=-0.0018288;
If (V >3.25) AND (V<=3.5)  Then
b0:=0.52368; b1:=-0.13958; b2:=0.013507; b3:=0.037992; b4:=-0.0064001;
b5:=0.017783;  b6:=-0.0032015;
If (V >3.5)  AND (V<=3.75) Then
b0:=0.49007; b1:=-0.11926; b2:=0.010445; b3:=0.037444; b4:=-0.0062475;
b5:=0.013224;  b6:=-0.0018931;
If (V >3.75) AND (V<=4.0)  Then
b0:=0.59578; b1:=-0.17358; b2:=0.017409; b3:=0.037163; b4:=-0.0061453;
b5:=0.0090755; b6:=-0.00081797;
If V>4.0            Then
b0:=0.0;   b1:=0.0;    b2:=0.0;   b3:=0.0;    b4:=0.0;    b5:=0.0;    b6:=0.0;
begin
d:=(b0+b1*V+b2*sqr(V))*x+(b3*V+b4*sqr(V))*sqr(x)*x+(b5*V+b6*sqr(V))*sqr(x)
*sqr(x)*x;
end;
end;

Function L(x,y,z:real):real;
var
r,Ah,Ak,D1,D2:Real;
begin
```

```
x:=ABS(x);
y:=ABS(y);
Num:=(sqr(x)-2*z*y*x+sqr(y));
If Num=0.0  Then r:=0.0;
If (1-sqr(z))=0.0 Then  r:=100
Else
r:=SQRT(Num/(1-sqr(z)));
If x=0.0   Then  Ah:=0.0;
If (y-(z*x))=0.0 Then Ah:=(Pi/2)
Else
Ah:=ARCTAN((x*SQRT(1-sqr(z)))/(y-(z*x)));
If y=0.0   Then  Ak:=0.0;
If (x-(z*y))=0.0 Then Ak:=(Pi/2)
Else
Ak:=ARCTAN((y*SQRT(1-sqr(z)))/(x-(z*y)));

if (z*ABS(x)) <= ABS(y)  Then
D1:=1/2*Q(r)*d(r,Ah)
Else
D1:=Q(ABS(x))-(1/2)*Q(r)*d(r,ABS(Ah));

If (z*ABS(y)) <= ABS(x)  Then
D2:=(1/2)*Q(r)*d(r,Ak)
Else
D2:=Q(ABS(y))-(1/2)*Q(r)*d(r,ABS(Ak));
begin
L:=D1+D2;
end;
end;

Function DT(x,y,z:real):real;
begin

If (x>=0) AND (y>=0) Then
DT:=Q(ABS(x))-L(ABS(x),ABS(y),(z));
If (x<0) AND (y>=0) Then
DT:=1-Q(ABS(x))-Q(ABS(y))+L(ABS(x),ABS(y),(-z));
If (x>=0) AND (y<0) Then
```

```
DT:=L(ABS(x),ABS(y),(-z));
If (x<0) AND (y<0) Then
DT:=Q(ABS(y))-L(ABS(x),ABS(y),(z));
end;

{***************First Integration Start From Here *************************}

Function FuncFirst(K1 : Real) : Real;

{----------------------------------------------------}
{-      This is the function to integrate          -}
{----------------------------------------------------}
begin

X1:=SNR*sin(2*pi*K1);
X2:=SNR*Rs*cos(2*pi*K1);
X3:=SNR*sin(2*pi*(K1+K2));
X4:=SNR*Rs*Cos(2*pi*(K1+K2));
F1:=((C2/(SIG*det_A))*((C1*X1)-X3));
F2:=((C2/(SIG*det_A))*(X1-(C1*X3)));
a1:=-(((1/SIG)*X2)+F1);
a2:=-(((1/SIG)*X4)+F2);
S1:=((1-sqr(Py))*g(a1,a2,Py))-(a1*Q((a1-(Py*a2))/SQRT(1-sqr(Py)))*Z(a2));
S2:=((a2*Z(a1)*P((a2-(Py*a1))/SQRT(1-sqr(Py))))-(((a1*a2)+Py)*DT(a1,a2,Py)));
TA:=((S1)+(S2));
FuncFirst :=(sqr(2*pi)*sqr(SIG))*g(X1,X3,C1)*(TA);
end; { Function First Integral }

Procedure   Adaptive_Gaus (LowerLimit, UpperLimit:Real; var Integral:real);

var
j:Integer;
Xr,Xm,dx:Real;
W,X:ARRAY[1..16]  OF Real;

begin
X[1] :=0.0483076656877383162348;
X[2] :=0.1444719615827964934851;
```

```
X[3] :=0.23928736225213707454S;
X[4] :=0.33186860228212764978O;
X[5] :=0.42135127613063534536 4;
X[6] :=0.50689990893222939002 4;
X[7] :=0.58771575724076232904 1;
X[8] :=0.66304426693021520097 5;
X[9] :=0.73218211874028968038 7;
X[10]:=0.79448379596794240696 3;
X[11]:=0.84936761373256997013 4;
X[12]:=0.89632115576605212396 5;
X[13]:=0.93490607593773968917 1;
X[14]:=0.96476225558750643077 4;
X[15]:=0.98561151154526833540 0;
X[16]:=0.99726386184948156354 5;
W[1] :=0.09654008851472780056 7;
W[2] :=0.09563872007927485941 9;
W[3] :=0.09384439908080456563 9;
W[4] :=0.09117387869576388471 3;
W[5] :=0.08765209300440381114 3;
W[6] :=0.08331192422694675522 2;
W[7] :=0.07819389578707030647 2;
W[8] :=0.07234579410884850622 5;
W[9] :=0.06582222277636184683 8;
W[10]:=0.05868409347853554714 5;
W[11]:=0.05099805926237617619 6;
W[12]:=0.04283589802222668065 7;
W[13]:=0.03427386291302143310 3;
W[14]:=0.02539206530926205945 6;
W[15]:=0.01627439473090567060 5;
W[16]:=0.00701861000947009660 0;

Xm:=0.5 * (Upperlimit + LowerLimit);
Xr:=0.5 * (Upperlimit - LowerLimit);
Integral:=0.0;

For j:=1 To 16 Do
Begin
```

```
dx :=Xr * X[j];
Integral :=Integral + W[j]*(FuncFirst(Xm + dx) + FuncFirst(Xm - dx))
End;
Integral :=Xr * Integral
End;

Procedure Adaptive_Limits_First(var Lowerlimit,UpperLimit:Real;
                    var Nlo,Nup:integer);
var count:integer;
K_fp,K_max,Lower,Upper,TralLow,TralUp:Real;
begin
Nlo:=0;
Nup:=0;

K_fp:= K2 - TRUNC(K2);
K_max:=0.5*(0.5 - K_fp);
IF K_max <= -(0.5)  Then  K_max:=K_max+1;

ING:=FuncFirst(K_max);

If (FuncFirst(K_max)< (1E-05))  Then

Begin
  Lowerlimit:=0.0;
  UpperLimit:=0.0;

  End
Else

 Begin
  Lower:=-(1/2);
  LowerLimit:=K_max + Lower;
  TralLow:=LowerLimit;
  Count:=0;
  While ((FuncFirst(TralLow)) < (ING*1E-05)) AND  (count <500)  Do
Begin
  LowerLimit:=TralLow;
  Lower:=(Lower/2);
```

```
  TralLow:=K_max + Lower;
  Count:=count + 1;
 End;
 Nlo:=count;

 Upper:=(1/2);
  UpperLimit:=K_max + Upper;
  TralUp:=UpperLimit;
  count:=0;
  While ((FuncFirst(TralUp)) < (ING*1E-05)) AND  (count <500)  Do
 Begin
  UpperLimit:=TralUp;
  Upper:=(Upper/2);
  TralUp:=K_max + Upper;
  Count:=count + 1;
 End;
 Nup:=count;
  End;
  End;

{****************Second Integration Start From
Here************************}

Function func(K2,K1 : Real) : Real;

{--------------------------------------------------}
{-      This is the function to integrate        -}
{--------------------------------------------------}
var
TA,a1,a2,X1,X2,X3,X4,F1,F2,S1,S2:Real;

begin
X1:=SNR*sin(2*pi*K1);
X2:=SNR*Rs*cos(2*pi*K1);
X3:=SNR*sin(2*pi*(K1+K2));
X4:=SNR*Rs*Cos(2*pi*(K1+K2));
F1:=((C2/(SIG*det_A))*((C1*X1)-X3));
F2:=((C2/(SIG*det_A))*(X1-(C1*X3)));
```

```
a1:=-(((1/SIG)*X2)+F1);
a2:=-(((1/SIG)*X4)+F2);
S1:=((1-sqr(Py))*g(a1,a2,Py))-(a1*Q((a1-(Py*a2))/SQRT(1-sqr(Py)))*Z(a2));
S2:=((a2*Z(a1)*P((a2-(Py*a1))/SQRT(1-sqr(Py))))-(((a1*a2)+Py)*DT(a1,a2,Py)));
TA:=((S1)+(S2));
func :=(sqr(2*pi)*sqr(SIG))*g(X1,X3,C1)*(TA);
end; { Function First Integral }

Function Y1(K2:Real):Real;
Var
K_fp,K_max_fir,Lowerfir,Upperfir,TralLowfir,TralUpfir,ING_fir:Real;
Begin

K_fp:= K2 - TRUNC(K2);
K_max_fir:=0.5*(0.5 - K_fp);
IF K_max_fir <= -(0.5)  Then  K_max_fir:=K_max_fir + 1;

ING_fir:=func(K2,K_max_fir);

If (ING_fir)< (1E-05)  Then

Begin
  Lowerlimitfir:=0.0;
 End

Else

 Lowerfir:=-(1/2);
 LowerLimitfir:=K_max_fir + Lowerfir;
 TralLowfir:=LowerLimitfir;

 While ((func(K2,TralLowfir)) < (ING_fir*1E-05))  Do
Begin
 LowerLimitfir:=TralLowfir;
 Lowerfir:=(Lowerfir/2);
 TralLowfir:=K_max_fir + Lowerfir;

End;
```

```
Y1:=LowerLimitFir;
End;

Function Y2(K2:Real):Real;
Var
K_fp,K_max_fir,Upperfir,TralUpfir,ING_fir:Real;
Begin

K_fp:= K2 - TRUNC(K2);
K_max_fir:=0.5*(0.5 - K_fp);
IF K_max_fir <= -(0.5)  Then  K_max_fir:=K_max_fir + 1;

ING_fir:=func(K2,K_max_fir);

If (ING_fir)< (1E-05)  Then

Begin
  Upperlimitfir:=0.0;
End

Else

  Upperfir:=(1/2);
 UpperLimitfir:=K_max_fir + Upperfir;
 TralUpfir:=UpperLimitfir;

 While ((func(K2,TralUpfir)) < (ING_fir*1E-05))  Do
Begin
 UpperLimitfir:=TralUpfir;
 Upperfir:=(Upperfir/2);
 TralUpfir:=K_max_fir + Upperfir;

End;

Y2:=UpperLimitFir;
End;
```

```
PROCEDURE quad3d(LowerLimitsec,UpperLimitsec: real; VAR Integral: real);
 PROCEDURE qgaus3(LowerLimit,UpperLimit: real; VAR Integral: real; n: integer);

var
j:Integer;
Xr,Xm,dx:Real;
W,X:ARRAY[1..16]  OF Real;

FUNCTION f(K2: real; n: integer): real;
    VAR
     Integral: real;
     K_max_sec,ING_Sec,Uppersec,LowerSec,Tralupsec,Trallowsec:Real;

    BEGIN
      IF n=1 THEN BEGIN
        glx := K2;
        qgaus3(y1(glx),y2(glx),Integral,2);
        f := Integral
      END
      ELSE f := func(glx,K2);
                              {Adaptive Limits For Secon Integration}
      K_max_sec:=0.5;
ING_sec:=func(K2,K_max_sec);

IF (LowerLimitsec > 0.5)  AND  (UpperLimitsec > 0.5)  THEN
Begin
LowerLimitsec:=LowerLimitsec;
Uppersec:=Xmax;
UpperLimitsec :=Uppersec - K_max_sec;
Tralupsec:=UpperLimitsec;
count:=0;
While ((func(k2,Tralupsec)) <  (ING_sec *1E-05)) AND (count < 500) DO
 Begin
 UpperLimitsec:=TralUpsec;
  Uppersec:=(Uppersec/2);
  TralUpsec:=Uppersec - K_max_sec  ;
  Count:=count + 1;
End;
```

```
End;


IF (LowerLimitsec < 0.5)   AND  (UpperLimitsec < 0.5)  Then
Begin
UpperLimitsec:=UpperLimitsec;
Lowersec:=Xmin;
LowerLimitsec:=K_max_sec - Lowersec;
Trallowsec:=LowerLimitsec;
count:=0;
While ((func(K2,TralLowsec)) <  (ING_sec *1E-05))  AND  (count < 500) DO
Begin
 LowerLimitsec:=TralLowsec;
 Lowersec:=(Lowersec/2);
 TralLowsec:=K_max_sec - Lowersec;
 Count:=count + 1;
End;
End

ELSE
  Begin
 Lowersec:=Xmin;
 LowerLimitsec:=K_max_sec - Lowersec;
 TralLowsec:=LowerLimitsec;
 Count:=0;
 While ((func(K2,TralLowsec)) < (ING_sec*1E-05)) AND  (count <500)  Do
Begin
 LowerLimitsec:=TralLowsec;
 Lowersec:=(Lowersec/2);
 TralLowsec:=K_max_sec - Lowersec;
 Count:=count + 1;
End;

Uppersec:=Xmax;
 UpperLimitsec:=Uppersec - K_max_sec  ;
 TralUpsec:=UpperLimitsec;
 count:=0;
 While ((func(K2,TralUpsec)) < (ING_sec*1E-05)) AND  (count <500)  Do
```

```
 Begin
  UpperLimitsec:=TralUpsec;
  Uppersec:=(Uppersec/2);
  TralUpsec:=Uppersec - K_max_sec  ;
  Count:=count + 1;
 End;
 End;
    END;
 begin

X[1] :=0.04830766568773831623 5;
X[2] :=0.14447196158279649348 5;
X[3] :=0.23928736225213707454 5;
X[4] :=0.33186860228212764978 0;
X[5] :=0.42135127613063534536 4;
X[6] :=0.50689990893222939002 4;
X[7] :=0.58771575724076232904 1;
X[8] :=0.66304426693021520097 5;
X[9] :=0.73218211874028968038 7;
X[10]:=0.79448379596794240696 3;
X[11]:=0.84936761373256997013 4;
X[12]:=0.89632115576605212396 5;
X[13]:=0.93490607593773968917 1;
X[14]:=0.96476225558750643077 4;
X[15]:=0.98561151154526833540 0;
X[16]:=0.99726386184948156354 5;
W[1] :=0.09654008851472780056 7;
W[2] :=0.09563872007927485941 9;
W[3] :=0.09384439908080456563 9;
W[4] :=0.09117387869576388471 3;
W[5] :=0.08765209300440381114 3;
W[6] :=0.08331192422694675522 2;
W[7] :=0.07819389578707030647 2;
W[8] :=0.07234579410884850622 5;
W[9] :=0.06582222277636184683 8;
W[10]:=0.05868409347853554714 5;
W[11]:=0.05099805926237617619 6;
W[12]:=0.04283589802222668065 7;
```

```
W[13]:=0.03427386291302143310З;
W[14]:=0.02539206530926205945б;
W[15]:=0.01627439473090567060S;
W[16]:=0.007018610009470096600;

Xm:=0.5 * (Upperlimit + LowerLimit);
Xr:=0.5 * (Upperlimit - LowerLimit);
Integral:=0.0;

For j:=1 To 16 Do
Begin

dx :=Xr * X[j];
Integral :=Integral + W[j]*(f(Xm + dx,n) + f(Xm - dx,n))
End;
Integral :=Xr * Integral
End;

BEGIN

  qgaus3(LowerLimitsec,UpperLimitsec,Integral,1);
END;

{----------------------------------------------------------------------}
{----------------------------------------------------------------------}

procedure setwindow(x1,y1,x2,y2:integer);
const
upleftcorner=#201;
horzbar    =#205;
uprightcorner=#187;
vertbar=#186;
lowleftcorner=#200;
lowrightcorner=#188;
var
i:integer;
begin
window(x1-1,y1-1,x2+1,y2+1);
```

```
clrscr;
window(1,1,80,25);
gotoxy(x1-1,y1-1);
write(upleftcorner);
for i:=x1 to x2 do
write(horzbar);
write(uprightcorner);
for i:=y1 to y2 do begin
gotoxy(x1-1,i);
write(vertbar);
gotoxy(x2+1,i);
write(vertbar);
end;
gotoxy(x1-1,y2+1);
write(lowleftcorner);
for i:=x1 to x2 do
write(horzbar);
write(lowrightcorner);
window(x1,y1,x2,y2)
end;
{------------------------------------------------------------------------}
{        WS/WB = The Ratio Of Signal To BandWidth Of Filter              }
{          SNR = Signal To Noise Ration                              }
{------------------------------------------------------------------------}

Procedure Input_Data;
begin
Writeln('');
Write('  WS/WB = ');
Read(Rs);
Writeln('');
Writeln('');
Write('    SNR = ');
read(SNR);
Writeln('');
Writeln('');
End;
```

```
{----------------------------------------------------------------------}
{----------------------------------------------------------------------}
Procedure ButterWorth;
begin
Assign(OutFile,'A:FILTER.TXT');
Rewrite(OutFile);
LeaveGraphic;
ClearScreen;
TextColor(Black);
TextBackGround(7);
Setwindow(25,5,45,12);
Input_Data;
gotoxy(1,25);
Write('Please Wait....');

K2:=0.1;
repeat
 T:=((2.0*PI*K2)/Rs);
 ButterWorth_Signal_Calculate(C1,C2,C3,C4,det_A,SIG,Py);
 Adaptive_Limits_first(LowerLimit,UpperLimit,Nlo,Nup);
 Adaptive_Gaus(LowerLimit,UpperLimit,Integral);

writeln(OutFile,K2:5:5,'   ',Integral:3:3);
 K2:=K2+0.1;
 Until K2>2.0;
 Close(OutFile);
 ClearScreen;
 InitGraphic;
end;

Procedure BrickWall;
begin
Assign(OutFile,'A:FILTER.TXT');
Rewrite(OutFile);
LeaveGraphic;
ClearScreen;
TextColor(Black);
TextBackGround(7);
```

```
 Setwindow(25,5,45,12);
 Input_Data;
 gotoxy(1,25);
 Write('Please Wait....');

  K2:=0.1;
 repeat
   T:=((2.0*PI*K2)/Rs);
   BrickWall_Signal_Calculate(C1,C2,C3,C4,det_A,SIG,Py);
   Adaptive_Limits_first(LowerLimit,UpperLimit,Nlo,Nup);
   Adaptive_Gaus(LowerLimit,UpperLimit,Integral);

 writeln(OutFile,K2:5:5,'   ',Integral:3:3);
  K2:=K2+0.1;
  Until K2>2.0;
  Close(OutFile);
  ClearScreen;
  InitGraphic;
 end;
 Procedure Probability_of_ButterWorth;
 Begin
 LeaveGraphic;
 ClearScreen;
 TextColor(Black);
 TextBackGround(7);
 Setwindow(54,2,75,10);
 Input_Data;
 setWindow(2,2,50,10);
 GetLimits(Xmin,Xmax);
 Setwindow(15,14,65,16);
 GoToXY(1,1);
 Writeln('Please Wait....');
 T:=((2.0*PI*K2)/Rs);
 ButterWorth_Signal_Calculate(C1,C2,C3,C4,det_A,SIG,Py);
  quad3d(Xmin,Xmax,Result);
  If RS=0.5 Then Result:=Result*4.0
  Else Result:=Result;
 Results_Area(Result);
```

```
 Repeat Until KeyPressed;
 Close(OutFile);
  ClearScreen;
  InitGraphic;
 end;

 Procedure Probability_of_BrickWall;
 Begin
 LeaveGraphic;
 ClearScreen;
 TextColor(Black);
 TextBackGround(7);
 Setwindow(54,2,75,10);
 Input_Data;
 setWindow(2,2,50,10);
 GetLimits(Xmin,Xmax);
 Setwindow(15,14,65,16);
 GoToXY(1,1);
 Writeln('Please Wait....');
 T:=((2.0*PI*K2)/Rs);
 BrickWall_Signal_Calculate(C1,C2,C3,C4,det_A,SIG,Py);
  quad3d(Xmin,Xmax,Result);
  If RS=0.5 Then Result:=Result*4.0
  Else Result:=Result;
 Results_Area(Result);
 Repeat Until KeyPressed;
 Close(OutFile);
  ClearScreen;
  InitGraphic;
 end;



begin { program Adaptive_Gauss_Quadrature_double_crossing }
 clrScr;

 CheckBreak := false;
 InitGraphic;
```

```
SetBreakOn;
SetMessageOn;
SetHeaderOn;
DefineWorld(2, 0, 199, 639, 0);
DefineWindow(10, 0, 0, XMaxGlb, YMaxGlb);
Intro;
Choice := 0;
repeat
 ClearScreen;
 DefineWorld(2, 0, 199, 639, 0);
 SetHeaderOn;
 DefineWindowIBM(5, 4, 5, 73, 180);
 DefineHeader(5,'THE MENUE ITEMS');
 SelectWindow(5);
 SetBackground(24);
 DrawBorder;
 DefineWindowIBM(6, 13, 10, 64, 40);
 DefineHeader(6,'TO EXIT THE DEMOSTRATION ');
 SelectWindow(6);
 SetBackground(0);
 DrawBorder;
 storewindow(6);
 DefineWindowIBM(7, 13, 40, 64, 110);
 DefineHeader(7,'CHOOSE YOUR FILTER');
 SelectWindow(7);
 SetBackground(0);
 DrawBorder;
 storewindow(7);
 DefineWindowIBM(8, 13, 110, 64, 170);
 DefineHeader(8,'To Find The Probability Of Zerocrossing');
 SelectWindow(8);
 SetBackground(0);
 DrawBorder;
 storewindow(8);

 Choice := MenuChoice(Choice);
 ClearScreen;
 case Choice of
```

```
    1 : ButterWorth;
    2 : BrickWall;
    3 : Plot_The_Graph;
    4 : Probability_of_ButterWorth;
    5 : Probability_of_BrickWall;
  end;
until Choice = 0;
LeaveGraphic;
End.(end Of The Program);
```

APPENDIX 7.A

IIR Filter Coefficient

The Z transform of the unit-sample response of an filter is Lin[1987]

$$H(z) = \prod_{i=0}^{N} \frac{(A_{0i} + A_{1i} Z^{-1} + A_{2i} Z^{-2})}{1 - B_{1i} Z^{-1} + B_{2i} Z^{-2}} \qquad (7.A.1)$$

Where the filter is realized as a series of biquads. The coefficient values are given as

■ **Second Order ButterWorth Filter**

a_{00}	+0.9546738		
a_{10}	+0.1909348	b_{10}	-1.7052101
a_{20}	+0.9546738	b_{20}	+0.7433956

■ **8th Order Butterworth Filter**

a_{00}	+0.9661901		
a_{10}	+0.1932380	b_{10}	-1.6248401
a_{20}	+0.9661901	b_{20}	+0.6611628
a_{01}	+0.9661901		
a_{11}	+0.1932380	b_{11}	-1.6678643
a_{21}	+0.9661901	b_{21}	+0.7051426
a_{02}	+0.9661901		
a_{12}	+0.1932380	b_{12}	-1.7536552
a_{22}	+0.9661901	b_{22}	+0.7928487
a_{03}	+0.9661901		
a_{13}	+0.1932380	b_{13}	-1.8799921
a_{23}	+0.9661901	b_{23}	+0.9220173

■ **Second Order Chebychev Filter**

a_{01}	+0.7270929		
a_{11}	+0.1454186	b_{10}	-1.8448801
a_{21}	+0.7270929	b_{20}	+0.8739652

8th Order Chebychev Filter

a_{01}	+0.3510437		
a_{11}	+0.7020873	b_{10}	-1.9532601
a_{21}	+0.3510437	b_{20}	+0.9554339
a_{02}	+0.3510437		
a_{12}	+0.7020873	b_{11}	-1.9483421
a_{22}	+0.3510437	b_{21}	+0.9622009
a_{03}	+0.3510437		
a_{13}	+0.7020873	b_{12}	-1.9442340
a_{23}	+0.3510437	b_{22}	+0.9746893
a_{04}	+0.3510437		
a_{14}	+0.7020873	b_{13}	-1.9486723
a_{24}	+0.3510437	b_{23}	+0.9910649