THE UNIVERSITY OF HULL

APPLICATION OF QUANTUM FIELD THEORY TO CONDENSED SYSTEMS

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ABSTRACT

We examine in detail the microscopic theories of Beliaev and Hugenholtz and Pines as applied to an interacting system of bosons in its ground state and demonstrate that these theories are strictly applicable to a zero density gas only. After providing a re-normalization of the time dependent perturbation theory of a many fermion system in its ground state, we reformulate the corresponding boson theory without ab initio approximations concerning the $\underline{k} = 0$ mode.

The resulting boson perturbation theory has a diagrammatic structure which is topologically identical with the corresponding fermion theory and provided the implicit assumptions concerning convergence are valid, we conclude that a finite fraction of the particles in an arbitrary interacting Bose gas at $T = 0K$ will not condense into the $\underline{k} = 0$ mode.

CONTENTS

INTRODUCTION

The remarkable success of time dependent perturbation theory in the many fermion problem is in marked contrast with its almost singular lack of success in the treatment of the corresponding boson problem. The reason for this situation can be traced to the fact that unlike the theory of fermions, the time dependent perturbation theory of bosons has not been formulated in a rigorous manner which can be treated exactly by graphical methods. The present work seeks to remove this deficiency and to provide the basis for a mathematically exact theory which is applicable to the many boson problem.

To this end, and in order to gain familiarity with some new techniques developed specifically for this purpose, we begin with a discussion of conventional Feynman-Dyson perturbation theory as applied to a system of interacting fermions at $T = 0K$, and proceed to reformulate the theory in terms of an effective interaction, Γ. The result is a highly summed and manifestly self-consistent version of conventional perturbation theory and all known applications of the theory may be obtained from zeroth or first order approximations in Γ. The techniques employed, although merely convenient in the fermion problem, turn out to be essential for the application of diagrammatic perturbation theory to the many boson problem.

In order to emphasize this, our initial discussion of the boson problem focusses attention on an expansion of the single particle condensate Green's function, obtained by employing a counting technique originally proposed by Brandow [1]. It is demonstrated that the apparently reasonable approximation employed by Brandow [1] and others [2] of ignoring diagrams of relative order $1/V$, where V is the volume of the system, is strictly only applicable to a zero-density

bose gas.

In view of this, we examine Beliaev's [3] formulation of the problem and in this way are led to the discovery of an error in his evaluation of the condensate Green's function. The arguments of Hugenholtz and Pines [4], which lead to the same result as Beliaev for the single particle Green's function are then briefly discussed.

This brings us to the central part of the thesis, in which the new techniques mentioned above, are employed in order to obtain an equation for the single particle condensate Green's function of a many boson system in its ground state. This equation allows us to demonstrate that the formalism of Beliaev and Hugenholtz and Pines although self-consistent, applies strictly to a zero gas only. We then describe an infinite hierarchy of self-consistent solutions, which do not suffer from this restriction.

Finally, we obtain a connected diagram expansion for the single particle Green's function of a system of interacting bosons in their ground state and demonstrate that it is topologically identical with the corresponding fermion expansion. The results suggest that an arbitrary interacting bose gas at $T = 0K$ does not possess a condensate.

CHAPTER 1

FERMION PERTURBATION THEORY

1.1. Introduction

The single particle Green's function $G^{(1)}$ of a system of interacting particles is a convenient quantity to calculate for several reasons. One of these is that once $G^{(1)}$ is known, the observables of greatest interest may be obtained, [5,6]. Another is that the Feynman rules for obtaining the n th order contribution to $G^{(1)}$ from perturbation theory are particularly simple and for this reason, $G^{(1)}$ will be the focus of our attention throughout this thesis.

When conventional perturbation theory [7,8] is employed to calculate $G^{(1)}$ for a system of interacting fermions in their ground state, the result is an infinite series of connected diagrams, whose sum is given by the well-known Dyson's equation for the fourier transform of the single particle Green's function $G^{(1)}(\underline{k},\omega)$,

$$G^{(1)}(\underline{k},\omega) \equiv G^{(1)}(K) = [\omega - \omega_{\underline{k}} - \Sigma(\underline{k},\omega)]^{-1} \qquad (1.1)$$

where

$$\omega_{\underline{k}} = \frac{\hbar K^2}{2m}$$

and Σ is known as the proper or irreducible self energy. In any practical calculation, $G^{(1)}$ is evaluated by summing an infinite sub-set of the diagrams which contribute to Σ, selected on the basis of physical arguments. However, outside of any explicit evaluation of Σ, it is well-known [9] that to every diagram in the expansion of Σ there corresponds an infinite sub-set of diagrams whose sum is self-consistently obtained by

replacing the 'bare' propagator lines in the diagram by 'clothed' propagator lines. These clothed lines are simply $G^{(1)}$ and the approximation of retaining only first order diagrams in the resulting series for Σ is the well-known Hartree-Fock approximation.

With all the propagator lines clothed, equation (1.1) becomes a self-consistent equation for $G^{(1)}$. The advantage of the 'clothed' form of Σ over the 'bare' form is demonstrated when one recognizes that first order diagrams of the former yield agreement with experiment in a wide range of applications, while first order diagrams of the latter do not. However, even when Σ is 'clothed', it is often necessary to sum an infinite sub-set of diagrams in order to obtain agreement with experiment. An example of this is the case of a hard sphere gas [10]. In situations of this type, it is usually the bare interaction lines which need to be 'clothed', and we ask ourselves if it is possible to perform further self-consistent summations of the perturbation series, without introducing approximations which are particular to a given calculation, in such a way that <u>all</u> practical calculations are reduced to the evaluation of a finite number of terms in the resulting perturbation series.

We shall demonstrate that this is indeed possible. In order to perform the actual calculation, it is convenient to introduce a technique which we developed originally in connection with the theory of bosons.

However, prior to this, it is convenient to give a brief account of conventional fermion perturbation theory.

1.2 Conventional Perturbation Theory

We shall consider a uniform system of N interacting fermions of mass m in their ground state $|\Psi_N\rangle$ and suppose that N and V become infinite, while the density $n = N/V$ remains finite. The Hamiltonian is assumed to take the form

$$H = H_0 + H_1 \tag{1.2}$$

where

$$H_0 = \int d^3x \, \psi^\dagger(\underline{x}) \left[-\frac{\hbar^2 \nabla^2}{2m} \right] \psi(\underline{x}) \tag{1.3}$$

and

$$H_1 = \tfrac{1}{2} \int d^3x \, d^3x' \, \psi^\dagger(\underline{x}) \psi^\dagger(\underline{x}') u(\underline{x}-\underline{x}') \psi(\underline{x}') \psi(\underline{x}) \tag{1.4}$$

$u(\underline{x}-\underline{x}')$ is an instantaneous, spin independent, two body interaction and the field operators are given by

$$\psi(\underline{x}) = \sum_{\underline{k}} V^{-1/2} e^{i\underline{k}.\underline{x}} a_{\underline{k}} \tag{1.5}$$

The subscripts $\underline{k}$ in equation (1.5) refer strictly to both spin and momentum, but since the effects of the former are easily accounted for by the Feynman rules which apply to the perturbation expansion we are about to obtain, we shall not consider spin explicity in what follows. The statistics of the problem are contained in the anti-commutation rules

$$[a_{\underline{k}}, a^\dagger_{\underline{k}'}]_+ \equiv a_{\underline{k}} a^\dagger_{\underline{k}'} + a^\dagger_{\underline{k}'} a_{\underline{k}} = \delta_{\underline{k},\underline{k}'}$$

and (1.6)

$$[a_{\underline{k}}, a_{\underline{k}'}]_+ = [a^\dagger_{\underline{k}}, a^\dagger_{\underline{k}'}]_+ = 0$$

where $a_{\underline{k}}$ $(a^{\dagger}_{\underline{k}})$ annihilates (creates) a particle with momentum $\hbar \underline{k}$.

The single particle Green's function $G^{(1)}(x,y)$ is defined by

$$iG^{(1)}(x,y) = \langle \mathcal{F}_N | T[\psi_H(x)\, \psi^{\dagger}_H(y)] | \mathcal{F}_N \rangle \qquad (1.7)$$

where $|\mathcal{F}_N\rangle$ is assumed normalized. The field operators appearing in equation (1.7) are in the Heisenberg picture and are defined by

$$\psi_H(x) \equiv \psi_H(\underline{x}, t_x) = e^{\frac{iHt_x}{\hbar}}\, \psi(\underline{x})\, e^{-\frac{iHt_x}{\hbar}} \qquad (1.8).$$

The time dependent perturbation expansion of $G^{(1)}$ is usually obtained by assuming that $|\mathcal{F}_N\rangle$ may be generated adiabatically from a non-degenerate, non-interacting, N particle ground state $|\Phi_N\rangle$, in accordance with the procedure of Gell-Mann and Low [11]. This allows us to write

$$iG^{(1)}(x,y) = \frac{\langle \Phi_N | T[\psi(x)\, \psi^{\dagger}(y)\, \hat{S}] | \Phi_N \rangle}{S_N} \qquad (1.9)$$

where

$$\hat{S} = \lim_{\varepsilon \to 0^+} T\left[\exp\left\{\frac{-i}{\hbar}\int_{-\infty}^{\infty} dt\, e^{-\varepsilon|t|} H_1(t)\right\}\right] \qquad (1.10)$$

and

$$S_N = \langle \Phi_N | \hat{S} | \Phi_N \rangle \qquad (1.11).$$

Equation (1.7) is in the Heisenberg picture whereas equation (1.9) is in the interaction picture. The interaction part of the Hamiltonian in the interaction picture is

$$H_1(t) \equiv e^{\frac{iH_0t}{\hbar}} H_1 e^{-\frac{iH_0t}{\hbar}}$$

$$= \tfrac{1}{2}\int d^3x_1\, d^3x_1'\, U(\underline{x}_1 - \underline{x}_1')\, \psi^\dagger(\underline{x}_1, t)\, \psi^\dagger(\underline{x}_1', t)\, \psi(\underline{x}_1', t)\, \psi(\underline{x}_1, t) \qquad (1.12)$$

where

$$\psi(\underline{x}, t) \equiv \psi(x) = \sum_{K=0} V^{-1/2} e^{i\underline{K}\cdot\underline{x}} e^{-i\omega_{\underline{K}} t} a_{\underline{K}} \qquad (1.13)$$

with

$$\omega_{\underline{K}} = \frac{\hbar K^2}{2m}.$$

In the case of fermions, all the single particle levels of the state $|\Phi_N\rangle$ with energies less than or equal to the Fermi energy ε_F^0 are occupied, while those with energies greater than ε_F^0 are unoccupied. The commutation rules in equations (1.6) allow us to write

$$a_{\underline{K}}^\dagger |\Phi_N\rangle = 0 \qquad \text{for} \quad K \leq K_F$$

which leads us to regard $a_{\underline{K}}^\dagger$ $(K \leq K_F)$ as an annihilation operator for the state $|\Phi_N\rangle$. Hence, the field operators may be written [8]

$$\psi(x) = \phi(x) + \beta^\dagger(x) \qquad (1.14)$$

and

$$\psi^\dagger(y) = \phi^\dagger(y) + \beta(y)$$

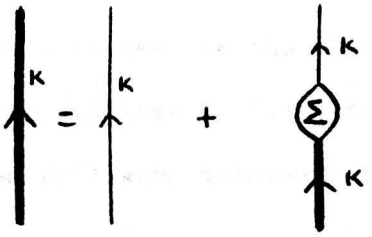
where

$$\beta|\Phi_N\rangle = \phi|\Phi_N\rangle = 0$$

The separation of the field operators expressed in equation (2.6) represents a canonical transformation to particles $(K > K_F)$ and holes $(K \leq K_F)$ and leads to great simplifications when Wick's theorem [12] is applied to equation (1.9), because averages of normal ordered products of uncontracted operators vanish. It is

this property which leads to the cancellation of disconnected diagrams in fermion perturbation theory.

The diagrammatic perturbation series obtained from equation (2.2) is well-known [5,6] and the end result can be expressed in terms of Dyson's equation in the form

$G(K) \equiv$ 

(1.15).

In equation (1.15), a thin solid line propagating 4-momentum $K\ (\equiv \underline{K}, \omega)$ is simply the fourier transform of the single particle non-interacting Green's function, given by [5,6]

$g(K) \equiv$ 

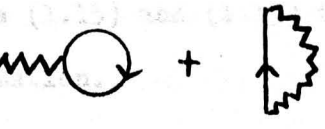
$= \dfrac{\Theta(\underline{K}-\underline{K}_F)}{\omega - \omega_{\underline{K}} + i\eta} + \dfrac{\Theta(\underline{K}_F-\underline{K})}{\omega - \omega_{\underline{K}} - i\eta}$ (1.16)

Written algebraically, equation (1.15) assumes the form

$$G(K) = g(K) + g(K)\,\Sigma(K)\,G(K)$$

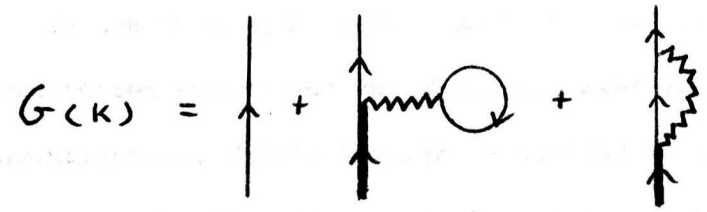
which is simply equation (1.1).

The first order contribution to Σ is

$\Sigma^{(1)} =$  (1.17)

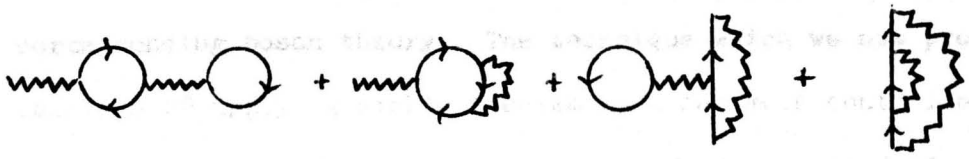
where a wavy line carrying a momentum K represents a factor $U(K)$,

which is simply the fourier transform of the two-body potential $U(\underline{x} - \underline{y})$. Hence, to first order, equation (1.15) yields

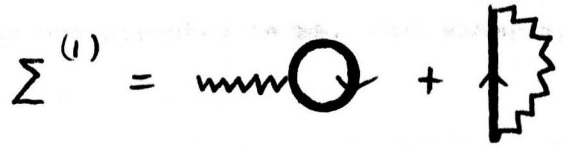


(1.18)

It is now easy to see how the 'bare' thin solid lines in equation (1.17) may be replaced by 'clothed' thick solid lines. To this end, consider the following diagrams which also contribute to Σ :



Obviously these and the corresponding higher order diagrams may be omitted from the expansion of Σ , provided equation (1.17) is written



(1.19)

This renormalization of thin solid lines occurs to all orders [9] and we end this discussion of conventional perturbation theory by noting that equations (1.15) and (1.19) together form the well-known Hartree-Fock approximation.

1.3 An expansion of the single particle Green's function in terms of an effective interaction

We are seeking a highly summed version of conventional perturbation theory and our starting point is equation (1.9). Conventionally, Wick's theorem is applied to this equation in a rather uncontrolled manner and the result is a proliferation of terms, which contribute to both the numerator and denominator. However, it is well-known [8] that the denominator of equation (1.9) serves to cancel exactly the disconnected diagrams which appear in the expansion of the numerator, a feature which is not present in the corresponding boson theory. The technique which we now propose, consists of applying Wick's theorem in a far more controlled manner and avoids completely the introduction of disconnected diagrams into the theory. We shall see that the resulting theory provides a link between time dependent perturbation theory and the equation of motion method [13].

We define an operator C such that when it acts on a string of operators, the result is a sum over all possible contractions, including uncontracted terms. For example,

$$C\left[\psi_{(x)}\,\psi^{\dagger}_{(y)}\right] = \psi_{(x)}\,\psi^{\dagger}_{(y)} + \overset{\bullet}{\psi}_{(x)}\,\overset{\bullet}{\psi}{}^{\dagger}_{(y)} \qquad (1.20)$$

where dots denote contractions. Representing the normal ordering operator by $\hat{N}$ allows us to write Wick's theorem in the symbolic form

$$T = \hat{N}\,C \qquad (1.21).$$

We now apply equation (1.21) to the numerator of equation (1.9)

and focus our attention on all possible contractions involving the operator $\psi(x)$. This is conveniently achieved by writing

$$C = C' C_{\psi S} C_{\psi\psi^+} \tag{1.22}$$

when applying equation (1.21) to equation (1.9). This yields

$$iG^{(1)}(x,y) = \frac{\langle \Phi_N | \hat{N}[C' C_{\psi S} C_{\psi\psi^+}\{\psi(x)\psi^+(y)\hat{S}\}] | \Phi_N\rangle}{S_N} \tag{1.23}$$

where $C_{\psi\psi^+}$ contracts $\psi(x)$ with $\psi^+(y)$ according to the prescription (1.20). Similarly, $C_{\psi S}$ contracts $\psi(x)$ with the field operators in $\hat{S}$, whilst C' performs the sum over all remaining contractions. Noting that $C_{\psi\psi^+}$, $C_{\psi S}$ and C' commute and allowing $C_{\psi\psi^+}$ to act yields

$$iG^{(1)}(x,y) = \psi(x)\psi^+(y) + \frac{\langle \Phi_N | \hat{N}[C' C_{\psi S}\{\psi(x)\psi^+(y)\hat{S}\}] | \Phi_N\rangle}{S_N} \tag{1.24}$$

Writing equation (1.10) in the form [5,6]

$$\hat{S} = 1 + \sum_{\nu=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{\nu!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_\nu\, e^{-\varepsilon(|\epsilon_1|+\cdots|\epsilon_\nu|)}\, T[H_1(\epsilon_1)\cdots H_1(\epsilon_\nu)] \tag{1.25}$$

we see that the last term on the right hand side (r.h.s.) of equation (1.24) may be written

$$\sum_{\nu=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{\nu!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_\nu\ \frac{\langle \Phi_N | \hat{N}[C' C_{\psi S}(H_1(\epsilon_1)\cdots H_1(\epsilon_\nu)\psi(x)\psi^+(y))] | \Phi_N\rangle}{S_N}$$

$$= \sum_{\nu=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{(\nu-1)!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_\nu \tag{1.26}$$

$$\times \frac{\langle \Phi_N | \hat{N}[C'(H_1^\bullet(\epsilon_1)\psi^\bullet(x)H_1(\epsilon_2)\cdots H_1(\epsilon_\nu)\psi^+(y))] | \Phi_N\rangle}{S_N}$$

where, from equation (1.12),

$$H_1^{\bullet}(t_1)\,\psi^{\bullet}(x) = \int d^3x_1\, d^3x_1'\, U(\underline{x}_1 - \underline{x}_1')\, \psi^{\bullet}(x)\, \psi^{\dagger\bullet}(\underline{x}_1, t_1)\, \psi^{\dagger}(\underline{x}_1', t_1)\, \psi(\underline{x}_1', t_1)\, \psi(\underline{x}_1, t_1)$$

Hence, equation (1.26) becomes

$$\int d^3x_1\, d^3x_1'\, U(\underline{x}_1 - \underline{x}_1') \sum_{\nu=0}^{\infty} \left(\frac{-i}{\hbar}\right)^{\nu} \frac{1}{(\nu-1)!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_\nu\, \psi^{\bullet}(x)\, \psi^{\dagger\bullet}(\underline{x}_1, t_1)$$

$$\times \frac{\langle \Phi_N | T[H_1(t_2) \cdots H_1(t_\nu)\, \psi^{\dagger}(\underline{x}_1, t_1)\, \psi(\underline{x}_1', t_1)\, \psi(\underline{x}_1, t_1)\, \psi^{\dagger}(y)] | \Phi_N \rangle}{S_N} \tag{1.27}$$

Writing equation (1.27) in the Heisenberg picture, we see that equation (1.24) takes the form

$$iG^{(1)}(x,y) = i\,g(x,y) + \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{\infty} dt_1 \int d^3x_1\, d^3x_1'\, U(\underline{x}_1 - \underline{x}_1')\, i\,g(x, x_1)\, e^{-\varepsilon|t_1|}$$

$$\times \langle \mathcal{F}_N | T[\psi_H^{\dagger}(\underline{x}_1', t_1)\, \psi_H^{\dagger}(y)\, \psi_H(\underline{x}_1', t_1)\, \psi_H(x_1, t_1)] | \mathcal{F}_N \rangle \tag{1.28}$$

where we have written the single particle non-interacting Green's function $i\,g(x,y) = \psi^{\bullet}(x)\, \psi^{\dagger\bullet}(y)$.

In order to write equation (1.28) in terms of diagrams we shall employ the following notation. An m particle Green's function, given by

$$(i)^m G^{(m)}(x_1 \cdots x_m, x_1' \cdots x_m') = \langle \mathcal{F}_N | T[\psi_H(x_1) \cdots \psi_H(x_m)\, \psi_H^{\dagger}(x_1') \cdots \psi_H^{\dagger}(x_m')] | \mathcal{F}_N \rangle$$

will be denoted by m heavy solid lines entering the points $x_1 \cdots x_m$ and m heavy solid lines leaving the points $x_1' \cdots x_m'$. These lines will have free ends. A special case of this rule is that of the

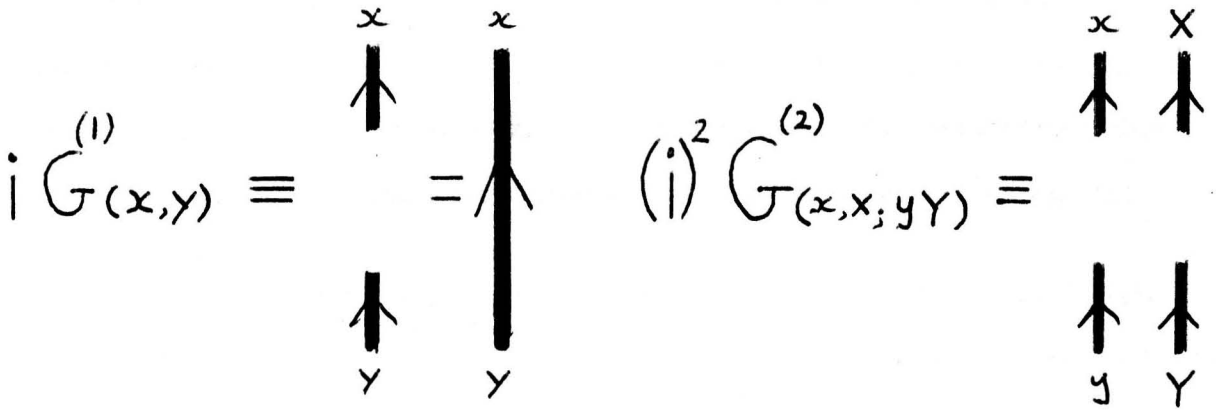


FIGURE 1

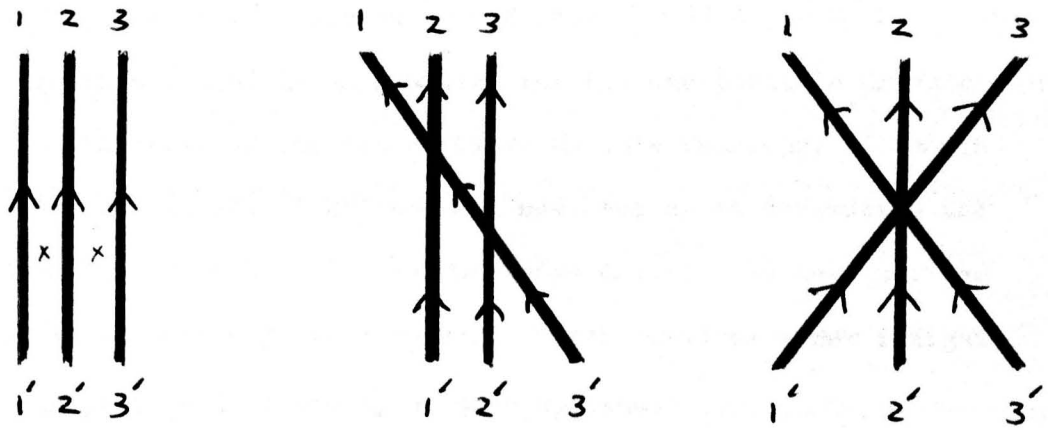
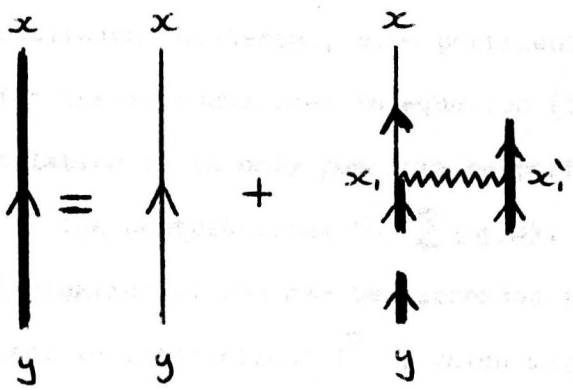


FIGURE 2

single particle Green's function $iG^{(1)}(x,y)$, which will be denoted by a thick solid line running continuously from y to x. Two examples are given in figure (1). A factor $ig(x,y)$ will be denoted by a thin solid line running from y to x. The remaining rules for interpreting diagrams constructed in this way are well-known and may be found in the literature [5,6].

Equation (1.28) may now be written



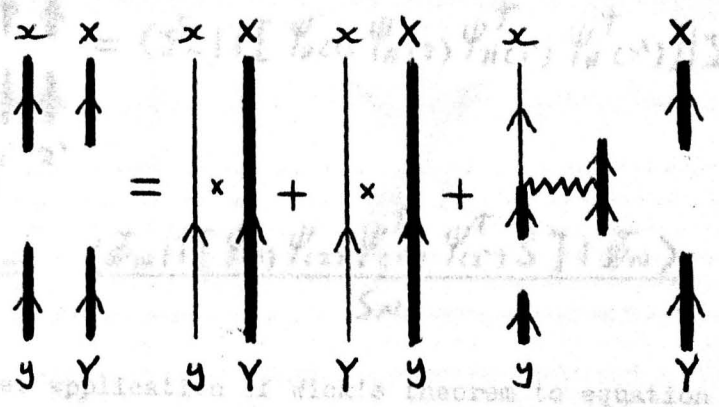
(1.29)

where $\underset{\sim}{x}$, ~~~~ $\underset{\sim}{x}'$ represents a factor $\left(\frac{-i}{\hbar}\right) U(\underset{\sim}{x}, -\underset{\sim}{x}')$

Equation (1.29) is an equation for the one particle Green's function in terms of the two particle Green's function. Although equations (1.28) and (1.29) are not new, the above derivation and presentation of them is to our knowledge unique. As demonstrated below, it is this different approach which provides a new insight into the perturbation theory of bose systems.

The same arguments leading to equation (1.29) may be used to obtain an equation for $G^{(2)}$ (and all higher order Green's functions). For example

$$(i)^2 G^{(2)}(x,X;y,Y) \equiv$$



(1.30)

which expresses $G^{(2)}$ in terms of $G^{(3)}$. Equations (1.29), (1.30) and higher order equations are the quantum analogue of the B.B.G.K.Y. hierarchy of equations for the classical distribution functions, [14], and may be obtained using the equation of motion method [13,15]. It has been noted [15,16] that further progress can only be made in general, by invoking some approximations on the two particle Green's function $G^{(2)}$ appearing in equation (1.29). An equivalent statement, more pertinent to the conventional perturbation theory summarized in equation (1.15), is that in any practical calculation it is only possible to perform the summation of a sub-set of the contributions to $\sum(\underline{k},\omega)$. We shall now demonstrate that equation (1.29) may be expressed in terms of a symmetrized 'effective interaction' Γ, which may be determined self-consistently. The self-consistent expression for the effective interaction is in the form of an infinite perturbation series (involving Γ) and the problem is reduced to that of choosing some suitable approximation for Γ. However, unlike conventional perturbation theory, it turns out this choice does not involve the summation of an infinite sub-set of contributions involving Γ. Rather we shall demonstrate that the zeroth and first order approximations to Γ are sufficient for most practical calculations.

In order to proceed we note that by definition of $G^{(2)}$ we have

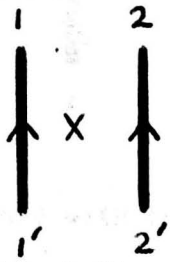
$$(i)^2 G^{(2)}_{(1,2;1',2')} \equiv \quad \begin{array}{cc} \uparrow_1 & \uparrow_2 \\ \uparrow_{1'} & \uparrow_{2'} \end{array} \quad = \langle \mathcal{F}_N | T[\psi_{H(1)}\psi_{H(2)}\psi^{\dagger}_{H(1')}\psi^{\dagger}_{H(2')}] | \mathcal{F}_N \rangle$$

$$= \frac{\langle \Phi_N | T[\psi_{(1)}\psi_{(2)}\psi^{\dagger}_{(1')}\psi^{\dagger}_{(2')}\hat{S}] | \Phi_N \rangle}{S_N} \qquad (1.31)$$

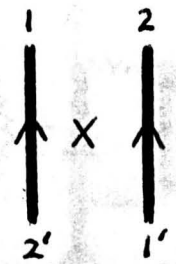
As mentioned earlier application of Wick's theorem to equation (1.31) leads, within conventional perturbation theory, to an infinite series

of connected diagrams. These diagrams fall into one of three categories:-

(i) Those diagrams connecting the point $1'$ to the point 1, but not to the points $2'$ or 2. Each of these diagrams is multiplied by a diagram connecting the point $2'$ to the point 2, but not to the points $1'$ or 1. The sum of all diagrams in this category is obviously

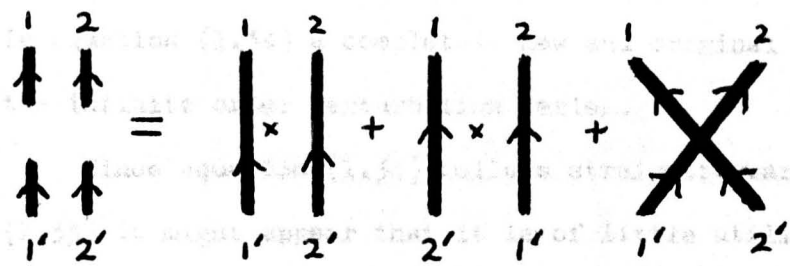


(ii) The second category is similar to the first apart from the interchange of the labels $1'$ and $2'$. Clearly the sum of diagrams in this category is



(iii) The third category contains all diagrams which are connected to all four points $1, 1', 2, 2'$. The sum of diagrams of this type may be represented by

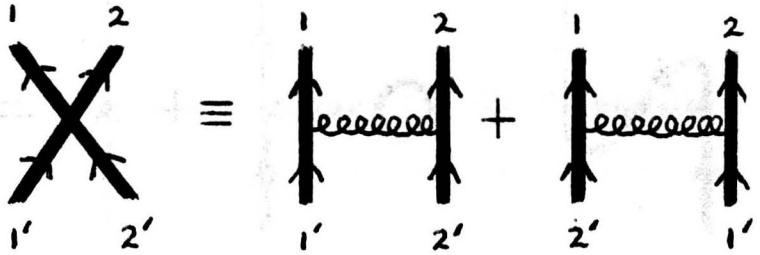
With this notation equation (1.31) may be written



(1.32).

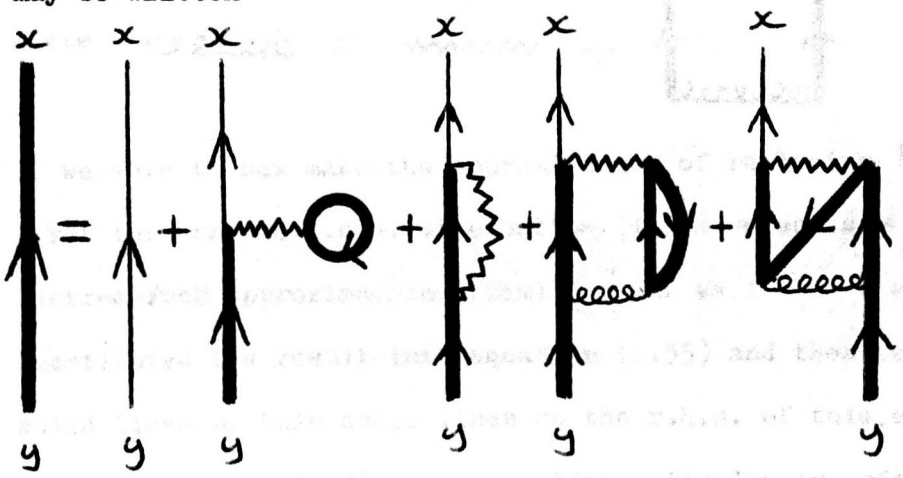
Note that we have ignored any minus one factors which arise from the anti-commutation rules, since these are automatically taken into account by the Feynman rules which apply to equation (1.29). We now

define an effective interaction Γ , denoted in the diagrams by ~~~~~~~~ , in terms of which we write the last term on the r.h.s. of equation (1.32) in the symmetrized form,



(1.33)

Equation (1.33) defines Γ and with this notation equation (1.29) may be written

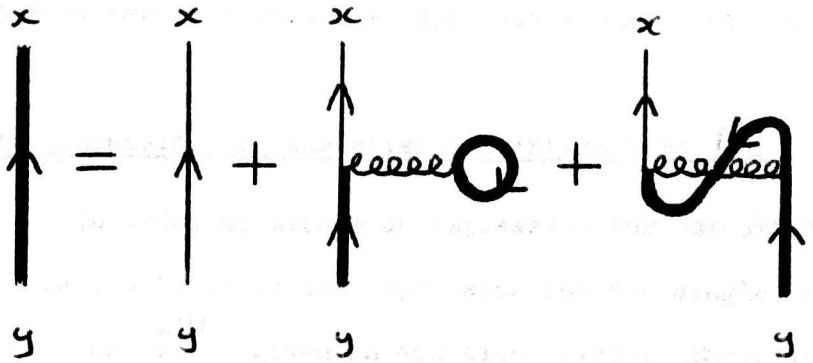


(1.34)

Although the decomposition expressed in equation (1.32) may be found in the literature [9,13,21] where it is often used to define the reducible vertex part, the expression for the last term on the r.h.s. of equation (1.32) in the form of equation (1.33) has rarely if ever been discussed. Our introduction of equation (1.33) provides in equation (1.34) a completely new and original way of representing the infinite order perturbation series.

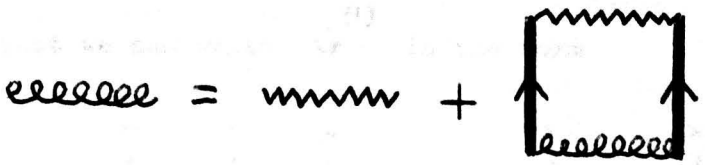
Since equation (1.34) follows straightforwardly from equation (1.33) it might appear that it is of little utility until we have obtained an expression for the effective interaction Γ . Although such an expression can be readily derived (as shown explicitly below) we wish first (as an interesting aside) to

demonstrate that equation (1.34) is itself an extremely useful equation. To this end we note that a <u>self-consistent</u> solution of of the latter takes the form



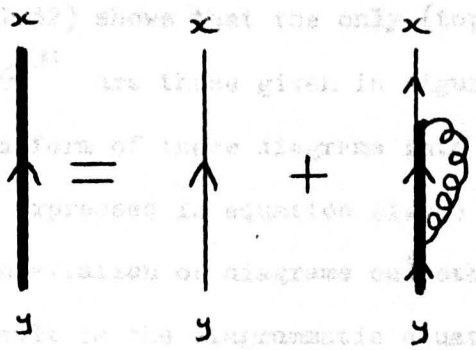
(1.35)

where



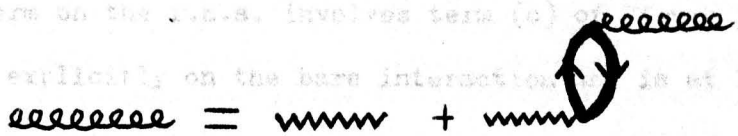
(1.36).

If we were to now make the approximation of replacing Γ by the first term on the r.h.s. of equation (1.36) we would obtain the Hartree-Fock approximation. Similarly if we iterated equation (1.36), substituted the result into equation (1.35) and then replaced thick solid lines by thin solid lines on the r.h.s. of this equation, we would obtain the ladder approximation. Finally in order to obtain what is essentially the ring approximation we must return to equation (1.34) and ignore the second and fifth terms on the r.h.s. If this is done, a <u>self-consistent</u> solution of the resulting equation is



(1.37)

where

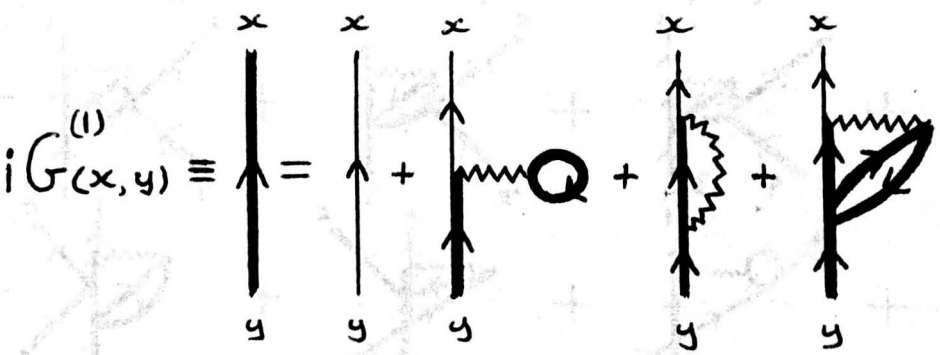


(1.38).

Iteration of equation (1.38) followed by the replacement of thick solid lines by thin ones then yields the desired result. This analysis shows that all the well-known methods of fermion perturbation theory can be regarded as approximations to self-consistent solutions.

1.4 An expression for the effective interaction Γ.

In order to obtain an expression for the effective interaction we return first to the expression for the single particle Green's function $G^{(1)}$ given in equation (1.29). Employing the represention for the two particle Green function $G^{(2)}$ given in equation (1.32) we see that we can write $G^{(1)}$ in the form



(1.39)

The next step in the argument requires us to employ equation (1.30) and expand the last term on the r.h.s., which involves the three particle Green function $G^{(3)}$. Employing an exactly analagous procedure to that which led to the expression for $G^{(2)}$ given by equation (1.32) shows that the only (topologically) distinct diagrams entering $G^{(3)}$ are those given in Figure 2. Substituting the symmetrized form of these diagrams into equation (1.30), and employing the result expressed in equation (1.39) for $G^{(1)}$ leads to a partial cancellation of diagrams on both sides of the ensuing equation. The end result is the diagrammatic equation given in Figure 3, where the last term on the r.h.s. involves term (c) of Figure 2, which does not depend explicitly on the bare interaction and is at least

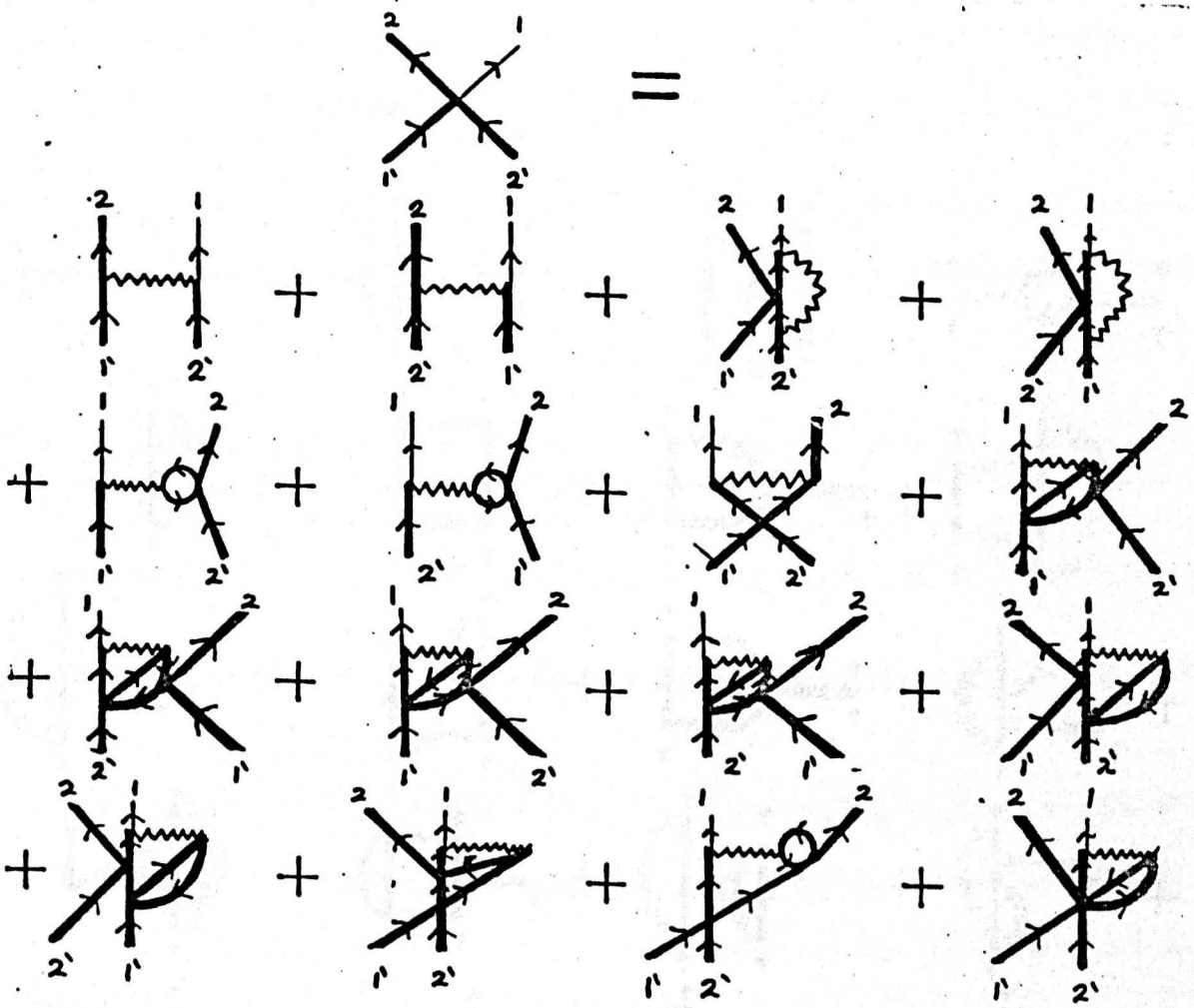


FIGURE 3

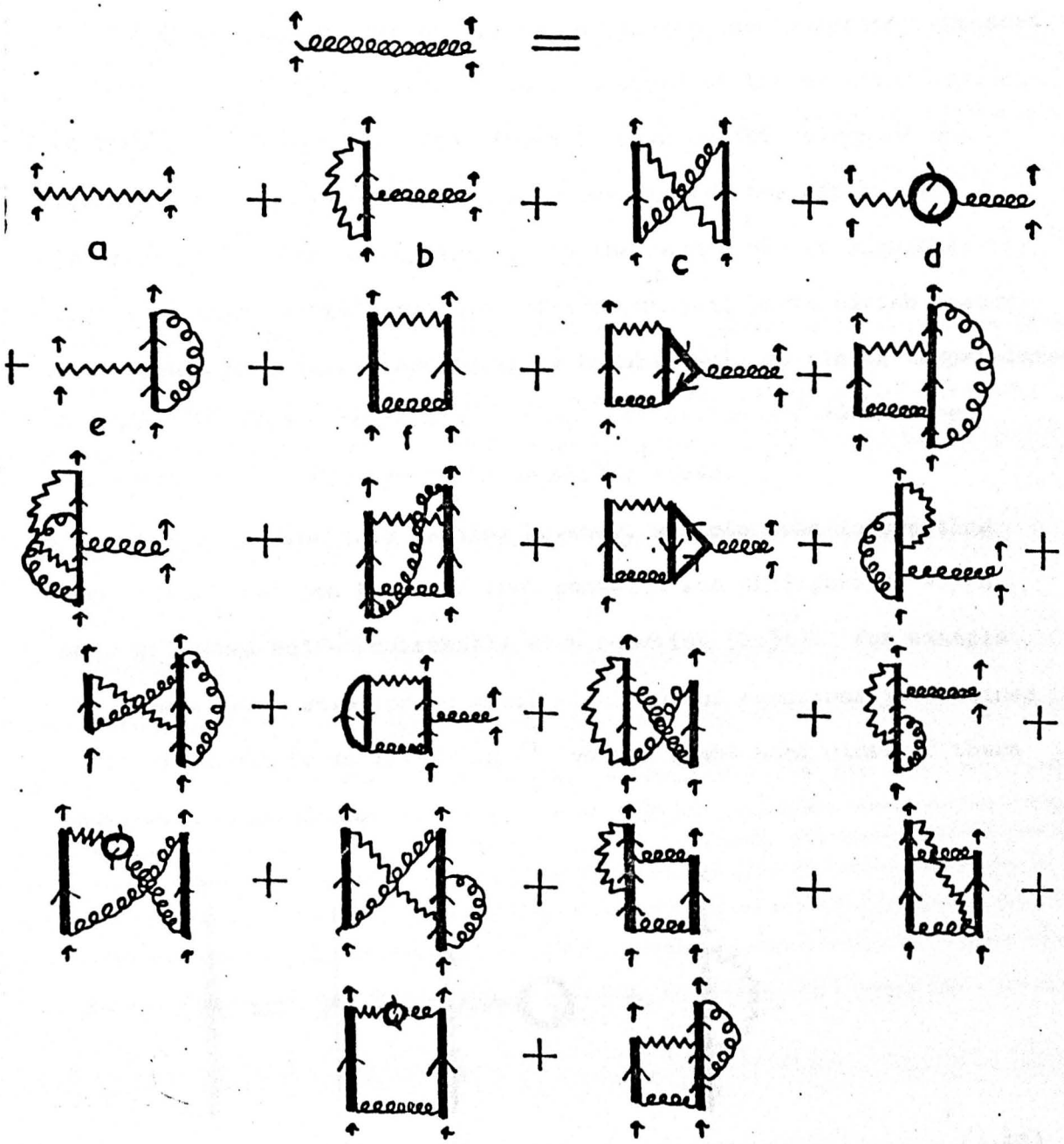
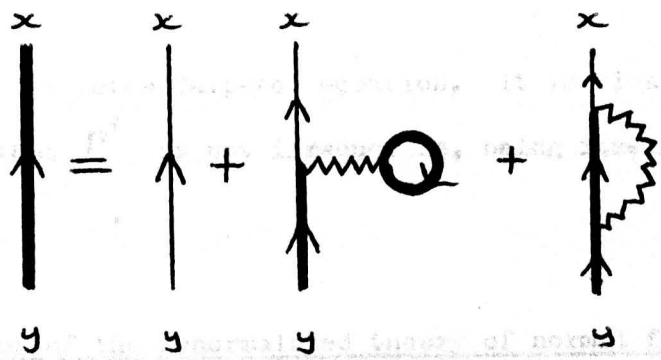


FIGURE 4

proportional to $(\Gamma)^3$. Although this term could be expanded further using the techniques described above we shall ignore it in what follows and restrict ourselves to second order in Γ .

With this approximation we now invoke the decomposition expressed in equation (1.33) and apply it to both sides of the equation appearing in Figure 3. Comparison of equivalent terms on both sides of the resultant equation shows that the expression for the effective interaction Γ can be represented by the form shown in Figure 4. This completes our reformulation of conventional perturbation theory and we note that these results apply equally well at finite temperatures, provided the finite temperature Green's functions are substituted for their $T = 0$ counter parts appearing above.

Before leaving this section however, we note some interesting conclusions that can be drawn from consultation of Figure 4, which must be solved self-consistently with equation (1.34). For example the zeroth order solution to these simultaneous equations is obtained by ignoring all terms involving Γ on the right hand sides of these equations, thus giving



(1.40).

Equation (1.40) is the well-known Hartree-Fock approximation. If we retain terms up to first order in Γ then we are left with equation (1.34) and the terms (a) to (f) on the r.h.s. of the equation appearing in Figure 4. In order to obtain the ladder approximation we insert the terms (a) and (f) only into the r.h.s. of equation (1.34).

Similarly to obtain what is essentially the ring approximation we substitute terms (a) and (d) of Figure 4 into the fourth term on the r.h.s. of equation (1.34).

Finally, we remark that the well-known Bethe-Salpeter equation [13] is contained in the equation in Figure 4. To see this explicitly to second order in Γ , define the quantity Γ' (denoted by ×××××××) to be the sum of diagrams a to e in Figure 4. Clearly then, the following diagrams are also to be found in Figure 4:

Generalizing Γ' to include higher order diagrams will obviously yield

which is the Bethe-Salpeter equation. It is clear from Figure 4 however that Γ' is not irreducible, being itself expressed in terms of Γ .

1.5 Discussion of the renormalized theory of normal fermi systems

In the equation of motion method of calculating the single particle Green's function, one must eventually tackle the problem of evaluating the two particle Green's function $G^{(2)}$. This has been dealt with in the literature by invoking various ad hoc factorization procedures [15,16,17] . In conventional perturbation

theory, the equivalent problem is resolved by summing an infinite sub-set of the contributions to the self energy Σ . What we have provided is a systematic technique for the treatment of $G^{(2)}$ (and higher order Green's functions) and the power of this formalism is demonstrated by the fact that the simplest of approximations yield many of the well-known results of fermion perturbation theory. It is to be emphasized that equations (1.29) and (1.34) are exact, while the equation in Figure 4 is correct only to second order in Γ . In the unlikely event that more diagrams are required, they may be readily obtained by employing the techniques described above. Thus, the application of the well-known Feynman rule, 'Draw all topologically distinct connected diagrams' may now be applied in a systematic manner and its application is no longer a chance affair in which diagrams may be missed.

A typical situation in which the perturbation theory discussed above may not be applied, occurs when the system enters a phase which exhibits long range order [9,18] . Two well-known examples are the superconducting phase and the ferromagnetic phase. In situations of this type, the order is characterized by the appearance, in the expansion of the single particle Green's function, of 'anomalous' propagators, [9,18,19,20] which disappear above the transition temperature. To complete our reformulation of fermion perturbation theory, we should now turn our attention to the perturbation theory of fermi systems exhibiting long range order characterized by the existence of anomalous propagators. However, since the primary objective of the present section has been to gain familiarity with the techniques involved, we defer a discussion of this problem to appendix A.

Finally, it should be noted that nowhere among the above diagrammatic manipulations have we considered the difficult question of convergence. In fact throughout the present thesis, this question will in the main be tacitly ignored. However, it is anticipated that such problems may be alleviated to some extent by the self-consistent nature of the manipulations involved, thus giving increased confidence in the validity of the results obtained.

CHAPTER 2

BOSON PERTURBATION THEORY (i): THE PARTICLE DENSITY IN THE CONDENSATE

2.1 Introduction

It is widely assumed that the density of particles n_0, in the $\underline{k} = 0$ mode of an interacting bose gas at $T = 0K$, is a finite fraction of the total particle density n. In fact, as used in the literature, the term condensate has become synonymous with exactly this concept. To be more precise, if we imagine that we know the distribution function $n_{\underline{k}}$ of an interacting bose gas, then a 'condensate' is said to exist if it is found that the $\underline{k} = 0$ mode is associated with a δ - function singularity in $n_{\underline{k}}$. The reason for dwelling at such great length on exactly what is meant by a condensate is that recent high energy neutron scattering experiments on superfluid ^4He [22] have yielded results which are consistent with the absence of such a singularity - i.e. are consistent with the absence of a condensate. Furthermore theoretical arguments due to Evan's [23] indicate that Bose-Einstein condensation in an interacting bose gas described by the grand-canonical ensemble cannot occur beyond the Hartree-Fock approximation.

In view of these recent criticisms of the validity of the existence of a condensate, it is relevant to review the arguments which led to the concept in the first place. For a system of non-interacting bosons, energy considerations alone show that when $T = 0K$, all the particles will occupy the $\underline{k} = 0$ mode. Similarly, by treating ^4He as a nearly perfect gas, London [36] argued that below a certain temperature T_1, particles will condense into the $\underline{k} = 0$ mode. The close agreement between his calculated value of T_1 and the critical temperature T_c of ^4He considerably strengthened belief in the correctness of this view. Since that time the existence of a

condensate in an interacting bose gas at $T=0K$ has been reasserted many times [4,6,24,25,26].

Starting from this assertion, Bogoliubov [26] argued that in the description of a low density bose gas at $T=0K$, the operators a_o and $a_o^\dagger$ for particles in the $\underline{K}=o$ mode may be replaced by the C - number $n_o^{1/2}$. Within this approximation, the single particle $\underline{K}=0$ Green's function $G_o^{(1)}(t-t')$ assumes the form

$$iG_o^{(1)}(t-t') = n_o e^{-i\mu/\hbar(t-t')} \tag{2.1}$$

where μ is the chemical potential. The resulting theory predicts a linear energy spectrum in the region $\underline{K} \triangleq 0$, in agreement with the original proposals of Landau [27] and also with the neutron scattering data [28].

Assuming the presence of a condensate, Hugenholtz and Pines [4] employed this same replacement of $\underline{K}=0$ operators by C - numbers to extend the ideas of Bogoliubov to finite densities. Within this formalism, Gavoret and Nozieres [29] demonstrated that an arbitrary interacting bose gas at $T=0K$, will have an energy spectrum which will vary linearly with $\underline{K}$ for small $\underline{K}$ and will pass through the origin. This is in agreement with the observed excitation spectrum of ^4He [28] and hence, we see that the existence of a condensate is a sufficient condition for the occurence of superfluidity. However, it should be noted that the evidence described above in support of a condensate in an interacting bose gas (i.e. $T_1 \triangleq T_c$ and a linear spectrum for small $\underline{K}$) is purely circumstantial.

A fundamentally different approach to the problem was developed by Beliaev [3], who purported to prove from first principles that equation (2.1) was a mathematically correct result (valid to order 1/V where V is the volume of the system) for an arbitrary interacting bose

gas in its ground state. This proof, if it is correct, provides direct evidence that an arbitrary interacting bose gas in its ground state possesses a condensate.

Within the formalism of Beliaev and Hugenholtz and Pines (BHP), n_o is only ever calculated self-consistently. However, we shall now demonstrate that the Beliaev formalism may be extended to provide an explicit evaluation of the particle density in the $\underline{k} = o$ mode, The result is quite startling and demonstrates that the common practice of ignoring diagrams of relative order $1/V$, can lead to inconsistencies in the boson problem.

2.2 The condensate Green's function

We consider a system of N interacting spin-less bosons of mass m, enclosed in a volume V and suppose N and V to become infinite while $n = N/V$ remains finite.

The Hamiltonian of the system is

$$H = H_o + H_1$$

where H_o describes a system of non-interacting bosons in their ground state $|\Phi_N\rangle$, where

$$|\Phi_N\rangle = \frac{1}{\sqrt{N!}} \left(a_o^\dagger\right)^N |o\rangle$$

The single particle condensate Green's function is defined by

$$i G_o^{(1)}(t-t') = \langle \Psi_N | T[\alpha_H(t)\, \alpha_H^\dagger(t')] | \Psi_N \rangle \tag{2.2}$$

where $|\Psi_N\rangle$ is the normalized ground state of the interacting system. Following Beliaev [3], we assume that $|\Psi_N\rangle$ is generated from $|\Phi_N\rangle$ by the adiabatic switching procedure of Gell-Mann and Low. This allows us to write

$$iG_0^{(1)}(t-t') = \frac{\langle \Phi_\omega | T[\alpha(t)\alpha^\dagger(t')\hat{S}] | \Phi_\omega \rangle}{S_\omega} \qquad (2.3)$$

where

$$S_\omega = \langle \Phi_\omega | \hat{S} | \Phi_\omega \rangle .$$

and we have inserted a 'hat' above the S in equation (2.3) to indicate that it is an operator. Equation (2.2) is in the Heisenberg picture, while equation (2.3) is in the interaction picture and we have

$$\alpha_H(t) \equiv e^{\frac{iHt}{\hbar}} \frac{a_0}{\sqrt{V}} e^{-\frac{iHt}{\hbar}}$$

and

$$\alpha(t) \equiv e^{\frac{iH_0t}{\hbar}} \frac{a_0}{\sqrt{V}} e^{-\frac{iH_0t}{\hbar}}$$

The S matrix and the interaction part of the Hamiltonian in the interaction picture, are given by equations (1.10) and (1.12), while the interaction picture field operators are given by equation (1.13). It is convenient to make the separation

$$\psi(\underline{x},t) = \alpha(t) + \sum_{\kappa \neq 0} V^{-1/2} e^{i\underline{\kappa}\cdot\underline{x}} e^{i\omega_{\underline{\kappa}}t} a_{\underline{\kappa}}$$

$$= \alpha(t) + \hat{\psi}'(\underline{x},t) \qquad (2.4)$$

and to define $\hat{S}'$ to be that quantity obtained from $\hat{S}$ by making

the separation (2.4) in the expansion (1.25) for $\hat{S}$ and averaging with respect to the vacuum of the operators $\hat{\psi}'$ and $\hat{\psi}^{+\prime}$. We may now follow Beliaev [3] and use Wick's theorem to write equation (2.3) in the form

$$i G_0^{(1)}(\epsilon-\epsilon') = \frac{\langle \Phi_\omega | \hat{N}\left[e^{\Delta}\left(\alpha(\epsilon)\alpha^{+}(\epsilon')\hat{S}'\right)\right] | \Phi_\omega \rangle}{S_\omega} \tag{2.5}$$

where e^{Δ} is a special case of the operator C defined in chapter 1 and Δ, which merely replaces a pair of operators $a_0 a_0^{+}$ by their contraction, is given by

$$\Delta = \frac{1}{V}\int dt\, dt'\, \Theta(\epsilon-\epsilon')\frac{\delta^2}{\delta\alpha(\epsilon)\delta\alpha^{+}(\epsilon')} \tag{2.6}$$

If λ is the sum of contributions from all connected vacuum loops, then [3] $\hat{S}'$ may be written

$$\hat{S}' = e^{\lambda} = e^{V\sigma}$$

where σ is a volume independent functional of α and α^{+}. Defining the quantities σ' and $\hat{S}'' = e^{\Delta} e^{V\sigma} = e^{V\sigma'}$, Beliaev proves to order $1/V$ that

$$e^{\Delta}\left[\alpha(\epsilon)\alpha^{+}(\epsilon')\hat{S}'\right] = \beta(\epsilon)\beta^{+}(\epsilon')\hat{S}'' \tag{2.7}$$

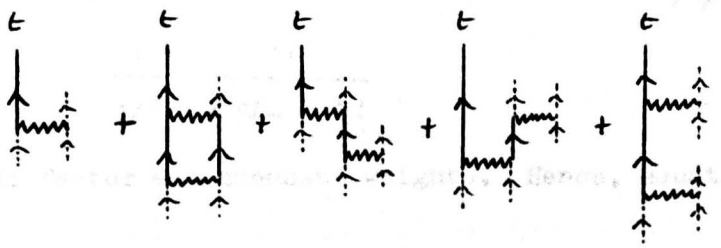
where

$$\beta(t) = \alpha(t) + \alpha_1(t) \quad \text{and} \quad \beta^\dagger(t) = \alpha^\dagger(t) + \alpha_1^\dagger(t) \tag{2.8}$$

the quantity $\alpha_1(t)$ being given by

$$\alpha_1(t) = \int dt' \, \theta(t-t') \frac{\delta \sigma}{\delta \alpha^\dagger(t')} \tag{2.9}.$$

To second order, $\alpha_1(t)$ is given by

$\alpha_1(t) =$ 

(2.10)

where a wavy line represents a factor $\left(\frac{-i}{\hbar}\right) U(\underline{x} - \underline{y})$, a dotted line entering (leaving) the point $x \equiv \underline{x}, t_x$ represents an operator $\alpha(t_x)$ $(\alpha^\dagger(t_x))$ and to order $1/V$, a solid line running from y to x represents a factor $i\, g(x,y)$ where

$$i\, g(x,y) = \langle 0 | T[\, \hat{\psi}(x), \hat{\psi}^\dagger(y) \,] | 0 \rangle \tag{2.11}.$$

Equation (2.5) may now be written

$$iG_0^{(1)}(t-t') = \frac{\langle \mathcal{F}_N | \hat{N}[\beta(t)\beta^\dagger(t')\hat{S}''] | \mathcal{F}_N \rangle}{S_N}$$

$$= \frac{1}{N!} \frac{\langle 0 | T[(a_0)^N \hat{N}\{\beta(t)\beta^\dagger(t')\hat{S}''\}(a_0^\dagger)^N] | 0 \rangle}{S_N} \qquad (2.12)$$

where the time associated with the N operators $a_0\ (a_0^\dagger)$ is greater (less) than the times associated with the operators $\beta, \beta^\dagger$ and $\hat{S}''$. Since the $\hat{N}$ operator ensures that contractions between operators α and $\alpha^\dagger$ in the curly brackets vanish, applications of Wick's theorem to equation (2.12) merely replaces the m pairs of operators $\alpha\alpha^\dagger$ in a given contribution to the product $\beta\beta^\dagger\hat{S}''$ by a factor

$$\frac{1}{V^m} \frac{N!}{(N-m)!}$$

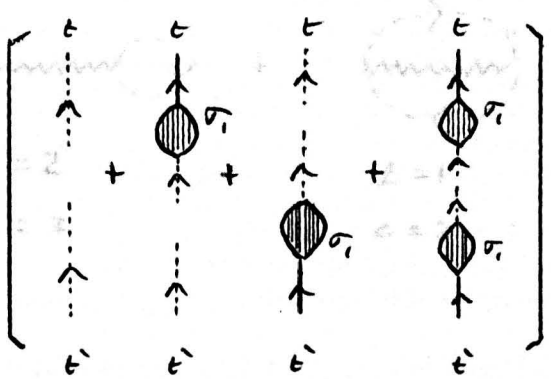
(we call this factor a condensate weight). Hence, equation (2.12) becomes

$$iG_0^{(1)}(t-t') = \frac{\bar{\beta}(t)\bar{\beta}^\dagger(t')S''}{S_N}$$

$$= \frac{\left\{\frac{N}{V} + \sqrt{\frac{N}{V}}[\bar{\alpha}_1(t) + \bar{\alpha}_1^\dagger(t')] + \overline{\alpha_1(t)\alpha_1^\dagger(t')}\right\} S''}{S_N} \qquad (2.13)$$

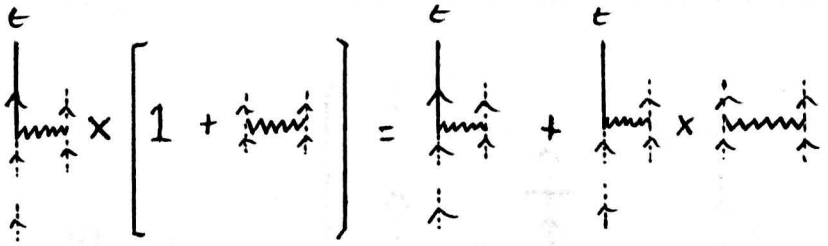
where for example, $\bar{\alpha}_1(t)$ is topologically identical with $\alpha_1(t)$ except that dotted lines no longer represent operators α and $\alpha^\dagger$. The final result takes the form

$$iG_0^{(1)}(t-t') = \frac{S''}{S_N}$$



(2.14)

which defines the self energy σ_1 . A diagram or product of diagrams containing a total of m pairs of dotted lines has associated with it a condensate weight $V^{-m} \frac{N!}{(N-m)!}$, hence although S'' is topologically identical with S_N , the two do not cancel. For example a contribution to the second term on the right of equation (2.14) is

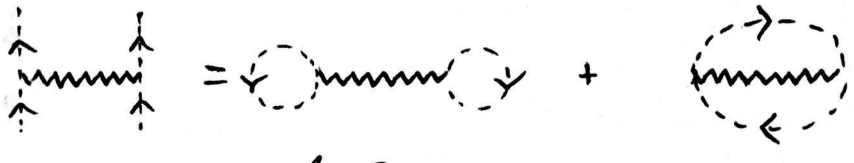


(2.15)

where the condensate weight associated with the second term on the right of equation (2.15) is $V^{-4} N(N-1)(N-2)(N-3)$. However, the zero and first order contributions to S_N are $1 +$  where the condensate weight associated with the first order contribution is $V^{-2} N(N-1)$. To overcome this problem of non-cancellation of disconnected diagrams, we resort to the counting procedure introduced by Brandow [1] .

2.3 Cancellation of disconnected diagrams

Brandow has shown that the condensate weights of disconnected diagrams may be evaluated by connecting dotted lines to each other in all possible ways to form continuous condensate lines and then imposing the rule 'associate a condensate weight $(-1)^{l+c} N^l V^{-c}$ with every diagram containing l loops and c condensate lines'. For example,



$l = 2$ $\quad$ $l = 1$

$c = 2$ $\quad$ $c = 2$

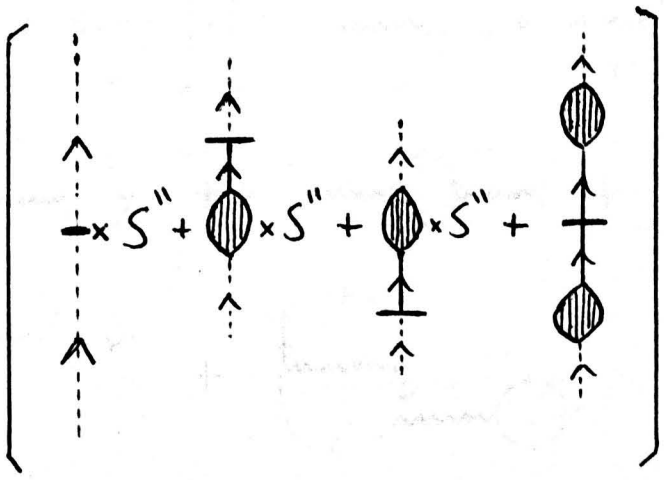
(2.16)

where the rule yields a condensate weight of

$$\frac{N^2}{V^2} - \frac{N}{V^2} = \frac{N(N-1)}{V^2} \quad .$$

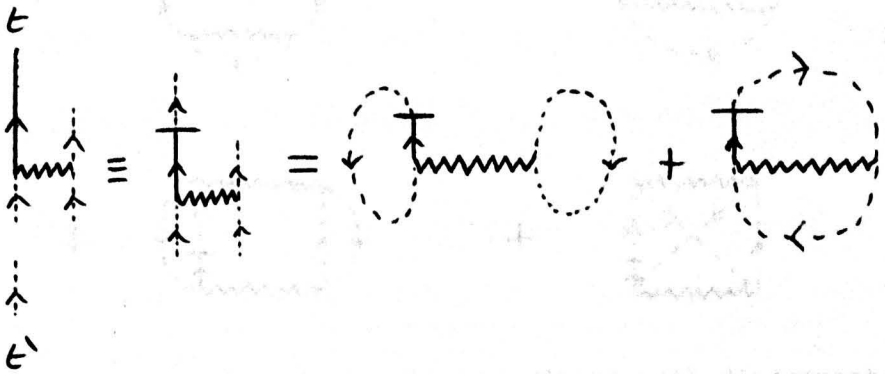
To apply this counting procedure to the connected diagrams in equation (2.14), we first re-write this equation in the form

$$iG_0^{(1)}(t-t') = \frac{1}{S_N}$$



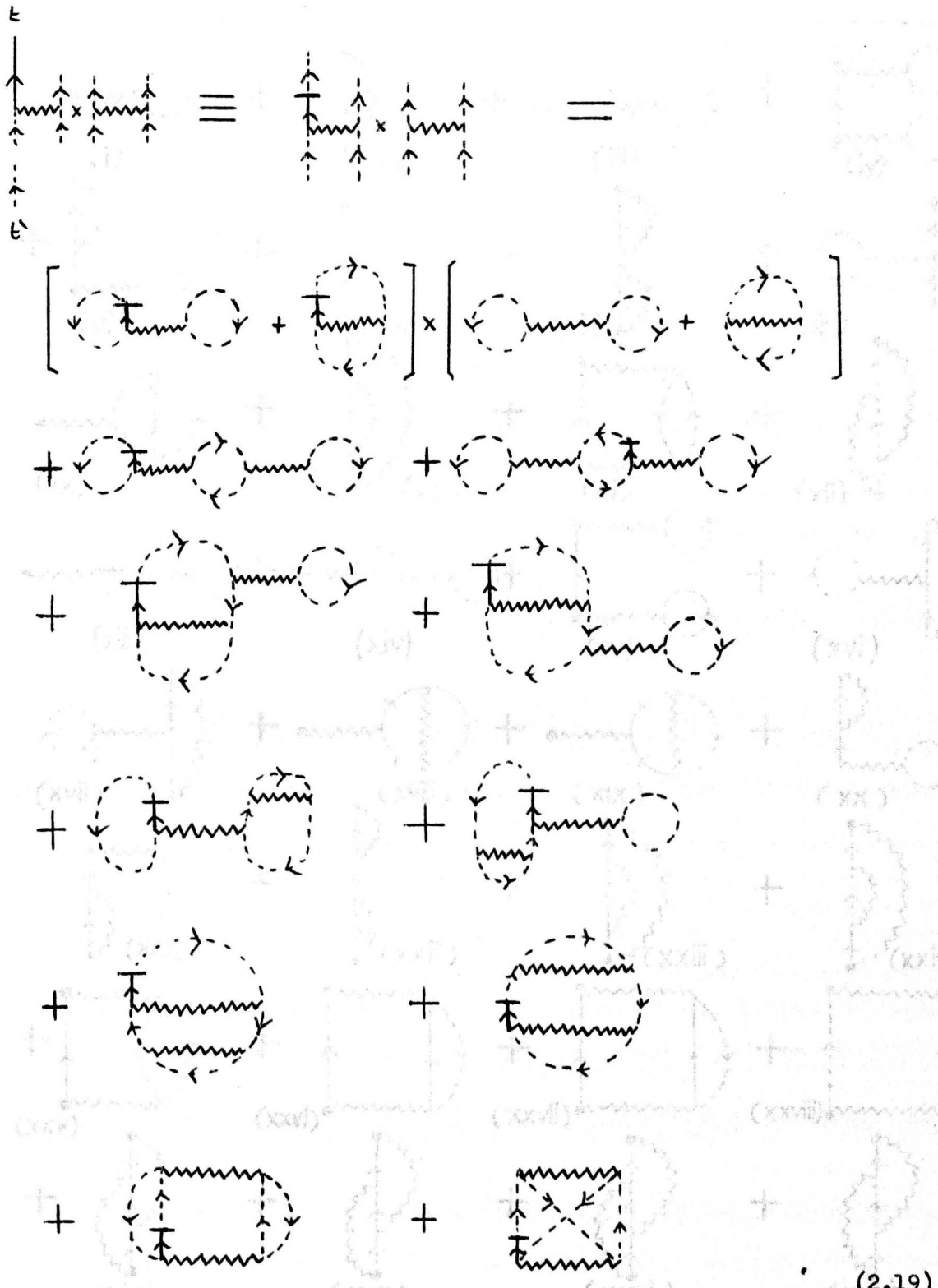
(2.17)

where we associate a time t with any line entering the horizontal bar and a time t with any line leaving it. The topology of connected diagrams is now identical with the topology of disconnected diagrams and hence we may use the counting procedure. For example,



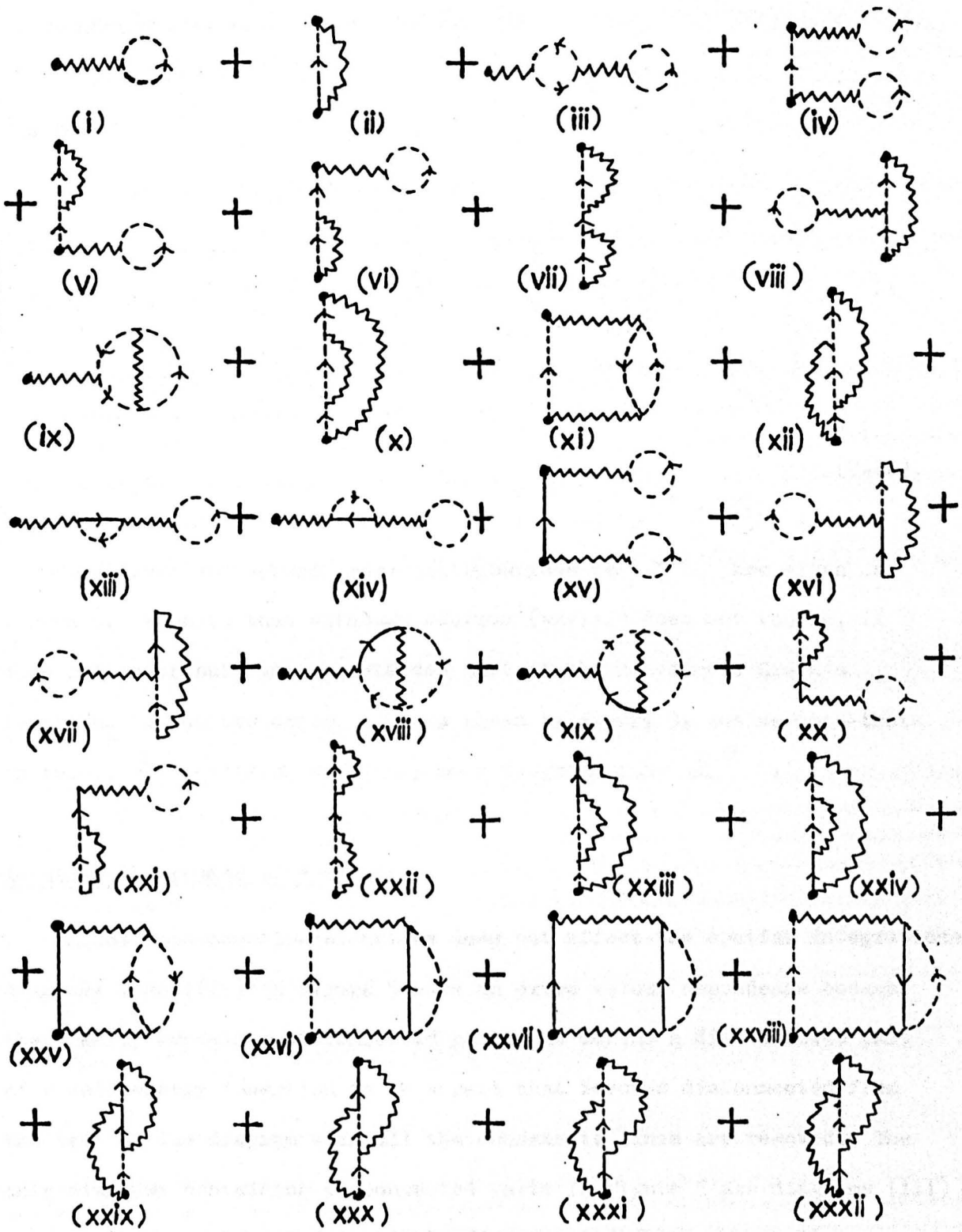
(2.18)

and



(2.19)

Since the condensate weight associated with disconnected diagrams is dependent on its internal structure only, cancellation of disconnected


(i)
(ii)
(iii)
(iv)
(v)
(vi)
(vii)
(viii)
(ix)
(x)
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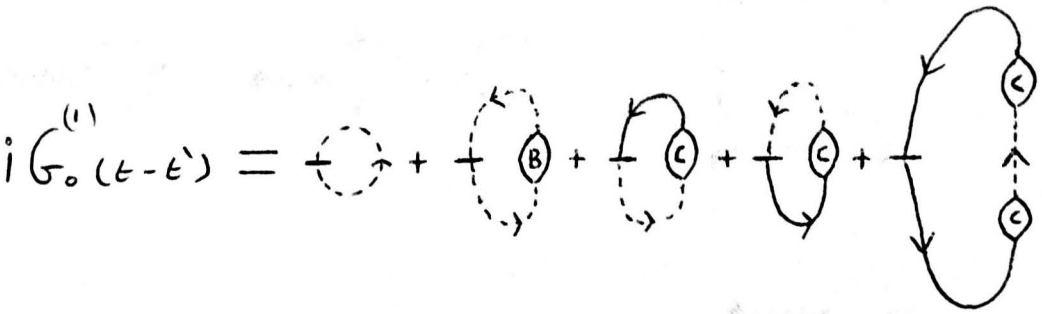
(xxiii)
(xxiv)
(xxv)
(xxvi)
(xxvii)
(xxviii)
(xxix)
(xxx)
(xxxi)
(xxxii)

FIGURE 5 .

diagrams by S_N in equation (2.17) now occurs. However, equation (2.19) demonstrates that this is at the cost of greatly increasing the number of connected diagrams. After changing dotted lines into condensate lines, and allowing cancellation of disconnected diagrams, equation (2.17) takes the form



(2.20)

which defines the self energies $\Sigma^B \equiv$ (B) and $\Sigma^C \equiv$ (C) .

The first and second order contributions to Σ^B are given in Figure 5. We note that although diagram (xxviii) does not vanish, it does not contribute to the retarded part of the condensate Green's function. To second order, Σ^C is given by Figure 5, but we note that in total, Σ^C contains very many more diagrams than Σ^B .

2.4 Relative magnitudes of diagrams

Since this counting technique does not affect the spatial integrations, diagrams like (iii) in Figure 5 have an extra volume dependence because they really contain a disconnected part. We define a disconnected part of a self energy insertion to be a part that becomes disconnected from the rest of the diagram when all the condensate lines are removed. The only diagrams containing disconnected parts in Figure 5 are diagrams (iii) to (x), which each contain a single disconnected part. We shall ignore the volume dependence of disconnected parts and multiply any diagram containing d disconnected parts by a factor V^d . The leading terms in Figure 5 are therefore the bubble diagrams i, iii, iv, xiii, xiv, xv and

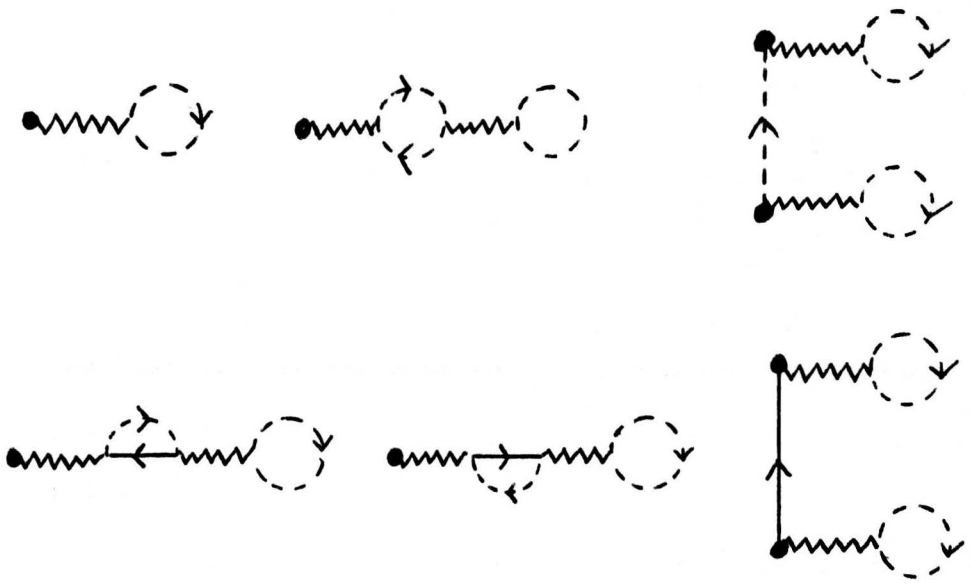
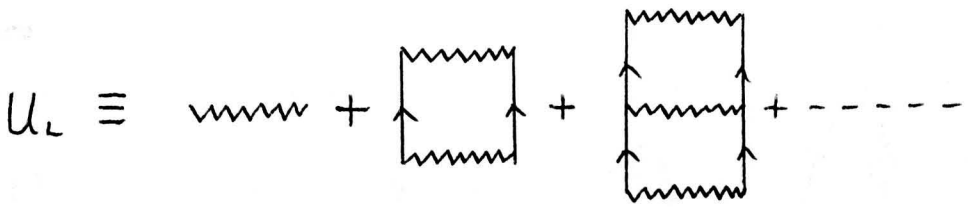


FIGURE 6

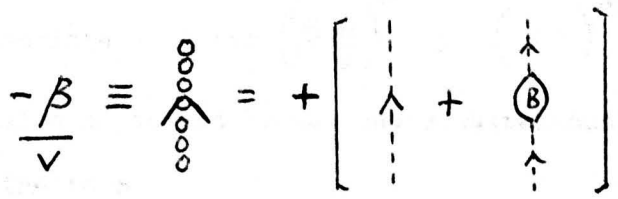
xxvii and all other terms are of relative order $1/V$. Indeed, bubble diagrams are the only diagrams ever considered by many workers [1,2] and therefore we shall consider here the summation of this sub-set of the right hand side of equation (2.20). Replacing the bare interactions in Figure 5, by the sum of ladders defined by



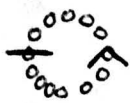
the leading contributions to Σ^{B} are to second order, as shown in Figure 6 where, in Figure 6, represents the sum of ladders U_L .

2.5 Renormalization of condensate lines

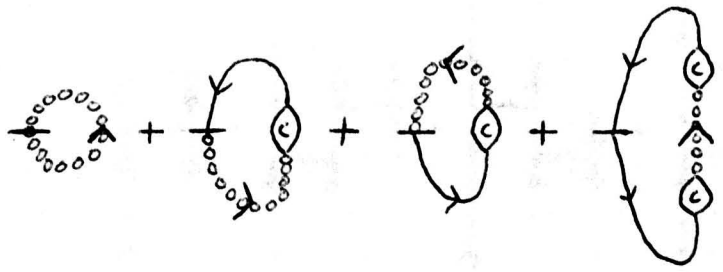
Consider the quantity β , defined by



(2.21)

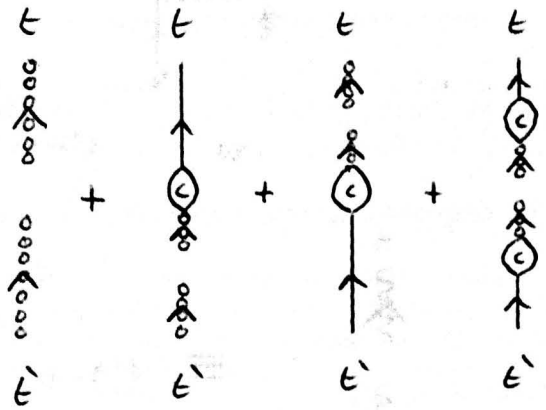
The sum of the first two terms of equation (2.20) is simply $n\beta$, which we denote by  . Now, β is obtained from the first term of equation (2.17) by allowing the dotted lines connected to the horizontal bar to pair with dotted lines in ζ'' . Since we carry out the same procedure with all of the dotted lines in the connected diagrams on the right hand side (r.h.s.) of equation (2.17), there will be associated with each pair of dotted lines in a connected diagram a term just equal to the

r.h.s. of equation (2.21). We may therefore omit all contributions to Σ^c that contain disconnected parts and reinterpret a condensate line as $-\frac{\beta}{v}$ rather than simply $-\frac{1}{v}$. This is only correct within the approximation of summing bubble diagrams and it is clear from equation (2.14) that $\beta = \frac{S_{N-1}}{S_N}$. Equation (2.20) now takes the form

$iG_0^{(1)}(t-t') =$ 

(2.22)

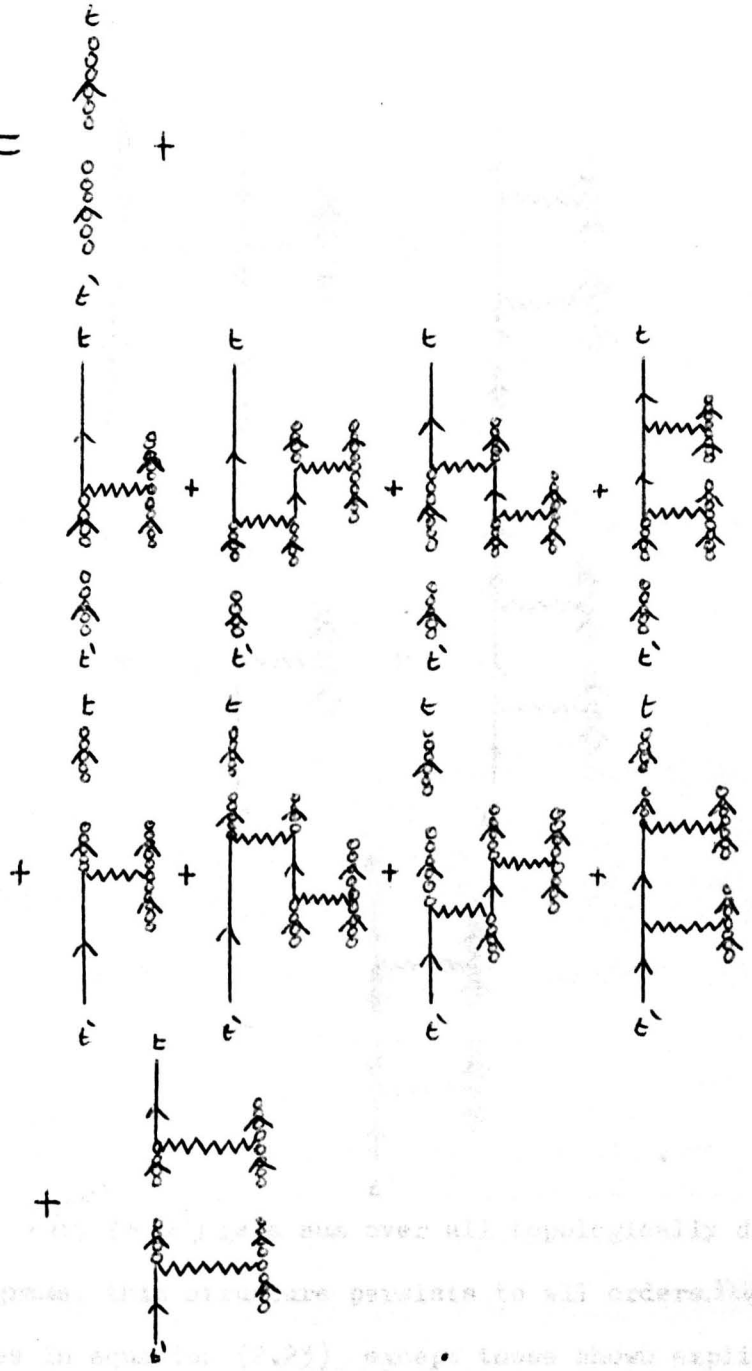
It is clear that for the present sub-set of diagrams, a diagram containing m renormalized condensate lines also contains m loops. We may therefore change the rule for obtaining condensate weights to 'associate a factor $\left(\beta \frac{N}{v}\right)^m = \left(\beta n\right)^m$ with any diagram containing m pairs of dotted lines' and simultaneously re-write equation (2.22) in the form

$iG_0^{(1)}(t-t') =$ 

(2.23)

which to second order is simply

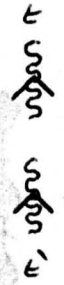
$$iG_0^{(1)}(\epsilon-\epsilon') =$$



(2.24)

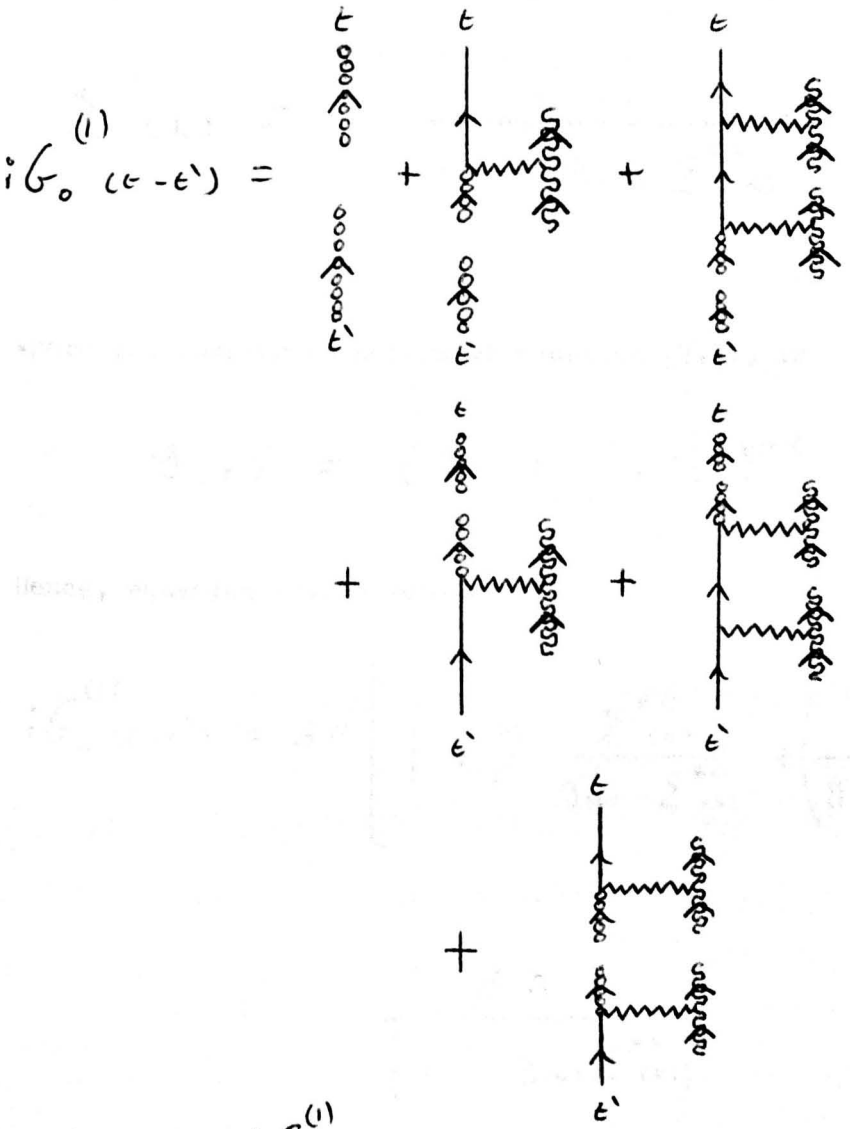
Denoting $iG_0^{(1)}(\epsilon-\epsilon')$ by

$$iG_0^{(1)}(\epsilon-\epsilon') \equiv$$



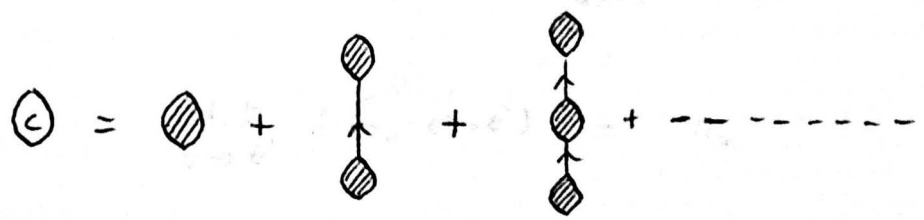
(2.25)

we see that to second order at least, equation (2.24) may be written



(2.26)

Indeed, since $iG_0^{(1)}(\epsilon-\epsilon')$ is a sum over all topologically distinct connected diagrams, this structure persists to all orders. Thus, all pairs of dotted lines in equation (2.23) except those shown explicity, renormalize in terms of the single particle condensate Green's function. The self energy Σ^c may be written in terms of a proper self energy Σ^{*c} denoted by ⬭ as follows



(2.27)

hence with the usual definition of a Fourier transform [5] we obtain

$$\Sigma^{c}_{(K)} = \frac{\Sigma^{*c}_{(K)}}{1 - g_{(K)}\Sigma^{*c}_{(K)}} \qquad (2.28)$$

where the Fourier transform of equation (2.11) is

$$g_{(K)} = (\omega - \omega_{\underline{K}} + i\eta)^{-1} \qquad . \qquad (2.29)$$

Hence, equation (2.23) becomes

$$iG_{0}^{(1)}(t-t') = \beta n \left[1 + 2\frac{\Sigma^{*c}_{(0)}}{g^{-1}_{(0)} - \Sigma^{*c}_{(0)}} + \left(\frac{\Sigma^{*c}_{(0)}}{g^{-1}_{(0)} - \Sigma^{*c}_{(0)}} \right)^{2} \right]$$

$$= \frac{\beta n}{\left[1 - g_{(0)}\Sigma^{*c}_{(0)} \right]^{2}} \qquad . \qquad (2.30)$$

An inspection of the r.h.s. of equation (2.30) shows that it is time independent, whereas it follows from the definition of $G_{0}^{(1)}$ that it is a function of $(t - t')$. These two statements are only mathematically compatible if both sides of the equation vanish. A fact which is consistent with the divergence of $g_{(0)}$. Now

$$\lim_{t \to t'} iG_{0}^{(1)}(t-t') = n_{0} \quad ,$$

hence we conclude that $n_{0} = 0$ in the present approximation.

2.6 Discussion

In the above, we have demonstrated that the self-consistent summation of 'ladder' and 'bubble' diagrams leads to an expression for the condensate Green's function, from which we deduce that $n_0 = 0$. Hence, we conclude that ladders and bubbles are not sufficient to describe a finite density bose gas and it is not permissible to neglect terms of relative order $1/V$ in the expansion of the condensate Green's function. In view of this, we turn our attention in the next chapter to a critical review of the formalisms of Beliaev and Hugenholtz and Pines. In this way, we are led to a new interpretation of equation (2.1) together with two self-consistency conditions which impose restrictions on any formalism in which this equation is employed.

Before ending this discussion we must emphasize that the results of this chapter merely represent a warning to be heeded in later work. The result $n_0 = 0$ should not be taken seriously, because of a possible lack of convergence in the perturbation series (see appendix D).

CHAPTER 3

BOSON PERTURBATION THEORY (ii). THE BHP FORMALISM

3.1 Introduction

It has been shown by Beliaev [3] how perturbation theory might be employed to calculate the single particle non-condensate Green's function G' of a many boson system, by considering the condensate as a kind of external field. The diagrammatic representation of the perturbation series contains incomplete vertices which may be completed by inserting pairs of dotted lines. Only connected diagrams need be considered in the expansion of G', provided m pairs of dotted lines are interpreted as an m particle condensate Green's function of the N-particle system. This result is exact.

In order to proceed, Beliaev purports to prove that for an arbitrary interacting bose gas at $T = 0K$, the m particle condensate Green's function $G_o^{(m)}$ may be written as a product of m single particle condensate Green's functions and that equation (2.1) is valid. In view of the calculation in chapter 2, we shall, in the present chapter, provide an examination of Beliaev's work, in an effort to determine its true range of validity.

It should be noted however that equation (2.1) and the factorization of $G_o^{(m)}$ can be obtained without recourse to perturbation theory, either by replacing the operators a_o and $a_o^\dagger$ which act on the $\underline{k} = o$ mode, by the c - number $\sqrt{n_o}$ [4], or by introducing the concept of a restricted average [25]. In both of these cases, it is assumed ab initio that a condensate exists in an interacting bose gas at $T = 0K$. In fact, as we shall demonstrate below these same arguments leads us to the conclusion that an interacting bose gas at $T = 0K$ <u>does not possess a condensate.</u>

3.2 Expansion of the single particle $\underline{K} \neq 0$ Green's function, G'.

In order to clarify the role played by the condensate Green's function $G_o^{(n)}$ in the expansion of the single particle non-condensate Green's function G', we shall briefly describe Beliaev's formulation of the problem of a finite density bose gas at $T = 0K$ in terms of Feynman graphs.

Employing the algebraic notation of chapter 2 in what follows, the single particle non-condensate (i.e. $\underline{K} \neq 0$) Green's function G', is given in the interaction picture by

$$i G'(x,y) = \frac{\langle \Phi_N | T[\, \psi'(x)\, \psi'^{\dagger}(y)\, \hat{S}\,] | \Phi_N \rangle}{S_N} \qquad (3.1)$$

In order to apply field theoretic techniques to equation (3.1), we note that $|\Phi_N\rangle$ is the vacuum of the operators ψ' and $\psi'^{\dagger}$. Thus, we apply the separation expressed in equation (2.4) to the field operators appearing in the expansion for $\hat{S}$ and substitute the result into equation (3.1). The averages of the time ordered products of the $\underline{K} \neq 0$ field operators (ψ' and $\psi'^{\dagger}$), which now appear in the numerator of equation (3.1) can now be performed with respect to their vacuum $|\Phi_N\rangle$. This averaging is most easily achieved by applying Wick's theorem to the products of $\underline{K} \neq 0$ operators and noting that averages of normal ordered products of uncontracted operators vanish.

The diagrams which contribute to the resulting perturbation expansion contain incomplete vertices, which may be completed by inserting dotted lines with free ends. In general, a diagram will contain m incoming and m outgoing dotted lines. Beliaev proves that the

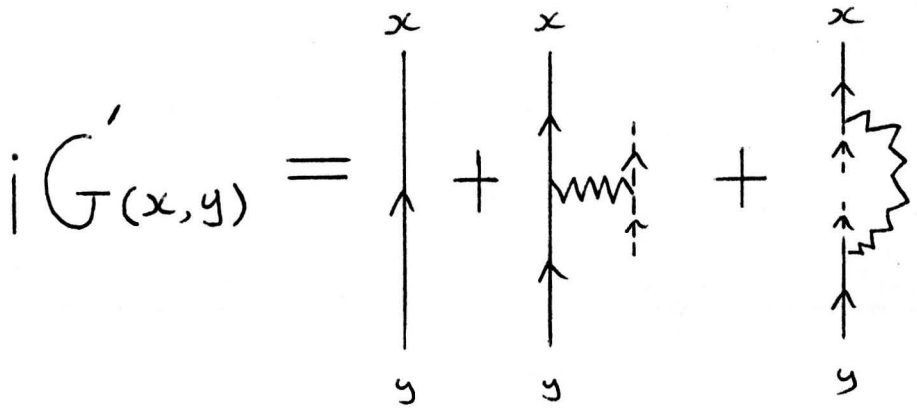


FIGURE 7

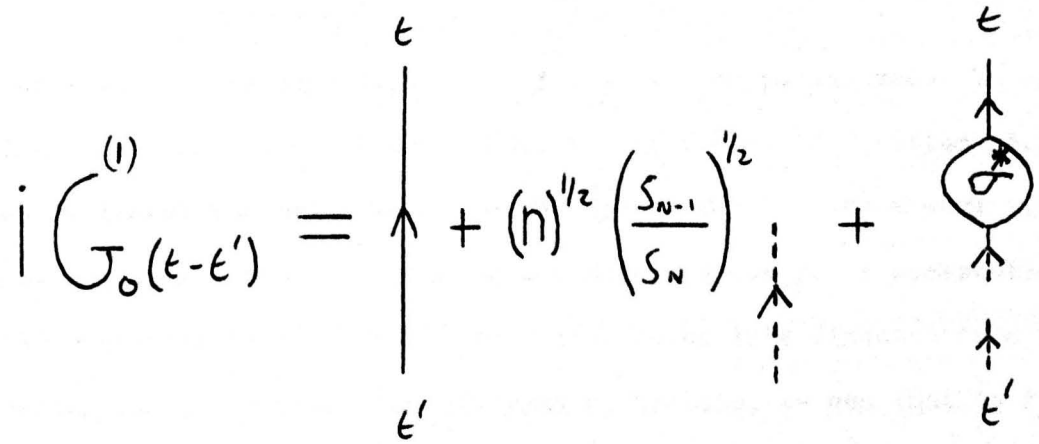


FIGURE 8

contribution from these dotted lines to a diagram containing m dotted lines entering (leaving) the points $t_1 \cdots t_m$ ($t_1' \cdots t_m'$) is

$$(i)^m G_0^m(t_1 \cdots t_m; t_1' \cdots t_m') = \frac{\langle \Psi_N | T[\alpha(t_1) \cdots \alpha(t_m) \alpha^\dagger(t_1') \cdots \alpha^\dagger(t_m') \hat{S}] | \Psi_N \rangle}{S_N} \tag{3.2}$$

which is simply an m particle $\underline{K} = 0$ Green's function of the N-particle system. In the Heisenberg picture, equation (3.2) takes the form

$$(i)^m G_0^{(m)}(t_1 \cdots t_m; t_1' \cdots t_m') = \langle \Psi_N | T[\alpha_H(t_1) \cdots \alpha_H(t_m) \alpha_H^\dagger(t_1') \cdots \alpha_H^\dagger(t_m')] | \Psi_N \rangle \tag{3.3}$$

where we have assumed that $|\Psi_N\rangle$ is normalized. In order to draw the diagrams which contribute to the r.h.s. of equation (3.1), we represent the potential $-i/\hbar\, U(\underline{x} - \underline{y})$ by a wavy line ($\underline{x}$ wwwww $\underline{y}$) joining the points $\underline{x}$ and $\underline{y}$. A contraction of two operators is represented by a continuous line directed from y to x. Noting that only connected diagrams contribute, we see that to first order, $iG'(x-y)$ is given by figure 7.

It is to be emphasised that the above results are <u>exact</u> and the problem of calculating the $\underline{K} \neq 0$ Green's function has been reduced to that of obtaining a general expression for the m-particle $\underline{K} = 0$ Green's functions $G_0^{(m)}$ of the interacting system. This is the point at which approximations are introduced into the theory [3], and the mathematical points raised are described in appendix (B). However, so that we may stress other, equally, important

mathematical features we proceed with the general argument. In order to make it possible for Feynman-Dyson perturbation theory to be applied to the present problem, we must find a way of decoupling the proper self-energies appearing in the expansion for G' . It is to be noted that one possible way of achieving this is to assume that $G_o^{(m)}$ takes the form

$$G_o^{(m)}(t_1 \cdots t_m; t_1' \cdots t_m') = F(t_1) \cdots F(t_m) F^*(t_1') \cdots F^*(t_m') \tag{3.4}$$

since then the value associated with any of the dotted lines appearing in a given proper self-energy is independent of the presence of dotted lines in the other self-energies appearing in the same diagram. Since the decomposition for $G_o^{(m)}$ of

$$(i)^m G_o^{(m)}(t_1 \cdots t_m; t_1' \cdots t_m') = (n_o)^m e^{-i\mu/\hbar[(t_1+\cdots+t_m)-(t_1'+\cdots+t_m')]} \tag{3.5}$$

is a special case of equation (3.4), and because equation (3.5) follows from equations (2.1), [3] , we see that the assumption that $G_o^{(1)}$ has the form of equation (2.1) can be viewed as a sufficient condition for the applicability of Feynman-Dyson perturbation theory to the problem. As is well-known, when equation (3.5) is applied to the expansion of G' the resulting series can be expressed in terms of three distinct proper self-energies (denoted by Σ_{11}, Σ_{12} , Σ_{21}) and may be summed to yield the well-known result [3,4] for the Fourier transform of G'

$$G'(\underline{k}, \omega+\mu/\hbar) \equiv G'(k+\mu/\hbar) = \frac{g^{-1}(-k+\mu/\hbar) - \Sigma_{11}(-k+\mu/\hbar)}{[g^{-1}(k+\mu/\hbar) - \Sigma_{11}(k+\mu/\hbar)][g^{-1}(-k+\mu/\hbar) - \Sigma_{11}(-k+\mu/\hbar)] - \Sigma_{12}(k+\mu/\hbar)\Sigma_{21}(k+\mu/\hbar)} \tag{3.6}$$

Since in appendix B, we show that the proof of Beliaev regarding the decomposition given in equation (2.1) (together with its generalization in equation (3.5)) is not generally valid, we shall consider next the derivation of an exact expression for $G_o^{(1)}$ from which we can obtain a <u>self-consistent</u> evaluation of this term. The validity of one such self-consistent solution for $G_o^{(1)}$, namely equation (2.1), introduces a restriction which is not evident in the Beliaev approach.

3.3 Calculation of the single particle $k = 0$ Green's function

From the definition of $G_o^{(1)}$, it follows that in the interaction picture we have

$$i G_o^{(1)}(t-t') = \frac{\langle \Phi_N | T[\alpha(t)\alpha^\dagger(t')\hat{S}] | \Phi_N \rangle}{S_N}$$

which can be written in the form

$$i G_o^{(1)}(t-t') = \frac{\langle \Phi_N | T[\alpha(t)\alpha(t')e^{V\sigma}] | \Phi_N \rangle}{S_N} \qquad (3.7)$$

where

$$S_N = \langle \Phi_N | T[e^{V\sigma}] | \Phi_N \rangle$$

As before, the quantity $e^{V\sigma}$ is given by

$$e^{V\sigma} = \langle 0 | T'[\hat{S}] | 0 \rangle \qquad (3.8)$$

where T' time orders the field operators ψ' and $\psi^{\dagger\prime}$ only, while treating the operators α and $\alpha^\dagger$ as $c-$ numbers. Hence, the quantity σ is a functional of α and $\alpha^\dagger$,

but not of ψ' and $\psi^{\dagger\prime}$.

In order to apply the techniques of chapter 1 to the present problem, we again define an operator C such that when it acts on a string of operators, the result is a sum over all possible contractions, including uncontracted terms. For example,

$$C[\alpha_{(\epsilon)}\,\alpha^{\dagger}_{(\epsilon')}] = \alpha_{(\epsilon)}\,\alpha^{\dagger}_{(\epsilon')} + \alpha^{\bullet}_{(\epsilon)}\,\alpha^{\dagger\bullet}_{(\epsilon')} \tag{3.9}$$

where dots denote contractions. Representing the normal ordering operator by $\hat{N}$ allows us again to write Wick's theorem in the symbolic form

$$T = \hat{N}\,C\,. \tag{3.10}$$

When equation (3.11) is substituted into equation (3.7) the operator C may be written in the form

$$C = C_{\alpha\alpha^{\dagger}}\,C_{\alpha\sigma}\,C'$$

where $C_{\alpha\alpha^{\dagger}}$ contracts α with $\alpha^{\dagger}$ according to the prescription appearing in equation (3.9), $C_{\alpha\sigma}$ contracts α with the operators appearing in σ and C' performs the sum over all remaining contractions. Hence, equation (3.7) may be written

$$iG_0^{(1)}(\epsilon-\epsilon') = \alpha^{\bullet}_{(\epsilon)}\,\alpha^{\dagger\bullet}_{(\epsilon')} + \frac{\langle\Phi_N|\hat{N}[C_{\alpha\sigma}C'(\alpha_{(\epsilon)}\alpha^{\dagger}_{(\epsilon')}e^{V\sigma})]|\Phi_N\rangle}{S_N}$$

$$= iG_0(\epsilon-\epsilon') + \frac{\langle\Phi_N|\hat{N}[C'(\alpha^{\bullet}_{(\epsilon)}\sigma^{\bullet}V\,\alpha^{\dagger}_{(\epsilon')}e^{V\sigma})]|\Phi_N\rangle}{S_N}$$

$$+ \frac{\langle\Phi_N|\hat{N}[C'(\alpha^{\dagger}_{(\epsilon')}e^{V\sigma})]\,\alpha_{(\epsilon)}|\Phi_N\rangle}{S_N} \tag{3.11}$$

where (3.12)

$$i g_0(\epsilon-\epsilon') = \frac{\Theta(\epsilon-\epsilon')}{V} \quad .$$

Writing

$$\alpha^{\circ}(\epsilon)\,\sigma^{\circ} V = \int_{-\infty}^{\infty} dt_1\, \Theta(\epsilon-\epsilon_1)\,\sigma_{(\epsilon_1)}\, e^{-\varepsilon|\epsilon_1|} \tag{3.13}$$

and noting that $\mathcal{C}'$ acts on all operators appearing in equation (3.11) yields

$$i G_0^{(1)}(\epsilon-\epsilon') = i g_0(\epsilon-\epsilon') + \int_{-\infty}^{\infty} dt_1\, \Theta(\epsilon-\epsilon_1)\, e^{-\varepsilon|\epsilon_1|} \frac{\langle \Phi_N | T[\sigma_{(\epsilon_1)} \alpha^{\dagger}_{(\epsilon')} e^{V\sigma}] | \Phi_N \rangle}{S_N}$$

$$+ \left(\frac{N}{V}\right)^{1/2} \frac{\langle \Phi_N | T[\alpha^{\dagger}_{(\epsilon')}\, e^{V\sigma}] | \Phi_{N-1} \rangle}{S_N} \tag{3.14}$$

where

$$\varepsilon \rightarrow 0^{+} \quad .$$

In the Heisenberg picture, the second term on the r.h.s. of equation (3.14) becomes simply

$$\int_{-\infty}^{\infty} dt_1\, \Theta(\epsilon-\epsilon_1)\, e^{-\varepsilon|\epsilon_1|} \langle \Psi_N | T[\sigma_{H(\epsilon_1)}\, \alpha^{\dagger}_{H(\epsilon')}] | \Psi_N \rangle \tag{3.15}$$

where all operators appearing in $\sigma_{H(\epsilon_1)}$ are in the Heisenberg picture. When converting the last term on the r.h.s. of equation (3.14) to the Heisenberg picture, we must take great care to isolate any time divergences appearing in the numerator and denominator. It is shown in appendix (C) that this term takes the form (to within a phase factor)

$$\left(\frac{N}{V}\right)^{1/2}\left(\frac{S_{N-1}}{S_N}\right)^{1/2}\langle \mathcal{F}_N|\,\alpha_H^{\dagger}(t')\,|\mathcal{F}_{N-1}\rangle \tag{3.16}$$

and hence, in the Heisenberg picture, equation (3.14) becomes

$$:G_0^{(1)}(t-t') = :g_0(t-t') + \left(\frac{N}{V}\right)^{1/2}\left(\frac{S_{N-1}}{S_N}\right)^{1/2}\langle \mathcal{F}_N|\,\alpha_H^{\dagger}(t')\,|\mathcal{F}_{N-1}\rangle$$

$$+\int_{-\infty}^{\infty} dt_1\,\theta(t-t_1)\,e^{-\varepsilon|t_1|}\langle \mathcal{F}_N|T[\sigma_H(t_1)\alpha_H^{\dagger}(t')]|\mathcal{F}_N\rangle \tag{3.17}$$

where

$$\alpha_H^{\dagger}(t') = e^{\frac{iHt'}{\hbar}}\frac{a_0}{\sqrt{V}}e^{-\frac{iHt'}{\hbar}}. \tag{3.18}$$

Diagrammatically, equation (3.17) takes the form of Figure 8, where we have drawn the 'extra' ingoing dotted line appearing in σ_H explicitly. (The choice of the 'extra' dotted line is arbitrary and cannot affect the arguments). This defines a proper self-energy σ^* which contains an equal number of incoming and outgoing dotted lines. Note that the last term in Figure 8 is a function of $t-t'$ while the second term is a function of t' only. We shall now take advantage of this property to obtain some self-consistency conditions.

In general, a given contribution to σ^* contains m incoming and m outgoing dotted lines. The corresponding contribution to the last term of Figure 8 contains $(m+1)$ such pairs of dotted lines. This corresponds to a factor $(i)^{m+1}\,G_0^{(m+1)}$ which contributes to the diagram as a whole. The self energy σ^* is only a mathematical

entity in its own right when it is independent of the two external lines drawn explicitly in Figure 8. When equation (3.5) applies this condition is satisfied because it permits us to write

$$(i)^{m+1} G_0^{(m+1)} = i G_0^{(1)} . (i)^m G_0^{(m)}$$

for all m. Within this approximation, we have

$$\sigma_H(\epsilon_1) = \int dt_2 \int d^3x_1 \, d^3x_2 \; \sigma^*(\underline{x}_1, t_1; \underline{x}_2, t_2) \, n_0^{1/2} \, e^{-i\mu t_2/\hbar}$$

and the last term on the r.h.s. of equation (3.17) becomes

$$n_0 \, e^{-i\mu(t-t')/\hbar} \; \frac{\sigma^*(0, \mu/\hbar)}{\mu/\hbar + i\eta} \tag{3.19}$$

where η is an infinitesimal positive constant and $\sigma^*(\underline{k}, \omega)$ is defined by

$$\sigma^*(\underline{x}_1, t_1; \underline{x}_2, t_2) = \int \frac{d\omega}{2\pi} \int \frac{d^3K}{(2\pi)^3} \, e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x}_2)} \, e^{-i\omega(t_1 - t_2)} \, \sigma^*(\underline{k}, \omega) .$$

Applying the approximation of equation (3.5) to the second term on the r.h.s. of equation (3.17) gives

$$\left(\frac{N}{V}\right)^{1/2} \left(\frac{S_{N-1}}{S_N}\right)^{1/2} n_0^{1/2} \, e^{i\mu t'/\hbar} . \tag{3.20}$$

Hence, for $t' > t$ equation (3.17) may be written

$$i G_0^{(1)}(t-t') = (n n_0)^{1/2} \left(\frac{S_{N-1}}{S_N}\right)^{1/2} e^{i\mu t'/\hbar} + n_0 \frac{\sigma^*(0, \mu/\hbar)}{\mu/\hbar} \, e^{-i\mu(t-t')/\hbar} . \tag{3.21}$$

However, employing equation (3.5) on the l.h.s. of this equation and comparing terms yields the result that, for $\mu \neq 0$,

$$\sigma^*(0, \mu/\hbar) = \mu/\hbar \tag{3.22}$$

and

$$(n n_0)^{1/2} \left(\frac{S_{N-1}}{S_N}\right)^{1/2} e^{i\mu t'/\hbar} = 0 \tag{3.23}$$

Application of the well-known arguments of Hugenholtz and Pines [4] and others [6] to the self-energy σ^* readily demonstrates that (see also appendix E)

$$\sigma^*(0, \mu/\hbar) = \Sigma_{11}(0, \mu/\hbar) - \Sigma_{12}(0, \mu/\hbar) \tag{3.24}$$

and hence, equation (3.22) is simply the celebrated Hugenholtz and Pines relation. As the latter follows simply from the self-consistency condition given in equation (3.22), we see that the linearity in the energy spectrum for small $\underline{k}$, (which follows from the Hugenholtz and Pines relation) is to be regarded, from the present viewpoint, as a measure of the self-consistency of the theory, rather than as evidence in support of any ab initio approximations concerning the $\underline{k} = 0$ Green's function.

The second self-consistency condition is more interesting, since it is not evident in the original formalism of Beliaev. Furthermore, since the Hugenholtz and Pines replacement of the operators a_0 and $a_0^\dagger$ by c-numbers implies the validity of equation (2.1), equation (3.23)

applies equally well to their formalism [4] . In the case of a non-interacting gas, the ratio S_{N-1}/S_N equals unity, while in the presence of an interaction H, we may write [4,30,31] within the Hugenholtz and Pines formalism,

$$S_N = \lim_{t \to \infty} \exp\left\{ \langle \Phi_N | U(t) | \Phi_N \rangle_{\text{connected}} \right\}$$

where $\langle \Phi_N | U(t) | \Phi_N \rangle_{\text{connected}}$ is the sum of all vacuum loops, given by [4,30,31]

$$\langle \Phi_N | U(t) | \Phi_N \rangle_{\text{connected}} = \sum_n \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\frac{1}{2}t}^{\frac{1}{2}t} dt_1 \cdots \int_{-\frac{1}{2}t}^{\frac{1}{2}t} dt_n$$

$$\times \langle \Phi_N | T[H_1(t_1) \cdots H_1(t_n)] | \Phi_N \rangle_{\text{connected}} .$$

In the limit $t \to \infty$ we find [4,30]

$$\langle \Phi_N | U(t) | \Phi_N \rangle_{\text{connected}} = -i/\hbar \, E_N t$$

and therefore

$$\frac{S_{N-1}}{S_N} = \frac{\lim_{t \to \infty} e^{-i/\hbar \, E_{N-1} t}}{\lim_{T \to \infty} e^{-i/\hbar \, E_N T}} . \qquad (3.25)$$

Hence the ratio S_{N-1}/S_N if it exists, cannot vanish.

This result, together with equation (3.23) shows that

$$n n_o = 0$$

$$\text{ie.} \quad n = 0 \quad \text{or} \quad n_o = 0 \quad . \tag{3.26}$$

Since neither of these possibilities is consistent with a finite density gas, we conclude that the condition $\mu \neq 0$ is invalid. This means that equation (2.1) strictly applies to a non-interacting gas of bosons only.

3.4 Discussion

We have demonstrated that the self-consistency of the Beliaev and Hugenholtz and Pines formalisms implies the existence of the two self-consistency conditions given in equations (3.22) and (3.23). The first of these conditions is the well-known Hugenholtz and Pines relation, and leads directly to a linear spectrum in the long wavelength limit. This demonstrates that a prediction of linearity in the energy spectrum in this limit is a measure of the self-consistency of the theory. However, the applicability of the theory to a given problem depends on both equations (3.22) and (3.23) being satisfied. We have shown that the latter condition cannot be satisfied by an interacting bose gas within the formalism of BHP, and therefore an arbitrary condensed bose system described by the BHP formalism is non-interacting. Hence, if the widely held view that the BHP formalism provides a valid description of an arbitrary interacting condensed bose gas, is correct, we conclude that an arbitrary interacting bose system is not condensed i.e. n_o is not a finite fraction of n. Such a conclusion is in agreement with

the finite temperature results of Evan's [23] and the neutron scattering data of Mook [22] .

An alternative interpretation of the above results is obtained when one notes that, within the Hugenholtz and Pines formalism and assuming that the limit of the ratio equals the ratio of the limits in equation (3.25),

$$\frac{S_{N-1}}{S_N} = \lim_{\epsilon \to \infty} e^{i\mu\epsilon/\hbar}$$

Hence this formalism can be made to apply to an interacting condensed bose gas (i.e. n_0 finite in equation (3.17)) by introducing a small positive imaginary part to the chemical potential. The full implications that such an ad hoc procedure would hold for the theory are not clear however and this possibility will not be discussed further.

CHAPTER 4

BOSON PERTURBATION THEORY (iii): AN INFINITE HIERARCHY OF SELF-CONSISTENT SOLUTIONS

4.1 Introduction

In chapters 2 and 3, we have demonstrated that it is not permissible in boson perturbation theory to neglect infinite series of diagrams of relative order $1/V$ and that the self-consistent solution of BHP strictly applies to a zero density gas only. The problem of keeping track of all connected diagrams, irrespective of their volume dependence, is enormous. As a result, it could be argued that until more powerful counting techniques are developed, we should content ourselves with self-consistent theories. Indeed, it is well known that such theories have been remarkably successful in many branches of physics as evidenced, for example, by the Hartree-Fock theory. In chapter 5, we shall reformulate the perturbation theory of bosons using techniques described in chapter 1. However, it is useful in the present chapter to consider an infinite hierarchy of self-consistent solutions to the problem which do not suffer from the restrictions of the BHP formalism.

The simplest of these solutions is investigated in some detail and the result is a model of a bose gas which can be viewed as comprising two interacting fluids which do not interact with each other. This leads to the 'prediction' of a new superfluid branch in the energy spectrum and it is shown that the existence of the latter offers an explanation of several well-known features of liquid ^4He which are difficult, if not impossible, to account for in terms of the normal well established excitation spectrum.

4.2 Choice of a restricted ensemble

It has been emphasized [9,25] that in situations where there exists long range order, the use of an unrestricted (e.g. canonical) ensemble leads to divergences, and a suitable restricted ensemble must be employed.

In the theory of bosons, many authors [3,4,25] restrict the ensemble by fixing the particle density n_0 in the $\underline{k} = 0$ mode. The assumption here is that n_0 is macroscopic and a common feature of these approximations is that they imply the validity of equation (3.5), which allows Feynman-Dyson perturbation theory to be applied to the problem. However, we have demonstrated that equation (3.5) applies strictly to a zero density gas only.

In view of the self-consistency conditions obtained in chapter 3 and their relation to the linearity of the energy spectrum in the long wavelength limit, it is desirable to develop a formalism which utilizes these conditions and yet still applies to an interacting bose gas. As emphasized in section (3.2), Feynman-Dyson perturbation theory can only be applied to Beliaev's expansion for the non-condensate Green's function G', in situations where the form of the m particle condensate Green's function $G_0^{(m)}$ allows for the decoupling of the self energies in the expansion for G'. Bearing this in mind, we propose the following method for restricting the ensemble.

(i) Write $G_0^{(m)}$ in the form

$$G_0^{(m)} = \tilde{G}_0^{(m)} + G_D^{(m)} \tag{4.1}$$

where $G_D^{(m)}$, when substituted into the expansion for G', decouples the self energies.

(ii) Ignore $\tilde{G}_0^{(m)}$ when interpreting the value to be associated with m pairs of dotted lines in Beliaev's expansion for G'.

Furthermore, we suggest the following functional forms for $G_0^{(m)}$ (each form corresponding to a different choice of r):-

$$G_0^{(m)} \equiv G_r^{(m)} \qquad r = 1, 2, 3, \ldots$$

where

$$(i)^m G_r^{(m)}(\epsilon_1 \ldots \epsilon_m, \epsilon_1' \ldots \epsilon_m') = \sum_{j=0}^{r} (A_j)^m e^{-i\frac{\Theta_j}{\hbar}[(\epsilon_1+\ldots+\epsilon_m)-(\epsilon_1'+\ldots+\epsilon_m')]} \tag{4.2}$$

and A_j and Θ_j are constants. Note that the possibility $r=0$ corresponds to the BHP formalism and has been excluded from equation (4.2).

4.3 The self-consistency conditions

In order to obtain the self-consistency conditions which apply to equation (4.2), we focus attention on Figure 8. Substitution of equation (4.2) into the last term on the r.h.s. of the equation in Figure 8 yields, for $\epsilon' > \epsilon$

$$i G_0^{(1)}(\epsilon-\epsilon') = \left(n\frac{S_{N-1}}{S_N}\right)^{1/2} \langle \mathcal{I}_N | \alpha_H^\dagger(\epsilon') | \mathcal{I}_{N-1} \rangle + \sum_{j=0}^{r} A_j \left[\frac{\sigma^j(0, \Theta_j/\hbar)}{\Theta_j/\hbar}\right] e^{-i\frac{\Theta_j}{\hbar}(\epsilon-\epsilon')} \tag{4.3}$$

where we have used the fact that the first term on the r.h.s. of the equation in Figure 8 vanishes for $\epsilon' > \epsilon$. Inserting a complete

set of eigenstates of H into the definition of $G_0^{(1)}$ given by equation (2.2) yields, for $\epsilon' > \epsilon$,

$$iG_0^{(1)}(t-t') = \sum_{j=0}^{\infty} \langle \Psi_N | \alpha_H^{\dagger}(t') | \Psi_{N-1}^{j} \rangle \langle \Psi_{N-1}^{j} | \alpha_H(t) | \Psi_N \rangle$$

$$= \sum_{j=0}^{\infty} B_j \, e^{-i\frac{\Delta_j}{\hbar}(t-t')} \tag{4.4}$$

where

$$B_j = \langle \Psi_N | \alpha^{\dagger} | \Psi_{N-1}^{j} \rangle \langle \Psi_{N-1}^{j} | \alpha | \Psi_N \rangle$$

and

$$\Delta_j = E_N^0 - E_{N-1}^{j}$$

so that

$$\Delta_0 = \mu \quad .$$

In equation (4.4), some of the Δ_j's may be equal. Say, for example

$$\Delta_j = \Delta_K = \cdots = \Delta_\ell$$

In this case, it is convenient to write

$$C_j = B_j + B_K + \cdots + B_\ell$$

so that equation (4.4) may be written in the form

$$iG_0^{(1)}(t-t') = \sum_{j=0} C_j \, e^{-i\frac{\Delta_j}{\hbar}(t-t')} \tag{4.5}$$

where the summation in equation (4.5) runs over all <u>distinct</u> values of Δ_j.

According to equation (4.1), equation (4.5) may be written

$$i G_0^{(1)} = i \tilde{G}_0^{(1)} + i G_D^{(1)} \tag{4.6}$$

where, within a given restricted ensemble (corresponding to a given r in equation (4.2), we have from equation (4.2),

$$i G_0^{(1)}(t-t') = \sum_{j=0}^{r} A_j e^{-i\frac{\Theta_j}{\hbar}(t-t')} \tag{4.7}$$

Selecting the r.h.s. of equation (4.7) from the sum on the r.h.s. of equation (4.5) and re-labelling the dummy indices in the remainder, allows equation (4.5) to be written

$$i G_0^{(1)}(t-t') = \sum_{K=0}^{\prime} C_K e^{-i\frac{\Delta_K}{\hbar}(t-t')} + \sum_{j=0}^{r} A_j e^{-i\frac{\Theta_j}{\hbar}(t-t')} \tag{4.8}$$

where $\Delta_K \neq \Theta_j$ for all K and j .

The self-consistency conditions may now be obtained by substituting equation (4.8) into equation (4.3). This yields

$$C_K = 0 \quad \text{for all } K \tag{4.9}$$

$$\left(n \frac{S_{N-1}}{S_N}\right)^{1/2} \langle \Psi_N | \alpha_H^\dagger(\epsilon') | \Psi_{N-1} \rangle = 0 \tag{4.10}$$

and

$$\sigma^j(0, \theta_j/t) = \theta_j/t \qquad j \leq r \,. \tag{4.11}$$

Note that σ^j has the same topology as σ^* of Figure 8, the difference being that in σ^j, m pairs of dotted lines are to be interpreted as a factor

$$A_j^m \, e^{-i\frac{\theta_j}{t}\left[(t_1+\cdots+t_m)-(t_1'+\cdots+t_m')\right]} \,.$$

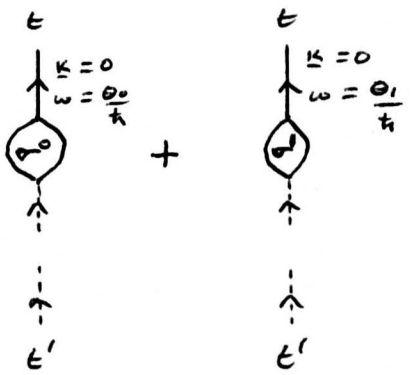
The arguments which led to equation (3.24) are readily generalized to yield

$$\sigma^j(0, \theta_j/t) = \sum_{11}^{j}(0, \theta_j/t) - \sum_{12}^{j}(0, \theta_j/t) \tag{4.12}$$

so that equation (4.11) may be regarded as a generalized Hugenholtz and Pines relation.

In order to illustrate the perturbation series for $G_0^{(1)}$ consider the ensemble characterized by $r = 1$ in equation (4.2). Taking into account equation (4.10), we see that the equation in Figure 8 becomes, to order $1/V$,

$$iG_0^{(1)}(t-t') =$$



(4.13)

In the rth ensemble there will be $r+1$ terms of the type shown on the r.h.s. of equation (4.13) which contribute to $G_0^{(1)}$ and in fact, as we shall see below, the expansion for G' has a similar structure. Before considering explicitly the structure of G', it is convenient at this point to examine the two remaining self-consistency conditions, namely equations (4.9) and (4.10), in more detail.

Equation (4.9) is of no obvious significance and will not be considered further. Equation (4.10) on the other hand is more important, because it indicates the presence of an energy gap in the excitation spectrum of a bose gas described by one of our restricted ensembles. That this is so will become clear later, but for the moment, we shall content ourselves with proving that equation (4.10) implies

$$\theta_j \neq \mu \quad \text{for all } j \tag{4.14}$$

Since n is assumed finite, equation (4.10) and the assumption $S_{N-1}/S_N \neq 0$ (see chapter 5) implies that

$$\langle \mathcal{F}_N | \alpha^{\dagger}_{H(t')} | \mathcal{F}_{N-1} \rangle = 0$$

i.e. $$e^{i\frac{\Delta_0}{\hbar}t'} \langle \mathcal{F}_N | \alpha^{+} | \mathcal{F}_{N-1} \rangle = 0 \quad .$$

This implies

$$\langle \mathcal{F}_N | \alpha^{+} | \mathcal{F}_{N-1} \rangle = 0$$

which demonstrates that

$$B_0 = 0 \quad ,$$

a result which is shown in chapter 5 to be of general validity. From this it follows that the term involving $\Delta_0 (= \mu)$ in equation (4.4) vanishes and hence in equation (4.8), we must have that

$$\Theta_j \neq \mu \quad \text{for all } j \quad . \tag{4.14}$$

4.4 The 'two fluid' approximation

In order to consider a definite example of equation (4.2), we shall employ the next simplest generalization of the BHP approximation, namely $r = 1$.

Writing

$$A_0 \equiv A \quad , \quad A_1 \equiv B \quad , \quad \Theta_0 \equiv \Theta \quad \text{and} \quad \Theta_1 \equiv \phi$$

yields for $r = 1$,

$$(i)^m G_0^{(m)} = A^m e^{-i\Theta/\hbar[(t_1 \cdots t_m) - (t_1' \cdots t_m')]} + B^m e^{-i\phi/\hbar[(t_1 \cdots t_m) - (t_1' \cdots t_m')]} \tag{4.15}$$

and

$$\Theta/\hbar = \Sigma_{11}^{\Theta}(0, \Theta/\hbar) - \Sigma_{12}^{\Theta}(0, \Theta/\hbar)$$

$$\phi/\hbar = \Sigma_{11}^{\phi}(0, \phi/\hbar) - \Sigma_{12}^{\phi}(0, \phi/\hbar) \quad . \qquad (4.16)$$

The non-condensate Green's function separates into a sum of two distinct contributions when equation (4.15) is employed in Beliaev's expansion for G', and we may write

$$G'(K) = G^{\Theta}(K) + G^{\phi}(K) \quad . \qquad (4.17)$$

Note that Σ_{11}^{Θ} and Σ_{12}^{Θ} (Σ_{11}^{ϕ} and Σ_{12}^{ϕ}) have the same topology as the self energies Σ_{11} and Σ_{12} which appear in the Beliaev formalism, except that m pairs of dotted lines are interpreted as the first (second) term on the r.h.s. of equation (4.15).

A trivial extension of the arguments [3,4] which led to equation (3.6) yields

$$G^{\Theta}(\underline{K}, \omega + \tfrac{\Theta}{\hbar}) = \frac{g^{-1}(-K + \Theta/\hbar) - \Sigma_{11}^{\Theta}(-K + \Theta/\hbar)}{[g^{-1}(K + \tfrac{\Theta}{\hbar}) - \Sigma_{11}^{\Theta}(K + \tfrac{\Theta}{\hbar})][g^{-1}(-K + \tfrac{\Theta}{\hbar} - \Sigma_{11}^{\Theta}(-K + \tfrac{\Theta}{\hbar})] - \Sigma_{12}^{\Theta}(K + \tfrac{\Theta}{\hbar})\Sigma_{21}^{\Theta}(K + \tfrac{\Theta}{\hbar})} \qquad (4.18)$$

with a similar equation for $G^{\phi}(\underline{K}, \omega + \phi/\hbar)$.

For an isotropic gas of bosons interacting through a hard sphere repulsive potential, Beliaev [32] has shown that within the approximation of summing ladder diagrams, the self energies Σ_{11} and Σ_{12}

appearing in equation (3.6) are to a first approximation independent of K and given by [5]

$$\Sigma_{11} = 2\Sigma_{12} = \frac{4\pi n_0 a \hbar^2}{m}$$

where a is the s - wave scattering length of the medium.

It is clear from equation (4.17) that the present approximation may be viewed as describing two interacting fluids, which do not interact (at least explicitly) with each other. With this in mind and in view of Beliaev's work [32], we shall, as a first approximation, regard the self energies Σ^{θ}_{ij} and Σ^{ϕ}_{ij} as constants, given by

$$\Sigma^{\theta}_{11} = 2\Sigma^{\theta}_{12} = \frac{4\pi A a_\theta \hbar^2}{m}$$

$$\Sigma^{\phi}_{11} = 2\Sigma^{\phi}_{12} = \frac{4\pi B a_\phi \hbar^2}{m}$$

where a_θ and a_ϕ are the s - wave scattering lengths of the two fluids and we have assumed for convenience that m is the same for both fluids. Equation (4.18) then becomes

$$G^{\theta}(\underline{k}, \omega + \theta/\hbar) = \frac{-g^{-1}_{(-k+\theta/\hbar)} + \Sigma^{\theta}_{11}}{\omega^2 - E^{\theta\,2}_{(\underline{k})}} \qquad (4.19)$$

where

$$E^{\theta}_{(\underline{k})} = \left[\omega^2_{\underline{k}} + 2\omega_{\underline{k}} (\Sigma^{\theta}_{11} - \theta/\hbar) + (\Sigma^{\theta}_{11} - \theta/\hbar)^2 - \Sigma^{\theta 2}_{12} \right]^{1/2}$$

$$= \left[\omega^2_k + 2\omega_k \Sigma^{\theta}_{12} \right]^{1/2} \qquad (4.20)$$

and we have employed the self-consistency relation (4.16).

Similarly

$$G^{\phi}_{(\underline{K},\omega+\phi/\hbar)} = \frac{-g^{-1}_{(-K+\phi/\hbar)} + \Sigma^{\phi}_{11}}{\omega^2 - E^{\phi\,2}_{(\underline{K})}} \qquad (4.21)$$

where

$$E^{\phi}_{(\underline{K})} = \left[\omega^2_{\underline{K}} + 2\omega_{K}\Sigma^{\phi}_{12} \right]^{1/2} . \qquad (4.22)$$

As expected, the self consistency of the present approximation (i.e. $\Gamma = 1$) leads to a phonon like excitation spectrum for small $\underline{K}$. However, in the present approximation, we find that there are two branches of the spectrum with slopes proportional to Σ^{θ}_{12} and Σ^{ϕ}_{12} in the linear region; i.e. proportional to A and B respectively within the approximation of summing ladders [32] .

Instead of becoming involved in detailed calculations within the present tentative model, we shall proceed to the next section where the qualitative features of the above results are discussed in relation to superfluid ^4He .

4.5 Comparison with experiment

Equation (4.14) tells us that

$$\theta \text{ and } \phi \neq \mu$$

which means that both branches of the energy spectrum may possess energy gaps. The magnitudes of these gaps are

$$\Delta_{\theta} = \mu - \theta \text{ and } \Delta_{\phi} = \mu - \phi$$

We shall assume in the present discussion that $\Theta \simeq \mu$ and hence that E^{Θ} represents the usual phonon spectrum of BHP, involving no energy gap. This still leaves the second branch of the spectrum described by equation (4.22).

It is clear that there exists an energy gap Δ of $(\phi - \mu)$ at the origin $(\underline{k} = o)$ between this branch and the usual phonon branch. Furthermore if $B \neq A$ in equation (4.15) the slopes (i.e. velocities) of the two branches will differ. Hence although it is tempting to try to identify this second 'branch' of the spectrum with the 'branch' observed several years ago by the Chalk River group in their neutron scattering experiments on ^4He [28], we would like to point out a more attractive alternative exists. Thus we suggest that the difference $(\phi - \mu)$ is small $\sim 1K$ and that the slopes of the two branches are such that they rapidly coalesce on the energy-momentum curve. As the energy difference between the two branches is so small it will not be 'resolved' in the single broad peaks observed in the neutron scattering data and associated with single phonon events. Thus its presence is consistent with such data and may also be responsible for the tentative conclusion reached by Phillips et al [33] that for very small k the spectrum appears to be concave upwards. What is more, one of the most puzzling features of the neutron scattering data, particularly evident at large momentum transfers, is the appearance of scattering intensity at energies below the one-phonon peak [28]. This is puzzling since in terms of the accepted excitation spectrum, there is no obvious way in which a phonon can decay so as to give scattering at lower energies. However, in terms of the present model, this can occur as a result of a neutron 'absorbing' a phonon from the normal branch and simultaneously emitting one or more

phonons of the second branch with momenta centred around $\underline{K} = 0$.

A related problem concerns the motion of a slowly moving massive particle at low temperatures $(< 1K)$. In terms of the usual excitation spectrum it is not easy to visualise how such a particle can lose energy and thus slow down since it cannot simultaneously satisfy the law of conservation of energy and momentum. This can readily be achieved in terms of the present model as a result of the particle scattering off a normal phonon and again simultaneously emitting several phonons of the second branch with momenta centred around $\underline{K} = 0$.

Yet another interesting possibility concerns the observed attenuation of ultrasonic waves where two maxima are observed at $0{\cdot}8\ K$ and the λ - point. It could be argued that the former peak arises from an exceptionally strong coupling between the ultrasonic wave and the thermally excited phonons of the second branch. At extremely low temperatures there are few such phonons but their number will increase rapidly as $\exp(-\beta\Delta)$ and will be appreciable when $KT \sim \Delta$ i.e. $T \sim 1K$. If the coupling between these phonons is large and increases rapidly with temperature, then at higher temperatures their lifetime becomes so short that their effect on the ultrasonic wave decreases with a concomitant decrease in the attenuation. In this connection it is to be noted that specific heat measurements at very low temperatures may reveal the existence of the second branch via a contribution which varies as $\exp(-\beta\Delta)$. However this assumes that the contribution from this branch is large enough to be separated from the usual T^3 part arising from the normal phonon gas [34] .

4.6 Discussion

The aim of the present chapter has been to demonstrate that the restrictions of the BHP formalism may be avoided by generalizing their choice of a restricted ensemble. We have deliberately chosen the restricted ensemble in such a way that the self-consistency conditions obtained in chapter 3 may be utilized. In this way, a phonon like energy spectrum in the long wavelength limit is guaranteed from the outset.

We have considered the next simplest alternative to that of BHP, which results in the prediction of a second branch of the excitation spectrum for small $\underline{K}$ values.

We have shown that the existence of this second branch in no way conflicts with the neutron scattering data. Indeed it could account for certain of the observed properties of the latter which are not readily interpreted in terms of the normal branch. We have also shown that the presence of the second branch permits a slowly moving massive body to lose energy and that this same branch may also contribute significantly to the observed peak in the ultrasonic attenuation at a temperature $\sim 1K$ Finally, we have noted that careful measurements of the specific heat at very low temperatures may enable the existence of the second branch to be experimentally established.

From the view point of the present work, additional branches of the energy spectrum result readily from different choices of the restricted ensemble and the self-consistency condition (4.12) indicates that each of these branches will be linear for small $\underline{K}$ (although their slopes will vary). It is not clear at this point which if any, of the self-consistent solutions given by equation (4.2) is relevant to

$4\hbar e$, which is why we have allowed the principle of simplicity to dictate our choice of ensemble.

In order that this uncertainty might eventually be removed, we shall now proceed from the rather qualitative work of the present chapter to a more rigorous formulation of the boson problem given in chapter 5.

CHAPTER 5

BOSON PERTURBATION THEORY (iv): A NEW FORMALISM

5.1 Introduction

In view of the fact that diagrams of relative order $1/V$ cannot be ignored, the counting technique of Brandow, which provides for the cancellation of disconnected diagrams in boson perturbation theory, is of little use and an alternative formulation of the problem is required. An examination of the techniques described in chapter 1, which consist of applying Wick's theorem in a more controlled manner, reveals that the introduction of disconnected diagrams into the theory may actually be avoided.

We shall now demonstrate that these techniques may be employed to provide a connected diagram expansion for the single particle Green's function which is topologically identical with the corresponding fermion expansion. Thus, the problem of the depletion effect, which has proved to be one of the main stumbling blocks in the theory of bosons is overcome without the introduction of ab initio approximations.

5.2 The irrelevance of normal ordered products

In common with chapters 2 and 3, we shall assume that $|\Psi_N\rangle$ is generated by the Gell-Mann and Low adiabatic switching procedure from the non-interacting ground state $|\Phi_N\rangle$, given by

$$|\Phi_N\rangle = \frac{(a_0^\dagger)^N}{\sqrt{N!}}\,|0\rangle \quad . \tag{5.1}$$

The problem which is immediately encountered when equation (5.1) is employed, is that $|\Phi_N\rangle$ is not the vacuum of the operators a_o^+ and a_o. Thus,

$$a_o|\Phi_N\rangle \neq 0 \quad .$$

The single particle $\underline{k} = o$ Green's function is given by,

$$iG_o^{(1)}(t-t') = \frac{\langle\Phi_N|T[\alpha(t)\alpha^+(t')\hat{S}]|\Phi_N\rangle}{S_N} \qquad (5.2)$$

and hence when Wick's theorem is applied to the r.h.s. of this equation, averages of normal ordered products of $\underline{k} = o$ operators do not vanish. However, we shall now demonstrate that the sum of all terms containing normal ordered products in the expansion for the single particle Green's function $G^{(1)}$ of a many boson system at $T = 0K$ must vanish.

In order to demonstrate this, we write Wick's theorem in the now familiar form

$$T = \hat{N}C \qquad (5.3)$$

and substitute equation (5.3) into equation (5.2)

Writing

$$C = C'C_{\alpha s}C_{\alpha\alpha^+}$$

and allowing $C \alpha \alpha^{\dagger}$ to act yields (c.f. equation (3.11)),

$$i G_0^{(i)}(t-t') = i g_0(t-t') + \frac{\langle \Phi_N | \hat{N}[c' C_{\alpha s}(\alpha(t) \alpha^{\dagger}(t') \hat{S})] | \Phi_N \rangle}{S_N} .$$

Noting that C' does not act on $\alpha(t)$ yields, on allowing $C_{\alpha s}$ to act,

$$i G_0^{(i)}(t-t') = i g_0(t-t') + \sqrt{n} \frac{\langle \Phi_{N-1} | N[c'(\alpha^{\dagger}(t') \hat{S})] | \Phi_N \rangle}{S_N}$$

$$+ \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{\infty} dt_1 \, e^{-\varepsilon |t-t_1|} \int d^3x_1 d^3x' \, U(\underline{x}_1 - \underline{x}') \, i g_0(t-t_1)$$

$$\times \langle \Phi_N | T[\psi_H^{\dagger}(\underline{x}', t_1) \alpha_H^{\dagger}(t') \psi_H(\underline{x}', t_1) \psi_H(\underline{x}_1, t_1)] | \Phi_N \rangle$$

(5.3)

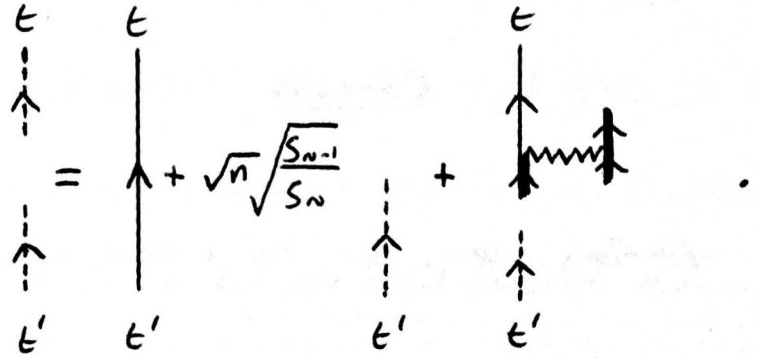
where the last term on the r.h.s. of equation (5.3) has been converted to the Heisenberg picture, in accordance with the techniques of chapter 1.

The second term on the r.h.s. of equation (5.3) has been converted to the Heisenberg picture in Appendix C, which leads us to write this equation in the form (to within a phase factor).

$$
\begin{aligned}
i G_0^{(1)}(\epsilon-\epsilon') = \quad & i\, g_0(\epsilon-\epsilon') \\
& + \sqrt{n\,\frac{S_{N-1}}{S_N}}\ \langle \mathcal{F}_N |\, \alpha_H^{\dagger}(\epsilon')\, | \mathcal{F}_{N-1} \rangle \\
& + \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{\infty} dt_1\, e^{-\varepsilon |t_1|} \int d^3x_1\, d^3x_1'\ i\, g_0(\epsilon-\epsilon_1)\, U(\underline{x}_1-\underline{x}_1') \\
& \times \langle \mathcal{F}_N | T\left[\psi_H^{\dagger}(\underline{x}_1', t_1)\, \alpha_H^{\dagger}(\epsilon')\, \psi_H(\underline{x}_1', t_1)\, \psi_H(\underline{x}_1, t_1) \right] | \mathcal{F}_N \rangle .
\end{aligned}
\tag{5.4}
$$

A trivial combination of the diagrammatic notation of equation (1.29) and Figure 8, enables equation (5.4) to be written as

$i G_0^{(1)}(\epsilon-\epsilon') \equiv$



(5.5)

We shall now demonstrate that the second term on the r.h.s. of equation (5.4) must in general vanish.

For $t' > t$, the first term on the r.h.s. of equation (5.4) vanishes, hence writing

$$\Delta_j = E_N^0 - E_{N-1}^j$$

and

$$A_j = \langle \mathcal{F}_N^0 | \alpha^+ | \mathcal{F}_{N-1}^j \rangle$$

with

$$| \mathcal{F}_N^0 \rangle \equiv | \mathcal{F}_N \rangle$$

we see that for $t' > t$, equation (5.4) becomes

$$iG_0^{(1)}(t-t') = \left(n \frac{S_{N-1}}{S_N}\right)^{1/2} e^{i\frac{\Delta_0}{\hbar}t'} A_0$$

$$+\left(\frac{-i}{\hbar}\right)\int dt_1 \int d^3x_1\, d^3x_1'\, e^{-\varepsilon|t_1|} U(\underline{x}_1 - \underline{x}_1') \frac{\Theta(t-t_1)}{V}$$

$$\times \sum_{j=0}^{\infty} e^{-i/\hbar\,\Delta_j(t_1-t')} A_j B_j(\underline{x}_1, \underline{x}_1') \qquad (5.6)$$

where we have inserted a complete set of eigenstates of H into the last term on the r.h.s. of equation (5.4) and written

$$i g_0(t-t') = \frac{\Theta(t-t')}{V}$$

$$B_j(\underline{x}_1, \underline{x}_1') = \langle \mathcal{F}_{N-1}^j | \psi^+(\underline{x}_1') \psi(\underline{x}_1') \psi(\underline{x}_1) | \mathcal{F}_N^0 \rangle$$

Assuming that it is permissible to interchange the summation and integration in equation (5.6) yields

$$i G_o^{(1)}(t-t') = \left(n \frac{S_{n-1}}{S_n}\right)^{1/2} e^{i/\hbar \Delta_0 t'} A_0$$

$$+ \sum_j A_j B_j \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{t} dt_1 \, e^{-\varepsilon|t_1|} e^{-i/\hbar \Delta_j (t_1 - t')} \tag{5.6}$$

where

$$B_j = \int d^3x_1 \, d^3x_1' \, \frac{U(\underline{x}_1 - \underline{x}_1')}{V} B_j(\underline{x}_1, \underline{x}_1') .$$

The infinitesimal positive constant ε in equation (5.6) ensures that the contribution to the definite integral from the lower limit vanishes. Performing the integration, then taking the limit $\varepsilon \to 0^+$ yields

$$i G_o^{(1)}(t-t') = \left(n \frac{S_{n-1}}{S_n}\right)^{1/2} A_0 e^{\frac{i\Delta_0 t'}{\hbar}} + \sum_j \frac{A_j B_j}{\Delta_j + i\varepsilon} e^{-i/\hbar \Delta_j (t-t')} . \tag{5.7}$$

For $t' > t$, the definition of $G_o^{(1)}$ yields

$$i G_o^{(1)}(t-t') = \sum_j A_j A_j^* e^{-i/\hbar \Delta_j (t-t')} . \tag{5.8}$$

Hence, equation (5.7) becomes

$$\sum_j \left[A_j A_j^* - \frac{A_j B_j}{\Delta_j + i\varepsilon} \right] e^{\frac{-i}{\hbar} \Delta_j (t-t')} = \left(n \frac{S_{n-1}}{S_n}\right)^{1/2} A_0 \, e^{\frac{i\Delta_0 t'}{\hbar}} \tag{5.9}$$

and we conclude that both sides of this equation must vanish.

In general therefore

$$\left(n\,\frac{S_{N-1}}{S_N}\right)^{1/2} A_0\, e^{i\frac{\Delta_0}{\hbar}\epsilon'} = 0 \quad . \tag{5.10}$$

Since n is assumed finite, this means that in general we must have

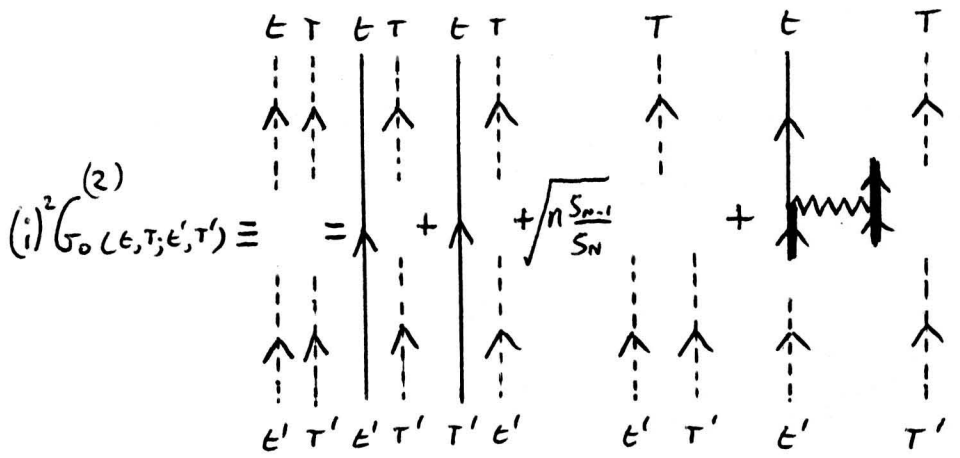
$$A_0 \sqrt{\frac{S_{N-1}}{S_N}} = 0 \quad . \tag{5.11}$$

This result tells us that the second term on the r.h.s. of equations (5.3) and (5.4) must vanish. Hence, had we employed the mnemonic 'Ignore normal ordered products in time dependent boson perturbation theory', we would have obtained the correct result for $G_0^{(1)}$. In order to demonstrate the general validity of this mnemonic, consider for example the two particle $\underline{K} = 0$ Green's function given by

$$(i)^2 G_0^{(2)}(\epsilon,T;\epsilon',T') = \langle \mathcal{F}_N | T[\alpha_H(\epsilon)\,\alpha_H(T)\,\alpha_H^+(\epsilon')\,\alpha_H^+(T')] | \mathcal{F}_N \rangle$$

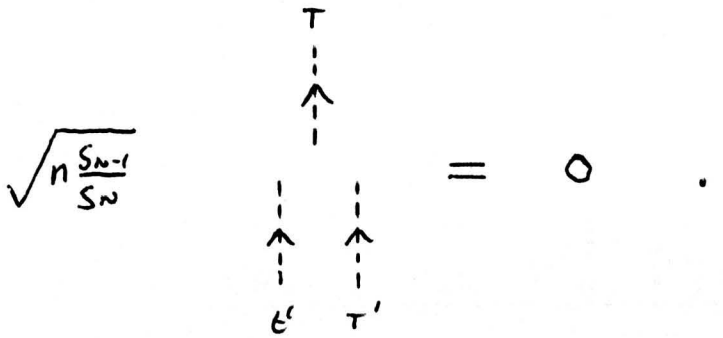
$$= \frac{\langle \mathcal{F}_N | T[\alpha(\epsilon)\,\alpha(T)\,\alpha^+(\epsilon')\,\alpha^+(T')\,\hat{S}] | \mathcal{F}_N \rangle}{S_N} \quad . \tag{5.12}$$

Applying to equation (5.12), the same arguments which led to equation (5.5) yields



(5.13)

and we are led to the conclusion



(5.14)

In other words, the correct result is again obtained if the mnemonic on the previous page is employed. Clearly, this result is of general validity, because whenever Wick's theorem is applied in the controlled manner described above to any Green's function appearing in boson perturbation theory, the resulting normal ordered terms lose their dependence on at least one time variable and must therefore vanish. Since the controlled application of Wick's theorem merely represents a convenient method of summing the diagrams obtained in the more conventional theory of chapter 2 and section 3.2, we have demonstrated that the sum of normal ordered terms in time dependent boson perturbation theory must in general vanish.

Before proceeding to formulate a new diagrammatic perturbation theory which is applicable to the boson problem, it is convenient at

this point to consider the existence or otherwise of a condensate in an interacting bose gas at $T = 0K$.

5.3 The question of a condensate

By definition, the $\underline{k} \neq 0$ single particle Green's function is given by

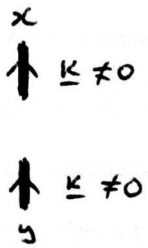
$$i G'(x,y) = \langle \mathcal{F}_N | T[\, \psi'_H(x)\, \psi'^{\dagger}_H(y)\,] | \mathcal{F}_N \rangle . \tag{5.15}$$

In the interaction picture this becomes

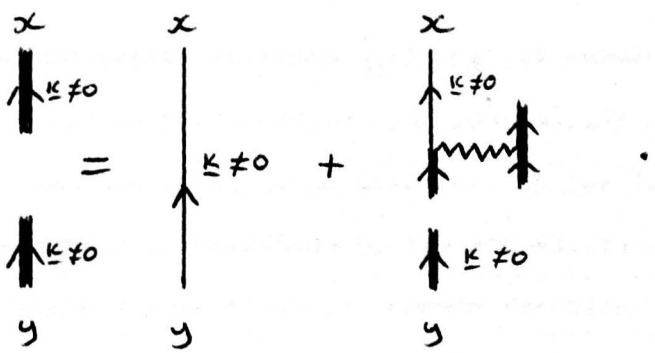
$$i G'(x,y) = \frac{\langle \mathcal{F}_N | T[\, \psi'(x)\, \psi'^{\dagger}(y)\, \hat{S}\,] | \mathcal{F}_N \rangle}{S_N} . \tag{5.16}$$

Since $|\mathcal{F}_N\rangle$ is the vacuum of the operators ψ' and $\psi'^{\dagger}$ the arguments which led to equation (1.29) of chapter 1 are valid and hence denoting $i G'(x,y)$ by

$i G'(x,y) \equiv$

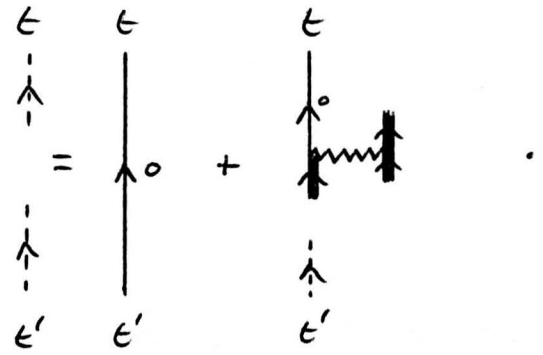


yields



(5.17)

In view of equation (5.17), equation (5.5) becomes

$i G_0^{(1)}(\epsilon-\epsilon') \equiv$  . (5.18)

A comparison of equations (5.17) and (5.18) reveals that

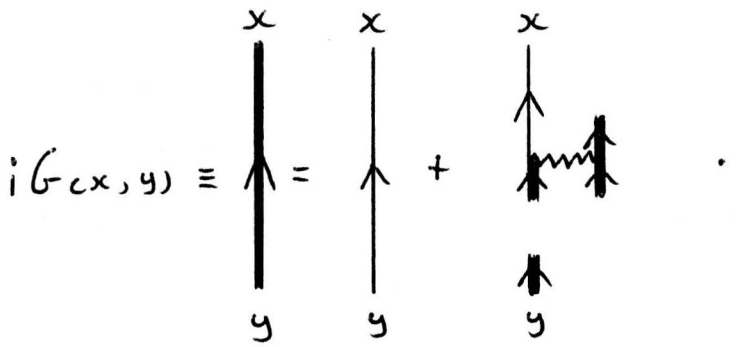
$$G_0^{(1)}(\epsilon-\epsilon') = \lim_{\underline{k}\to 0} G'(\underline{k}, \epsilon-\epsilon') \quad . \tag{5.19}$$

Since the particle density $n_{\underline{k}}$ in the state $\underline{k}$ is simply $\lim_{\epsilon'\to\epsilon^+} i G(\underline{k}, \epsilon-\epsilon')$, equation (5.19) yields

$$n_0 = \lim_{\underline{k}\to 0} n_{\underline{k}} \quad . \tag{5.20}$$

Hence, assuming that $n_{\underline{k}}$ is a continuous function of $\underline{k}$ for small $\underline{k}$, there will be no δ - function singularity in the particle distribution function of an interacting bose gas in the state $|\Psi_N\rangle$, i.e. there will be no condensate. This is consistent with the neutron scattering data of Mook [22] and the theoretical arguments of Evans [23] . It should be noted however that equation (5.20) does not exclude the possibility of a generalized or smeared condensate of the type discussed in the literature [35] .

A comparison of equations (5.17) and (5.18) shows that we may write the whole single particle Green's function $G^{(1)} = G_0^{(1)} + G'$ in the form of equation (1.29)



(5.21)

In terms of mathematics, equation (5.21) is simply

$$iG(x,y) = ig(x,y) + \left(\frac{-i}{\hbar}\right)\int_{-\infty}^{\infty} dt_1\, e^{-\varepsilon|6.1|}\int d^3x_1\, d^3x_1'\, U(\underline{x}_1 - \underline{x}_1')$$

$$\times ig(\underline{x},\underline{x}_1)\langle \mathcal{F}_\infty | T[\psi_H^+(x_1')\psi_H^+(y)\psi_H(x_1')\psi_H(x_1)] | \mathcal{F}_\infty\rangle$$

(5.22)

which is identical with equation (1.28) of the fermion problem.

In view of the similarity between equations (5.21) and (1.29) it is perhaps not too surprising that (as demonstrated below) the perturbation expansion for $G^{(1)}$ in the boson problem turns out to be topologically identical with the corresponding expansion in the fermion problem. The derivation of the boson perturbation expansion is non-trivial however and in fact we shall find that a canonical transformation is required in order to obtain it. In order to illustrate the new techniques which we shall employ to generate the boson perturbation expansion, we shall consider in the following section, the problem of obtaining the infinite order fermion perturbation series directly from equation (1.29).

5.4 Generation of the conventional fermion perturbation series

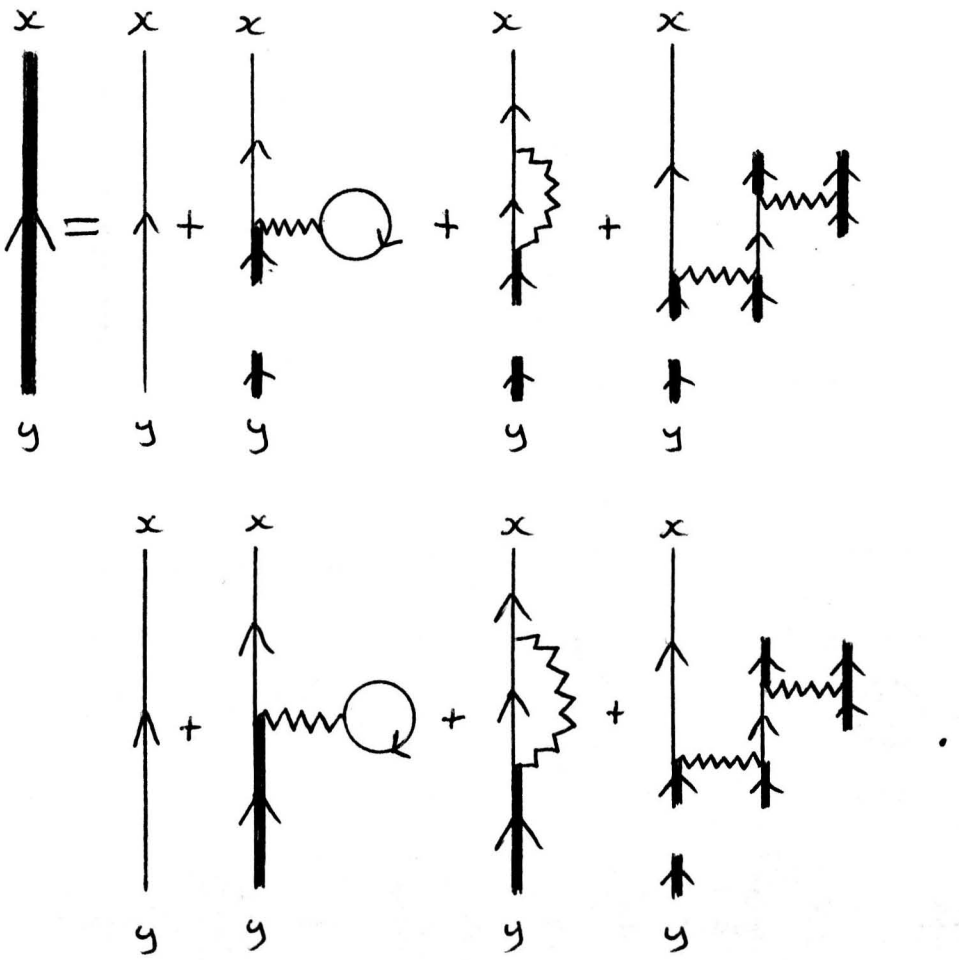
In fermion perturbation theory, the canonical transformation to particles and holes in equation (1.14) ensures that averages of normal ordered products of uncontracted operators vanish. An additional feature of this transformation is that it allows $ig(x,y)$ to propagate both forwards (particles) and backwards (holes) in time, a property which as we shall see in section (5.5) is vital for the generation of the perturbation series.

In order to obtain a perturbation series from equation (1.29), we focus attention on the matrix element appearing in the corresponding equation (1.28). In the interaction picture, this matrix element is

$$\frac{\langle \Phi_N | T[\, \psi^{+}_{(\underline{x}',\epsilon_1)} \psi^{+}_{(y)} \psi_{(\underline{x}',\epsilon_1)} \psi_{(\underline{x}_1,\epsilon_1)} \hat{S}\,] | \Phi_N \rangle}{S_N} \,. \tag{5.23}$$

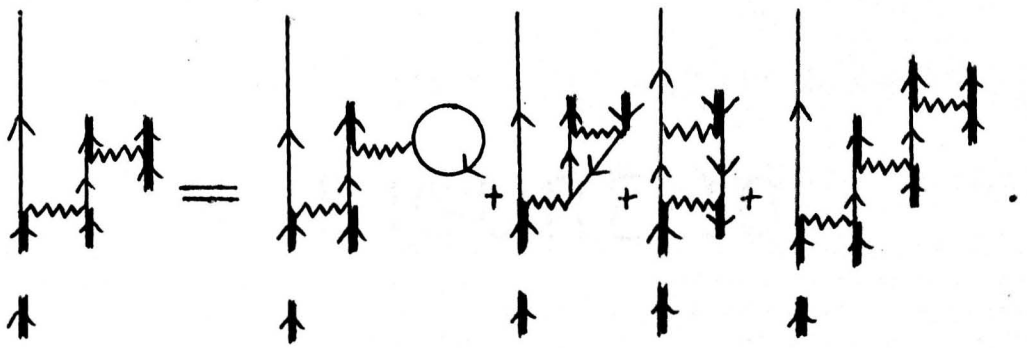
We shall generate the perturbation expansion in stages. The first stage consists of selecting one of the operators appearing in the above matrix element and considering all the possible contractions which may involve the chosen operator. This is similar to the process which led to equation (1.29), where, by selecting the operator $\psi_{(x)}$ we re-wrote equation (1.9) in the form of equation (1.23).

In equation (5.23), we shall for convenience select the operator $\psi^{+}_{(\underline{x}',\epsilon_1)}$, which may contract with $\psi_{(\underline{x}',\epsilon_1)}$, $\psi_{(\underline{x}_1,\epsilon_1)}$ and $\hat{S}$. Employing the techniques which led to equation (1.29) readily demonstrates that equation (1.29) becomes



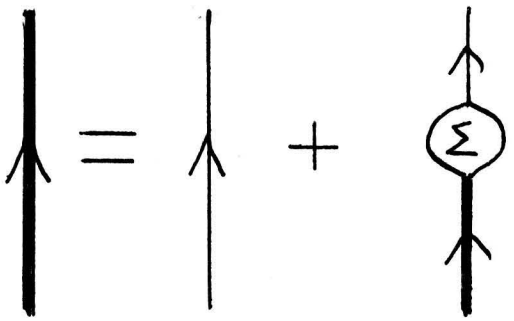
(5.24)

The next state in this iterative procedure consists of arbitrarily selecting one of the six operators in the last term on the r.h.s. of equation (5.24) and considering all the possible contractions which may involve it. For example, the last term on the r.h.s. of equation (5.24) may be written



(5.25)

The middle two terms on the r.h.s. of equation (5.24) have the topology of the last term on the r.h.s. of equation (1.15) and the ultimate goal



(5.26)

where to second order,

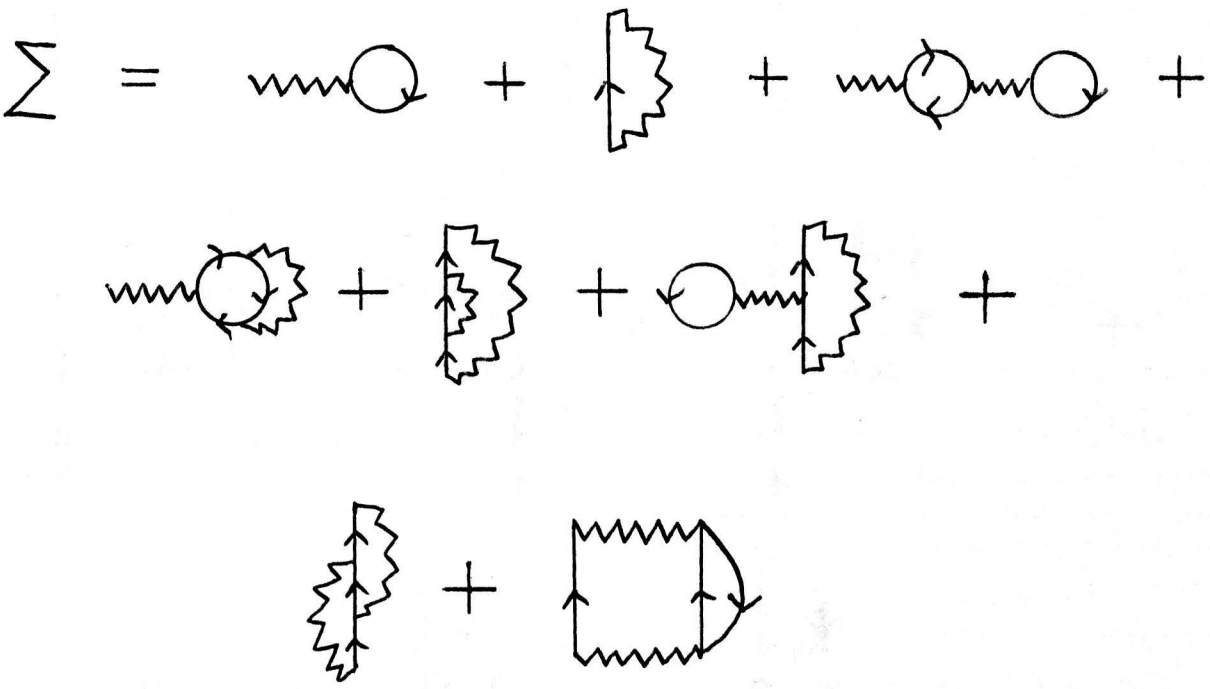


FIGURE 10

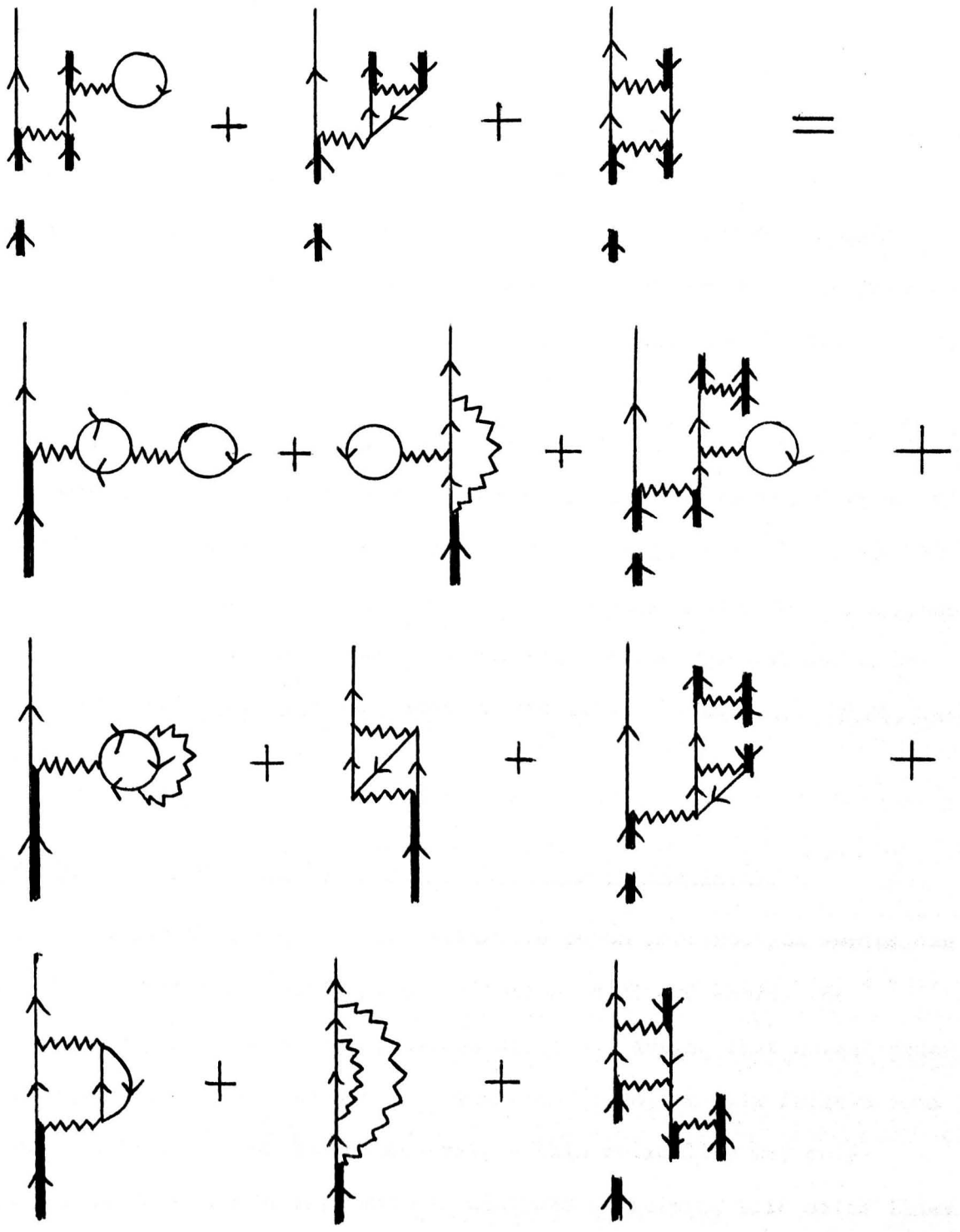


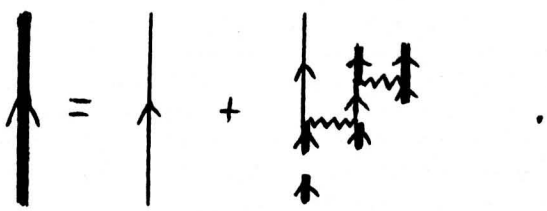
FIGURE 9

of our iterative procedure is to write the last term on the r.h.s. of equation (5.24) in this same form. The first three terms on the r.h.s. of equation (5.25) will assume the correct form after a pair of thick solid lines with free ends have been eliminated from these diagrams. Iterating these terms one stage further yields Figure 9, which demonstrates that, to second order in the bare interaction, equation (5.24) takes the form of Figure 10.

The latter figure demonstrates explicitly how the clothing of bare propagator lines occurs in the theory and it is easy to see that repeating this iterative procedure to infinity yields the conventional perturbation series found in the literature [7,8] . The assumption of convergence is self-evident in the present derivation, because terms which do not have the topology of the last term on the r.h.s. of equation (5.26) have been ignored at infinity.

5.5 Problems with the formation of a boson perturbation series

As a first attempt at generating the boson perturbation series, we shall take equation (5.21) as our starting point and employ the techniques described in the previous section. Noting that normal ordered terms may be ignored, we see that equation (5.24) readily follows from equation (5.21). For bosons however, a thin solid line may only propagate forwards in time and all diagrams containing thin solid lines at equal times or running backwards in time must be omitted. Hence, for bosons, equation (5.24) becomes



(5.27)

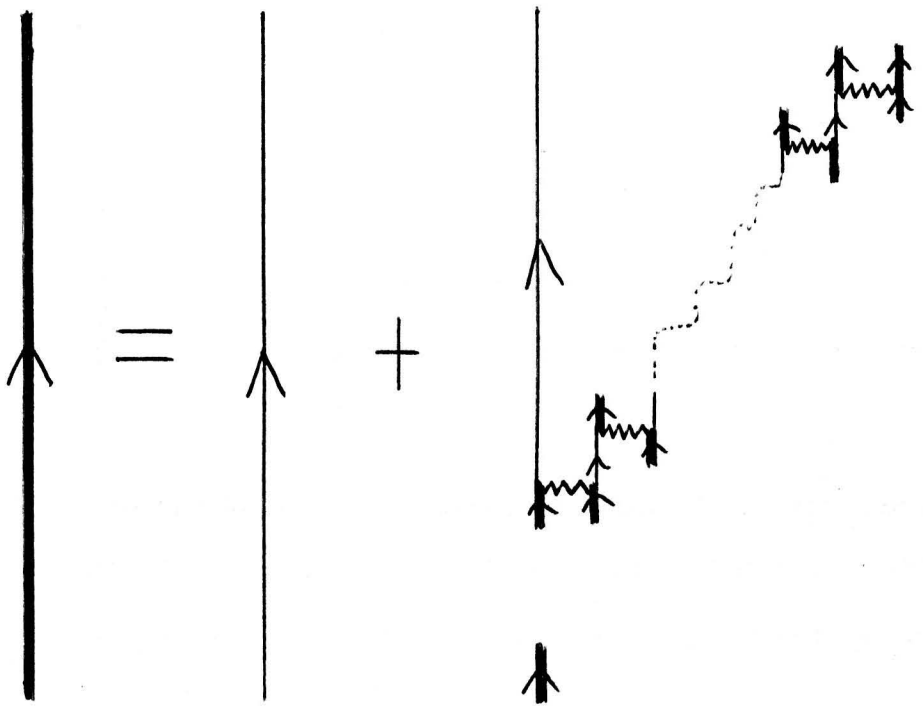
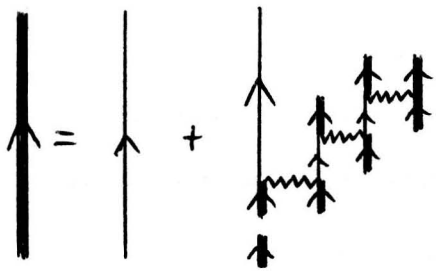


FIGURE 11

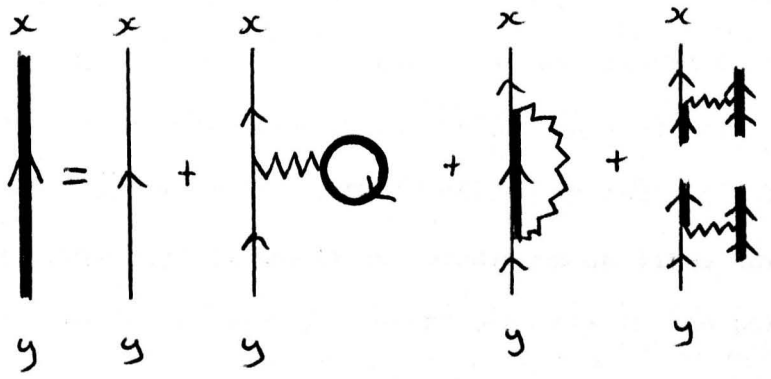
A glance at equation (5.25) reveals that this equation may be further iterated to yield



(5.28)

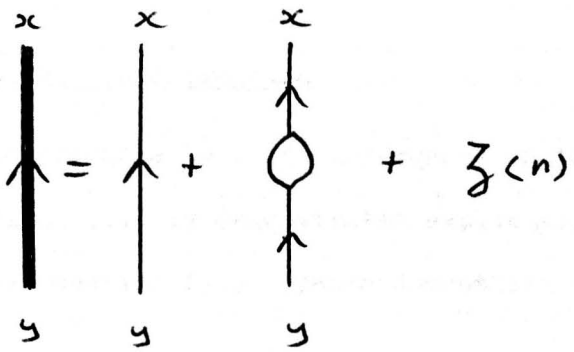
Clearly the topology of equation (5.26) of Figure 10 is not forthcoming. Indeed, an arbitrarily large number of iterations of the type which led to equation (5.28) yields Figure 11, which indicates a lack of convergence in the present iterative procedure.

Of course, it is possible to iterate equation (5.21) in a different manner. For example, after one iteration, equation (5.21) may be written in the form



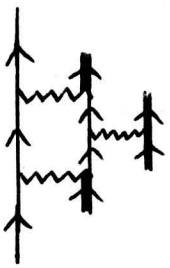
(5.30)

Iteration of equation (5.30) demonstrates that $G^{(1)}$ may be written (after n iterations say)



(5.31)

where $\mathcal{Z}(n)$ represents terms with the topology of the last term on the r.h.s. of equation (5.30), consisting of two parts separately connected to the points x and y. In view of equation (5.29) however, it is not clear that $\mathcal{Z}(n)$ may be ignored in the limit $n \to \infty$ and furthermore, even if $\mathcal{Z}(n)$ could be ignored in this limit, not all contributions to the second term on the r.h.s. of equation (5.31) have the topology of the diagrams in Figure 10. A typical contribution to equation (5.31) of this type is



which involves a two particle Green's function $G^{(2)}$. Since the purpose of the present iterative procedure is to remove $G^{(2)}$ from the theory (c.f. equation (5.21)) the topology of equation (5.31) is clearly unsatisfactory. For this reason, we must resort to the canonical transformation described in the following section.

The essence of the transformation is that it allows thin solid lines to propagate both forwards and backwards in time, while still preserving the irrelevance of normal ordered products in the perturbation series. Thus the problems of the present section are removed and the techniques of section (5.4) may be applied to yield a perturbation series which is identical with that of the fermion problem.

5.6 A canonical transformation

The probable lack of convergence in the perturbation series obtained in section (5.4) is demonstrated explicitly in Figure 11. In the fermion series of section (5.3) such a demonstration of non-convergence was not

possible, because thin solid lines were not constrained to run forwards in time only. Of course, this does not mean that the fermion perturbation series converges, it simply means that we cannot prove that it does not converge. In the present section, we shall reformulate the perturbation theory of bosons in such a way that the non-convergence of the perturbation series obtained cannot be demonstrated. The theory obtained is thus on the same footing as the corresponding fermion theory.

To this end, consider the simple canonical transformation

$$\alpha \equiv \beta^{+} \quad \text{and} \quad \alpha^{+} \equiv \beta \qquad . \tag{5.32}$$

Equations (5.32) may be considered to be a transformation from particles, to holes in the condensate, although the analogy with holes in the fermi sea should be treated with caution.

In order to proceed, we shall first prove that normal ordered products may still be ignored when the transformation (5.32) is employed . Our starting point is again equation (5.2) and we shall follow closely the analysis of section (5.2).

The normal ordered product of two operators β and β^{+} is given by

$$\hat{N}[\beta\beta^{+}] \equiv \beta^{+}\beta \qquad . \tag{5.33}$$

Thus, writing

$$\beta^{\cdot}(t')\beta^{+\cdot}(t) \equiv T[\beta(t')\beta^{+}(t)] - \hat{N}[\beta(t')\beta^{+}(t)]$$

yields

$$\beta^{\cdot}(t')\beta^{+\cdot}(t) = \frac{-\Theta(t'-t)}{V} \equiv i\ell_{0}(t-t') \quad \text{say.} \tag{5.34}$$

Applying the transformation (5.32) to equation (5.2) yields

$$i G_0^{(1)}(\epsilon-\epsilon') = \frac{\langle \Phi_N | T[\beta^+(\epsilon)\beta(\epsilon')\hat{S}] | \Phi_N \rangle}{S_N} \quad . \tag{5.35}$$

The analysis of section (5.2) which led to equation (5.3) now yields

$$i G_0^{(1)}(\epsilon-\epsilon') = i\ell_0(\epsilon-\epsilon') + \sqrt{n}\,\frac{\langle \Phi_{N+1} | T[\beta(\epsilon')\hat{S}] | \Phi_N \rangle}{S_N}$$

$$+\left(\frac{-i}{\hbar}\right)\int_{-\infty}^{\infty} dt_1\, e^{-\varepsilon|t_1|}\int d^3x_1\, d^3x_1'\, U(\underline{x}_1-\underline{x}_1')\, i\ell_0(\epsilon-\epsilon_1)$$

$$\times \langle \Phi_N | T[\psi_H^+(\underline{x}_1',t_1)\beta_H(\epsilon')\psi_H(\underline{x}_1',t_1)\psi_H(\underline{x}_1,t_1)] | \Phi_N \rangle .$$

(5.36)

Notice the difference between the second terms on the r.h.s. of equations (5.3) and (5.36). This is due to the re-definition of a normal ordered product given in equation (5.33). An analysis similar to that given in Appendix C readily shows that the second term on the r.h.s. of equation (5.36) may be written

$$\left(n\,\frac{S_{N+1}}{S_N}\right)^{1/2} \langle \Phi_{N+1} | \beta_H(\epsilon') | \Phi_N \rangle$$

and hence, equation (5.36) becomes

$$i G_0^{(1)}(\epsilon-\epsilon') = i\ell_0(\epsilon-\epsilon') + \left(n\,\frac{S_{N+1}}{S_N}\right)^{1/2} \langle \Phi_{N+1} | \alpha_H^+(\epsilon') | \Phi_N \rangle$$

$$+\left(\frac{-i}{\hbar}\right)\int_{-\infty}^{\infty} dt_1\, e^{-\varepsilon|t_1|}\int d^3x_1\, d^3x_1'\, U(\underline{x}_1-\underline{x}_1')\, i\ell_0(\epsilon-\epsilon_1)$$

$$\times \langle \Phi_N | T[\psi_H^+(\underline{x}_1',t_1)\alpha_H^+(\epsilon')\psi_H(\underline{x}_1',t_1)\psi_H(\underline{x}_1,t_1)] | \Phi_N \rangle .$$

(5.37)

The analysis which led from equation (5.3) to equation (5.11), when applied to equation (5.37) yields

$$\left(\frac{S_{N+1}}{S_N}\right)^{1/2} \langle \Psi_{N+1} | \alpha^+ | \Psi_N \rangle = 0 \quad . \tag{5.38}$$

Since we are working in the limit $N \to \infty$ equations (5.38) holds equally well for an $N-1$ particle system i.e.,

$$\left(\frac{S_N}{S_{N-1}}\right)^{1/2} A_0 = 0 \quad . \tag{5.39}$$

Again we find that normal ordered terms may be ignored, and comparison of equations (5.11) and (5.39) yields

$$A_0 = 0 \quad . \tag{5.40}$$

These equations are also consistent with the result obtained within the Hugenholtz and Pines formalism in section (3.3), that

$$\frac{S_{N-1}}{S_N} \neq 0$$

Note that equation (5.40), which is simply

$$\langle \Psi_N | \alpha^+ | \Psi_{N-1} \rangle = 0 \tag{5.41}$$

supports the result of section (5.3) that there will be no condensate in an interacting bose gas. This follows from the work of Beliaev [3] where it is argued that when the number of particles in the $\underline{k} = 0$ mode

is a finite fraction of N , we should find that

$$a_0^+ |\mathcal{F}_{N-1}\rangle \simeq \sqrt{n_0}\, |\mathcal{F}_N\rangle \quad .$$

Since this is certainly not the case according to equation (5.41), we conclude that there is no condensate.

In order to generate the perturbation series, we introduce the transformation (5.32) and employ the techniques of section (5.4). A thin solid line running from y to x is now to be interpreted as

$$\begin{aligned} i\,g(x,y) &= \psi'_{(x)}\,\psi^{+\prime}_{(y)} + i\,\ell_0(t_x - t_y) \\ &\equiv i\,g'(x,y) + i\,\ell_0(t_x - t_y) \end{aligned} \tag{5.42}$$

where ψ' and ψ'^{+} are $\underline{k} \neq 0$ field operators.

The fourier transform of equation (5.42) is

$$g(k) = \frac{\delta_{\underline{k},0}}{\omega - \omega_k - i\eta} + \frac{1 - \delta_{\underline{k},0}}{\omega - \omega_k + i\eta} \tag{5.43}$$

and we now see that a thin solid line may run both forwards and backwards in time. Since normal ordered terms are to be ignored, the graphical analysis of section (5.4) applies equally well to the present section. In particular, Figure 10 remains correct for bosons whilst Figure 11 no longer applies.

Evidently, the convergence in the boson series is extremely slow, because each individual contribution to Σ is of order $1/V$, while the sum (equation 5.26) must be of order unity. The former observation follows , because a thin solid line running backwards in time, or at

equal times, is of order $1/V$ and every contribution to Σ contains at least one such line. However, the topology of the bare perturbation series is identical with that of the corresponding fermion series, as a result each thin solid line may be 'clothed' in the manner described in chapter 1. In this way we obtain a renormalized version of the self-energy Σ in which each term is of order unity, because now, only thick solid lines appear in a given contribution to Σ .

The final result is the well-known Dyson equation

$$G(k) = \left[\omega - \omega_k - \Sigma(k,\omega)\right]^{-1} \tag{5.44}$$

and the analysis of chapter 1 and Appendix A may be invoked to express the self-energy appearing in equation (5.44) in terms of self-consistently determined effective interactions.

Before leaving the present section and going on to consider the application of perturbation theory to a bose system which exhibits long range order, it should be noted that the present theory applies to a real interacting system of bosons only. That this is so follows from the implicit assumption that $\mu \neq 0$, which was employed to eliminate normal ordered terms from the theory. For this reason, equation (5.44) will not yield the Green's function of a finite density non-interacting system of bosons in the limit $\Sigma \to 0$.

5.7 Application to liquid ^4He

We shall consider in the present section the perturbation theory of an interacting system of bosons at $T = 0K$, exhibiting long range order characterized by the existence of anomalous propagators. These propagators are assumed to play the role of an order parameter and therefore vanish

above the transition temperature. We will not consider the question of what physically constitutes the order parameter (although in the absence of a condensate, the pairing of bosons seems to be a likely candidate), but merely assume that the anomalous propagators may be represented topologically by the notation of equation A3.

The diagrammatic manipulations found in Appendix A and elsewhere [9] now apply automatically to the boson problem. In particular, equations (A10) and (A11) yield

$$G(K) = \frac{g^{-1}(-K) - \Sigma_{11}(-K)}{(g^{-1}(K) - \Sigma_{11}(K))(g^{-1}(-K) - \Sigma_{11}(-K)) - \Sigma_{12}(K)\Sigma_{21}(K)} \quad . \tag{5.45}$$

Instead of entering into detailed numerical calculations, we shall demonstrate that the simplest of approximations on equation (5.45) yields some agreement with experiment. For convenience, we shall switch to the Heisenberg picture of the Hamiltonian

$$\hat{K} = \hat{H} - \mu \hat{N} \quad .$$

This is readily achieved by writing

$$g^{-1}(\underline{K}, \omega) = \omega - \omega_{\underline{K}} + \mu/\hbar$$

in equation (5.45) and allows us to measure energies relative to μ . Assuming as in chapter 5 that (for small $\underline{K}$ and ω at least) the self energies are frequency independent, equation (5.45) yields

$$G(K) = \frac{\omega + \omega_{\underline{K}} - \mu/\hbar + \Sigma_{11}}{\omega^2 - E^2(\underline{K})} \tag{5.46}$$

where

$$E_{(\underline{K})} = \left[\omega_{\underline{K}}^2 + 2\omega_{\underline{K}}(\Sigma_{11} - \mu) + (\Sigma_{11} - \mu)^2 + \Sigma_{12}^2 \right]^{1/2} . \quad (5.47)$$

Now, the particle density in the state $\underline{K}$ is

$$n_{\underline{K}} = -\frac{1}{2\pi i} \lim_{\eta \to 0^+} \int d\omega \, e^{i\omega\eta} G(\underline{K}, \omega)$$

Coupling this with equation (5.46) followed by an elementary integration yields

$$n_{\underline{K}} = \frac{\omega_{\underline{K}} + \mu - \Sigma_{11}}{2E_{(\underline{K})}} - 1 . \quad (5.48)$$

In the case of superfluid liquid ^4He, the excitation spectrum is known to be linear in the long wavelength limit [28] which implies via equation (5.47) that for small $\underline{K}$

$$\mu = \Sigma_{11} - \Sigma_{12} . \quad (5.49)$$

In the present context, the relation (5.49) is semi-empirical, unlike the Hugenholtz and Pines relation of chapter 3 which is exact with their (zero density) formalism. Equations (5.48) and (5.49) show that $n_{\underline{K}}$ is proportional to $1/K$ in the limit $\underline{K} \to 0$ and hence we expect a plot of $K^2 n_{\underline{K}} \to K$ for ^4He to pass through the origin. This is in agreement with the neutron scattering data [22,23,35].

Further progress cannot be made without an explicit evaluation of the self-energies. However, since the aim of the present thesis has been to provide a rigorous, general framework in terms of which the problem may be clearly formulated, we shall terminate the discussion at this point and leave the question of the detailed structure of the self-energies to form the subject of a future investigation.

CONCLUSION

The aim of perturbation theory is to express a real but insoluble problem in terms of a hypothetical but soluble one. In conventional perturbation theory this is emphasized by the fact that only bare interaction (i.e. $u(\underline{x},\underline{y})$) and propagator (i.e. $g(x,y)$) lines, corresponding to the hypothetical system, appear in the diagrammatic perturbation series, and any practical calculation consists of summing an infinite subset of these diagrams.

It is well-known that the theory may be improved by replacing bare propagator lines by the exact propagator ($G^{(1)}(x,y)$) of the interacting system. In this way, the original expansion of the self-energy in powers of g and u, is replaced by an expansion in powers of G and u, and the retention of only first order (i.e. Hartree-Fock) diagrams, yields agreement with experiment in a wide range of situations. The renormalization is incomplete however, and in some situations it is still necessary to perform the summation of an infinite subset of the above diagrams.

We have shown in the present thesis how the renormalization may be completed, by expanding the self-energy in powers of G and an effective interaction Γ. The result is not a perturbation theory in the true sense, because the real problem is now expressed in terms of itself rather than a hypothetical problem. However, a given practical calculation is now reduced to that of solving a self-consistent set of equations containing a <u>finite</u> number of terms and evidently the summation of an infinite subset of diagrams is no longer required. This is a highly desirable feature as far as numerical calculations are concerned, since the former problem lends itself to solution by computer, while the latter does not. Furthermore the formalism itself is directly amenable to the construction of effective interactions between particles, a topic of

great interest in nuclear physics and theories based on the concept of a pseudopotential - such theories have not been considered here since they would take us too far afield from our main theme.

In order to obtain the above result we developed an entirely new approach to fermion perturbation theory in which the power of diagrammatic analysis is emphasized by the fact that without the use of diagrams, the self consistent manipulations involved in the formalism would not have been possible. The power of the present approach rests on the concept of employing Wick's theorem in a controlled manner, and then reverting to the Heisenberg picture. This new technique allows us to avoid from the outset the introduction of disconnected diagrams into the theory (c.f. section (5.4)). Furthermore it eliminates normal ordered products from the theory of bosons. The latter feature is extremely important, since it is the presence of normal ordered products, together with the associated depletion effect, which prevents an exact formulation of the boson problem.

Until now, there has been no microscopic theory of a many boson system in its ground state, which has not invoked ab initio approximations in order to overcome the problem of the depletion effect. Two of these approximations, namely those of ignoring diagrams of relative order $1/V$ and replacing the $\underline{K} = 0$ operators by C - numbers, have been shown to be strictly only applicable to a zero density gas. An infinite hierarchy of self consistent solutions to the problem of an interacting bose gas have been proposed; the simplest of these 'predicting' a new branch in the excitation spectrum which is linear for small $\underline{K}$. Clearly, any number of such branches is possible within the proposed hierarchy of solutions, which emphasizes that linearity in the energy spectrum is to be regarded more as a measure of the self consistency of the theory than

as a measure of the validity of assumptions concerning the $\underline{K} = 0$ mode. Indeed, even the possibility of a band structure in the excitation spectrum for small $\underline{K}$ is indicated by the above solutions. The range of applicability of these results is not known however and in chapter 5, we turned our attention to a more rigorous solution of the problem.

The theory which is provided in chapter 5 is an exact analytic solution of the problem of an interacting bose gas at $T = 0K$. It makes no approximations or assumptions, apart from the unavoidable one concerning the convergence of the perturbation expansion. In relation to this point, we note that in the absence of a canonical transformation, section (5.5) indicates a lack of convergence of the perturbation series. The effect of the introduction of the transformation (5.32) is to provide a 'bare' perturbation series in which each term is individually of order $1/v$. Obviously, terms of order $1/v$ cannot be ignored in such a series, whose sum is of order unity. However, the summation techniques of chapter 1 and Appendix A may be employed and the final result is identical with that of fermion perturbation theory. Hence we may reasonably assume that the resulting series does indeed converge.

At first sight it may seem strange that a canonical transformation can introduce convergence to a previously non-convergent perturbation series. However, such a transformation merely provides an alternative expansion of the Green's function $G^{(1)}(x,y)$. In relation to this point we note that an analogous situation is to be found in the following two series, both of which sum self-consistently to yield $1/3$.

$$1/3 = (1+2)^{-1} = 1 - 2 + 4 - 8 + \cdots$$

$$1/3 = 1/2\,(1+1/2)^{-1} = 1/2\left[1 - 1/2 + 1/4 - 1/8 + \cdots\right]$$

An important feature of the present work is its relevance to the question of the existence of a condensate in an interacting bose gas at $T = 0\,K$. We have shown on quite general grounds that

$$\langle \Psi_{N-1} | a_0^+ | \Psi_N \rangle = 0 \tag{6.1}$$

which indicates that the addition of a particle to the $\underline{k} = 0$ mode of the interacting system in its ground state, produces an excited state of the system. Since it has been argued [3] that the addition of a particle to the $\underline{k} = 0$ mode of a condensed system will not excite the system, equation (6.1) indicates that an interacting system of bosons at $T = 0\,K$ does not possess a condensate.

In section (5.7), we considered the perturbation theory of a system of bosons exhibiting long range order characterized by the existence of anomalous propagators. The assumption that the self-energies were continuous functions of $\underline{k}$ for small $\underline{k}$, again led to the conclusion that an interacting bose gas will not possess a condensate. This immediately poses the question of what physically constitutes the order parameter in liquid ^4He . Unfortunately, perturbation theory does not provide the answer to this question, but it is interesting to note that (as shown in section (5.7)) the very assumption that an order parameter exists in the absence of Bose-Einstein condensation provides a correlation between the linearity of the energy spectrum and the $k^2 n_{c(k)} \to k$ curve of superfluid ^4He , for small $\underline{k}$.

The excitation spectrum cannot be calculated without an explicit evaluation of the self-energies and until such a calculation is performed, it is not possible to determine which, if any of the self-

consistent solutions proposed in chapter 4 are applicable to superfluid ^4He . Although we have not concerned ourselves with such numerical calculations in the present work, we note in conclusion that the general results of Appendix A should prove to be a useful point from which a self-consistent evaluation of the self-energies may proceed.

APPENDIX A

PERTURBATION THEORY IN THE PRESENCE OF ANOMALOUS PROPAGATORS

A.1 Introduction

The adiabatic switching on of the interaction is most conveniently achieved by writing the Hamiltonian in the form

$$H = H_0 + g H_1 \tag{A1}$$

where

$$g = \lim_{\varepsilon \to 0} e^{-\varepsilon |t|} \tag{A2}$$

The basic hypothesis of the perturbation theory discussed in Chapter 1 is the continuity of all physical quantities as functions of the coupling constant g in the interval $0 \leq g \leq 1$.

An equivalent hypothesis [13] consists of assuming the continuity of all quantities as functions of the density n, (if n is small, it is physically evident that g is small). Such a hypothesis obviously eliminates any possibility of a phase change. The most convenient way of avoiding this problem and generating the ground state of an interacting system which is known a priori to have undergone a transition, is to introduce a small symmetry breaking term α into the free part of the Hamiltonian, H_0 and to take the limit $\alpha \to 0$ after the adiabatic switching on procedure has been carried out [9].

The perturbation series thus generated contains an infinite number of terms of order α, which must be summed self-consistently before the limit $\alpha \to 0$ is taken. In this limit, equations (1.29) and (1.30) are still valid, but the appearance of 'anomalous' propagators

in the perturbation series means that our treatment of the two and three particle Green's functions appearing in these equations requires some modification.

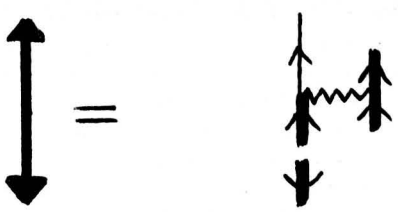
The anomalous propagators, which we shall call G_{12} and G_{21} and denote by

$G_{12}(K) \equiv$  $G_{21}(K) \equiv$  (A3)

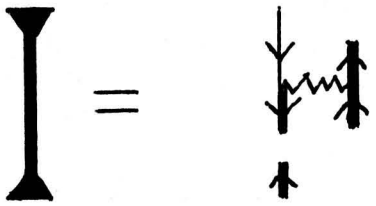
have various definitions, depending on the particular system under consideration [9] . However, since almost all of our manipulations are of a diagrammatic nature, the final results apply to any system whose anomalous propagators can be written in the diagrammatic form of equation (A3), whatever their definition. An example, which is useful to bear in mind throughout this appendix is

$$\begin{aligned} i G_{12}(x,y) &= \langle \mathcal{F} | T[\psi_H(x)\, \psi_H(y)] | \mathcal{F} \rangle \\ i G_{21}(x,y) &= \langle \mathcal{F} | T[\psi_H^{\dagger}(x)\, \psi_H^{\dagger}(y)] | \mathcal{F} \rangle \end{aligned} \qquad \text{(A4)}$$

Equations (A4) are useful, because they describe a superconducting system of spin-less fermions [19] or alternatively a superfluid system of spin-less bosons within the BHP formalism. The arguments leading to equations (1.29) and (1.30) yield, in the limit $\alpha \to 0$,

 (A4)

and



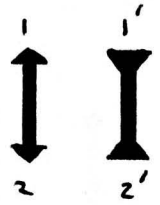
(A5)

and we now proceed to solve equations (1.29), (A4) and (A5) self-consistently to first order in the effective interaction.

A.2 The two particle Green's functions

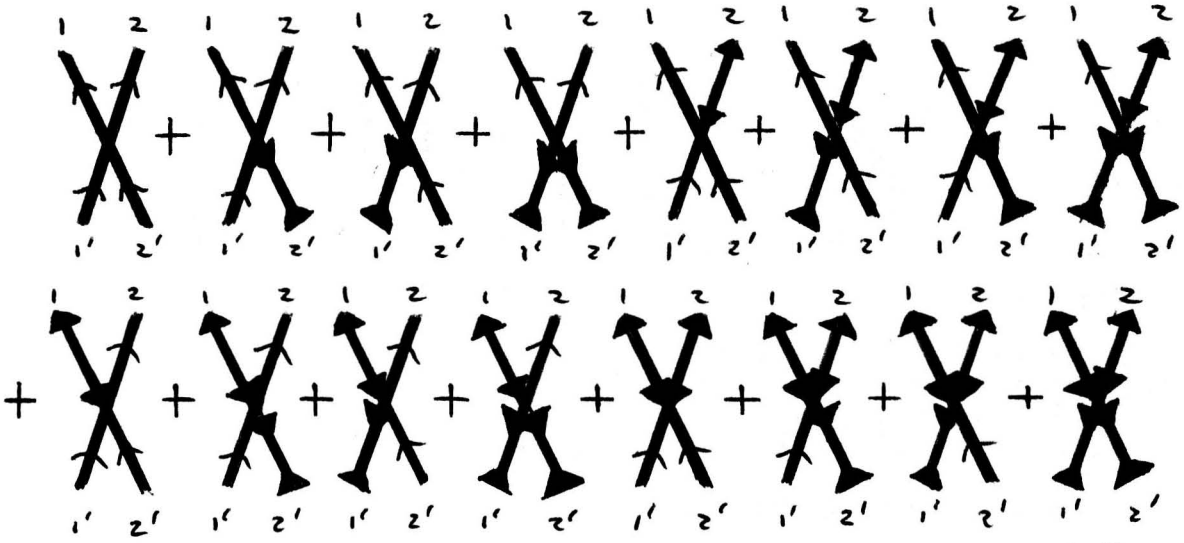
In the presence of anomalous propagators, an examination of the perturbation series [3,4,9] reveals that equation (1.32) must be generalized as follows:

i) In addition to the first two terms on the r.h.s. of equation (1.32), we must include the term



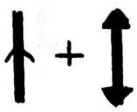
ii) The only restriction on the interference term (i.e. the last term on the r.h.s. of equation (1.32) is that lines must enter (leave) the points 1 and 2 ($1'$ and $2'$). Hence, propagators of the type

may be attached to the points 1 and 2 ($1'$ and $2'$) and the generalization of the last term on the r.h.s. of equation (1.32) is the following sum of sixteen diagrams.

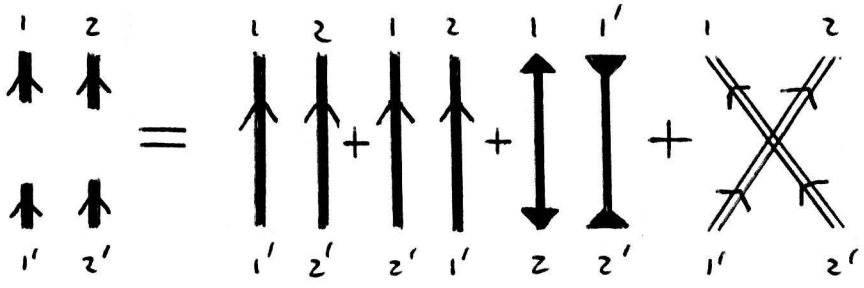


(A6)

The sum of terms (A6) may conveniently be represented by the single diagram below,

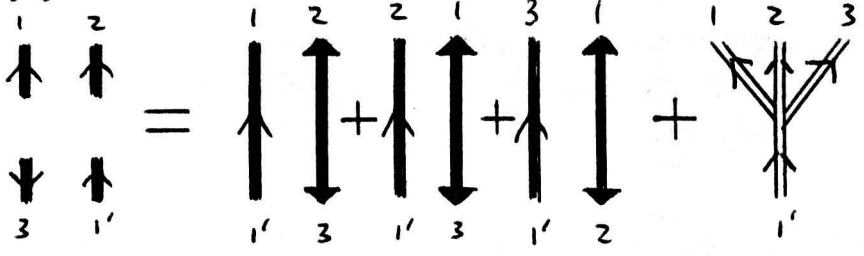
provided we always bear in mind that the generalized propagator $G \equiv$ represents either  or  . Fortunately this presents no ambiguity in the present work, because we shall find that at least one end of G is always attached to either an external point (e.g. $1, 2, 1', 2'$ of (A6)) or a bare interaction line (represented by ) and the following rule determines which of the above two possibilities are to be chosen. 'A line G entering (leaving) either an external point or a vertex of the bare interaction is to be interpreted as  .'

With this notation, the generalization of equation (1.32) takes the form



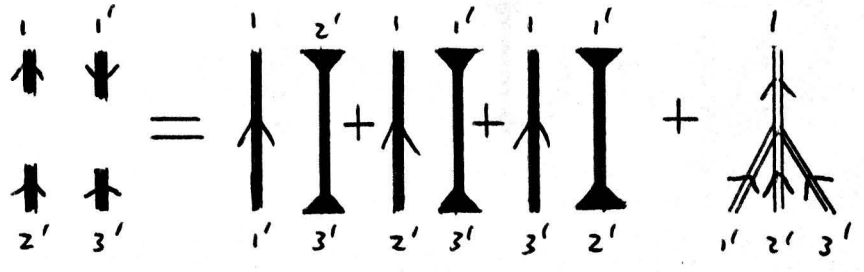
(A7)

In order to obtain Dyson's equation for $G^{(1)}$, we require an expression for the averages of four Heisenberg operators appearing on the r.h.s. of equations (A4) and (A5). The arguments which led to equation (A7) readily yield



(A8)

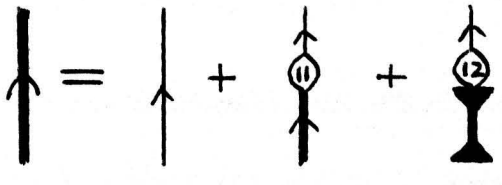
and



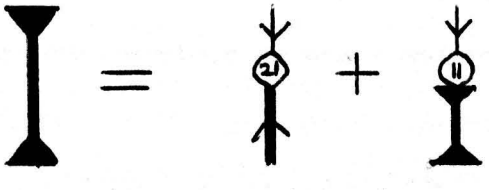
(A9)

A.3 Dyson's equation and the self-energies

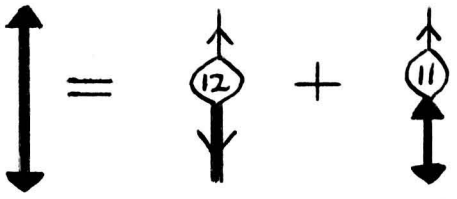
Before proceeding to introduce effective interactions into the theory, it is convenient at this point to obtain Dyson's equation for the single particle Green's function $G^{(1)}$. Substituting equations (A7), (A8) and (A9) into equations (1.29), (A4) and (A5) respectively, yields in the limit $\alpha \to 0$,



(A10)

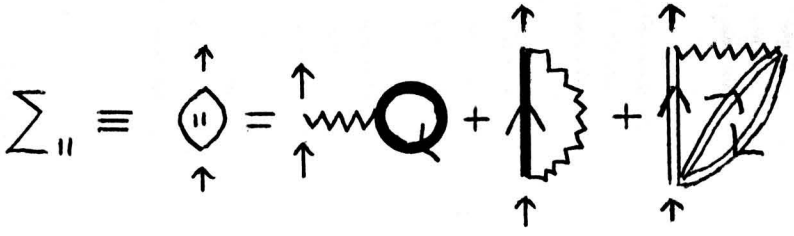


(A11)

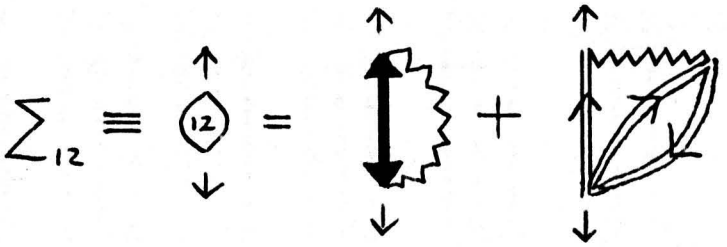


(A12)

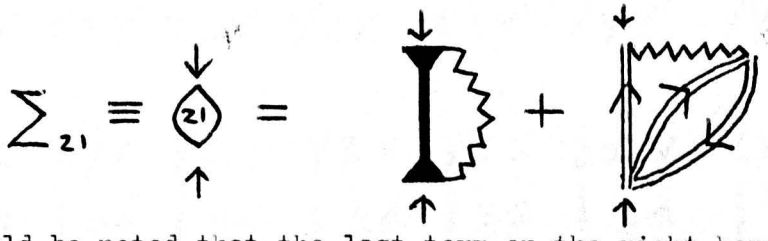
which agrees with Mattuck and Johansson, [9] . The self-energies are given by



(A13)



(A14)



(A15)

and it should be noted that the last term on the right hand sides of these equations are <u>not</u> identical, as will become clear later, when we introduce the effective interactions.

Equations (A10) to (A15) are exact. However, although equations (A10) to (A12) are to be found in the literature [3,4,9] , equations (A13) to (A15) from an entirely new and compact expression for the self-energies and we shall proceed by evaluating these terms in a self-consistent manner.

A.4 <u>A self-consistent expression for the interference term</u>

In order to reduce the number of diagrams, it is convenient to proceed as far as possible without introducing any effective interactions. In this section therefore we shall self-consistently obtain an expression

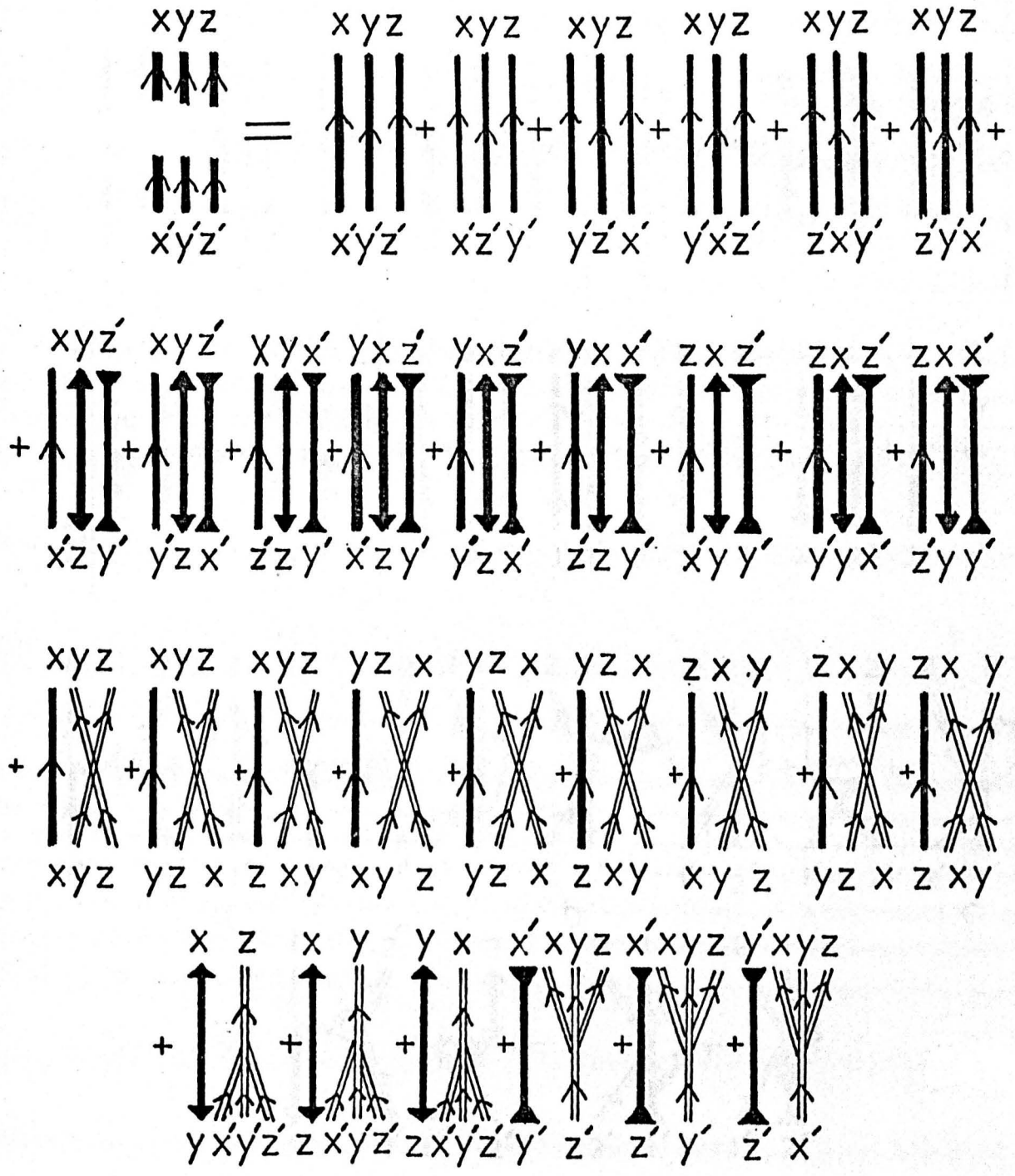

xyz
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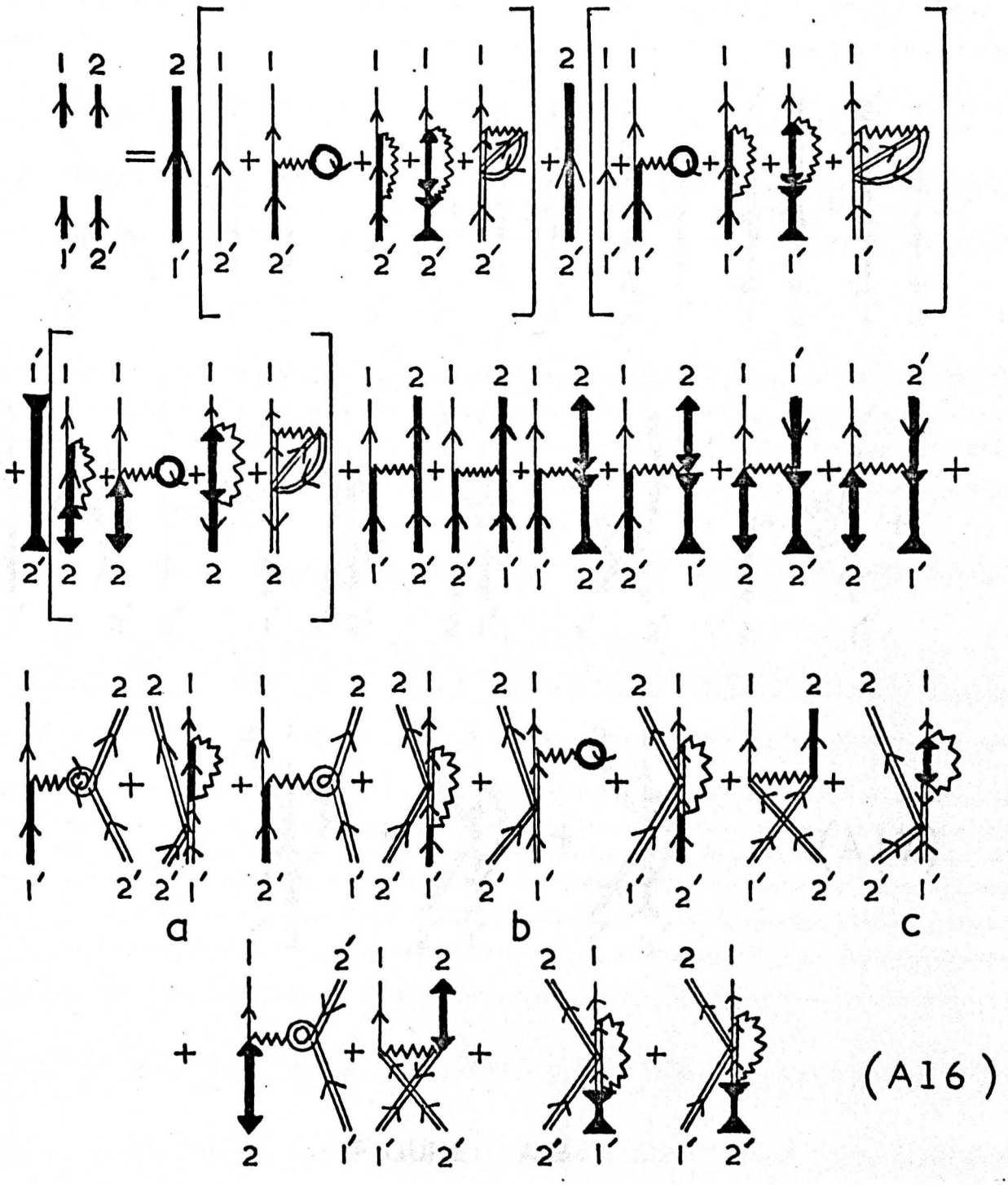
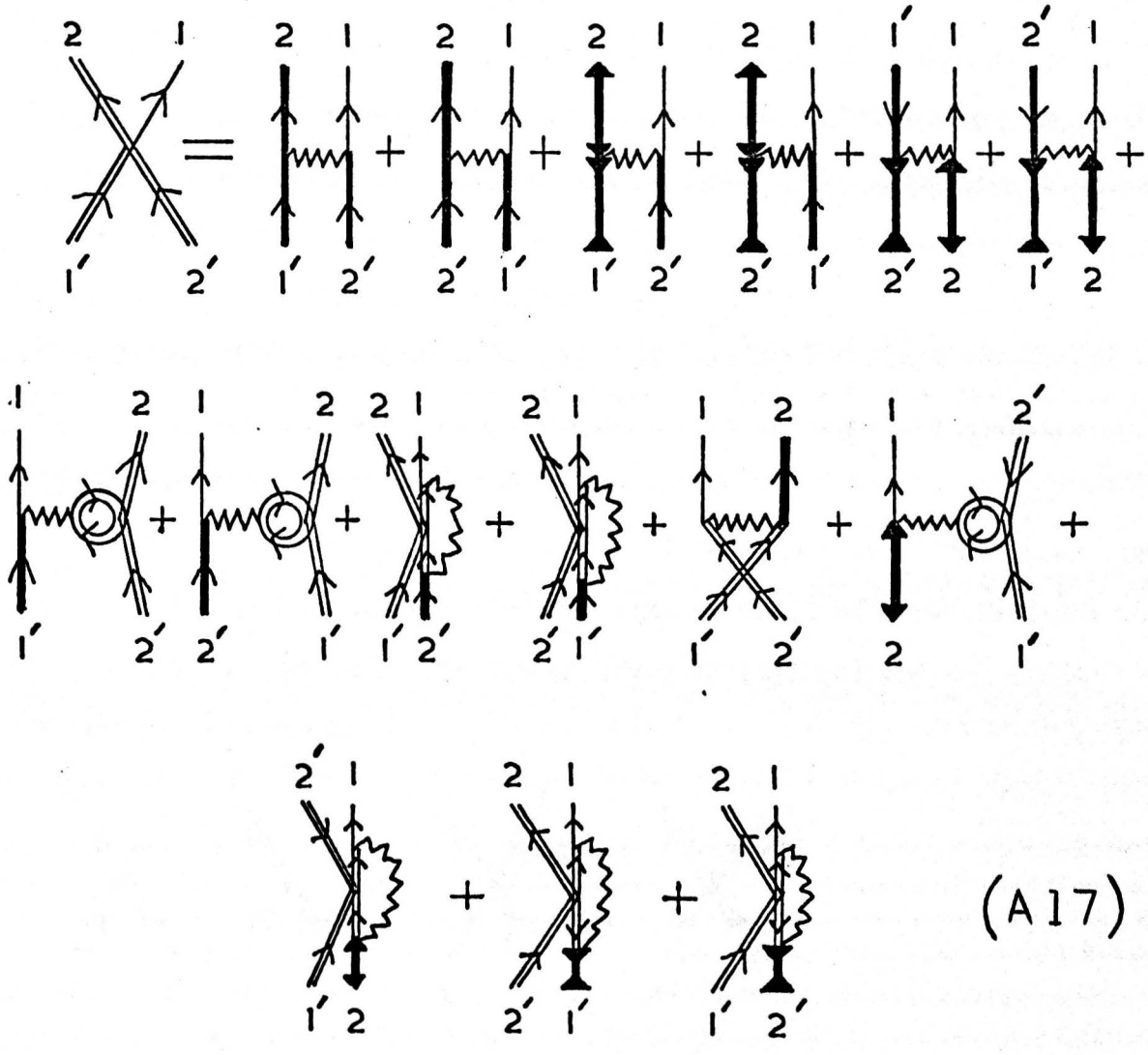
FIGURE A1



(A16)

FIGURE A 2



(A17)

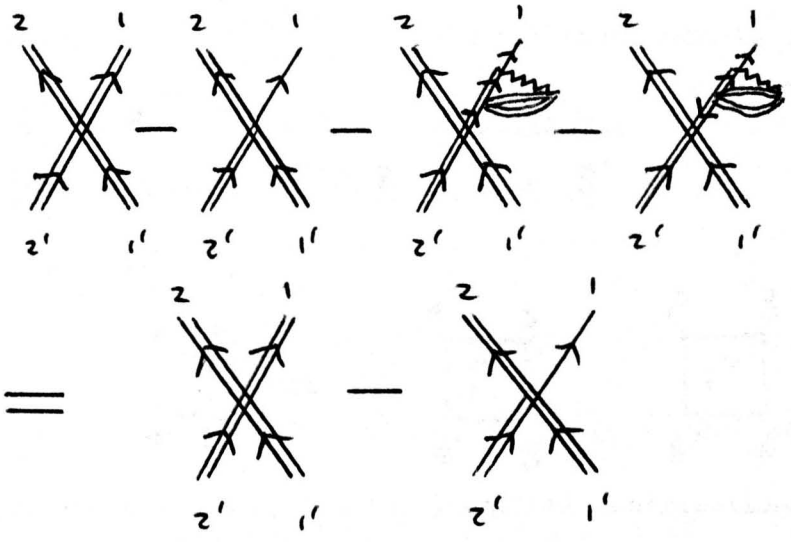
FIGURE A 3

for the interference term  and defer the introduction of the effective interaction to the next section.

Following the method detailed in chapter 1, we begin by writing down an expression for the three particle Green's function appearing in equation (1.30). In chapter 1 we found that terms first order in the effective interaction were sufficient to cover most situations of interest and as a result in this appendix, we shall restrict our evaluation of the self-energies to this approximation.

Within this approximation, the three particle Green's function takes the form of Figure (A1). Substitution of Figure (A1) into equation (1.30) yields equation (A16) of Figure (A2).

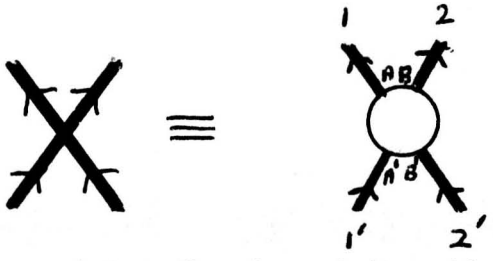
From equations (A10) to (A15), it follows that the first three terms on the r.h.s. of equations (A16) are simply the first three terms on the r.h.s. of (A7). Similarly, the sum of diagrams (a), (b) and (c) of Figure A2 is given by

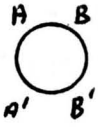


to first order in the effective interaction. Hence, a comparison of Figure A2 and equation (A7) yields equation (A17) of Figure (A3). We are now in a position to introduce the effective interaction, a task which is accomplished in the next section.

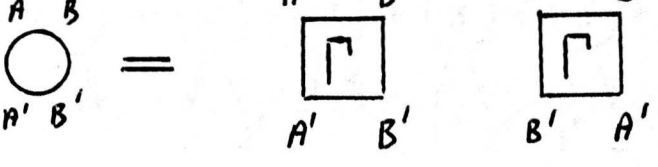
A.5 The effective interactions and their self-consistent evaluation

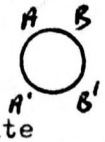
Before attempting to write  in terms of a two body effective interaction, it is useful to consider the reasons which enabled us to write  in terms of a single interaction Γ in the situation where anomalous propagators did not exist. This term may be written

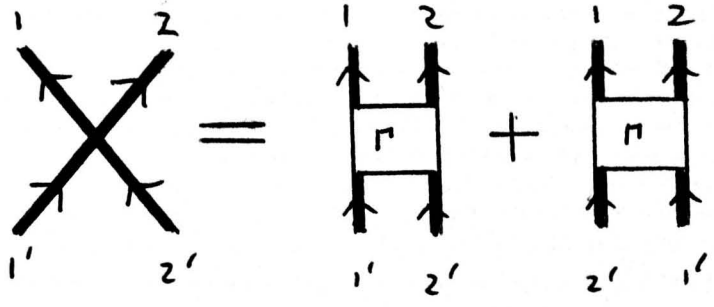


where  contains n bare interaction lines and $2(n-1)$ propagator lines  . We shall give the external points A and B (A' and B') the name exit (entrance) points.

A given contribution to  may be arranged to have two parallel solid lines  running vertically from one of the entrance points to one of the exit points. In general, to every contribution to  with solid lines running from A' to A and B' to B , there is an identical contribution with solids lines running from A' to B and B' to A . Hence  may be written



where only one of the two identical contributions to  now contributes to Γ . Thus, it is permissible to write



(A15)

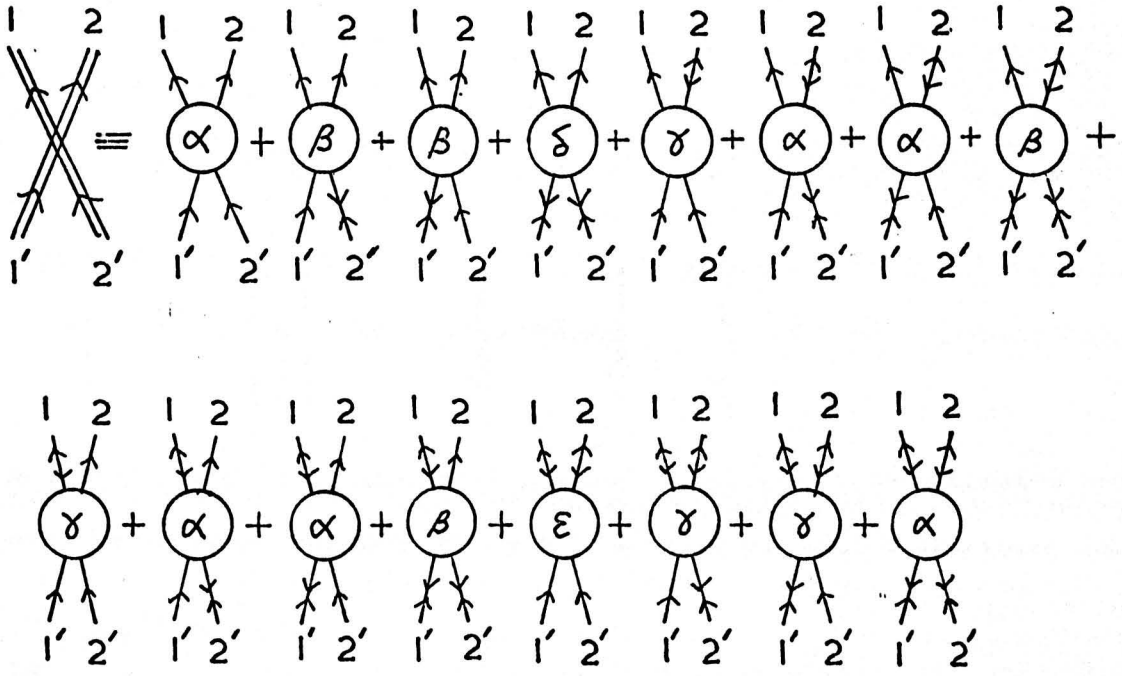


FIGURE A 4

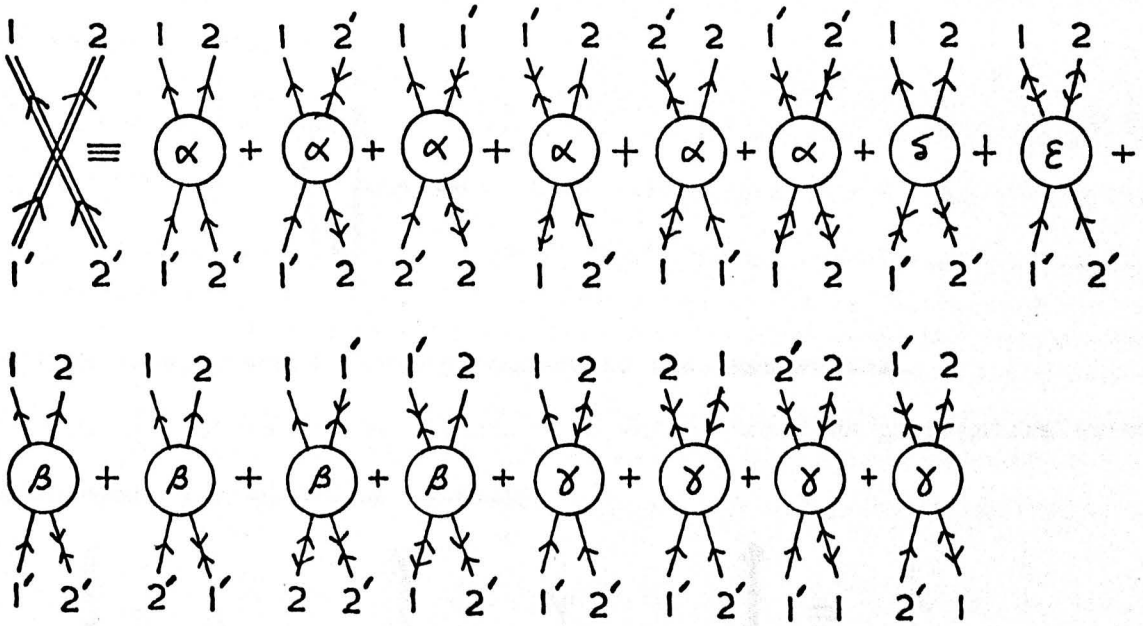
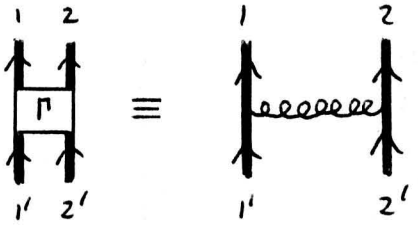
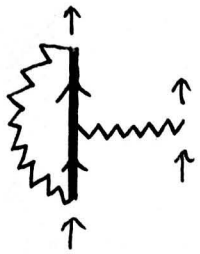


FIGURE A 5

where in  for example, there is a thick solid line running vertically from $2'$ to 2 and a corresponding line running from $1'$ to 1. Since the r.h.s. and l.h.s. of such a term contains one exit and one entrance point only, we may write

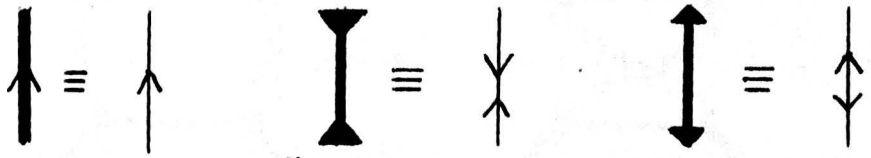


because the points at which the external lines are to be connected are uniquely given by the rule 'Connect  to the exit point which is connected by a vertical solid line in Γ to the entry point of the line  '. The only situation which this rule does not cover is that where an exit and entrance point occur at the same vertex, as in the following diagram.

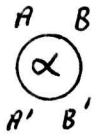


In this case, 2 and $2'$ simply connect to the same vertex.

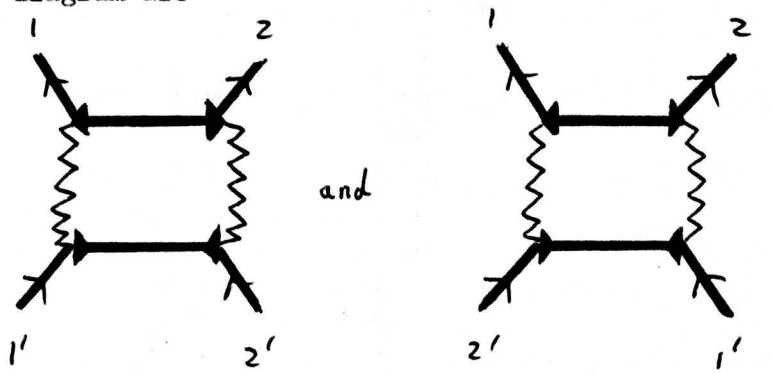
We now return to the situation in which anomalous propagators exist and temporarily adopt the notation



With this notation,  may be written in the form of Figure (A4), which defines the 'interactions' $\alpha, \beta, \gamma, \delta, \varepsilon$. This may be written in the more suggestive form of Figure (A5) where the possibility of vertical parallel lines connecting the external points is self evident.

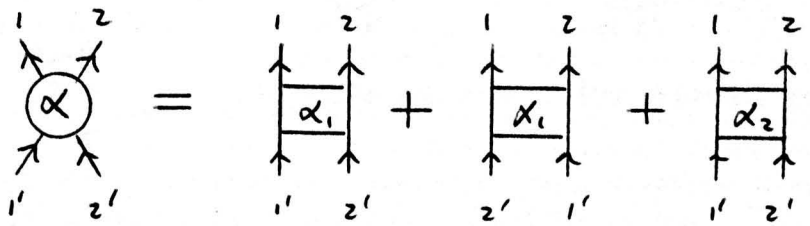
Consider the particular interaction  which has two entrance and two exit points. To every contribution in which a vertical string of solid lines connects A′ to A and another string connects B′ to B, there corresponds another diagram in which A′ is connected to B and B′ to A. For example, to diagram (a) below, there corresponds diagram (b).

In the present case however, unlike the situation where anomalous propagators are absent, this does not exhaust every type of diagram. All such remaining diagrams are identified by the fact that they contain a string of solid lines joining A to B and A′ to B′. Two examples of this type of diagram are



(A19)

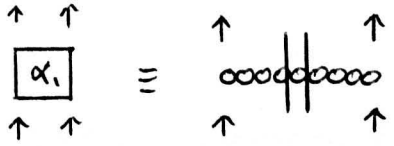
Hence  is given by



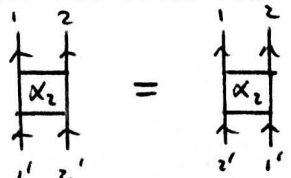
(A20)

Note that we have chosen not to permute the labels 1 and 2 in the last term on the r.h.s. of equation (A20), since as will become evident later, this allows the interactions to be written in a more compact form. One aspect of such a procedure is to give rise to a double counting of certain diagrams in $\boxed{\alpha_2}$. This turns out to be of no consequence however, since it is taken account of in a self-consistent manner in the course of the calculation.

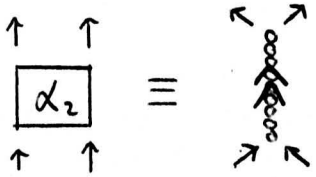
The entrance and exit points are uniquely defined in $\boxed{\alpha_1}$. As a result of this, just as in the case where anomalous propagators do not exist, we can write this term in the more 'compressed' notation.



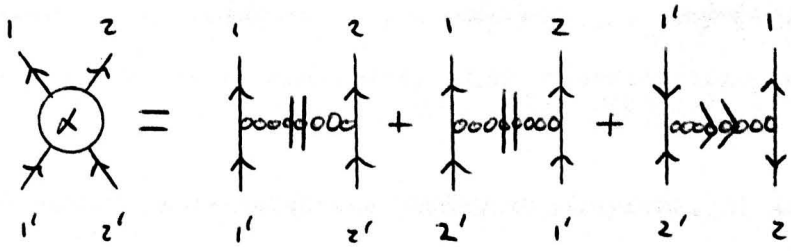
This situation does not occur for $\boxed{\alpha_2}$, but since



we may write α_2 in the form



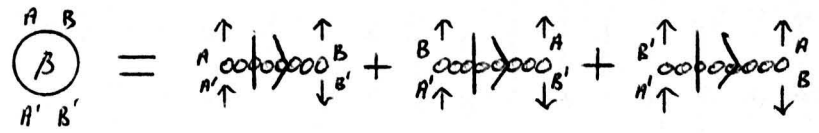
Hence, equation (A20) becomes



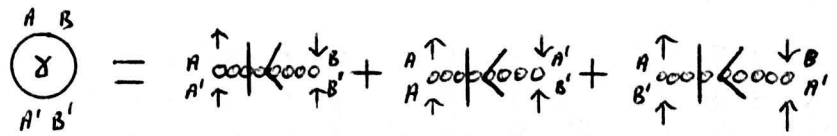
(A21)

The arguments which led to equation (A21) are readily applied to each of the terms  ,  ,  , and  and since any one of the external points (e.g. A) may

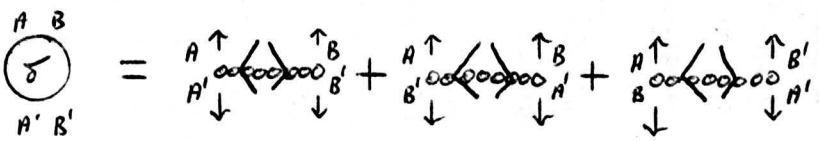
'connect' to any one of the other three external points (i.e. B A′ B′), we obtain three contributions to each of the above entities when they are written in terms of effective two body interactions. The final result is



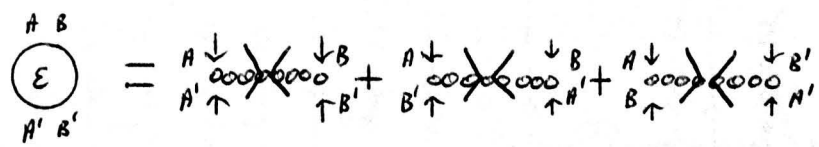
(A22)



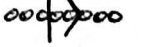
(A23)



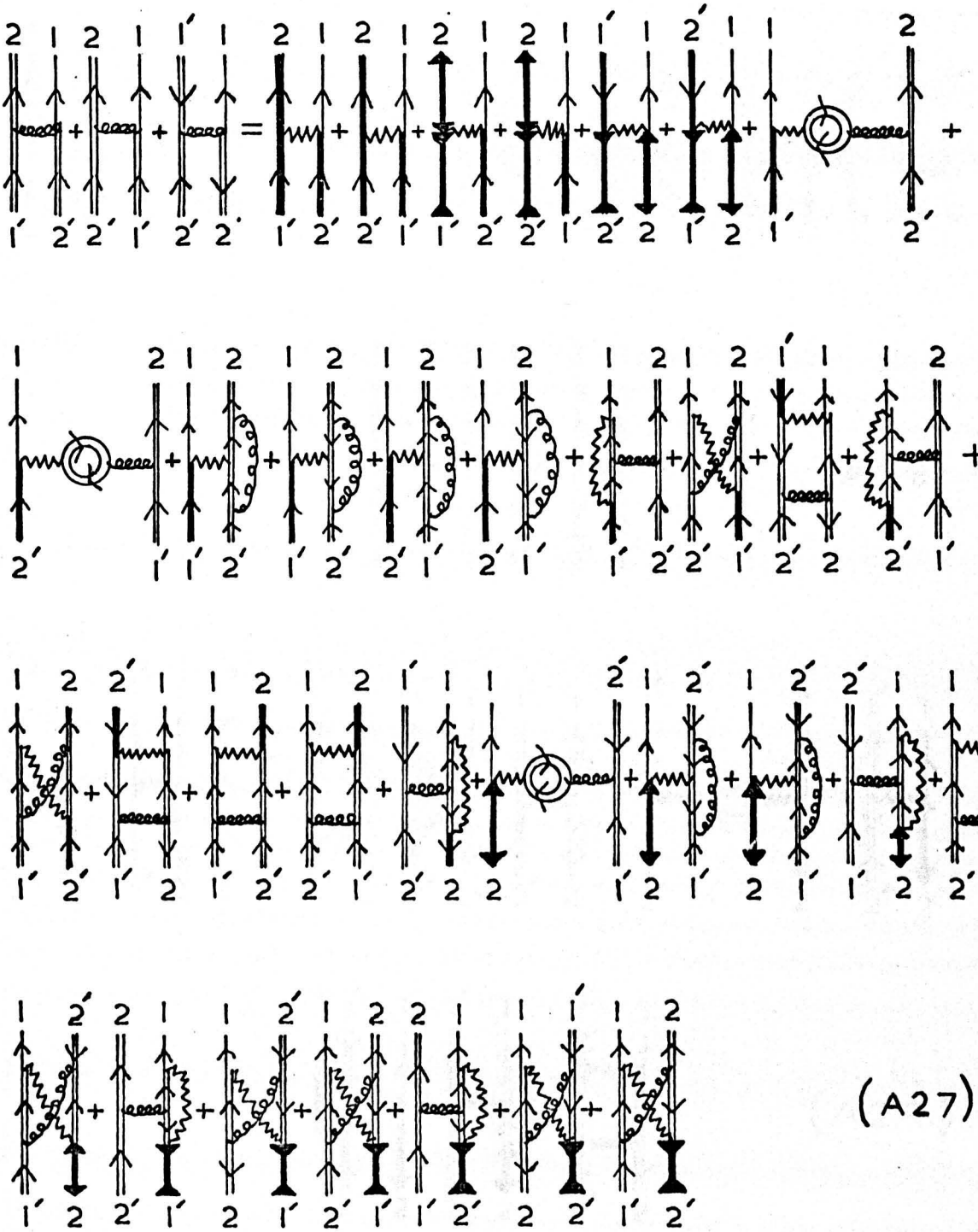
(A24)



(A25)

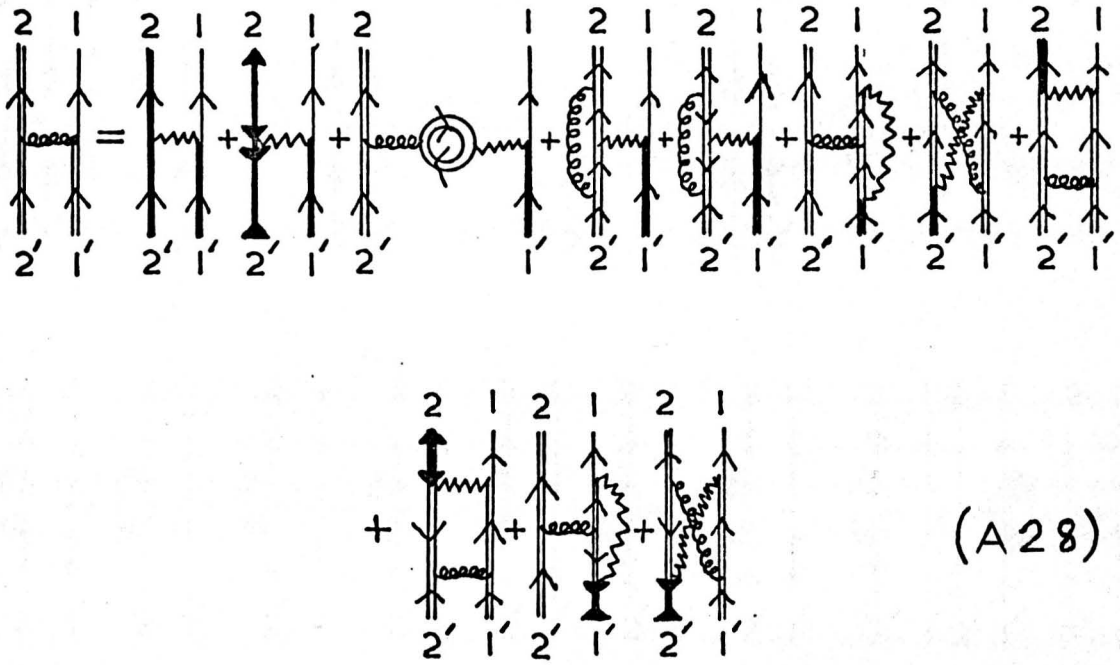
Equations (A21) to (A25) define six effective two body interactions $\alpha_1, \alpha_2, \beta, \gamma, \delta, \varepsilon$, denoted by  ,  ,  ,  ,  ,  , respectively and we see that there are fourty eight contributions to  from these interactions.

In order to decrease the number of diagrams, it is convenient to denote the set of interactions $\alpha_1, \alpha_2, \beta, \gamma, \delta, \varepsilon$ by the symbol $\Gamma \equiv$  . We may then write symbolically,

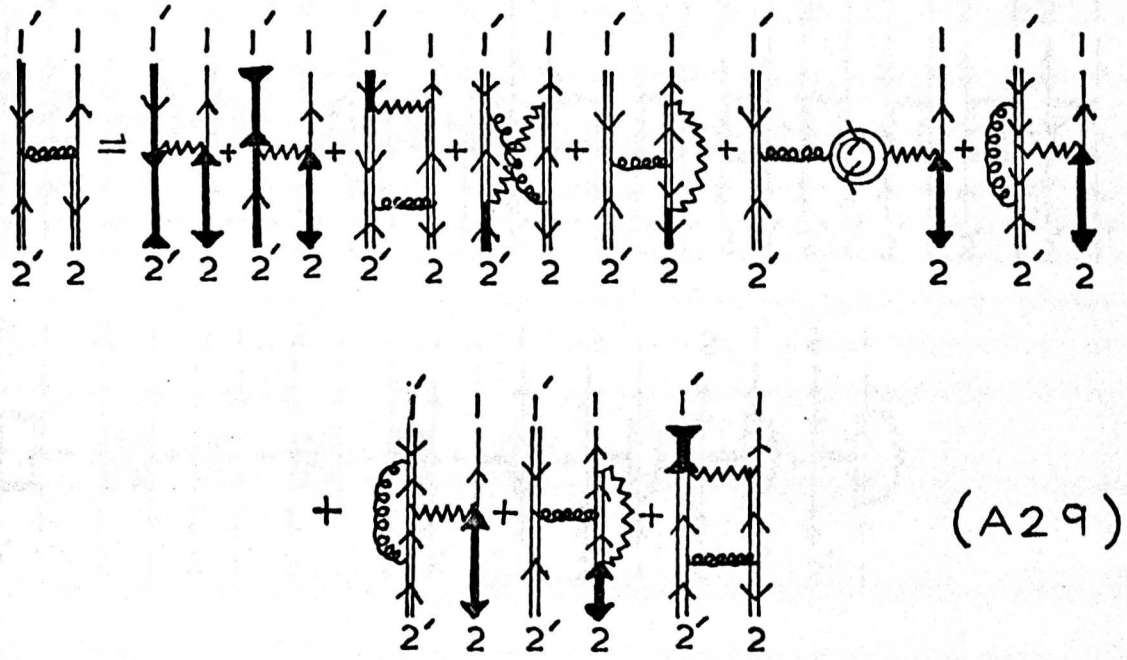


(A27)

FIGURE A6



(A28)



(A29)

FIGURE A7

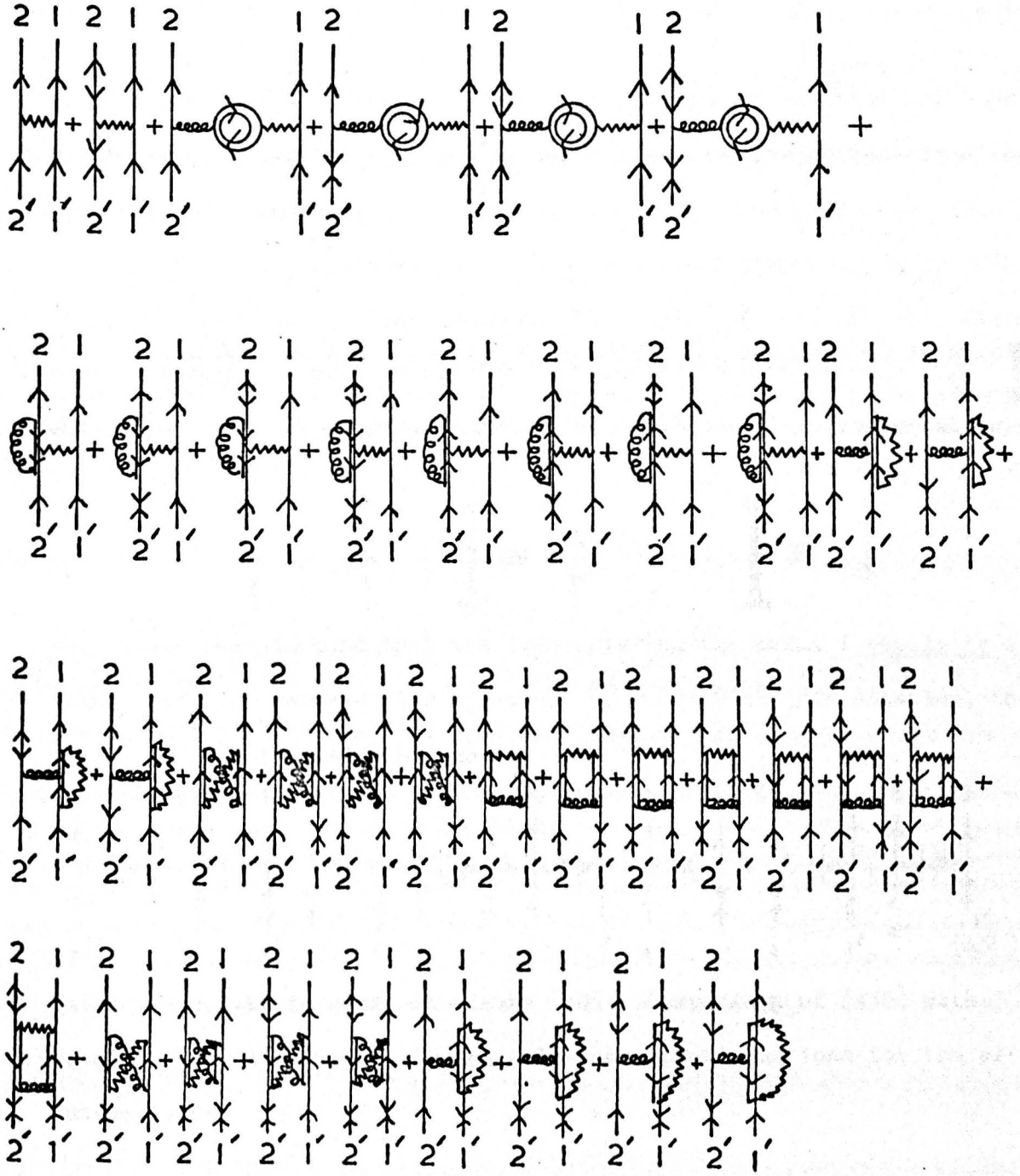
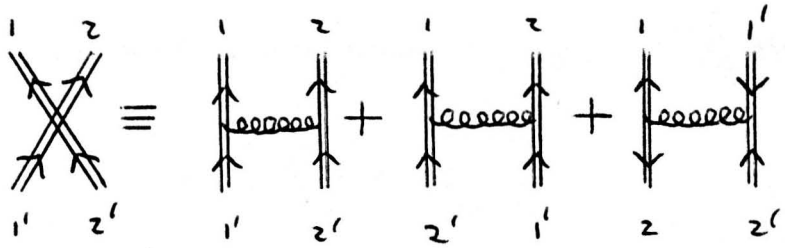


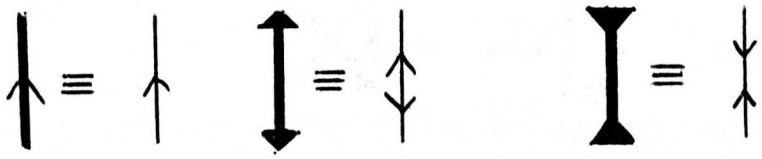
FIGURE A8



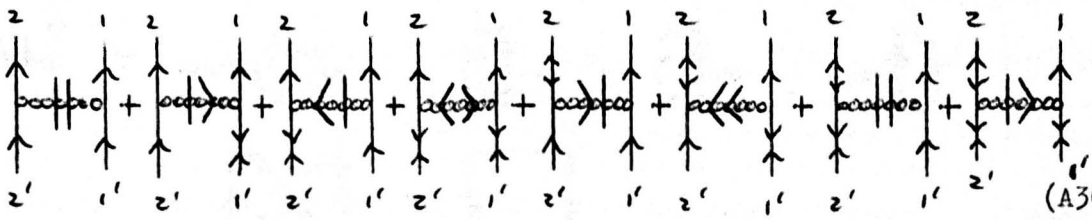
(A26)

With this notation, equation (A17) takes the form of equation (A27) of Figure (A6). Identifying terms on the r.h.s. then yields equations (A28) and (A29) of Figure (A7).

In order to proceed, the l.h.s. of equations (A28) and (A29) must be written in terms of the interactions α_1, α_2, β etc. and the external legs  must be written in terms of the propagators  and  . For convenience, we shall revert to the notation,

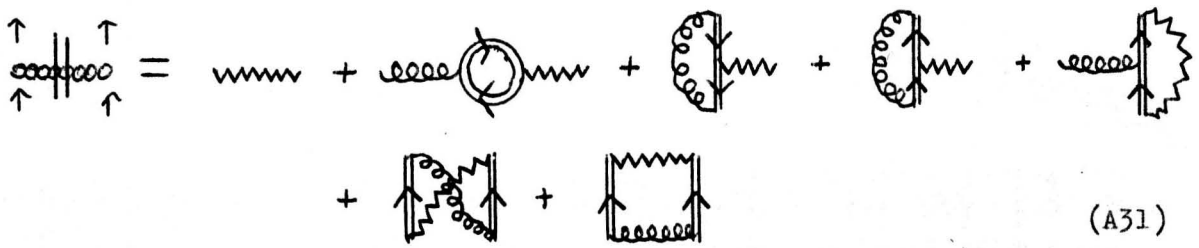


and simply bear in mind that the line entering the point 1 really is a thin solid line representing a factor $i\mathcal{G}$. With this notation, the l.h.s. of equation (A28) becomes

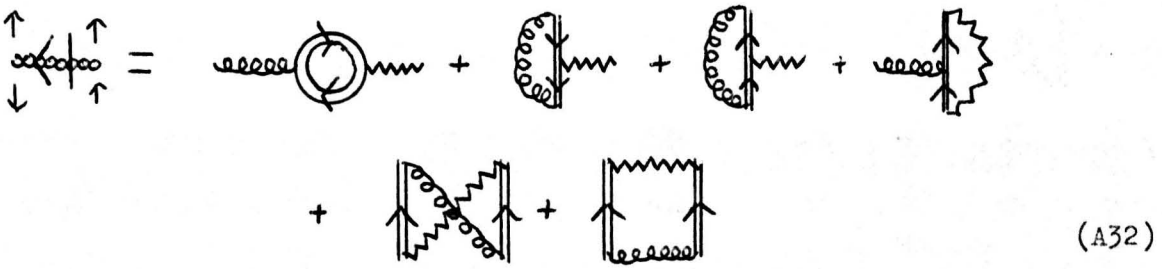


(A30)

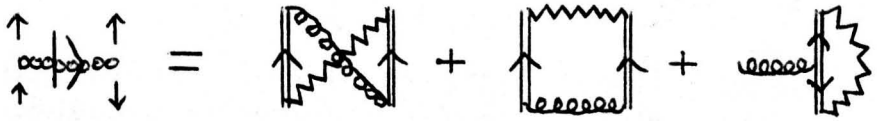
while the r.h.s. is given in Figure (A8). Comparison of (A30) with Figure (A8) yields the following self-consistent equations for the effective interactions.



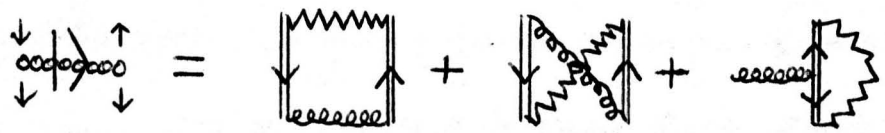
(A31)



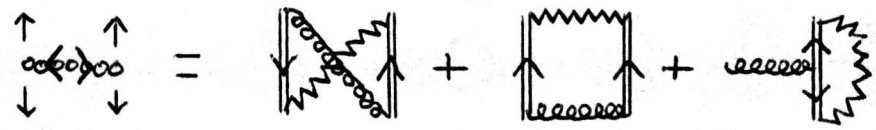
(A32)



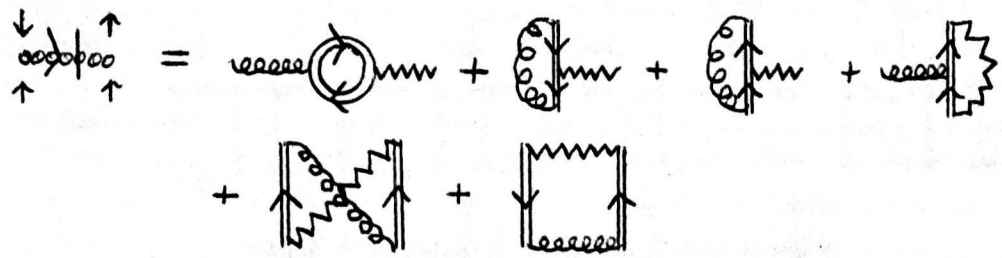
(A33)



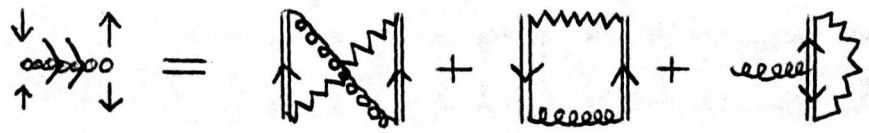
(A34)



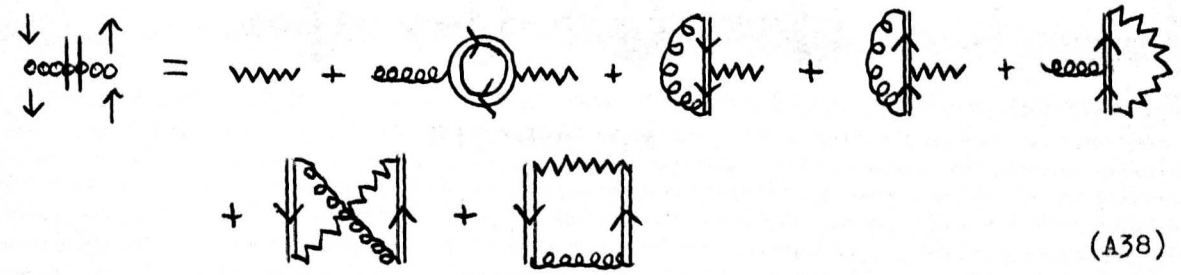
(A35)



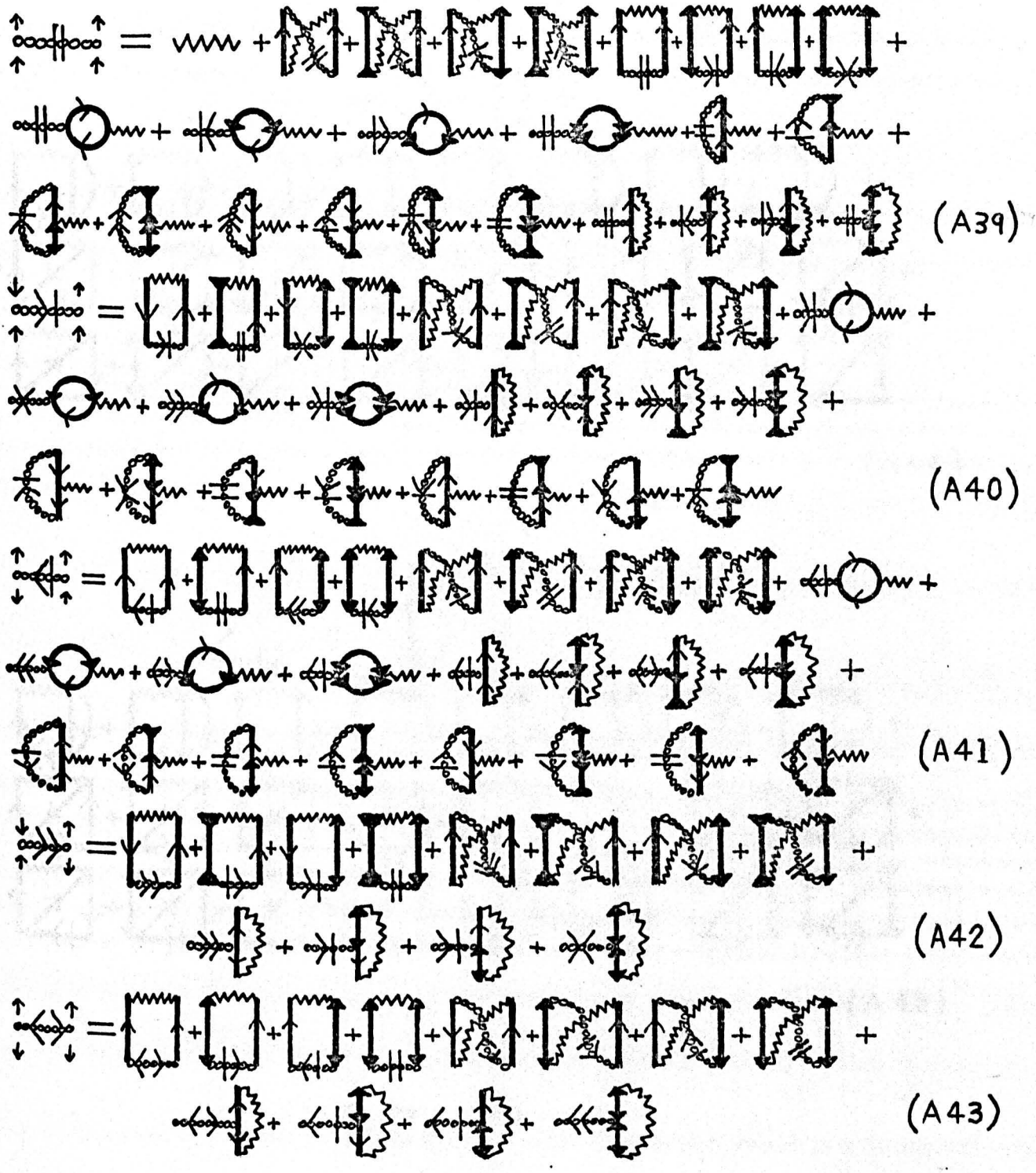
(A36)



(A37)



(A38)



(A39)

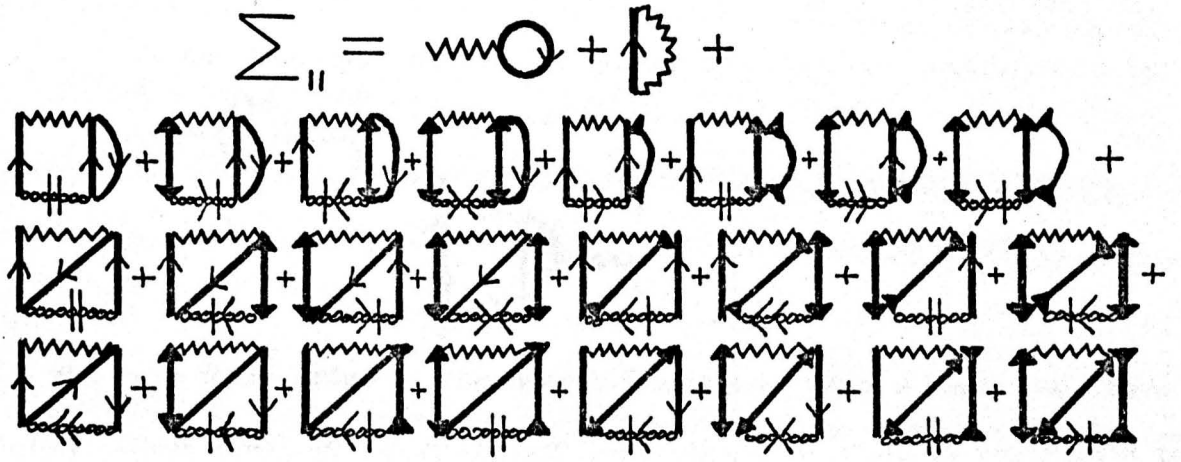
(A40)

(A41)

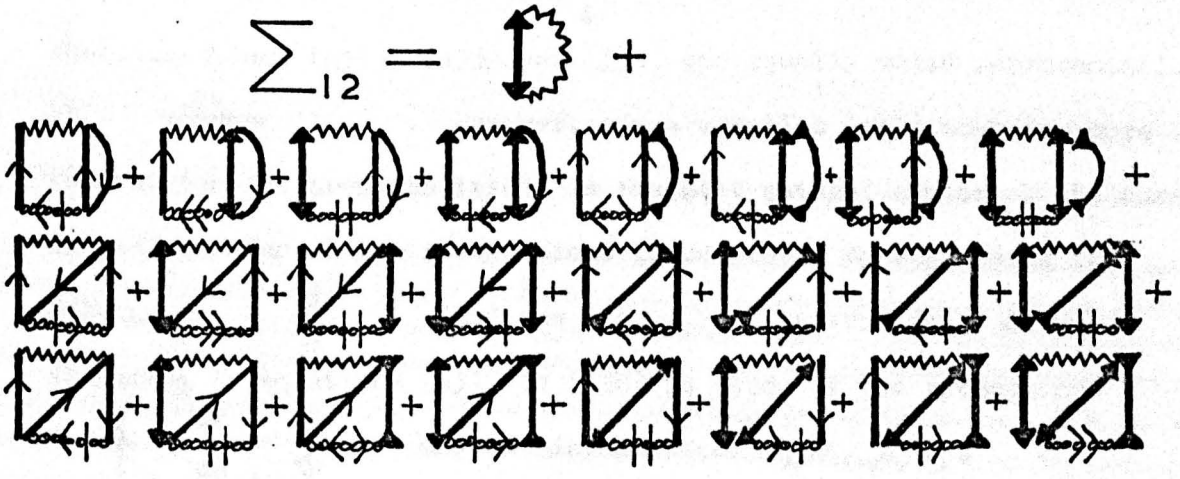
(A42)

(A43)

FIGURE A9



(A44)

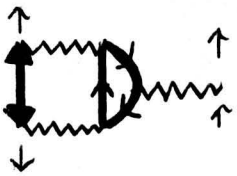


(A45)

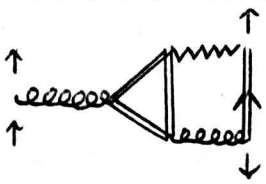
FIGURE A10

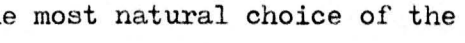
Notice for example the discrepancy between equation (A32) and equations (A33) and (A34). This discrepancy occurs because we are working to a finite (i.e. first) order in the effective interaction.

To see this more clearly, consider the simplest contribution to  , given by



which is third order in the bare interaction. Such a term would have to be second order in the effective interaction in order to contribute to equation (A33) and would have the form



Thus, equations (A32), (A33) and (A34) are equally valid representations of  . However, since equation (A32) contains more information, it presents itself as the most natural choice of the three. The calculation of the interactions is completed by expressing the propagator  and the generalized interaction  appearing in equations (A31) to (A38) in terms of the propagators  and the interactions $\alpha_1, \alpha_2, \beta, \gamma, \delta, \varepsilon$. The result is given in Figure A9 and we remark that the same result follows from equation (A29). Substitution of the equations in Figure A9 into equations (A13) and (A14) yields the equations in Figure (A10) for the self energies Σ_{11} and Σ_{12} . Obviously, Σ_{21} is obtained from Σ_{12} by reversing the arrows. This completes our self-consistent renormalization of the perturbation series in the presence of anomalous propagators.

A.6 Comments on the self-consistent renormalization procedure

The diagrammatic tour de force presented in this appendix demonstrates the enormous power of diagrammatic perturbation theory. Within this formalism, it has been possible to define certain classes of diagrams according to their topological structure and to identify these terms in any given equation. Such a procedure would be impossible without the use of diagrams.

The method described of self-consistently defining and then identifying the various terms appearing in the perturbation expansion has enabled the self energies to be summed to first order in the effective interactions; the latter themselves being determined to first order. A given practical calculation is thus reduced to that of self-consistently evaluating a subset of the diagrams contained in equations (A39) to (A45), selected on the basis of physical arguments.

APPENDIX B

In this appendix, we wish to briefly raise some points with regard to the question of the general validity of the arguments of Beliaev [3] based on the neglect of terms of relative order $1/V$. To this end we focus attention on the quantity $e^{V\sigma'}$, defined by Beliaev to be

$$e^{V\sigma'} = C\left[e^{V\sigma}\right] \tag{B1}$$

where $e^{V\sigma}$ and C are given by equations (3.8) and (3.9) respectively. As noted by Beliaev, σ' is independent of the volume, V and it is also known [30,31] that the phase of σ' diverges like $L\epsilon \underset{\epsilon\to 0}{}^{1/2}$ Introducing a quantity L, defined by

$$\hat{N}\left[e^{V\sigma'}\right] = e^{VL}$$

we see that L is independent of the volume, but has a divergent phase. The relevance of this is that in the work of Beliaev [3] we come across terms of the form

$$\sum_K b_K (\alpha^\dagger)^K e^{VL} (\alpha)^K \tag{B2}$$

in which we wish to transfer the operator $(\alpha)^K$ to the left of the operator e^{VL}. In order to carry out this procedure, Beliaev states that the following commutation relation holds for any function $y(\nu)$,

$$\alpha\, y(\nu) = \left(y + 1/V \frac{\partial y}{\partial \nu}\right)\alpha \tag{B3}$$

where $\nu = (\alpha^\dagger \alpha)$

The fact that this relation is only correct to order $1/V$ is not evident in the work of Beliaev and we shall now demonstrate that the use of this relation is not justified in his proof.

Starting from the relation

$$\alpha(\nu - 1/V) = \nu\alpha$$

it readily follows that

$$(\alpha)^m \, y(\nu - m/V) = y(\nu)(\alpha)^m$$

Since L is a function of ν, we have

$$(\alpha)^K \, e^{VL(\nu - K/V)} = e^{VL(\nu)} (\alpha)^K \qquad \text{(B4)}$$

A Taylor expansion of the term $L(\nu - \frac{K}{V})$ yields

$$L(\nu - K/V) = L(\nu) - \frac{K}{V}\frac{\partial L}{\partial \nu} + L' \quad \text{say}$$

This serves to define L' which is seen to consist of an infinite number of terms of progressively increasing powers of $1/V$, the leading term going as $(1/V)^2$. Hence we can re-write equation (B4) as

$$(\alpha)^K e^{VL'} e^{-K\frac{\partial L}{\partial \nu}} e^{VL} = e^{VL}(\alpha)^K \qquad \text{(B5)}$$

Employing equation (B3) leads Beliaev to write

$$(\alpha)^K e^{-K\frac{\partial L}{\partial \nu}} e^{VL} = e^{VL}(\alpha)^K \qquad \text{(B6)}$$

The arguments of Beliaev rest on the validity of this equation. However, the correct form of equation (B6) is equation (B5) which may be written

$$(\alpha)^K \left[1 + VL' + \frac{(VL')^2}{2!} + \cdots\right] e^{-K\frac{\partial L}{\partial \nu}} e^{VL} = e^{VL}(\alpha)^K \qquad \text{(B7)}$$

It is tempting to replace the term in the parentheses on the l.h.s. of equation (B7) by unity, because the largest contribution to is of order $1/V$. However, as pointed out by Hugenholtz and Pines [4]

and indeed, by Beliaev himself, it is not permissible to neglect a string of terms which are each of order $1/V$ when their sum is multiplied by a term which is exponentially large in powers of V. Thus, equation (B6) cannot be rigorously justified. Yet another point which should be noted is that L' itself diverges like $\lim_{z\to 0} 1/z$ and this casts further doubt on the general validity of equation (B6).

One final point regarding the validity of ignoring terms of relative order $1/V$, concerns the fact that in the first step of Beliaev's proof of a 'disentangling theorem' he, in effect, ignores a series of terms corresponding to

$$\lim_{\kappa\to 0} G'(\underline{\kappa}, \mu/\hbar)$$

where $G'(\kappa, \omega)$ is given in equation (3.6) of the text. The self-consistency condition expressed in equation (3.22) when coupled with the result of equation (3.24) shows that $\lim_{\kappa\to 0} G'(\underline{\kappa}, \mu/\hbar)$ diverges like $\lim_{\kappa\to 0} 1/\kappa$. Hence, although the individual terms in this series are all of order $1/V$ their sum cannot be ignored.

It is because of reasons of this nature that we do not regard the disentangling theorem of Beliaev [3] to be a mathematically exact result correct to order $1/V$. The importance of the theorem lies in the fact that it is used to prove that

$$i G_0^{(1)}(\epsilon-\epsilon') = F(\epsilon) F^*(\epsilon')$$

This result, coupled with the fact that we know on general grounds that $i G_0^{(1)}(\epsilon-\epsilon')$ is a function of $(\epsilon-\epsilon')$, leads to the conclusion that

$$i G_0(\epsilon-\epsilon') = n_0 e^{-i\mu/\hbar(\epsilon-\epsilon')}$$

We emphasise that we do not regard this as a mathematically exact equation, correct to order $1/v$, however, as described in the body of the text, it is a self-consistent solution to the problem.

APPENDIX C

The equation [11]

$$\frac{|\Psi_N\rangle}{\langle\Phi_N|\Psi_N\rangle} = \frac{U_\varepsilon(0,\pm\infty)\,|\Phi_N\rangle}{\langle\Phi_N|U_\varepsilon(0,\pm\infty)|\Phi_N\rangle} \tag{C1}$$

where

$$U_\varepsilon(t,t_0) = T\left[\exp\left\{\frac{-i}{\hbar}\int_{t_0}^{t} dt_1\, e^{-\varepsilon|t_1|} H_1(t_1)\right\}\right]$$

allows us to write

$$\frac{\langle\Psi_N|\alpha_H^+(t')|\Psi_{N-1}\rangle}{\langle\Psi_N|\Phi_N\rangle\langle\Phi_{N-1}|\Psi_{N-1}\rangle} = \frac{\langle\Phi_N|T[\alpha^+(t')\,\hat{S}]|\Phi_{N-1}\rangle}{\langle\Psi_N|U(\infty,0)|\Phi_N\rangle\langle\Phi_{N-1}|U(0,-\infty)|\Psi_{N-1}\rangle} \tag{C2}$$

and

$$\frac{\langle\Psi_N|\Psi_N\rangle}{\langle\Psi_N|\Phi_N\rangle\langle\Phi_N|\Psi_N\rangle} = \frac{S_N}{\langle\Phi_N|U(\infty,0)|\Psi_N\rangle\langle\Psi_N|U(0,-\infty)|\Phi_N\rangle} \tag{C3}$$

It is important to remember that the numerators and denominators on the l.h.s. of equation (C2) and (C3) separately exist, while those on the r.h.s. do not. Similarly as defined in equation (C1) the entity $|\Psi_N\rangle$ need not be normalized. From equations (C2) and (C3) we readily obtain

$$\frac{\langle\Phi_N|T[\alpha_{(t')}^+\hat{S}]|\Phi_{N-1}\rangle}{S_N} = A\,\langle\widetilde{\Psi_N}|\alpha_H^+(t')|\widetilde{\Psi_{N-1}}\rangle \tag{C4}$$

where

$$|\tilde{\Psi}_{N-1}\rangle = \frac{|\Psi_{N-1}\rangle}{(\langle\Psi_{N-1}|\Psi_{N-1}\rangle)^{1/2}}$$

$$\langle\tilde{\Psi}_N| = \frac{\langle\Psi_N|}{(\langle\Psi_N|\Psi_N\rangle)^{1/2}}$$

and

$$A = \frac{\langle\Phi_N|\Psi_N\rangle}{\langle\Phi_{N-1}|\Psi_{N-1}\rangle} \cdot \left(\frac{\langle\Psi_{N-1}|\Psi_{N-1}\rangle}{\langle\Psi_N|\Psi_N\rangle}\right)^{1/2} \frac{\langle\Phi_{N-1}|U(0,-\infty)|\Phi_{N-1}\rangle}{\langle\Phi_N|U(0,-\infty)|\Phi_N\rangle} \quad (C5)$$

$$= B \times C \times D \quad \text{say}$$

However, it follows from equation (C3) and the related equation for that

$$\frac{S_{N-1}}{S_N} = B B^* C^2 D D^*$$

where, e.g. $B^* = \left(\text{complex conjugate of } B\right) = \dfrac{\langle\Psi_N|\Phi_N\rangle}{\langle\Psi_{N-1}|\Phi_{N-1}\rangle}$

Writing

$$B = |B| e^{i\theta}$$

$$D = |D| e^{i\alpha}$$

then

$$B^* = |B| e^{-i\theta} = B e^{-i2\theta}$$

$$D^* = D e^{-2i\alpha}$$

Hence

$$\frac{S_{N-1}}{S_N} e^{2i(\theta+\alpha)} = B^2 C^2 D^2$$

i.e.

$$A = BCD = \pm \left(\frac{S_{N-1}}{S_N}\right)^{1/2} e^{i(\theta+\alpha)}$$

This means that we can rewrite equation (C4) in the form

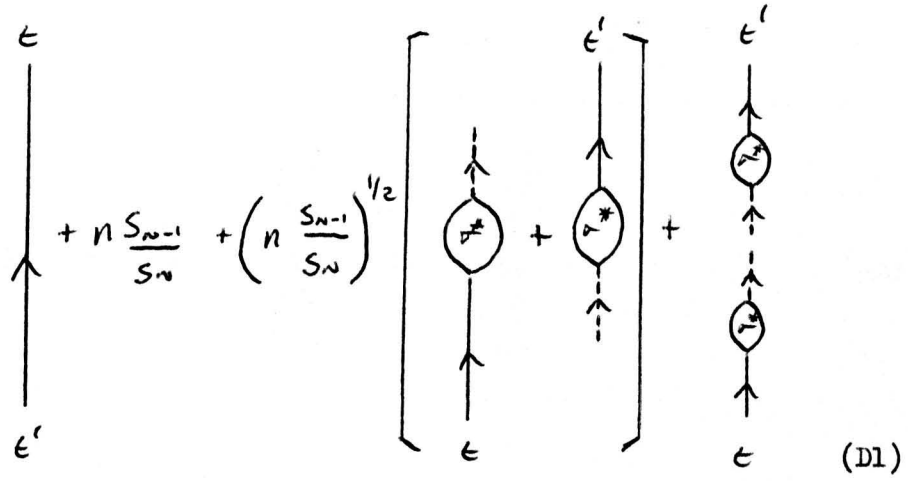
$$\frac{\langle \mathcal{F}_N | T[\alpha^+_{(\epsilon')} \hat{S}] | \mathcal{F}_{N-1} \rangle}{S_N} = \pm \left(\frac{S_{N-1}}{S_N} \right)^{1/2} e^{i(\theta+\alpha)} \langle \mathcal{F}_N | \alpha^+_{H(\epsilon')} | \mathcal{F}_{N-1} \rangle \qquad (C6)$$

In this equation we have replaced the expression $|\tilde{\mathcal{F}}_{N-1}\rangle$, for example, by $|\mathcal{F}_{N-1}\rangle$ since in the body of the text it is implicitly assumed that $|\mathcal{F}_{N-1}\rangle$ is normalized. The expression in equation (C6) is employed in the body of the script apart from the phase factor which does not influence the arguments.

APPENDIX D

In order to demonstrate the possible lack of convergence of the perturbation series appearing in chapter 2, we focus attention on Figure 8 and equation (3.11). An examination of the possible contractions involving $\alpha^{+}(t')$ readily shows that Figure 8 may be written

$$iG_0^{(1)}(t-t') =$$

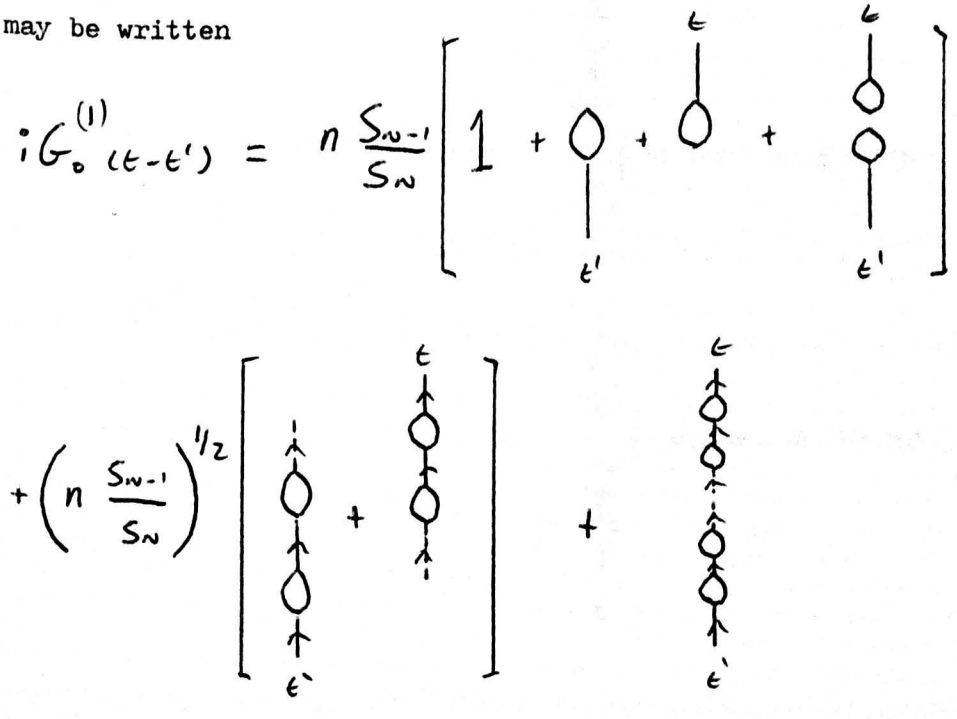


(D1)

Note that m pairs of dotted lines represents a factor $(i)^m G_0^{(m)}$ in equation (D1) and to order $1/V$, we may omit the first term on the r.h.s. of this equation.

Focussing attention on the possible contractions involving the dotted lines shown explicitly in equation (D1) readily shows that this equation may be written

$$iG_0^{(1)}(t-t') = n\frac{S_{N-1}}{S_N}$$



Repeating this iterative procedure to infinity yields

$$iG_0^{(1)}(t-t') = n\frac{S_{n-1}}{S_n}\Big[\;|\; + \;\overset{y}{\bigcirc}\; + \;\bigcirc\; + \;\bigcirc\;\Big] + X(t) + X^*(t') + Z(t,t')$$

(with t at the top and t' at the bottom of each bracketed diagram)

where

$$y(k) \equiv \bigcirc = \bigcirc + \bigcirc\!\!\overset{k}{\cdot} + \bigcirc\!\!\overset{k}{\cdot}\!\!\overset{k}{\cdot} + \cdots \qquad \text{(D3)}$$

$$X(t) = \left(n\frac{S_{n-1}}{S_n}\right)^{1/2}$$ (vertical chain, t at top) ∞ no. of terms (D4)

and

$Z(t,t') =$ (D5)

(vertical chain, t at top) ∞ no. of terms

(vertical chain, t' at bottom) ∞ no. of terms.

The assumed convergence in chapter 2 led us to neglect the terms x and z . However, we shall now demonstrate that this step may not be justified. Consider the second term on the r.h.s. of equation (D3). This term contains two self energies(σ) , but these self energies are not mathematical entities in their own right, because we have not yet employed a suitable decoupling approximation. For convenience, we shall assume the validity of equation (3.5) and apply it to equations (D2) to (D5).

Since the self energies are now decoupled, we may sum the r.h.s. of equation (D3) to yield

$$g_{(k)} = \frac{\sigma^*_{(k)}}{1 - \sigma^*_{(k)} g_{(k)}}$$

Hence, the term in the square parentheses of equation (D2) takes the form

$$n \frac{S_{N-1}}{S_N} \left[1 + \frac{g_{(0)} \sigma^*_{(0)}}{1 - \sigma^*_{(0)} g_{(0)}} \right]^2$$

which vanishes. This is the result obtained in chapter 2. Note that the term

which appears in equation (D2) is, outside of any decoupling approximation, a function of $\epsilon - \epsilon'$, while the assumed validity of equation (3.5) renders this term independent of time. Turning our attention to the terms x and z and bearing in mind the self consistency

condition equation (3.22) reveals (for $\mu \neq 0$),

$$\chi(\epsilon) = \left(n n_0 \frac{S_{N-1}}{S_N} \right)^{1/2} e^{-i\mu/2\epsilon}$$

and

$$\zeta(\epsilon - \epsilon') = n_0 e^{-i\mu/2(\epsilon - \epsilon')}$$

which demonstrates a lack of convergence within the approximation of equation (3.5)

We have not demonstrated a general lack of convergence in the above perturbation series since the above result relies upon the use of equation (3.5), the validity of which is suspect. However, this appendix together with chapter 2 provide a note of caution which must be borne in mind when dealing with 'bare' perturbation series of the Brandow type [1].

APPENDIX E

E.1 The Hugenholtz and Pines relation

We shall now demonstrate that the relation (3.24) follows straight forwardly from the manner in which σ^*, Σ_{11} and Σ_{12} are constructed.

Within the approximation of equation (3.5), we may write

$$S_N = \langle \Phi_N | T[e^{V\sigma}] | \Phi_N \rangle$$

where σ is the sum of all connected vacuum loops and is a functional of α and α^+. Let $\bar{\sigma}$ have the same topology as σ, the difference being m pairs of dotted lines in $\bar{\sigma}$ represent a factor $(i)^m G_0^{(m)}$ (given by equation (3.5)). Let $\bar{\sigma}_m$ be the sum of all contribution to $\bar{\sigma}$ containing m pairs of dotted lines. Then

$$\bar{\sigma} = \sum_m \bar{\sigma}_m$$

The self energy $\sigma^*(0,\mu/t)$ is obtained by first removing a single arbitrarily chosen ingoing dotted line from $\bar{\sigma}$ and then removing in all possible ways an outgoing dotted line. Hence,

$$\sigma^*(0,\mu/t) = \sum_m \frac{m}{n_0} \bar{\sigma}_m \qquad \text{(E1)}$$

$\Sigma_{11}(0,\mu/t)$ is obtained by removing an ingoing and outgoing dotted line in all possible ways, so that

$$\Sigma_{11}(0,\mu/t) = \sum_m \frac{m^2}{n_0} \bar{\sigma}_m \qquad \text{(E2)}$$

Finally, $\Sigma_{12}(0,\mu/t)$ is obtained by removing a pair of ingoing dotted lines in all possible ways, to yield

$$\Sigma_{12}(0,\mu/t) = \sum_m \frac{m(m-1)}{n_0} \quad \overline{\sigma} \tag{E3}$$

Equations (E1), (E2) and (E3) combine to yield equation (3.24) of the text and hence the Hugenholtz and Pines relation

$$\Sigma_{11}(0,\mu/t) - \Sigma_{12}(0,\mu/t) = \mu/t \tag{E4}$$

Note that this derivation has been performed entirely within the formalism of Beliaev, and is therefore unique

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