THE UNIVERSITY OF HULL

THE INVERSE SCATTERING PROBLEM AT FIXED ENERGY AND OFF-SHELL SCATTERING AMPLITUDES

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CONTENTS

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CHAPTER 1

INTRODUCTION

In the study of three-particle interactions at low energies by means of the Faddeev equations (Faddeev (1961)) it is necessary to know the two-body scattering amplitudes both on and off the energy shell. The on-shell amplitudes correspond to physical states and can be measured experimentally, whereas the off-shell amplitudes are generally unknown. Since the off-shell amplitudes are not uniquely specified by those on-shell, it is necessary to make some assumptions regarding the nature of the two-body interaction.

The simplest assumption which can be made is that the interaction is given by an energy-independent separable potential (Lovelace (1964)). A more physically realistic assumption is that the interaction is given by a local, energy-independent potential and this is the assumption we shall make, together with conditions on the short-range nature of the potential.

The off-shell amplitude is determined by the Lippman-Schwinger equation

$$T = V + VG_o T = V + TG_o V \tag{1.1}$$

where

$$G_o = \left(k^2 - H_o + i\varepsilon\right)^{-1}$$

is the free particle Green's function, H_o is the free particle Hamiltonian, V is the potential and k^2 is the energy. Using the partial-wave expansion

$$\langle p_1|T|p_2\rangle = \left(4\pi p_1 p_2\right)^{-1}\sum_{\ell=0}^{\infty}(2\ell+1)\langle p_1|T_\ell|p_2\rangle P_\ell(\hat{p}_1\cdot\hat{p}_2) \tag{1.2}$$

and a similar expansion for the Born approximation $\langle p_1|V|p_2\rangle$, we obtain the partial-wave Lippman-Schwinger equation

$$T_\ell = V_\ell + V_\ell G_o T_\ell \tag{1.3}$$

where there is a different G_o for each partial wave ℓ.

Therefore, in determining the off-shell amplitudes, the potential is not involved directly but only via the Born approximations to the off-shell partial-wave amplitudes

$$\begin{aligned} V_\ell(p_1^2, p_2^2) &= -\frac{\pi}{2p_1p_2} \langle p_1 | V_\ell | p_2 \rangle \\ &= -\int_0^\infty j_\ell(p_1 r)\, j_\ell(p_2 r)\, V(r)\, r^2\, dr \end{aligned} \tag{1.4}$$

However, because of the existence of the potential, the partial-wave amplitudes $V_\ell(p_1^2,p_2^2)$ are not independent and in this thesis we investigate relationships between the amplitudes which enable the off-shell Born approximations to be determined directly from the on-shell Born approximations $V_\ell(k^2) = V_\ell(k^2,k^2)$.

For the Born approximation to the off-shell scattering amplitude we use the notation

$$\begin{aligned} f^B(-q^2) &= -2\pi^2 \langle \underline{p}_1 | V | \underline{p}_2 \rangle \\ &= \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(p_1^2, p_2^2)\, P_\ell(\hat{p}_1 \cdot \hat{p}_2) \end{aligned} \tag{1.5}$$

with $q = |\underline{p}_1 - \underline{p}_2|$, or in terms of the potential

$$f^B(-q^2) = -\frac{1}{q} \int_0^\infty \sin(qr)\, V(r)\, r\, dr \tag{1.6}$$

The justification for deriving (1.6) from (1.5) is given in Section 1.1.

On-shell, we have $p_1^2 = p_2^2 = k^2$ and

$$V_\ell(k^2) = \delta_\ell^B(k^2)/k \tag{1.7}$$

where $\delta_\ell^B(k^2)$ is the Born approximation to the ℓ'th phase shift.

In Chapters 2, 3 and 4 we consider the inverse scattering problem at fixed energy (Newton (1962) and Sabatier (1966,1967)), that is, determining the underlying potential directly from all the phase shifts at a given energy k^2. This approach is more practical than the Gel'fand-Levitan approach (1951), starting from one phase shift at all energies, since the latter requires a knowledge of the phase shifts at high energies where potential theory is not valid. The problem of finding the potential from all the phase shifts at one energy has also been treated by Wheeler (1955), Regge (1959) and Martin and Targonski (1961) but none of these approaches are as general as that of Newton (1962). However, Sabatier (1966) has shown that this generality leads to an infinite set of potentials associated with each set of phase shifts at a given energy and hence that the off-shell extrapolation obtained from (1.6) is not unique. This lack of uniqueness can be removed (Sabatier (1966)) by selecting the potential with the shortest range from amongst this infinite set, although as shown in Chapter 3 this only gives correct values for $f^{B}(-q^2)$ when $q^2 < 4k^2$.

A review of the inverse scattering problem at fixed energy is given in Chapter 2 in terms of the generalisation of Newton's method derived by Sabatier (1967). In Chapter 3 the inverse problem is treated in the Born approximation and since a general solution to the equations has not been found, we consider two special cases which can be solved explicitly. For these two special cases, expressions are derived for the Born approximation $f^{B}(-q^2)$ in terms of the phase shifts and it is found that both cases produce incorrect results for $f^{B}(-q^2)$ when $q^2 > 4k^2$. The reason for this incorrect behaviour is that neither of the potentials is short-range in the sense of (1.20) and hence the $V_m(k^2)$ only determine $V_\ell(p^2)$ for $p^2 \leqslant k^2$ as described in Chapter 5.

In Chapter 4 we consider approximations to Newton's potential where N (=0,1,2) phase shifts are treated exactly and the rest are treated to first order only. Expressions are derived for the approximate potential in terms of the phase shifts and numerical results are given for each approximation when the input phase shifts are those of a unit-range Yukawa potential.

As mentioned previously, the partial-wave amplitudes are not independent and in Sections 5.1 and 5.2 we derive relations between the Born approximations to the on-shell amplitudes $V_\ell(k^2)$. It is shown that $V_\ell(k^2)$ can be expressed in terms of either one $V_m(q^2)$ at all energies or all the $V_m(q^2)$ at any energy $q^2 \geqslant k^2$, the expressions being explicit sums or integrals over the given amplitudes. The conditions on the potential that are required for these results to be valid are discussed in Section 1.1. The results are most useful for short-range potentials, that is, ones for which $V(r)$ falls off exponentially as $r \to \infty$, so we also consider this class of potentials and the consequent behaviour of the partial-wave amplitudes associated with them. In Section 5.3 it is shown that a low-energy expansion of the amplitudes can be obtained for short-range potentials and that the region of convergence of this expansion is directly related to the range of the potential. Finally, in Section 5.4, we consider a high-energy expansion of the amplitudes which is valid for a class of potentials which includes sums of Yukawa and exponential potentials. For such a potential it is shown that the high-energy expansion enables $V_\ell(k^2)$ to be calculated from all the $V_m(q^2)$ at energy $q^2 < k^2$.

Therefore by using the high-energy expansion in conjunction with the low-energy expansion, it is possible to calculate $V_\ell(k^2)$ for all k^2 from all the $V_m(q^2)$ at any energy q^2.

In Chapter 6 we consider the off-shell amplitudes $V_\ell(p_1^2,p_2^2)$ and derive an expression for these amplitudes in terms of all the on-shell amplitudes $V_m(k^2)$ at energy k^2. The expression obtained converges for all p_1^2 and p_2^2 such that $p_1 + p_2 \leqslant 2k$ and for short-range potentials the region of convergence is extended slightly. For the particular choice $k = \frac{1}{2}(p_1 + p_2)$, a simpler expression is obtained which converges rapidly and involves precisely the same functions as were required for the low-energy expansion.

Therefore, given all the $V_m(q^2)$ at any energy q^2, we can calculate $V_\ell(p_1^2,p_2^2)$ for any p_1^2 and p_2^2 by first determining all the $V((p_1 + p_2)^2/4)$ from either the low-energy or high-energy expansions and then using the procedure described in Chapter 6.

1.1 Conditions on the potential

In this section we derive properties of the partial-wave amplitudes which are required in the following chapters.

We assume the existence of a potential V(r) such that

$$\int_0^\infty |V(r)| r^2 \, dr < \infty \tag{1.8}$$

For such a potential, the Born approximations to the partial-wave amplitudes (1.4) and

$$V_\ell(k^2) = -\int_0^\infty j_\ell^2(kr) \, V(r) \, r^2 \, dr \tag{1.9}$$

are well defined for p_1^2, $p_2^2 \geqslant 0$ and $k^2 \geqslant 0$ respectively by virtue of the equality (Abramowitz and Stegun (1968) ; 10.1.50)

$$\sum_{\ell=0}^\infty (2\ell+1) \, j_\ell^2(z) = 1 \tag{1.10}$$

which gives

$$|j_\ell(z)| \leqslant (2\ell+1)^{-\frac{1}{2}} \qquad (\ell \geqslant 0) \tag{1.11}$$

Setting $q = |\mathbf{p}_1 - \mathbf{p}_2|$ we have from (1.4) and (1.5)

$$\begin{aligned} f^B(-q^2) &= -\int_0^\infty \left\{ \sum_{\ell=0}^\infty (2\ell+1) \, j_\ell(p_1 r) \, j_\ell(p_2 r) \, P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \right\} V(r) \, r^2 \, dr \\ &= -\int_0^\infty q^{-1} \sin(qr) \, V(r) \, r \, dr \end{aligned} \tag{1.12}$$

where we have used (Abramowitz and Stegun (1968) ; 10.1.45)

$$\frac{\sin(\lambda R)}{\lambda R} = \sum_{\ell=0}^\infty (2\ell+1) \, j_\ell(\lambda a) \, j_\ell(\lambda b) \, P_\ell(\cos\theta) \tag{1.13}$$

with $R = (a^2 + b^2 - 2ab\cos\theta)^{\frac{1}{2}}$. For potentials of the form (1.8), the legitimacy of interchanging the order of integration and summation in deriving (1.12) is assured by their absolute convergence since

$$|\sin(x)/x| \leqslant 1 \tag{1.14}$$

for real x.

Therefore the scattering amplitude $f^B(-q^2)$ is well defined both on and off the energy shell for potentials belonging to the class (1.8).

From (1.4), (1.11) and (1.12) we see that

$$\left| V_\ell(p_1^2, p_2^2) \right| \leq \left| f^B(0) \right| / (2\ell+1) \tag{1.15}$$

and so the amplitudes are bounded and tend to zero as $\ell \to \infty$. Therefore, by the Lebesgue dominated convergence theorem (Bartle (1966), Chapter 5), we can take limits under the integral in (1.4) and obtain

$$V_\ell(0, p^2) = V_\ell(p^2, 0) = \delta_{\ell,0} f^B(-p^2) \tag{1.16}$$

and

$$\lim_{p_1^2 \to \infty} V_\ell(p_1^2, p_2^2) = \lim_{p_2^2 \to \infty} V_\ell(p_1^2, p_2^2) = 0 \tag{1.17}$$

since $j_\ell(0) = \delta_{\ell,0}$ and $j_\ell(\infty) = 0$. Note that on-shell (1.16) becomes

$$V_\ell(0) = \delta_{\ell,0} f^B(0)$$

Also, provided $k^2 \neq 0$, we can differentiate (1.9) under the integral sign to give

$$k^2 V_\ell'(k^2) = -\int_0^\infty k \, j_\ell(kr) \, j_\ell'(kr) \, V(r) \, r^3 \, dr \tag{1.18}$$

which is well defined for $k^2 > 0$ since $z j_\ell(z)$ is bounded for fixed ℓ and

$$\left| j_\ell'(z) \right| = \left| \ell j_{\ell-1}(z) - \ell(\ell+1) j_{\ell+1}(z) \right| (2\ell+1)^{-1} < 1 \tag{1.19}$$

Equation (1.18) also holds at $k^2 = 0$ in the sense that $\lim_{k^2 \to 0} k^2 V_\ell'(k^2) = 0$.

These properties of the $V_\ell(p_1^2, p_2^2)$ are all that are required to derive the basic equations in Chapters 5 and 6. However we often restrict the potential further by assuming that it is a 'short-range' potential. We define a short-range potential to be one such that

$$\lim_{r \to \infty} \exp(\alpha r) V(r) = 0 \tag{1.20}$$

for some $\alpha > 0$. The range of the potential is taken as the reciprocal of the least upper bound of such α.

For any such potential we have

$$\int_0^\infty r^{n+2} V(r)\, dr < \infty$$

for all $n \geqslant 0$ and therefore we see from (1.4) that as p_1^2 or $p_2^2 \to 0$

$$V_\ell(p_1^2, p_2^2) = O\left((p_1 p_2)^\ell\right) \tag{1.21}$$

where we have used the bound (Abramowitz and Stegun (1968) ; 9.1.62)

$$|j_\ell(z)| \leqslant \frac{|2z|^\ell \ell!}{(2\ell+1)!} \exp\left[|\mathfrak{Im}\, z|\right] \tag{1.22}$$

Using (1.22) for $r \leqslant R$ and $|V(r)| \leqslant A\exp(-\alpha r)$ for $r > R$, where A and R are positive constants independent of ℓ, gives

$$\begin{aligned} |V_\ell(p_1^2, p_2^2)| \leqslant{} & |2p_1 p_2 R^2|^\ell \left[\frac{\ell!}{(2\ell+1)!}\right]^2 \int_0^R |V(r)| r^2\, dr \\ & + \frac{\alpha A}{2(p_1 p_2)^2} \left| Q_\ell'\left(\frac{\alpha^2 + p_1^2 + p_2^2}{2p_1 p_2}\right)\right| \end{aligned} \tag{1.23}$$

where we have used (Watson (1966) ; 13.22.2)

$$\int_0^\infty e^{-at} j_\nu(bt) j_\nu(ct)\, dt = \frac{1}{2bc} Q_\nu\left(\frac{a^2+b^2+c^2}{2bc}\right) \tag{1.24}$$

From the asymptotic behaviour of the Legendre functions of the second kind (Abramowitz and Stegun (1968) ; 8.10.5, see also (5.35)) we see that for fixed p_1^2 and p_2^2

$$V_\ell(p_1^2, p_2^2) \sim e^{-\ell\sigma} \tag{1.25}$$

as $\ell \to \infty$, with $\cosh\sigma = (\alpha^2 + p_1^2 + p_2^2)/2p_1 p_2$.

For complex k^2 we see from (1.9), (1.22) and the Lebesgue dominated convergence theorem that $V_\ell(k^2)$ is analytic in k^2 provided

$$2|\mathfrak{Im}\, k| < \alpha \tag{1.26}$$

and hence that the series expansion of $V_\ell(k^2)$ about $k^2 = 0$ will converge for at least $k^2 < \alpha^2/4$. Therefore, for a short-range potential of range α^{-1}, the low-energy expansion described in Section 5.3 will certainly converge for $k^2 < \alpha^2/4$.

For the Yukawa potential $V(r) = -A\exp(-\mu r)/r$ we see from (1.9) and

(1.24) that

$$V_\ell(k^2) = \frac{A}{2k^2} Q_\ell\left(1 + \frac{\mu^2}{2k^2}\right) \tag{1.27}$$

and hence using (Abramowitz and Stegun (1968) ; 8.6.19)

$$Q_\ell(x) = \frac{1}{2} P_\ell(x) \log\left(\frac{x+1}{x-1}\right) - \sum_{m=1}^{\ell} \frac{1}{m} P_{m-1}(x) P_{\ell-m}(x) \tag{1.28}$$

we have

$$V_\ell(k^2) = \log k^2 \sum_{i=1}^{\ell+1} k^{-2i} \alpha_{i,\ell} + \sum_{i=1}^{\infty} k^{-2i} \beta_{i,\ell} \tag{1.29}$$

Therefore the 'high-energy' expansion

$$V_\ell(k^2) = \sum_{i=1}^{\infty} k^{-2i} \left(\alpha_{i,\ell} \log k^2 + \beta_{i,\ell}\right)$$

will certainly be valid for any sum of Yukawa potentials and also for potentials of the form $V(r) = r^j \exp(-\mu r)$ with $j > -1$, since these can be obtained from Yukawas by differentiating with respect to μ. Consequently the high-energy expansion described in Section 5.4 will be valid for at least all potentials of the form

$$V(r) = \sum_i \sum_{j \geq -1} A_{i,j}\, r^{-j} \exp(-\mu_i r)$$

These properties of the partial-wave amplitudes are all that are required in the following chapters and the only conditions which have been assumed are those regarding the short range nature of the potential.

CHAPTER 2

REVIEW OF THE INVERSE SCATTERING PROBLEM AT FIXED ENERGY

This chapter consists mainly of a review of the results obtained by Newton (1962) and Sabatier (1966 and 1967) in their work on determining a local potential from the knowledge of all the phase shifts at a fixed energy. The notation used and formulae derived form the basis for Chapters 3 and 4 in which several approximations to the inverse scattering problem are investigated.

The first section of this chapter is concerned with the general theory of the inverse scattering problem and it is shown that an infinite set of constants (at each energy) are all that are required to determine the underlying local potential. A method of relating these constants to the phase shifts is described in the second section thereby completing the formal solution of the inverse problem.

2.1 General Theory

We consider elastic scattering of spinless particles by a spherically symmetric local potential. It is assumed that all the phase shifts at a fixed energy k^2 are known and that an underlying local potential does exist. The method to be used in constructing the potential is the one due to Sabatier (1967) as it results in a much wider class of potentials than the original method of Newton (1962). The essential difference between the two approaches is that Newton considered the summation in (2.5) to be over integer values only, whereas Sabatier (1967) extended the technique to allow the summation to be over any subset of $[-\frac{1}{2},\infty[$. As well as including the potentials of Newton as a special case, it will be seen at the end of this chapter that this wider class of potentials includes all those which are analytic in a neighbourhood of the origin.

The problem is to determine the potential which, when inserted in the radial Schrödinger equation, reproduces the original set of phase shifts at the given energy k^2.

Defining the differential operator

$$D_0(z) = z^2\left(\frac{\partial^2}{\partial z^2}+1\right) \tag{2.1}$$

the zero-potential radial Schrödinger equation can be written as

$$D_0(kr)\,u_\mu(kr) = \mu(\mu+1)\,u_\mu(kr) \tag{2.2}$$

The regular solutions of (2.2) subject to the boundary conditions

$$\lim_{r\to 0} u_\mu(kr)\,(kr)^{-\mu-1}\,\Gamma(\mu+\tfrac{3}{2})\,2^{\mu+1}\,\pi^{-\frac{1}{2}} = 1 \tag{2.3}$$

are

$$\begin{aligned} u_\mu(kr) &= \sqrt{\pi kr/2}\; J_{\mu+\frac{1}{2}}(kr) \\ &= kr\, j_\mu(kr) \end{aligned} \tag{2.4}$$

In order that the $u_\mu(kr)$ be regular solutions of (2.2) we see from (2.3) that we must have $\mu > -1$. In fact in the remainder of this chapter we assume for simplicity that $\mu \geqslant 0$.

The regular solutions $u_\mu(kr)$ are used to define the function

$$g(k,r,r') = k\sum_\mu c_\mu(k^2)\,u_\mu(kr)\,u_\mu(kr') \tag{2.5}$$

where the constants $c_\mu(k^2)$ are as yet unspecified. From (2.2) it can be seen that $g(k,r,r')$ satisfies the differential equation

$$D_0(kr)\,g(k,r,r') = D_0(kr')\,g(k,r,r') \tag{2.6}$$

and also that

$$g(k,0,r') = g(k,r,0) = 0 \tag{2.7}$$

It is assumed throughout that μ is a discrete index, but with only slight modification we could take μ to be a continuous index in which case the summation in (2.5) would be replaced by an integral.

The function $g(k,r,r')$ is now used to define a function $K(k,r,r')$

via the integral equation

$$K(k,r,r') = g(k,r,r') - \int_0^r dr_1\,(kr_1^2)^{-1} K(k,r,r_1)\,g(k,r_1,r') \tag{2.8}$$

In order to ensure convergence of (2.5) and hence the existence and uniqueness of a solution to (2.8), Sabatier (1967) imposed the following bound on the c_μ for any μ :-

$$c_\mu < B\,\alpha^{-2\mu}\,\mu^{\beta}\,(\mu!)^2 \tag{2.9}$$

for some α and $\beta > 0$, and with B a constant.

With this bound on the c_μ Sabatier (1967) showed that the only solution to the homogeneous form of (2.8) was the trivial solution and hence that the solution of (2.8) was unique.

It is now possible to define the function $\xi(k,r,r')$ by

$$\begin{aligned}\xi(k,r,r') = {} & D(k,r)\,K(k,r,r') \\ & - D_0(kr')\,K(k,r,r')\end{aligned} \tag{2.10}$$

where

$$D(k,r) = D_0(kr) - r^2 V(r) \tag{2.11}$$

and

$$V(r) = -\frac{2}{kr}\frac{d}{dr}\left[\frac{K(k,r,r)}{r}\right] \tag{2.12}$$

By straightforward differentiation, integration by parts and the use of (2.6) and (2.7) it can be seen that the righthand side of (2.10) satisfies the homogeneous version of (2.8). Therefore, as (2.8) possesses a unique solution we have

$$\xi(k,r,r') \equiv 0 \tag{2.13}$$

and hence from (2.10)

$$D(k,r)\,K(k,r,r') = D_0(kr')\,K(k,r,r') \tag{2.14}$$

Also, from (2.8) and (2.7), it can be seen that

$$K(k,r,0) = K(k,0,r') = 0 \tag{2.15}$$

Finally the functions $\phi_\mu(k,r)$ are defined by

$$\phi_\mu(k,r) = u_\mu(kr) - \int_0^r \frac{dr'}{k(r')^2} K(k,r,r')\, u_\mu(kr') \tag{2.16}$$

Application of the differential operator D(k,r) to (2.16) yields

$$D(k,r)\,\phi_\mu(k,r) = \mu(\mu+1)\,\phi_\mu(k,r) \tag{2.17}$$

and from (2.3) and (2.15)

$$\phi_\mu(k,0) = 0 \tag{2.18}$$

Therefore the functions $\phi_\mu(k,r)$ defined by (2.16) are the regular solutions of the radial Schrödinger equation under the potential V(r) defined by (2.12).

Upon inserting (2.5) into (2.8) we obtain

$$K(k,r,r') = k\sum_\mu c_\mu(k^2)\, \mathcal{K}(k,r)\, u_\mu(kr') \tag{2.19}$$

with

$$\mathcal{K}(k,r) = u_\mu(kr) - \int_0^r \frac{dr_1}{k(r_1)^2} K(k,r,r_1)\, u_\mu(kr_1) \tag{2.20}$$

and hence from comparison with (2.16)

$$\mathcal{K}(k,r) = \phi_\mu(k,r) \tag{2.21}$$

This allows (2.19) to be rewritten as

$$K(k,r,r') = k\sum_\mu c_\mu(k^2)\,\phi_\mu(k,r)\, u_\mu(kr') \tag{2.22}$$

and so after substituting for $K(k,r,r_1)$ and interchanging the order of integration and summation, (2.20) reduces to

$$\phi_\mu(k,r) = u_\mu(kr) - k\sum_\nu L_{\mu,\nu}(kr)\, c_\nu(k^2)\, \phi_\nu(k,r) \tag{2.23}$$

where

$$L_{\mu,\nu}(kr) = \int_0^{kr} \frac{dx}{x^2} u_\mu(x) u_\nu(x) \tag{2.24}$$

The definition of the potential V(r) in (2.12) can also be simplified using (2.22) to give

$$V(r) = -\frac{2}{r}\frac{d}{dr}\left[\frac{1}{r}\sum_\mu c_\mu(k^2) u_\mu(kr) \phi_\mu(k,r)\right] \tag{2.25}$$

In principle, equations (2.23), (2.24) and (2.25) provide the solution to the inverse scattering problem at fixed energy. The integral in (2.24) can be evaluated for $\mu \neq \nu$ due to the fact that the $u_\mu(kr)$ satisfy the differential equations

$$\frac{d^2}{dx^2} u_\mu(x) + \left(1 - \frac{\mu(\mu+1)}{x^2}\right) u_\mu(x) = 0$$

and therefore

$$\frac{d}{dx}\left[W(u_\mu(x), u_\nu(x))\right] = \frac{\nu(\nu+1) - \mu(\mu+1)}{x^2} u_\mu(x) u_\nu(x)$$

where W(a,b) is the Wronskian of a and b. Hence, as the $u_\mu(x)$ are regular solutions, we obtain

$$L_{\mu,\nu}(kr) = \frac{W(u_\mu(kr), u_\nu(kr))}{\nu(\nu+1) - \mu(\mu+1)} \tag{2.26}$$

for $\mu \neq \nu$.

If the constants $c_\nu(k^2)$ are known, the regular solutions $\phi_\mu(k,r)$ of the radial Schrödinger equation can be found by solving the system of linear equations in (2.23). Once the $\phi_\mu(k,r)$ have been determined, the potential can easily be calculated from (2.25).

Therefore the only step remaining in the solution of the inverse scattering problem is that of obtaining the constants $c_\nu(k^2)$ from all the phase shifts at the given energy k^2.

2.2 Relating the $c_\mu(k^2)$ to the phase shifts

In the previous section the underlying potential in the inverse scattering problem was obtained in terms of a set of, as yet unspecified, constants $c_\mu(k^2)$. As the only quantities assumed known are all the phase shifts $\delta_\nu(k^2)$ at the fixed energy k^2, it is necessary to determine a relationship between these phase shifts and the $c_\mu(k^2)$. In practice, we only have knowledge of the phase shifts for $\nu = \ell$ where ℓ is a non-negative integer, but as no additional work is involved we consider the more general case where $\nu \in \mathbb{R}$.

The phase shifts are defined in terms of the asymptotic behaviour of the regular solutions of the radial Schrödinger equation via

$$\phi_\mu(k,r) \sim A_\mu \sin\left(kr - \frac{\mu\pi}{2} + \delta_\mu\right) \tag{2.27}$$

as $r \to \infty$. The properties of the constants A_μ can easily be obtained by comparison of $\phi_\mu(k,r)$ with the physical wave functions $\psi_\mu(k,r)$ which are normalised such that

$$\psi_\mu(k,r) \sim e^{i\delta_\mu} \sin\left(kr - \frac{\mu\pi}{2} + \delta_\mu\right) \tag{2.28}$$

as $r \to \infty$, and

$$\psi_\mu(k,0) = 0 \tag{2.29}$$

Due to the normalisation condition (2.3) we have as $r \to \infty$ (see, for example, De Alfaro and Regge, 1965)

$$\begin{aligned} \phi_\mu(k,r) &= \phi(\lambda,k,r) \\ &\sim \frac{\sqrt{\pi}}{2i}\left[f_\lambda(k)e^{ikr} - f_\lambda(-k)e^{-ikr}\right]\frac{k^\mu}{2^{\mu+1}\Gamma(\mu+\frac{3}{2})} \end{aligned} \tag{2.30}$$

where $\lambda = \mu + \frac{1}{2}$, $f_\lambda(k)$ are the Jost functions

$$f_\lambda(k) = W\left(f(\lambda,k,r), \phi(\lambda,k,r)\right) \tag{2.31}$$

and $f(\lambda,k,r)$ are the solutions of the radial Schrödinger equation for which

$$\lim_{r\to\infty} e^{ikr} f(\lambda,k,r) = 1 \tag{2.32}$$

The definition of the S matrix

$$S(\lambda, k) = e^{2i\delta_\mu(k^2)}$$
$$= \frac{f_\lambda(k)}{f_\lambda(-k)} e^{i\pi(\lambda - \frac{1}{2})} \tag{2.33}$$

allows (2.28) to be rewritten as

$$\psi_\mu(k,r) \sim \frac{e^{\frac{i\mu\pi}{2}}}{2i f_\lambda(-k)} \left[e^{ikr} f_\lambda(k) - e^{-ikr} f_\lambda(-k) \right] \tag{2.34}$$

Therefore from (2.30) we obtain

$$\phi_\mu(k,r) = \sqrt{\pi}\, k^\mu e^{-\frac{i\mu\pi}{2}} \frac{f_\lambda(-k)\ \psi_\mu(k,r)}{2^{\mu+1}\,\Gamma(\mu+\frac{3}{2})} \tag{2.35}$$

and the constants A_μ in (2.27) are given by

$$A_\mu = e^{i\delta_\mu(k^2)} k^\mu \sqrt{\pi}\, \frac{e^{-\frac{i\mu\pi}{2}} f_\lambda(-k)}{2^{\mu+1}\,\Gamma(\mu+\frac{3}{2})} \tag{2.36}$$

Use of (2.33) and the equality

$$f_\lambda^*(k) = f_\lambda(-k) \tag{2.37}$$

for λ and k real (De Alfaro and Regge ,1965) readily yields the fact that the constants A_μ are real, that is

$$A_\mu^* = A_\mu \tag{2.38}$$

In order to relate the $c_\mu(k^2)$ to the phase shifts, Sabatier (1967) made the assumption that the asymptotic behaviour of the $c_\mu(k^2)$ was such that

$$|c_\mu(k^2)| < B\mu^{\frac{1}{3}-\varepsilon} \tag{2.39}$$

for some constant B and $\varepsilon > 0$. With this constraint Sabatier (1967) proved that the series in (2.23) are uniformily convergent as $r \to \infty$ and hence can be replaced by their asymptotic behaviour (2.27),

$$u_\mu(kr) \sim \sin(kr - \tfrac{\pi\mu}{2}) \tag{2.40}$$

and

$$L_{\mu,\nu}(kr) \sim L^{(\infty)}_{\mu,\nu} = \begin{cases} \dfrac{\sin[\frac{\pi}{2}(\nu-\mu)]}{(\nu+\frac{1}{2})^2 - (\mu+\frac{1}{2})^2} & \nu \neq \mu \\[2ex] \dfrac{\pi}{2(2\nu+1)} & \nu = \mu \end{cases} \tag{2.41}$$

(The above result for $\nu = \mu$ is obtained directly from (2.24) and is valid for $\mu > -\frac{1}{2}$; see, for example, Abramowitz and Stegun (1968) : 11.4.6)

Substituting these asymptotic forms into (2.23) we obtain

$$A_\mu \sin\left(kr - \frac{\mu\pi}{2} + \delta_\mu\right) = \sin\left(kr - \frac{\mu\pi}{2}\right) - k\sum_\nu L^{(\infty)}_{\mu,\nu}\, c_\nu(k^2)\, A_\nu \sin\left(kr - \frac{\nu\pi}{2} + \delta_\nu\right) \qquad (2.42)$$

and after equating coefficients of $\exp(ikr)$

$$A_\mu e^{i(\delta_\mu - \frac{\mu\pi}{2})} = e^{-i\frac{\mu\pi}{2}} - k\sum_\nu L^{(\infty)}_{\mu,\nu}\, c_\nu(k^2)\, A_\nu\, e^{i(\delta_\nu - \frac{\nu\pi}{2})} \qquad (2.43)$$

The real nature of the A_μ now allows the real and imaginary parts of (2.43) to be separated, giving

$$A_\mu = \cos\delta_\mu - k\sum_\nu L^{(\infty)}_{\mu,\nu}\, c_\nu(k^2)\, A_\nu \cos\left[\delta_\nu - \delta_\mu - \frac{\pi}{2}(\nu - \mu)\right] \qquad (2.44)$$

and

$$\sin\delta_\mu = -k\sum_\nu L^{(\infty)}_{\mu,\nu}\, c_\nu(k^2)\, A_\nu \sin\left[\delta_\nu - \delta_\mu - \frac{\pi}{2}(\nu - \mu)\right] \qquad (2.45)$$

Equations (2.44) and (2.45) provide the required relationships between the phase shifts and the coefficients $c_\mu(k^2)$. Given the phase shifts at energy k^2 these equations, in principle, yield the $c_\mu(k^2)$ which in turn determine the potential V(r) via equations (2.23) and (2.25).

Unfortunately, due to their complexity, it is not possible in practice to obtain numerical solutions to these sets of equations and so it is necessary to investigate various approximations to them. These approximations to the inverse scattering problem form most of the work described in Chapters 3 and 4. In Chapter 3 the problem is considered in the Born approximation whilst in Chapter 4 approximation methods are described where several phase shifts are treated exactly and the remainder are treated to first order only.

Throughout this chapter the index μ has been allowed to take any real value not less than zero. Since, in practice, the only known quantities are the phase shifts δ_ℓ , for ℓ an integer, it is necessary to consider the range of values of the indices for which solutions to (2.44), (2.45) and (2.23) are meaningful. Sabatier (1967) proved that if the μ's are rational and include all the integers, then it is possible to choose arbitrarily, subject only to some very weak constraints, all the c_μ for which μ is not an integer and hence solve the problem.

Furthermore, on restricting attention to potentials such that rV(r) is analytic in $|r| < \Gamma$, Sabatier (1967) has proved that the function K(k,r,r) in (2.12) can be expanded as

$$K(k,r,r) = k\sum_{\ell=0}^{\infty} c_\ell(k^2)\, u_\ell(kr)\, \phi_\ell(k,r) + k\sum_{\ell=0}^{\infty} c_{\ell+\frac{1}{2}}(k^2)\, u_{\ell+\frac{1}{2}}(kr)\, \phi_{\ell+\frac{1}{2}}(k,r) \tag{2.46}$$

convergent in $|r| < \Gamma$, and that this expansion is unique.

Therefore, under the assumption that rV(r) is analytic in $|r| < \Gamma$, we need only consider solutions which correspond to indices taking integer and half-odd-integer values.

CHAPTER 3

THE INVERSE SCATTERING PROBLEM IN THE BORN APPROXIMATION

In this chapter solutions to the inverse scattering problem are obtained by treating the problem in the Born approximation. It is shown in the first section that the problem is greatly simplified in this approximation although it is still not possible to solve the equations and find the most general potentials corresponding to ν taking both integer and half-odd-integer values. Two special cases for which solutions can be found are considered in the second section and it is seen that one of these results in a one-parameter family of potentials associated with each set of phase shifts at a given energy. The asymptotic behaviour of this family of potentials allows just one to be selected which falls off faster than $r^{-2+\varepsilon}$.

In the final section the potentials derived from these two special cases are used to determine the off-shell scattering amplitude in Born approximation. When the input phase shifts are taken to be those of a single Yukawa potential, it is shown that these off-shell amplitudes are very different from those of the original Yukawa potential.

3.1 The Born approximation

Following the method described by Sabatier (1967), we consider solutions to equations (2.23), (2.25),(2.44) and (2.45) in the 'linear approximation' ; that is, only first order terms in the phase shifts and the $c_\nu(k^2)$ are retained. In this approximation, equations (2.44) and (2.45) reduce to

$$A_\mu = 1 - \sum_\nu L^{(\infty)}_{\mu,\nu}\, a_\nu \cos\left[\tfrac{\pi}{2}(\nu-\mu)\right] \tag{3.1}$$

and

$$\delta_\mu(k^2) = \sum_\nu L^{(\infty)}_{\mu,\nu}\, a_\nu \sin\left[\tfrac{\pi}{2}(\nu-\mu)\right] \tag{3.2}$$

upon setting

$$a_\mu = k\, c_\mu(k^2)\, A_\mu \cos(\delta_\mu(k^2))$$
$$= k\, c_\mu(k^2)\, A_\mu \qquad (3.3)$$

As we are dealing only to first order in the $c_\mu(k^2)$, it can be seen from (3.3) that the a_μ are also first order small and hence that (3.1) and (3.3) become

$$A_\mu = 1 \qquad (3.4)$$

and

$$a_\mu = k\, c_\mu(k^2) \qquad (3.5)$$

The expression for the potential is also greatly simplified in this approximation. From (2.23) and (2.25) we see that the dependence on the solutions $\phi_\mu(k,r)$ is removed, yielding

$$V(r) = -\frac{2}{r}\frac{d}{dr}\left[\frac{1}{r}\sum_\mu c_\mu(k^2)\, u_\mu^2(kr)\right] \qquad (3.6)$$

Since the only known quantities are the physical phase shifts $\delta_\ell(k^2)$, it is necessary to restrict the range of values for μ in (3.2) to $\mu = \ell$, an integer. Having done this, it is then possible to determine the a_ν, for ν non-integer, from the $\delta_\ell(k^2)$ and solve the problem. In the process of determining the a_ν, it may be found that (3.2) does not have a unique solution (c.f. Newton's potential). If this occurs, it may be necessary to impose constraints on the potential so that one particular set of a_ν can be selected.

In particular, allowing ν to take only integer and half-odd-integer values we obtain a solution corresponding to the potential, described at the end of Chapter 2, for which $rV(r)$ is analytic in $|r| < \Gamma$. It is this class of potentials which is to be considered in the remainder of this chapter since, in potential scattering theory, it is a desirable feature that the potential be analytic.

Sabatier (1967) has proved that this linear approximation is the same as the usual Born approximation. The Born approximation to the phase shifts is given by

$$\delta_\ell^B(k^2) \simeq \tan \delta_\ell^B(k^2)$$
$$= k\int_0^\infty r^2 j_\ell^2(kr)\, V(r)\, dr$$
$$= k^{-1}\int_0^\infty u_\ell^2(kr)\, V(r)\, dr \tag{3.7}$$

which, on replacing the potential by (3.6), becomes

$$\delta_\ell^B(k^2) = -2k^{-1}\int_0^\infty r^{-1} u_\ell^2(kr) \frac{d}{dr}\left[r^{-1} \sum_\mu c_\mu(k^2)\, u_\mu^2(kr)\right] dr$$
$$= -2k \sum_\mu \left\{ c_\mu(k^2) \int_0^\infty q^{-1} u_\ell^2(q) \frac{d}{dq}\left[q^{-1} u_\mu^2(q)\right] dq \tag{3.8}$$

The integral in (3.8) can be evaluated (see, for example, Sabatier (1967) Appendix B) giving

$$\delta_\ell^B(k^2) = -\sum_\mu a_\mu \frac{\sin^2\left[\frac{\pi}{2}(\mu-\ell)\right]}{\mu(\mu+1) - \ell(\ell+1)}$$
$$= \sum_\mu L_{\ell,\mu}^{(\infty)}\, a_\mu \sin\left[\frac{\pi}{2}(\mu-\ell)\right] \tag{3.9}$$

which is identical to (3.2) for the physical phase shifts.

Therefore, in the Born approximation, the inverse scattering problem reduces to finding solutions of the linear system of equations

$$\delta_\ell(k^2) = \sum_{p=0,\frac{1}{2}}^{\infty} B_{\ell,p}\, a_p \tag{3.10}$$

where

$$B_{\ell,p} = \frac{\sin^2\left[\frac{\pi}{2}(p-\ell)\right]}{(p+\frac{1}{2})^2 - (\ell+\frac{1}{2})^2} \tag{3.11}$$

for $\ell = 0,1,2,\cdots\cdots$ and $p = 0,\frac{1}{2},1,1\frac{1}{2},\cdots\cdots$. The potential can then be obtained directly from (3.5) and (3.6).

3.2 The Potentials of Newton and Sabatier

Although (3.5), (3.6) and (3.10) in principle provide the solution to the inverse scattering problem in Born approximation, there still remains the question of the existence and uniqueness of a solution to (3.10).

Initially it appears that a solution can be obtained by truncating the system (3.10) after N terms and numerically solving the reduced system of equations. This turns out to be an unsatisfactory approach since the elements

$B_{\ell,\ell+\frac{1}{2}}$ fall off very slowly (as ℓ^{-1}) and as $N \to \infty$ the determinant of B tends to zero. In fact, for N = 15, $\det(B) \simeq 10^{-36}$.

As the general inverse of B has not been found, we consider two special cases for which explicit inverses have been constructed. These two cases are obtained by setting

(i) $c_p = 0$ for p = m, an integer

and (ii) $c_p = 0$ for $p = m + \frac{1}{2}$, a half-odd-integer

and they produce the potential originally investigated by Newton (1962), and the 'even potential' of Sabatier (1967), respectively.

Newton's Potential

After removing all half-odd-integer values of the index p, we see that (3.10) can be rewritten in the form

$$\delta_\ell(k^2) = \sum_{p=0}^{\infty} M_{\ell,p}^{(\frac{1}{2})} a_p \tag{3.12}$$

where we have introduced the matrix $M^{(\alpha)}$ defined by

$$M_{\ell,m}^{(\alpha)} = \begin{cases} \dfrac{1}{(m+\alpha)^2-(\ell+\alpha)^2} & \text{for } (\ell-m) \text{ odd} \\ 0 & \text{for } (\ell-m) \text{ even} \end{cases} \tag{3.13}$$

The two-sided inverse $\gamma^{(\alpha)}$ of $M^{(\alpha)}$ and a vector $\underline{v}^{(\alpha)}$ which is annihilated by $M^{(\alpha)}$ have been constructed by Sabatier (1966). Explicitly

$$V_{2\ell}^{(\alpha)} = \frac{2\Gamma(\ell+\alpha)\Gamma(\ell+\frac{1}{2})(2\ell+\alpha)}{\pi\,\Gamma(\ell+1)\,\Gamma(\ell+\alpha+\frac{1}{2})}$$

$$V_{2\ell+1}^{(\alpha)} = 0 \tag{3.14}$$

$$\gamma_{\ell,m}^{(\alpha)} = 0 \qquad \text{for } (\ell-m) \text{ even} \tag{3.15}$$

and

$$\gamma_{2\ell,2m+1}^{(\alpha)} = -\gamma_{2m+1,2\ell}^{(\alpha)} = V_{2\ell}^{(\alpha)}\,T_{\ell,m}^{(\alpha)}\,V_{2m}^{(\alpha)}$$

where

$$T_{\ell,m}^{(\alpha)} = \frac{(2m+2\alpha)(2m+1)(2m+\alpha+1)}{(2m+\alpha)(2\ell+2m+2\alpha+1)(2\ell-2m-1)} \tag{3.16}$$

and these formulae are certainly valid for $\alpha > -\frac{1}{2}$.

Since the matrix $iM^{(\alpha)}$ is Hermitean, it appears to have paradoxical properties - the existence of an inverse and a zero eigenvalue. Redmond (1964) investigated these properties for the case $\alpha = 0$ and showed that the source of the paradox was the existence of the eigenvector $\underset{\sim}{v}^{(0)}$ which was orthogonal to all other eigenvectors but which was not normalisable and did not belong to a complete set of eigenvectors. The analysis also showed that the vector $\underset{\sim}{v}^{(0)}$ was unique to within a constant multiplier. Sabatier (1966) extended the uniqueness proof to $\underset{\sim}{v}^{(\frac{1}{2})}$ and for the two special cases being considered, these are the only vectors which will be required.

The system of equations in (3.12) can now be inverted, giving

$$a_p = \sum_{\ell=0}^{\infty} \gamma_{p,\ell}^{(\frac{1}{2})} \delta_\ell (k^2) + \lambda v_p^{(\frac{1}{2})} \tag{3.17}$$

with λ a constant. Therefore, when the a_p are substituted in (3.5) and (3.6), we obtain a family of potentials parameterised by λ for each set of phase shifts. Even though a unique solution does not exist in this case, it is still possible to single out a potential of particular interest by considering the asymptotic behaviour of this family of potentials.

With the assumption that the phase shifts tend to zero faster than $\ell^{-3-\varepsilon}$ as $\ell \to \infty$, Sabatier (1966) showed that the family of potentials behaved asymptotically as

$$V(r) \sim -2\pi^{-\frac{1}{2}} (\lambda - \beta) r^{-\frac{3}{2}} \cos (2r - \tfrac{\pi}{4}) + O(r^{\varepsilon - 2}) \tag{3.18}$$

where

$$\beta = \sum_{p=0}^{\infty} v_p^{(\frac{1}{2})} \tan \delta_{2p} - \sum_{p=0}^{\infty} a_{2p+1} \sigma_{2p+1} \tag{3.19}$$

and

$$\sigma_{2p+1} = \sum_{m=0}^{\infty} v_{2m}^{(\frac{1}{2})} \tan \delta_{2m} M_{2m,2p+1}^{(\frac{1}{2})} \tan \delta_{2p+1} \tag{3.20}$$

Note that ε in these formulae refers to a positive number which can be made arbitrarily small but not equal to zero, but that it does not necessarily have the same value each time it occurs.

Therefore, from (3.18), we see that there is only one potential within each family which goes to zero faster than $r^{-2+\varepsilon}$ as $r \to \infty$ and that all the other potentials have a damped oscillating tail. This potential is characterised by the condition $\lambda = \beta$ and has been called the 'special potential' by Sabatier (1966) but in this and the following chapter we refer to it as 'Newton's potential'. Although β in (3.19) depends on a_{2p+1}, we see from (3.17) and (3.14) that it can be expressed in a closed form independent of both $\underset{\sim}{a}$ and λ.

It should be noted that the condition $\lambda = \beta$ is valid for the general solution of Newton's potential and not just for its Born approximation. In fact, this condition, together with equations (3.19) and (3.20), will be used in Chapter 4 when we investigate other approximations to Newton's potential.

In the Born approximation we see from (3.19) and (3.20) that Newton's potential is determined by

$$\lambda = \sum_{p=0}^{\infty} V_{2p}^{(\frac{1}{2})} \delta_{2p}(k^2) \tag{3.21}$$

and therefore from (3.5) and (3.17)

$$c_p(k^2) = k^{-1} \sum_{\ell=0}^{\infty} \gamma_{p,\ell}^{(\frac{1}{2})} \delta_\ell(k^2) + k^{-1} V_p^{(\frac{1}{2})} \sum_{\ell=0}^{\infty} V_{2\ell}^{(\frac{1}{2})} \delta_{2\ell}(k^2) \tag{3.22}$$

A procedure has been written to calculate the $c_p(k^2)$ from any set of phase shifts at energy k^2 (it is assumed that only the first N phase shifts are non-zero). As a simple practical test we have taken as $\delta_0, \delta_1, \dots, \delta_{10}$ the Born approximations to the Yukawa potential $V(r) = -\exp(-r)/r$ and set $\delta_\ell \equiv 0$ for $\ell > 10$. Table 3.1 contains some results for this example where several initial energies are considered, and it can be seen that the $c_\ell(k^2)$ rapidly approach a constant value at each energy. This behaviour can be seen directly from (3.22), (3.14), (3.15) and (3.16) since, as $p \to \infty$

$$c_p(k^2) \to 4(\pi k)^{-1} \sum_{\ell=0}^{\infty} V_{2\ell}^{(\frac{1}{2})} \delta_{2\ell}(k^2) \tag{3.23}$$

The results shown in Table 3.1 will be used in Section 3.3 to calculate the Born approximation to the off-shell scattering amplitude.

Sabatier's Even Potential

Returning to the second special case of the inverse of B, we see that, on setting $c_p = 0$ for all integer values of p, (3.10) becomes

$$S_\ell(k^2) = \sum_{p=0}^{\infty} L^{(\infty)}_{\ell,p+\frac{1}{2}} \, a_{p+\frac{1}{2}} \sin\left[\tfrac{\pi}{2}\left(p-\ell+\tfrac{1}{2}\right)\right] \tag{3.24}$$

In the Born approximation it is easy to see that this case produces an even potential, since, from (3.6)

$$V(r) = -\frac{2}{kr}\frac{d}{dr}\left[\frac{1}{r}\sum_{p=0}^{\infty} a_{p+\frac{1}{2}}(k^2)\, u^2_{p+\frac{1}{2}}(kr)\right] \tag{3.25}$$

and

$$\begin{aligned} u^2_{p+\frac{1}{2}}(kr) &= \frac{\pi kr}{2} J^2_{p+1}(kr) \\ &= \alpha r^{2p+1} \times \left(\text{sum of terms } \beta_n r^{2n}\right) \end{aligned}$$

where α and β_n are constants.

In order to simplify the notation we introduce the matrix E defined by

$$\begin{aligned} E_{\ell,p} &= L^{(\infty)}_{\ell,p+\frac{1}{2}} \sin\left[\tfrac{\pi}{2}\left(p-\ell+\tfrac{1}{2}\right)\right] \\ &= \tfrac{1}{2}\left[\left(\ell+\tfrac{1}{2}\right)^2 - p^2\right]^{-1} \end{aligned} \tag{3.26}$$

and the vector $\hat{c}$ such that

$$\begin{aligned} \hat{c}_{p+1} &= a_{p+\frac{1}{2}} \\ \text{with} \quad \hat{c}_0 &= 0 \end{aligned} \tag{3.27}$$

Rewriting (3.23) in terms of E and $\hat{c}$ we obtain

$$S_\ell(k^2) = -\sum_{p=0}^{\infty} E_{\ell,p}\,\hat{c}_p \tag{3.28}$$

From the definition of $M^{(\alpha)}$ in (3.13) we see that

$$\begin{aligned} E_{\ell,p} &= -2\left[(2p)^2 - (2\ell+1)^2\right]^{-1} \\ &= -2\, M^{(0)}_{2\ell+1,2p} \end{aligned} \tag{3.29}$$

and as $\gamma^{(\alpha)}$ is a two-sided inverse of $M^{(\alpha)}$, the inverse η of E is given by

$$\begin{aligned} \eta_{\ell,p} &= -\tfrac{1}{2}\gamma^{(0)}_{2\ell+1,2p} \\ &= \begin{cases} 4\pi^{-2} & \text{for } \ell = 0 \\ \dfrac{-8(2p+1)^2\pi^{-2}}{(2\ell)^2 - (2p+1)^2} & \text{for } \ell \neq 0 \end{cases} \end{aligned} \tag{3.30}$$

Similarly, the vector $\underset{\sim}{s}$ which is annihilated by E is given by

$$s_p = \begin{cases} \frac{1}{2} & \text{for } p = 0 \\ 1 & \text{for } p \neq 0 \end{cases} \tag{3.31}$$

and therefore it is possible to invert (3.28), giving

$$\hat{c}_p = -\sum_{\ell=0}^{\infty} \eta_{p,\ell}\, \delta_\ell(k^2) + \beta s_p \tag{3.32}$$

with β a constant.

From the condition $\hat{c}_0 = 0$, we see that

$$\beta = 8\pi^{-2} \sum_{\ell=0}^{\infty} \delta_\ell(k^2) \tag{3.33}$$

and therefore

$$c_{p+\frac{1}{2}}(k^2) = -k^{-1} \sum_{\ell=0}^{\infty} \eta_{p+1,\ell}\, \delta_\ell(k^2) + 8\pi^{-2} k^{-1} s_{p+1} \sum_{\ell=0}^{\infty} \delta_\ell(k^2) \tag{3.34}$$

A procedure, analogous to the one for Newton's potential, has been written to calculate the $c_{p+\frac{1}{2}}$ from any set of phase shifts at a given energy. Table 3.2 contains some results for the example where we take as $\delta_0, \delta_1, \ldots\ldots, \delta_{10}$ the Born approximations to the Yukawa potential $V(r) = -\exp(-r)/r$ and set $\delta_\ell \equiv 0$ for $\ell > 10$. As in the case of Newton's potential, the values rapidly approach a constant value at each energy; in fact, from (3.34), (3.30) and (3.31) we see that as $p \to \infty$

$$c_{p+\frac{1}{2}}(k^2) \longrightarrow 8\pi^{-2} k^{-1} \sum_{\ell=0}^{\infty} \delta_\ell(k^2) \tag{3.35}$$

In the next section we describe a method of determining the Born approximation to the off-shell scattering amplitude directly from the phase shifts, by assuming that the potential is of the form given by (3.6).

3.3 <u>The Born approximation to the off-shell scattering amplitude</u>

The Born approximation to the off-shell scattering amplitude is defined as

$$\begin{aligned} f^B(-q^2) &= -2\pi^2 \langle \underset{\sim}{p} | V | \underset{\sim}{p}' \rangle \\ &= q^{-1} \int_0^{\infty} \sin(qr)\, V(r)\, r\, dr \end{aligned} \tag{3.36}$$

where $q = |\underset{\sim}{p} - \underset{\sim}{p}'|$.

Assuming that the potential is given by (3.6) we can rewrite (3.36) as

$$f^B(-q^2) = 2q^{-1}\int_0^\infty \sin(qr)\,\frac{d}{dr}\Big[r^{-1}\sum_\mu c_\mu(k^2)\,u_\mu^2(kr)\Big]\,dr \tag{3.37}$$

For Newton's potential and the even potential of Sabatier we introduce the notation $f_N^B(-q^2)$ and $f_E^B(-q^2)$ respectively. In the case of Newton's potential we see from (3.23) that $c_\rho(k^2) \to \bar{c}$, a constant, as $p \to \infty$. Therefore, on setting $c_\rho = \bar{c} + d_\rho$, (3.37) becomes

$$\begin{aligned} f_N^B(-q^2) = {} & 2q^{-1}\bar{c}\int_0^\infty \sin(qr)\,\frac{d}{dr}\Big[r^{-1}\sum_{\ell=0}^\infty u_\ell^2(kr)\Big]\,dr \\ & + 2q^{-1}\int_0^\infty \sin(qr)\,\frac{d}{dr}\Big[r^{-1}\sum_{\ell=0}^\infty d_\ell\,u_\ell^2(kr)\Big]\,dr \end{aligned} \tag{3.38}$$

provided that the c_ρ are bounded. The integral in the first part of (3.38) can be evaluated (Abramowitz and Stegun (1968) ; 10.1.52, 5.2.1 and 4.3.143) giving

$$\begin{aligned} f_N^B(-q^2) = {} & \frac{\bar{c}k}{4q}\log\Big(\frac{q+2k}{q-2k}\Big)^2 \\ & + 2q^{-1}\int_0^\infty \sin(qr)\,\frac{d}{dr}\Big[r^{-1}\sum_{\ell=0}^\infty d_\ell\,u_\ell^2(kr)\Big]\,dr \end{aligned} \tag{3.39}$$

for $q \neq 2k$. The point $q = 2k$ produces a singularity in both parts of (3.38) and will not be considered further. After integration by parts and use of the estimate (Underhill (1970))

$$\sum_{\ell=0}^\infty d_\ell\,u_\ell^2(x) < C + C'\,|\log(x)| \tag{3.40}$$

for ℓd_ℓ bounded, (3.39) reduces to

$$f_N^B(-q^2) = \frac{\bar{c}k}{4q}\log\Big(\frac{q+2k}{q-2k}\Big)^2 + 2\int_0^\infty r^{-1}\cos(qr)\sum_{\ell=0}^\infty d_\ell\,u_\ell^2(kr)\,dr \tag{3.41}$$

Provided $|d_\ell| < A\ell^{-\frac{5}{3}-\varepsilon}$ as $\ell \to \infty$ it is possible to interchange the order of integration and summation in (3.41). The integral can then be expressed in a standard form (Bateman (1954) ; 19.2.32) and hence

$$f_N^B(-q^2) = \begin{cases} \dfrac{\bar{c}k}{2q}\log\Big(\dfrac{q+2k}{2k-q}\Big) - \displaystyle\sum_{\ell=0}^\infty d_\ell\,Q_\ell\Big(1-\frac{q^2}{2k^2}\Big) & q^2 < 4k^2 \\[2ex] \dfrac{\bar{c}k}{2q}\log\Big(\dfrac{q+2k}{q-2k}\Big) + \displaystyle\sum_{\ell=0}^\infty (-1)^\ell d_\ell\,Q_\ell\Big(\frac{q^2}{2k^2}-1\Big) & q^2 > 4k^2 \end{cases} \tag{3.42}$$

We can perform a similar analysis for Sabatier's even potential, since from (3.35) we see that $c_{p+\frac{1}{2}}(k^2) \to \bar{e}$, a constant, as $p \to \infty$. Therefore, on setting $c_{p+\frac{1}{2}} = \bar{e} + g_p$, we obtain

$$f_E^B(-q^2) = \begin{cases} \frac{\bar{e}}{2} Q_{-\frac{1}{2}}\left(1-\frac{q^2}{2k^2}\right) - \sum_{\ell=0}^{\infty} g_\ell\, Q_{\ell+\frac{1}{2}}\left(1-\frac{q^2}{2k^2}\right) & q^2 < 4k^2 \\ 0 & q^2 > 4k^2 \end{cases} \tag{3.43}$$

provided $|g_\ell| < B\ell^{-\frac{1}{2}-\varepsilon}$. The result that $f_E^B(-q^2) = 0$, for $q^2 > 4k^2$, is in agreement with Sabatier (1967).

A procedure has been written to calculate $f_N^B(-q^2)$ and $f_E^B(-q^2)$, for any value of q^2, from (3.42) and (3.43) given the coefficients $c_\ell(k^2)$ and $c_{\ell+\frac{1}{2}}(k^2)$. For $q^2 < 4k^2$ the series in (3.42) and (3.43) converge slowly and results were obtained by summing the first 200 terms in each series. For $q^2 > 4k^2$ the series in (3.42) converges very rapidly and in order to obtain 6 figure accuracy, only the first 30 terms were found necessary.

Tables 3.3 and 3.4 show results for several values of q^2 when the coefficients c_ℓ and $c_{\ell+\frac{1}{2}}$ are taken as the ones in Tables 3.1 and 3.2 respectively (these were obtained from the phase shifts to the Yukawa potential $V(r) = -\exp(-r)/r$). From these tables it can be seen that the results are only correct for the region $q^2 < 4k^2$. For $q^2 > 4k^2$ we have $f_E^B(-q^2) = 0$ from (3.43), and upon considering the limit $q^2 \to \infty$ in (3.42) we see that

$$\begin{aligned} f_N^B(-q^2) &\simeq 2k^2 q^{-2}(\bar{c} + d_0) \\ &\simeq 2k^2 q^{-2} c_0(k^2) \end{aligned} \tag{3.44}$$

The correct asymptotic behaviour for the Yukawa potential is q^{-2} whereas from Table 3.1 we find that $2k^2 c_0(k^2) = 0.072$, 0.360 and 0.553 for $k = 0.2$, 0.6 and 1.0 respectively. This behaviour can easily be seen in Figure 3.1 where $f_N^B(-q^2)$ is plotted for $q^2 < 10.0$ and $k = 0.2$, 0.6 and 1.0.

Having determined $f^B(-q^2)$ from all the phase shifts at a given energy k^2, it is now possible to obtain the Born approximation to the on-shell partial wave amplitudes $V_\ell(p^2)$ for any value of p^2. On shell, the partial-wave expansion of $f^B(-q^2)$ is given by

$$f^B(-q^2) = \sum_{\ell=0}^{\infty}(2\ell+1)\, P_\ell(\mu)\, V_\ell(p^2) \tag{3.45}$$

where $q^2 = 2p^2(1-\mu)$ and $\mu = \cos\theta$. Therefore

$$\begin{aligned} V_\ell(p^2) &= \tfrac{1}{2}\int_{-1}^{1} P_\ell(\mu)\, f^B(-2p^2(1-\mu))\, d\mu \\ &= -(2p)^{-2}\int_0^{4p^2} P_\ell(1-q^2/2p^2)\, f^B(-q^2)\, dq^2 \end{aligned} \tag{3.46}$$

which can be evaluated numerically for any value of p^2 from the previously calculated $f_N^B(-q^2)$ and $f_E^B(-q^2)$. For $p^2 = k^2$, (3.46) reduces to $V_\ell(k^2) = \delta_\ell(k^2)/k$ when either (3.42) or (3.43) are substituted for $f^B(-q^2)$, by virtue of the integral (Abramowitz and Stegun (1968) ; 8.14.8)

$$\int_{-1}^{1} P_\ell(x)\, Q_\mu(x)\, dx = \begin{cases} \dfrac{1-\cos(\mu\pi-\ell\pi)}{(\ell-\mu)(\ell+\mu+1)} & \mu \neq \ell \\ 0 & \mu = \ell \end{cases} \tag{3.47}$$

Since, in the example considered, $f_N^B(-q^2)$ and $f_E^B(-q^2)$ were only correct for $q^2 < 4k^2$, it follows from (3.46) that $V_\ell(p^2)$ will only be correct for $p < k$. The behaviour of $V_\ell(p^2)$ for $p > k$ is illustrated in Table 3.5 where results are given for the case $k^2 = 1.0$ and $p^2 = 2.0$.

It is to be expected that $f_N^B(-q^2)$ and $f_E^B(-q^2)$, together with any other $f^B(-q^2)$ derived from the inverse problem, be correct for $q^2 < 4k^2$ since, in this region, the phase shifts at energy k^2 uniquely determine $f^B(-q^2)$ (and $V_\ell(q^2)$ for $q^2 < k^2$) as described in Chapter 5.

Because of the discrepancy in $V_\ell(p^2)$ for $p > k$, it follows that the off-shell extrapolations from the $V_\ell(p^2)$ obtained from (3.46) and (3.42) or (3.43), will also differ from the true values. For example, using (6.8) we find that $V_0(4,1) = 0.1184$ for Newton's potential based on the $V_\ell(0.36)$ for the Yukawa potential $V(r) = -\exp(-r)/r$. Using Sabatier's even potential we obtain $V_0(4,1) = 0.0238$ whereas the true value for the Yukawa potential is 0.2012. It also follows that the off-shell T matrix will exhibit similar discrepancies which will increase as the energy k^2 is decreased.

The origin of the discrepancies lies in the long-range nature of both Newton's potential and the even potential of Sabatier. In fact, on setting

$c_\mu = A$, a constant, for all μ, we see from (3.6) and (3.25) that $V(r) \sim r^{-2}$ as $r \to \infty$. As in nature we are concerned with short-range interactions, the use of either of these potentials is unphysical. In particular, the off-shell extension of the scattering amplitude derived by Underhill (1970), which is based on Newton's potential, is likely to be unreliable.

Although, in the Born approximation, these two special cases have resulted in long-range potentials it is possible that the family of solutions to (3.10) includes one which corresponds to a short-range potential. If the inverse of the matrix B in (3.10) can be determined and a solution to the equations found by demanding that the potential be short-range, then the Born approximation to the off-shell scattering amplitude can be obtained from (3.37) without direct reference to the potential.

ℓ	$c_\ell(k^2)$				
	k = 0.1	k = 0.2	k = 0.6	k = 1.0	k = 2.0
0	0.97097	0.89404	0.49974	0.27640	0.09474
1	1.47073	1.39080	0.91406	0.57295	0.22861
2	1.23173	1.18203	0.87533	0.61905	0.29263
3	1.28709	1.21974	0.86396	0.62324	0.32393
4	1.24240	1.17800	0.83631	0.60736	0.33555
5	1.26405	1.19714	0.83345	0.59820	0.33863
6	1.24549	1.18093	0.82384	0.58853	0.33703
7	1.25686	1.19020	0.82426	0.58421	0.33396
8	1.24672	1.18140	0.81983	0.57971	0.33041
9	1.25371	1.18717	0.82063	0.57796	0.32703
10	1.24733	1.18166	0.81810	0.57570	0.32422
11	1.25206	1.18558	0.81882	0.57495	0.32165
12	1.24768	1.18180	0.81718	0.57367	0.32027
13	1.25109	1.18464	0.81778	0.57332	0.31829
98	1.24850	1.18217	0.81514	0.56947	0.30883
99	1.24853	1.18222	0.81516	0.56948	0.30883

Table 3.1 : $c_\ell(k^2)$ for Newton's potential from $\delta_0(k^2)$, $\delta_1(k^2)$, ..., $\delta_{10}(k^2)$, the Born approximations to the Yukawa potential $V(r) = -\exp(-r)/r$.

ℓ	$c_{\ell+\frac{1}{2}}(k^2)$				
	k = 0.1	k = 0.2	k = 0.6	k = 1.0	k = 2.0
0	0.00000	0.00000	0.00000	0.00000	0.00000
1	1.05540	0.98756	0.60782	0.36186	0.13548
2	0.85957	0.84354	0.67964	0.48890	0.22681
3	0.82455	0.79993	0.64988	0.50540	0.27172
4	0.81351	0.78645	0.62554	0.49510	0.29131
5	0.80857	0.78070	0.61219	0.48232	0.29785
6	0.80593	0.77770	0.60512	0.47245	0.29806
7	0.80435	0.77590	0.60116	0.46569	0.29551
8	0.80332	0.77476	0.59875	0.46119	0.29205
9	0.80263	0.77398	0.59718	0.45819	0.28855
10	0.80213	0.77342	0.59610	0.45614	0.28540
11	0.80176	0.77008	0.59313	0.45469	0.28275
12	0.80148	0.77270	0.59473	0.45364	0.28063
13	0.80126	0.77245	0.59428	0.45285	0.27941
98	0.80004	0.77109	0.59180	0.44878	0.26865
99	0.80004	0.77109	0.59180	0.44878	0.26865

Table 3.2 : $c_{\ell+\frac{1}{2}}(k^2)$ for the even potential of Sabatier from $\delta_0(k^2)$, $\delta_1(k^2), \cdots\cdots, \delta_{10}(k^2)$, the Born approximations to the Yukawa potential $V(r) = -\exp(-r)/r$.

	$f_N^B(-q^2)$				$f^B(-q^2)$
q \ k	0.2	0.6	1.0	2.0	$=(1+q^2)^{-1}$
0.1	0.99010	0.99010	0.99009	0.98549	0.99010
0.2	0.96154	0.96154	0.96154	0.95887	0.96154
0.3	0.91743	0.91743	0.91744	0.91702	0.91743
0.5	0.35433	0.80000	0.79999	0.80192	0.80000
0.7	0.15976	0.67114	0.67114	0.67161	0.67114
0.9	0.09296	0.55249	0.55248	0.55137	0.55249
1.0	0.07452	0.50000	0.49999	0.49894	0.50000
1.1	0.06112	0.45249	0.45249	0.45197	0.45249
1.3	0.04333	0.28021	0.37175	0.37242	0.37175
1.5	0.03235	0.18950	0.30769	0.30820	0.30770
1.9	0.02003	0.10925	0.21692	0.21648	0.21692
2.4	0.01250	0.06591	0.11136	0.14796	0.14793
3.0	0.00798	0.04133	0.06667	0.10028	0.10000

Table 3.3 : $f_N^B(-q^2)$ from $\delta_0(k^2), \delta_1(k^2), \ldots\ldots, \delta_{10}(k^2)$, the Born approximations to the Yukawa potential $V(r) = -\exp(-r)/r$.

q \ k	$f_E^B(-q^2)$ 0.2	0.6	1.0	2.0	$f^B(-q^2) = (1+q^2)^{-1}$
0.1	0.99010	0.99010	0.99009	0.98550	0.99010
0.2	0.96154	0.96154	0.96154	0.95887	0.96154
0.3	0.91743	0.91743	0.91744	0.91702	0.91743
0.5		0.79999	0.79999	0.80192	0.80000
0.7		0.67114	0.67114	0.67161	0.67114
0.9		0.55248	0.55248	0.55137	0.55249
1.0		0.49999	0.49999	0.49894	0.50000
1.1		0.45248	0.45249	0.45197	0.45249
1.3			0.37175	0.37242	0.37175
1.5			0.30769	0.30820	0.30770
1.9			0.21692	0.21648	0.21692
2.4				0.14796	0.14793
3.0				0.10028	0.10000

Table 3.4 : $f_E^B(-q^2)$ from $\delta_0(k^2)$, $\delta_1(k^2)$, ..., $\delta_{10}(k^2)$, the Born approximations to the potential $V(r) = -\exp(-r)/r$. $f_E^B(-q^2) \equiv 0$ for $q > 2k$.

ℓ	$V_\ell(1)$	Estimated $V_\ell(2)$		Exact $V_\ell(2)$
		Newton's Pot$^\ell$.	Even Pot$^\ell$.	
0	0.4024	0.2579	0.2012	0.2747
1	0.1035	0.1023	0.1265	0.0933
2	0.0318	0.0370	0.0428	0.0376
3	0.0104	0.0141	0.0049	0.0162
4	0.0035	0.0078	0.0059	0.0072
5	0.0012	0.0042	0.0093	0.0033
6	0.0004	0.0012	0.0021	0.0015
7	0.0002	0.0002	-0.0031	0.0007

Table 3.5 : Values of the Born approximations $V_\ell(k^2)$ at energy $k^2 = 2$ for the Newton-Sabatier potentials, compared with those for the Yukawa potential $V(r) = -\exp(-r)/r$, to which the former are fitted at energy $k^2 = 1$.

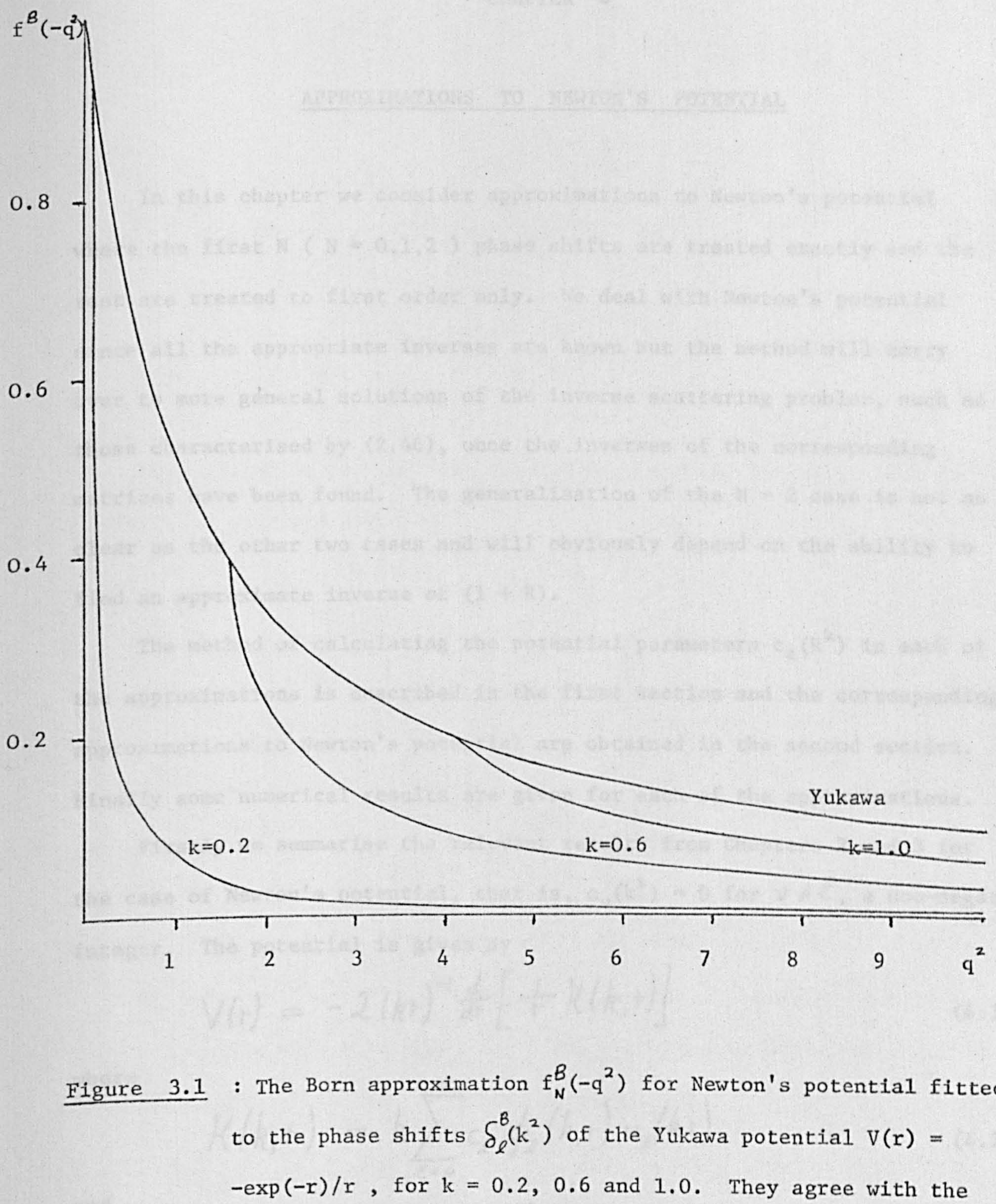

Figure 3.1 : The Born approximation $f_N^{B}(-q^2)$ for Newton's potential fitted to the phase shifts $\delta_\ell^{B}(k^2)$ of the Yukawa potential $V(r) = -\exp(-r)/r$, for k = 0.2, 0.6 and 1.0. They agree with the Yukawa curve, $f^{B}(-q^2) = (q^2 + 1)^{-1}$, for $q < 2k$.

CHAPTER 4

APPROXIMATIONS TO NEWTON'S POTENTIAL

In this chapter we consider approximations to Newton's potential where the first N (N = 0,1,2) phase shifts are treated exactly and the rest are treated to first order only. We deal with Newton's potential since all the appropriate inverses are known but the method will carry over to more general solutions of the inverse scattering problem, such as those characterised by (2.46), once the inverses of the corresponding matrices have been found. The generalisation of the N = 2 case is not as clear as the other two cases and will obviously depend on the ability to find an approximate inverse of (1 + R).

The method of calculating the potential parameters $c_\ell(k^2)$ in each of the approximations is described in the first section and the corresponding approximations to Newton's potential are obtained in the second section. Finally some numerical results are given for each of the approximations.

Firstly we summarise the relevant results from Chapters 2 and 3 for the case of Newton's potential, that is, $c_\nu(k^2) = 0$ for $\nu \neq \ell$, a non-negative integer. The potential is given by

$$V(r) = -2(kr)^{-1}\frac{d}{dr}\left[\frac{1}{r}K(k,r)\right] \tag{4.1}$$

where

$$K(k,r) = k\sum_{\ell=0}^{\infty} c_\ell\, \phi_\ell(k,r)\, u_\ell(kr) \tag{4.2}$$

and

$$\phi_\ell(k,r) = u_\ell(kr) - k\sum_{m=0}^{\infty} L_{\ell,m}(kr)\, c_m\, \phi_m(k,r) \tag{4.3}$$

with

$$L_{\ell,m}(z) = \int_0^z j_\ell(y)\, j_m(y)\, dy \tag{4.4}$$

The coefficients c_ℓ are such that

$$(kc_\ell)^{-1} = -\frac{\pi}{2(2\ell+1)} + \left(1 - \sum_{m=0}^{\infty} M_{\ell,m} \tan\delta_m a_m\right) a_\ell^{-1} \tag{4.5}$$

with

$$\sum_{m=0}^{\infty} M_{\ell,m} a_m + \sum_{m=0}^{\infty} \tan\delta_\ell M_{\ell,m} \tan\delta_m a_m = \tan\delta_\ell \tag{4.6}$$

and

$$M_{\ell,m} = \begin{cases} [(m-\ell)(m+\ell+1)]^{-1} & (m-\ell) \text{ odd} \\ 0 & (m-\ell) \text{ even} \end{cases} \tag{4.7}$$

From (3.13) we see that the inverse of M and the vector $\underset{\sim}{v}$ which is annihilated by M are given by

$$\begin{aligned} v_{2\ell} &= (4\ell+1)\, 16^{-\ell} \left[\binom{2\ell}{\ell}\right]^2 \\ v_{2\ell+1} &= 0 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} M^{-1}_{\ell,m} &= 0 \qquad\qquad (\ell-m) \text{ even} \\ M^{-1}_{2\ell,2m+1} &= -M^{-1}_{2m+1,2\ell} = T_{\ell,m}\, v_{2\ell}\, v_{2m} \end{aligned} \tag{4.9}$$

where

$$T_{\ell,m} = \frac{(4m+3)(2m+1)^2}{(4m+1)(2\ell-2m-1)(2\ell+2m+2)} \tag{4.10}$$

In order to simplify the analysis we use matrix notation with Δ the diagonal matrix such that $\Delta_{\ell,\ell} = \tan\delta_\ell$ and $\underset{\sim}{e}$ the column vector with all its elements unity. With this notation, (4.6) becomes

$$M\underset{\sim}{a} + \Delta M \Delta \underset{\sim}{a} = \Delta \underset{\sim}{e} \tag{4.11}$$

the general solution to which is

$$\underset{\sim}{a} = (1+R)^{-1} M^{-1} \Delta \underset{\sim}{e} + \lambda (1+R)^{-1} \underset{\sim}{v} \tag{4.12}$$

with

$$R = M^{-1} \Delta M \Delta \tag{4.13}$$

and λ a constant.

It is possible for some choices of Δ that the matrix (1 + R) will be singular, although we have not been able to construct such a Δ (with Δ real), and in the following sections we assume that (1 + R) is non-singular.

For Newton's potential, that is, the one which falls off faster than $r^{-2+\epsilon}$, we have from (3.19)

$$\lambda = \underset{\sim}{v}^T \Delta \left(\underset{\sim}{e} - M \Delta \underset{\sim}{a} \right) \tag{4.14}$$

which ensures that

$$\lim_{\ell \to \infty} \left(a_{2\ell+1} - a_{2\ell} \right) = 0 \tag{4.15}$$

Note that (4.14) involves only the odd a's which are given by (4.12) independently of λ , since $R_{\ell,m} = 0$ unless (ℓ - m) is even. Therefore λ is given explicitly and the problem of constructing Newton's potential consists essentially of inverting the matrix (1 + R) and solving the equations in (4.3).

4.1 The Potential Parameters

In this section we calculate the coefficients c_ℓ for Newton's potential in the approximations where N (= 0,1,2) phase shifts are treated exactly and the rest are treated to first order only.

N = 0 :

We consider all the phase shifts to be small, so that we are essentially working in the Born approximation. From (4.13) we see that R is second order small and may be neglected. Therefore

$$\underset{\sim}{a}^{(0)} = M^{-1} \Delta \underset{\sim}{e} + \lambda^{(0)} \underset{\sim}{v} \tag{4.16}$$

where

$$\lambda^{(0)} = \underset{\sim}{v}^T \Delta \underset{\sim}{e} \tag{4.17}$$

and hence, by (4.5)

$$k c_\ell^{(0)} = \left[-\frac{\pi}{2(2\ell+1)} + \frac{1}{a_\ell^{(0)}} \right]^{-1} \tag{4.18}$$

In each of these equations the errors are of order δ_ℓ^3, so that we are in fact working to second order in the phase shifts. To first order, (4.18) reduces to $k\ c_\ell^{(0)} = a_\ell^{(0)}$ which is the Born approximation described in Chapter 3.

This approximation satisfies the requirement (4.15) exactly, since, from (4.8)

$$\lim_{\ell \to \infty} V_{2\ell} = 4\pi^{-1} \tag{4.19}$$

and from (4.10)

$$\lim_{\ell \to \infty} T_{m,\ell} = -1$$

and

$$\lim_{\ell \to \infty} T_{\ell,m} = 0 \tag{4.20}$$

for fixed m.

<u>N = 1</u> :

We now treat δ_0 exactly and all the other phase shifts to first order. Since $M_{0,0} = 0$, we see from (4.13) that $R_{i,j} = \tan\delta_0 \sum_{m=1}^{\infty} d_m \tan\delta_m$, i.e. R is first order small and so $(1 + R)^{-1} = 1 - R$. Also, to first order, we see that

$$\Delta M \Delta M^{-1} \Delta \simeq \beta \Delta \tag{4.21}$$

where

$$\begin{aligned} \beta &= \tan\delta_0 \sum_{m=0}^{\infty} M_{0,2m+1} \tan\delta_{2m+1} M^{-1}_{2m+1,0} \\ &= \tan\delta_0 \left(\tfrac{3}{4} \tan\delta_1 + \tfrac{7}{64} \tan\delta_3 + \tfrac{11}{256} \tan\delta_5 + \cdots\cdots \right) \end{aligned} \tag{4.22}$$

a first order small quantity.

Therefore, from (4.12), we obtain for the odd a's (which do not depend on λ)

$$\left(\underset{\sim}{a}^{(1)} \right)_{ODD} = \left(1 - M^{-1} \Delta M \Delta \right) M^{-1} \Delta \underset{\sim}{e}$$

$$\simeq (1-\beta) M^{-1} \Delta \underset{\sim}{e} = (1-\beta) (\underset{\sim}{a}^{(0)})_{ODD} \tag{4.23}$$

and hence from (4.14)

$$\lambda^{(1)} \simeq \underset{\sim}{v}^T \Delta \underset{\sim}{e} - (1-\beta) \underset{\sim}{v}^T \Delta M \Delta M^{-1} \Delta \underset{\sim}{e}$$

$$\simeq \underset{\sim}{v}^T \Delta \underset{\sim}{e} - (1-\beta) \underset{\sim}{v}^T \beta \Delta \underset{\sim}{e} \tag{4.24}$$

From (4.22) we know that β is first order small and so

$$\lambda^{(1)} \simeq (1-\beta) \underset{\sim}{v}^T \Delta \underset{\sim}{e}$$

$$= (1-\beta) \lambda^{(0)} \tag{4.25}$$

Therefore the even a's are given by

$$(\underset{\sim}{a}^{(1)})_{EVEN} \simeq (1-\beta) M^{-1} \Delta \underset{\sim}{e} + \lambda^{(1)} (1 - M^{-1} \Delta M \Delta) \underset{\sim}{v}$$

$$\simeq (1-\beta) (\underset{\sim}{a}^{(0)})_{EVEN} - (1-\beta) \lambda^{(0)} M^{-1} \Delta M \Delta \underset{\sim}{v}$$

and hence

$$\underset{\sim}{a}^{(1)} \simeq (1-\beta) \underset{\sim}{a}^{(0)} - \tan \delta_0 \, R \underset{\sim}{v} \tag{4.26}$$

since $v_{2\ell+1} = 0$, $v_0 = 1$ and $R_{2\ell+1,2m} = R_{2m,2\ell+1} = 0$.

From the definition of R we see that

$$(R\underset{\sim}{v})_{2\ell} = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} M^{-1}_{2\ell,2m+1} \tan \delta_{2m+1} M_{2m+1,2j} \tan \delta_{2j}$$

$$= \tan \delta_0 \sum_{m=0}^{\infty} v_{2\ell} v_{2m} T_{\ell,m} \tan \delta_{2m+1} M_{2m+1,0} \tag{4.27}$$

to first order in $\tan \delta_j$, $j \neq 0$.

Introducing the vector $\underset{\sim}{m}$ equal to the first column of the matrix M, we see that (4.27) gives

$$R\underset{\sim}{v} = \tan \delta_0 M^{-1} \Delta \underset{\sim}{m} \tag{4.28}$$

and hence from (4.26)

$$M \underset{\sim}{a}^{(1)} = (1-\beta) \Delta \underset{\sim}{e} - \tan^2 \delta_0 \Delta \underset{\sim}{m} \tag{4.29}$$

which, with (4.11), yields

$$M \Delta \underset{\sim}{a}^{(1)} = \beta \underset{\sim}{e} + \tan^2 \delta_0 \underset{\sim}{m} \tag{4.30}$$

provided $\delta_\ell \neq 0$ for all ℓ.

Using (4.30) we see that the coefficients c_ℓ in (4.5) are given by

$$k c_\ell = \left[-\frac{\pi}{2(2\ell+1)} + (1-\beta-\tan^2\delta_0 \, M_{\ell,0})/a_\ell^{(1)} \right]^{-1} \tag{4.31}$$

which is valid even if $\delta_j = 0$ for some j.

We complete the approximation by removing the remaining negligible terms in (4.26), so that

$$a_{2\ell}^{(1)} = a_{2\ell}^{(0)} - \tan^2\delta_0 \, v_{2\ell} \sum_{m=0}^{\infty} M_{0,2m+1} \tan\delta_{2m+1} M^{-1}_{2m+1,0}$$
$$- \tan^2\delta_0 \sum_{m=0}^{\infty} M^{-1}_{2\ell,2m+1} \tan\delta_{2m+1} M_{2m+1,0}$$

i.e.

$$a_{2\ell}^{(1)} = a_{2\ell}^{(0)} + \tan^2\delta_0 \sum_{m=0}^{\infty} v_{2\ell} v_{2m} \tan\delta_{2m+1} \left[\frac{T_{\ell,m}}{(2m+1)(2m+2)} - \frac{(4m+3)}{(4m+1)(2m+2)^2} \right] \tag{4.32}$$

and

$$a_{2\ell+1}^{(1)} = a_{2\ell+1}^{(0)} - \tan^2\delta_0 \sum_{m=0}^{\infty} M_{0,2m+1} \tan\delta_{2m+1} M^{-1}_{2m+1,0} M^{-1}_{2\ell+1,0}$$

i.e.

$$a_{2\ell+1}^{(1)} = a_{2\ell+1}^{(0)} - \tan^2\delta_0 \sum_{m=0}^{\infty} v_{2\ell} v_{2m} \tan\delta_{2m+1} \frac{T_{0,m} T_{0,\ell}}{(2m+1)(2m+2)} \tag{4.33}$$

Using the limits (4.19) and (4.20) we readily see that (4.32) and (4.33) satisfy the requirement (4.15) and hence that the resultant potential will fall off faster than $r^{-2+\varepsilon}$.

<u>N = 2</u> :

In this approximation both δ_0 and δ_1 are treated exactly and we see from (4.13) that R is no longer small. However, we show that it is still possible to invert (1 + R) explicitly to first order in the remaining phase shifts.

Let $\bar{R}$ represent the matrix R when all the phase shifts except δ_0 and δ_1 are set to zero. We see from (4.13) that the only non-zero elements of $\bar{R}$ are

$$\bar{R}_{2\ell,0} = M^{-1}_{2\ell,1} \tan\delta_1 M_{1,0} \tan\delta_0$$

and

$$\bar{R}_{2\ell+1,1} = M^{-1}_{2\ell+1,0} \tan\delta_0 M_{0,1} \tan\delta_1 \tag{4.34}$$

Therefore

$$\bar{R}^2 = \alpha \bar{R} \tag{4.35}$$

where

$$\alpha = M_{0,1} M_{1,0}^{-1} \tan\delta_0 \tan\delta_1$$
$$= \tfrac{3}{4} \tan\delta_0 \tan\delta_1 \tag{4.36}$$

and it follows that

$$(1+\bar{R})^{-1} = 1 - (1+\alpha)^{-1} \bar{R} \tag{4.37}$$

We now set $R = \bar{R} + \epsilon$ and work to first order in ϵ, so that

$$(1+R)^{-1} = (1+\bar{R})^{-1} - (1+\bar{R})^{-1}\epsilon\,(1+\bar{R})^{-1} \tag{4.38}$$

To first order in δ_i ($i > 1$) we have from (4.13)

$$\begin{aligned}
\epsilon_{2j,2\ell+1} &= \epsilon_{2j+1,2\ell} = 0 \\
\epsilon_{2j,2\ell} &= M^{-1}_{2j,1} \tan\delta_1 M_{1,2\ell} \tan\delta_{2\ell} \qquad (\ell \neq 0) \\
\epsilon_{2j,0} &= \sum_{m=1}^{\infty} M^{-1}_{2j,2m+1} \tan\delta_{2m+1} M_{2m+1,0} \tan\delta_0 \\
\text{and}\quad \epsilon_{2j+1,2\ell+1} &= M^{-1}_{2j+1,0} \tan\delta_0 M_{0,2\ell+1} \tan\delta_{2\ell+1} \qquad (\ell \neq 0) \\
\epsilon_{2j+1,1} &= \sum_{m=1}^{\infty} M^{-1}_{2j+1,2m} \tan\delta_{2m} M_{2m,1} \tan\delta_1
\end{aligned} \tag{4.39}$$

From (4.34) and (4.39) we see that the only non-zero elements of $\bar{R}\epsilon\bar{R}$ are

$$(\bar{R}\epsilon\bar{R})_{2\ell,0} = \sum_{m=0}^{\infty} \bar{R}_{2\ell,0}\, \epsilon_{0,2m} \bar{R}_{2m,0}$$
$$= \bar{R}_{2\ell,0} (\epsilon_{0,0}\alpha + \epsilon_{1,1}\alpha)$$

and

$$(\bar{R}\epsilon\bar{R})_{2\ell+1,1} = \sum_{m=0}^{\infty} \bar{R}_{2\ell+1,1}\, \epsilon_{1,2m+1} \bar{R}_{2m+1,0}$$
$$= \bar{R}_{2\ell+1,1} (\epsilon_{0,0}\alpha + \epsilon_{1,1}\alpha)$$

and so

$$\bar{R}\epsilon\bar{R} = \alpha(\epsilon_{0,0} + \epsilon_{1,1})\bar{R} \tag{4.40}$$

Therefore, by virtue of (4.40), it is possible to rewrite (4.38) in the simpler form

$$
\begin{aligned}
(1+R)^{-1} &= 1 - \epsilon + (1+\alpha)^{-1}\left[\bar{R}\epsilon + \epsilon\bar{R} - \bar{R}\right] \\
&\qquad - \alpha(1+\alpha)^{-2}(\epsilon_{0,0} + \epsilon_{1,1})\bar{R} \\
&\simeq 1 - \epsilon + (1+\alpha)^{-1}\left[R^2 - \alpha\bar{R} - \bar{R}\right] \\
&\qquad - \alpha(1+\alpha)^{-2}(\epsilon_{0,0} + \epsilon_{1,1})\bar{R}
\end{aligned}
\tag{4.41}
$$

i.e.

$$
\begin{aligned}
(1+R)^{-1} &\simeq 1 - \left[1 + \alpha(1+\alpha)^{-2}(\epsilon_{0,0} + \epsilon_{1,1})\right]R \\
&\qquad + (1+\alpha)^{-1}R^2
\end{aligned}
\tag{4.42}
$$

since $\epsilon_{0,0}$ and $\epsilon_{1,1}$ are first order in δ_ℓ, $\ell > 1$.

In proceeding from (4.41) to (4.42) we have introduced terms of order ϵ^2 and although (4.42) is easier to handle than (4.41), it can lead to a set of a's which violate (4.15) and consequently a potential which has an oscillating $r^{-\frac{3}{2}}$ tail of order ϵ^2.

Having obtained $(1 + R)^{-1}$ we can now determine the odd a's directly from (4.12). Straightforward but tedious analysis gives the odd a's explicitly as

$$
a^{(2)}_{2\ell+1} = (1+\alpha)^{-1} M^{-1}_{2\ell+1,0} \tan\delta_0 + \sum_{m=2}^{\infty} \gamma_{2\ell+1,m} \tan\delta_m \tag{4.43}
$$

where the γ's are given by

$$
\gamma_{2\ell+1,2m+1} = -(1+\alpha)^{-2} \tan^2\delta_0 \, M^{-1}_{2\ell+1,0} M_{0,2m+1} M^{-1}_{2m+1,0}
$$

and

$$
\gamma_{2\ell+1,2m} = \left[1 - \frac{2\alpha M_{2m,1}}{(1+\alpha)}\right]\left[M^{-1}_{2\ell+1,2m} - \frac{2\alpha M^{-1}_{2\ell+1,0} M^{-1}_{1,2m}}{3(1+\alpha)}\right] \tag{4.44}
$$

The odd a's can now be inserted in (4.14) to give the approximation to λ as

$$
\lambda^{(2)} = (1+\alpha)^{-1} \tan\delta_0 + \sum_{m=2}^{\infty} \lambda^{(2)}_m \tan\delta_m \tag{4.45}
$$

with

$$
\lambda^{(2)}_{2m+1} = -(1+\alpha)^{-2} \tan^2\delta_0 \, M_{0,2m+1} M^{-1}_{2m+1,0}
$$

and

$$
\lambda^{(2)}_{2m} = \left[1 - \frac{2\alpha M_{2m,1}}{(1+\alpha)}\right]\left[V_{2m} - \frac{2\alpha M^{-1}_{1,2m}}{3(1+\alpha)}\right] \tag{4.46}
$$

Finally, using (4.45) and (4.12) we obtain the even a's as

$$a_{2\ell}^{(2)} = (1+\alpha)^{-2} M_{2\ell,1}^{-1}\left[\tfrac{2}{3}\alpha \tan\delta_0 + (1+\alpha)\tan\delta_1\right] + (1+\alpha)^{-1}\tan\delta_0\, v_{2\ell} + \sum_{m=2}^{\infty} \gamma_{2\ell,m} \tan\delta_m \tag{4.47}$$

where the $\gamma_{2\ell,m}$ are explicit, though complicated, expressions in $\tan\delta_0$, $\tan\delta_1$, M, M^{-1}, and $\underset{\sim}{v}$ but not involving any summations.

In these forms, equations (4.43) and (4.47) explicitly show the terms corresponding to the case where $\delta_\ell = 0$ for all $\ell > 1$.

As the $\gamma_{i,k}$ are tedious to calculate it is often more convenient in practice to keep some of the terms of order ϵ^2 rather than working exactly to order ϵ. For instance, it is computationally easier to use expression (4.42) to calculate $(1 + R)^{-1}$ than it is to use (4.41), although as mentioned earlier this can lead to a set of a's violating (4.15).

Having obtained the a's from (4.43) and (4.47) it is a simple process to complete the approximation by determining the c's from (4.5).

4.2 The Approximate Potentials

Having calculated the coefficients c_ℓ to the desired accuracy as described in the previous section, we must now solve equation (4.3) to the same accuracy in order that an approximation to Newton's potential can be determined from (4.2) and (4.1).

For Newton's potential we see from (4.5) that $c_\ell \to$ constant, as $\ell \to \infty$, since the a's were constructed such that condition (4.15) was satisfied. As $\phi_\ell = u_\ell + O(\delta_\ell)$ we have, from (4.3),

$$\sum_{m=0}^{\infty} L_{\ell,m}(kr)\, c_m\, \phi_m(k,r) = O(\delta_\ell) \tag{4.48}$$

and as $m \to \infty$, for fixed ℓ and kr, $L_{\ell,m}(kr) \sim (mkr)^{-1}\left(\frac{ekr}{2m+1}\right)^{m+1}$.

Therefore, since in these approximations we are working to first order in δ_m for $m \geqslant N$, we can replace (4.3) by

$$\phi_\ell = u_\ell - k\sum_{m=0}^{N-1} L_{\ell,m}\, c_m\, \phi_m - k\sum_{m=N}^{\infty} L_{\ell,m}\, c_m\, u_m \tag{4.49}$$

This gives a set of N linear equations for $\phi_0, \phi_1, \ldots, \phi_{N-1}$ which then explicitly give ϕ_ℓ for $\ell \geqslant N$.

Substituting (4.49) in (4.2) gives

$$K(k,r) = K_0 - k\sum_{\ell=0}^{N-1} c_\ell S_\ell (\phi_\ell - u_\ell) \tag{4.50}$$

where

$$S_\ell = k\sum_{m=0}^{\infty} L_{\ell,m} c_m u_m \tag{4.51}$$

and

$$K_0 = k\sum_{\ell=0}^{\infty} c_\ell u_\ell (u_\ell - S_\ell) \tag{4.52}$$

From (4.49) we see that the ϕ_ℓ for $\ell < N$ are given by

$$\sum_{m=0}^{N-1} (\delta_{\ell,m} + L_{\ell,m} c_m)(\phi_m - u_m) + S_\ell = 0 \tag{4.53}$$

and hence K(k,r) can be obtained for any value of r by solving (4.53) for $\phi_\ell - u_\ell$ ($\ell < N$) and performing the summation in (4.50).

Although this process can be performed for any N we are here only concerned with the cases N = 0, 1 and 2 corresponding to the approximations to the c's given in the previous section.

<u>N = 0</u> :

When all the phase shifts are small (4.50) becomes

$$K(k,r) = K_0 \tag{4.54}$$

In this case all the c's are small, so that we see that the errors in (4.54) are of order δ_ℓ^3 ,as were the errors in the corresponding approximations (4.18). In fact the Born approximation to Newton's potential described in Chapter 3 was obtained by using

$$K(k,r) = k\sum_{\ell=0}^{\infty} c_\ell u_\ell^2 \tag{4.55}$$

<u>N = 1</u> :

Here all the phase shifts except δ_0 are considered to be small and ϕ_0 is given, by (4.53), as

$$(1 + L_{0,0} c_0)(\phi_0 - u_0) + \delta_0 = 0 \tag{4.56}$$

and hence from (4.50)

$$K(k,r) = K_0 + k c_0 \delta_0^2 [1 + L_{0,0} c_0]^{-1} \tag{4.57}$$

N = 2 :

When δ_0 and δ_1 are the only phase shifts which are not small, we obtain from (4.53)

$$\phi_0 - u_0 = [\delta_1 L_{0,1} c_1 - \delta_0 (1 + L_{1,1} c_1)] D$$

and

$$\phi_1 - u_1 = [\delta_0 L_{0,1} c_0 - \delta_1 (1 + L_{0,0} c_0)] D \tag{4.58}$$

where

$$D = [(1 + L_{0,0} c_0)(1 + L_{1,1} c_1) - L_{0,1}^2 c_0 c_1]^{-1} \tag{4.59}$$

Therefore (4.50) gives

$$K(k,r) = K_0 + k[(1 + L_{0,0} c_0) c_1 \delta_1^2 + (1 + L_{1,1} c_1) c_0 \delta_0^2 - 2 L_{0,1} c_0 c_1 \delta_0 \delta_1] D \tag{4.60}$$

For each of these approximations it is a straightforward process to obtain the potential from K(k,r) via (4.1); for instance, by using numerical differentiation techniques.

As an illustration of these three approximations to Newton's potential we consider the example of the Yukawa potential $V(r) = -\exp(-r)/r$. The input phase shifts are taken as those of the Yukawa at energy $k^2 = 1$ and the resultant potentials are calculated for various values of r. In Tables 4.1 and 4.2 we give results for r in the range 0.1 to 2.2 and 2.8 to 10.0 respectively. As can be seen from these results the approximate potentials oscillate about the true values and that the results become progressively better as more phase shifts are treated exactly. The exact values for the input phase shifts used in these calculations were derived from results of Stern (1969) and they have the values :-

$$\tan \delta_0(1) = 0.52058$$

$$\tan \delta_1(1) = 0.11209$$

$$\tan \delta_2(1) = 0.03271$$

$$\tan \delta_3(1) = 0.01055$$

and $$\tan \delta_\ell(1) = \tan \delta_\ell^B(1) \qquad \text{for } \ell > 3$$

where $\delta_\ell^B(k)$ is the Born approximation to the ℓ'th phase shift.

r	Approximations to -V(r)			exp(-r)/r
	N=0	N=1	N=2	
0.1	7.03925	13.96851	14.22785	9.04837
0.2	3.19689	5.92616	6.01806	4.09365
0.3	1.90939	3.34484	3.38875	2.46939
0.4	1.26473	2.11490	2.14008	1.67580
0.5	0.88035	1.41732	1.43515	1.21306
0.6	0.62856	0.98089	0.99679	0.91469
0.7	0.45440	0.69049	0.70727	0.70941
0.8	0.33016	0.48927	0.50837	0.56166
0.9	0.24019	0.34612	0.36818	0.45174
1.0	0.17488	0.24264	0.26781	0.36788
1.2	0.09478	0.11250	0.14312	0.25100
1.4	0.05798	0.04501	0.07917	0.17614
1.6	0.04638	0.01360	0.04884	0.12619
1.8	0.04846	0.00314	0.03712	0.09183
2.0	0.05646	0.00429	0.03498	0.06767
2.2	0.06526	0.01099	0.03674	0.05037

Table 4.1 : Approximations to Newton's potential obtained from the phase shifts to the potential $V(r) = -\exp(-r)/r$ at energy $k^2 = 1.0$, when N (=0,1,2) phase shifts are treated exactly and the rest are considered to first order only.

r	Approximations to $-V(r)\times10^2$			$10^2 \frac{e^{-r}}{r}$
	N=0	N=1	N=2	
2.8	7.18696	3.22779	3.71191	2.17179
3.2	5.61272	3.51266	2.57550	1.27382
3.6	3.28980	2.87038	1.03011	0.75899
4.0	1.21701	1.72359	-0.17899	0.45789
4.4	0.57386	0.57386	-0.63796	0.27903
4.8	-0.16198	-0.17323	-0.40412	0.17145
5.2	0.28152	-0.36616	0.15625	0.10609
5.6	0.84732	-0.13921	0.06298	0.06603
6.0	1.12694	0.22098	0.75777	0.04131
6.4	0.98364	0.46057	0.51895	0.02596
6.8	0.54763	0.46762	0.09145	0.01638
7.2	0.08407	0.27984	-0.27293	0.01037
7.8	-0.17652	0.02684	-0.39622	0.00658
8.0	-0.15368	-0.15143	-0.25903	0.00419
8.4	0.06905	-0.17827	0.01586	0.00268
8.8	0.31773	-0.07217	0.25349	0.00171
9.2	0.43694	0.07859	0.32702	0.00110
9.6	0.37186	0.18071	0.21965	0.00071
10.0	0.17876	0.18375	0.01449	0.00045

Table 4.2 : Approximations to Newton's potential obtained from the phase shifts to the potential $V(r) = -\exp(-r)/r$ at energy $k^2 = 1.0$, when N (=0,1,2) phase shifts are treated exectly and the rest are considered to first order only.

CHAPTER 5

THE BORN APPROXIMATION TO ON-SHELL AMPLITUDES

In this chapter relations between the Born approximations to the on-shell partial-wave amplitudes are derived. These relations can then, in principle, be used to determine the off-shell amplitudes as described in Chapter 6.

By assuming that the potential satisfies $\int_0^\infty |V(r)| r^2 dr < \infty$, we show that knowledge of one partial-wave amplitude (in Born approximation) at all energies implies knowledge of all others at all energies, and that knowledge of all these amplitudes at any one energy implies knowledge of them all at lower energies. These relations are derived in Section 5.1 and 5.2.

Upon restricting the potential further, such that it is short-range, we show in Section 5.3 that a low-energy series expansion of the amplitudes can be obtained which can be used as a parameterisation of the potential. It is shown that this series expansion leads to an expression for $V_\ell(k^2)$ in terms of $V_m(q^2)$ which converges for all $k^2 \leqslant q^2$. This expression, which is derived independently in Section 5.2 and shown to converge for a short distance outside $k^2 \leqslant q^2$ for short-range potentials, is used to evaluate $V_\ell(k^2)$ in this extended region for Yukawa potentials of range 1 and 1/7.

As the methods described in the first three sections can in general only be used to evaluate the partial-wave amplitudes at lower energies, we investigate a 'high energy' expansion of the amplitudes in Section 5.4. This expansion, which is valid for any potential of the form (5.68), is shown to converge very rapidly for Yukawa potentials and that small errors in the input $V_m(q^2)$ have little effect on the calculated $V_\ell(k^2)$.

5.1 Relations between Born approximations for different angular momenta

In this section we show that for any potential satisfying

$$\int_0^\infty |V(r)|\, r^2\, dr < \infty$$

the on-shell Born approximations $V_\ell(k^2)$ can be expressed in terms of an integral of any other $V_m(q^2)$, the integral running from $q^2 = 0$ to k^2 if $\ell > m$ and from $q^2 = k^2$ to ∞ if $\ell < m$.

On-shell, the Born approximation to the scattering amplitude is given in terms of the partial-wave amplitudes by

$$f^B\left(-2k^2(1-\mu)\right) = \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(k^2)\, P_\ell(\mu) \tag{5.1}$$

and so

$$\begin{aligned} \frac{\partial}{\partial k^2} f^B &= -2(1-\mu)\,(f^B)' \\ &= \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell'(k^2)\, P_\ell(\mu) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{\partial}{\partial \mu} f^B &= 2k^2\,(f^B)' \\ &= \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(k^2)\, P_\ell'(\mu) \end{aligned} \tag{5.3}$$

where

$$(f^B)' = \frac{d}{dx} f^B(x) \tag{5.4}$$

Therefore

$$(\mu+1)\, k^2 \frac{\partial}{\partial k^2} f^B = (\mu^2-1) \frac{\partial}{\partial \mu} f^B \tag{5.5}$$

or in terms of the partial-wave amplitudes

$$(\mu+1)\, k^2 \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell'(k^2)\, P_\ell(\mu) = (\mu^2-1) \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(k^2)\, P_\ell'(\mu) \tag{5.6}$$

which can be written as

$$\begin{aligned} k^2 \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell'(k^2)\, P_\ell(\mu) &+ k^2 \sum_{\ell=0}^{\infty} V_\ell'(k^2)\left[(\ell+1) P_{\ell+1}(\mu) + \ell P_{\ell-1}(\mu)\right] \\ &= \sum_{\ell=0}^{\infty} V_\ell(k^2)\, \ell(\ell+1)\left[P_{\ell+1}(\mu) - P_{\ell-1}(\mu)\right] \end{aligned} \tag{5.7}$$

since

$$(2\ell+1)(1-\mu^2)\,P_\ell'(\mu) = \ell(\ell+1)\left[P_{\ell-1}(\mu) - P_{\ell+1}(\mu)\right]$$

and

$$(2\ell+1)\mu\,P_\ell(\mu) = (\ell+1)\,P_{\ell+1}(\mu) - \ell\,P_{\ell-1}(\mu) \tag{5.8}$$

for $\ell \geqslant 0$.

The orthogonality of the Legendre polynomials enables the coefficients of $P_\ell(\mu)$ to be equated, giving

$$k^2\left[(2\ell+1)V_\ell'(k^2) + \ell V_{\ell-1}'(k^2) + (\ell+1)V_{\ell+1}'(k^2)\right]$$
$$= \ell(\ell-1)V_{\ell-1}(k^2) - (\ell+1)(\ell+2)V_{\ell+1}(k^2) \tag{5.9}$$

that is

$$(\ell+1)W_\ell = -\ell W_{\ell-1} \tag{5.10}$$

with

$$W_\ell = k^2 V_{\ell+1}'(k^2) + k^2 V_\ell'(k^2) + (\ell+2)V_{\ell+1}(k^2) - \ell V_\ell(k^2) \tag{5.11}$$

Therefore, by repeated application of (5.10), we obtain $(\ell+1)W_\ell = 0$ for $\ell \geqslant 0$ and hence

$$k^{2\ell+2}\left[\frac{d}{dk^2}\left(k^{-2\ell}V_\ell(k^2) + k^{-2\ell}V_{\ell+1}(k^2)\right)\right] + (2\ell+2)V_{\ell+1}(k^2) = 0 \tag{5.12}$$

which, when integrated from k^2 to ∞, gives

$$V_\ell(k^2) = (2\ell+2)k^{2\ell}\int_{k^2}^{\infty} q^{-2\ell-2}\,V_{\ell+1}(q^2)\,dq^2 - V_{\ell+1}(k^2) \tag{5.13}$$

for $\ell \geqslant 0$, since $V_\ell(q^2) \to 0$ as $q^2 \to \infty$.

Similarly, rewriting (5.11) in the form

$$\frac{d}{dk^2}\left[k^{2\ell+2}V_{\ell-1}(k^2) + k^{2\ell+2}V_\ell(k^2)\right] = 2\ell\,k^{2\ell}\,V_{\ell-1}(k^2) \tag{5.14}$$

and integrating from 0 to k^2, gives

$$V_\ell(k^2) = 2\ell\,k^{-2\ell-2}\int_0^{k^2} q^{2\ell}\,V_{\ell-1}(q^2)\,dq^2 - V_{\ell-1}(k^2) \tag{5.15}$$

for $\ell \geqslant 1$, since $k^2V_\ell(k^2) \to 0$ as $k^2 \to 0$.

These equations may be generalised to give

$$V_\ell(k^2) = (\ell+m+1)\, k^{2\ell} \int_{k^2}^{\infty} q^{-2\ell-2} P_{m-\ell-1}^{(2\ell+1,1)}\left(1-\frac{2k^2}{q^2}\right) V_m(q^2)\, dq^2$$

$$+ (-1)^{m-\ell} V_m(k^2) \qquad (0 \leqslant \ell < m) \tag{5.16}$$

and

$$V_\ell(k^2) = (\ell+m+1)\, k^{-2m-4} \int_{0}^{k^2} q^{2m+2} P_{\ell-m-1}^{(2m+1,1)}\left(1-\frac{2q^2}{k^2}\right) V_m(q^2)\, dq^2$$

$$+ (-1)^{m-\ell} V_m(k^2) \qquad (0 \leqslant m < \ell) \tag{5.17}$$

where the Jacobi polynomials $P_n^{(a,b)}(x)$ are given by

$$P_n^{(a,b)}(x) = \frac{(a+n)!}{(a+b+n)!} \sum_{\ell=0}^{n} \frac{(a+b+n+\ell)!\,(x-1)^\ell}{\ell!\,(n-\ell)!\,(a+\ell)!\,2^\ell} \tag{5.18}$$

The proof of (5.16) is by induction on ℓ. The validity of (5.16) for $m = \ell+1$ is established by (5.13) and we assume that (5.16) holds for a given $\ell < m$. Therefore for $\ell - 1$ we have from (5.13), (5.16) and (5.18)

$$V_{\ell-1}(k^2) = 2\ell(\ell+m+1)\, k^{2\ell-2} \int_{k^2}^{\infty}\int_{q^2}^{\infty} \frac{V_m(p^2)}{p^{2\ell+2}} \sum_{n=0}^{m-\ell-1} \frac{q^{2n}\,(m+\ell+n+1)!\,(-1)^n\, dp^2\, dq^2}{p^{2n}\, n!\,(m-\ell-n-1)!\,(2\ell+n+1)!}$$

$$+ 2\ell\, k^{2\ell-2} \int_{k^2}^{\infty} (-1)^{m-\ell} q^{-2\ell} V_m(q^2)\, dq^2 - V_\ell(k^2)$$

$$= 2\ell\, k^{2\ell-2} \sum_{n=0}^{m-\ell-1} \frac{(m+\ell+n+1)!\,(-1)^n}{(n+1)!\,(m-\ell-n-1)!\,(2\ell+n+1)!} \Bigg[\int_{k^2}^{\infty} q^{-2\ell} V_m(q^2)\, dq^2$$

$$- k^{2n+2} \int_{k^2}^{\infty} q^{-2(n+\ell+1)} V_m(q^2)\, dq^2 \Bigg] - V_\ell(k^2)$$

$$+ 2\ell\, k^{2\ell-2} \int_{k^2}^{\infty} (-1)^{m-\ell} q^{-2\ell} V_m(q^2)\, dq^2 \qquad (m \geqslant \ell+1) \tag{5.19}$$

after integrating by parts. Use of (5.16) and the identity (Abramowitz and Stegun (1968) ; 22.4.1)

$$P_{m-\ell}^{(2\ell,0)}(-1) = \sum_{p=0}^{m-\ell} (-1)^p \frac{(\ell+m+p)!}{p!\,(2\ell+p)!\,(m-\ell-p)!} = (-1)^{m-\ell} \qquad (m \geqslant \ell) \tag{5.20}$$

then gives

$$
\begin{aligned}
V_{\ell-1}(k^2) &= k^{2\ell-2}\int_{k^2}^{\infty} q^{-2\ell}\, V_m(q^2)\Bigg[\frac{(m+\ell)!}{(m-\ell)!\,(2\ell-1)!} \\
&\qquad - \sum_{n=0}^{m-\ell-1}\frac{(m+\ell+n+1)!\,(-1)^n}{(n+1)!\,(m-\ell-n-1)!\,(2\ell+n)!}\left(\frac{k^2}{q^2}\right)^{n+1}\Bigg]dq^2 + (-1)^{m-\ell+1}V_m(k^2) \\
&= (\ell+m)\,k^{2\ell-2}\int_{k^2}^{\infty} q^{-2\ell}\, P_{m-\ell}^{(2\ell-1,1)}\left(1-\frac{2k^2}{q^2}\right)V_m(q^2)\,dq^2 \\
&\qquad + (-1)^{m-\ell+1}\,V_m(k^2)
\end{aligned}
\tag{5.21}
$$

thereby completing the proof of (5.16).

The proof of (5.17) can be performed similarly, or by noting that (5.15) is obtained from (5.13) (and (5.17) from (5.16)) under the transformation

$$
\begin{aligned}
\ell &\to -\ell-1 \\
m &\to -m-1 \\
\int_{k^2}^{\infty} &\to \int_{k^2}^{0}
\end{aligned}
\tag{5.22}
$$

and that

$$
P_{\ell-m-1}^{(-2\ell-1,1)}\left(1-\frac{2k^2}{q^2}\right) = \left(\frac{k^2}{q^2}\right)^{\ell-m-1} P_{\ell-m-1}^{(2m+1,1)}\left(1-\frac{2q^2}{k^2}\right) \qquad (\ell>m) \tag{5.23}
$$

which can be proved directly from (5.18).

From (5.16) we see that as $k^2 \to 0$, $V_\ell(k^2) = O(k^{2\ell})$ for short-range potentials, and so the equations exhibit the appropriate threshold behaviour for these types of potentials. The consistency of (5.17) with this threshold behaviour is assured by virtue of the identity

$$
\sum_{p=0}^{\ell-m-1}(-1)^{\ell-m+p+1}\frac{(\ell+m+p+1)!}{p!\,(\ell-m-1-p)!\,(2m+p+1)!\,(m+p+r+2)} = 1 \qquad (m \le r < \ell) \tag{5.24}
$$

which, for $r = m$, follows directly from (5.18). The validity of (5.24) for $m < r < \ell$ can be seen by considering

$$
I_r = \int_0^1 \lambda^{r-m}(1-x)^{\ell-m-1}\frac{d}{d\lambda}(1-\lambda x)^{m-\ell}\,d\lambda \tag{5.25}
$$

Expanding (5.25) as a series in x we see that the coefficient of $x^{\ell+m+1}$ in (5.25) is the same as the left hand side of (5.24). Integration by parts of (5.25) and use of the recurrence relation (Gradshteyn (1965) ; 2.111.2)

$$\int \frac{\lambda^n \, d\lambda}{(1-x\lambda)^m} = \frac{n}{(n-m+1)x} \int \frac{\lambda^{n-1} \, d\lambda}{(1-x\lambda)^{m-1}} - \frac{\lambda^n}{(1-x\lambda)^{m-1}(n-m+1)x} \tag{5.26}$$

then shows that the coefficient of $x^{\ell+m+1}$ is unity, thus proving (5.24).

Although (5.16) can in principle be used to obtain any $V_\ell(k^2)$ from $V_m(q^2)$ for $m > \ell$, it involves an integral from $q^2 = k^2$ to ∞ and this high-energy behaviour of $V_m(q^2)$ is usually unknown. Therefore we now show that, for any short-range potential, the integral in (5.16) can be replaced by an integral running from $q^2 = 0$ to k^2 at the expense of introducing a polynomial in k^2 with $m - \ell$ undetermined coefficients.

Since $V_\ell(k^2) = O(k^{2\ell})$ as $k^2 \to 0$, for any short-range potential, we define

$$\beta_\ell = \lim_{k^2 \to 0} k^{-2\ell} V_\ell(k^2) \tag{5.27}$$

and so it follows from (5.16) that

$$\beta_\ell = (2\ell+2)\binom{\ell+m+1}{m-\ell-1} \int_0^\infty q^{-2\ell-2} V_m(q^2)\, dq^2 \qquad (0 \leq \ell < m) \tag{5.28}$$

Therefore

$$(m+\ell+1)\, k^{2\ell} \int_0^\infty q^{-2\ell-2} P^{(2\ell+1,1)}_{m-\ell-1}\left(1 - \frac{2k^2}{q^2}\right) V_m(q^2)\, dq^2$$

$$= \sum_{n=0}^{m-\ell-1} (-1)^n k^{2\ell+2n} \frac{(\ell+m+n+1)!}{n!\,(2\ell+n+1)!\,(m-\ell-n-1)!} \int_0^\infty q^{-2(\ell+n+1)} V_m(q^2)\, dq^2$$

$$= \sum_{n=\ell}^{m-1} \beta_n k^{2n} (-1)^{n-\ell} \binom{2n+1}{n-\ell} \tag{5.29}$$

where we have used (5.18); and hence (5.16) can be recast as

$$V_\ell(k^2) = \sum_{n=\ell}^{m-1} (-1)^{n-\ell} \binom{2n+1}{n-\ell} \beta_n k^{2n} + (-1)^{m-\ell} V_m(k^2)$$
$$- (\ell+m+1)\, k^{2\ell} \int_0^{k^2} q^{-2\ell-2} P^{(2\ell+1,1)}_{m-\ell-1}\left(1 - \frac{2k^2}{q^2}\right) V_m(q^2)\, dq^2 \tag{5.30}$$

for $0 \leq \ell < m$.

The last term on the right hand side of (5.30) is of order k^{2m} and so the first term gives the first $m - \ell$ terms in the series expansion of $V_\ell(k^2)$. This series expansion of $V_\ell(k^2)$ is discussed further in Section 5.3 where a generalisation of (5.30) is also derived.

It should be noted that by definition the constants β_n are independent of ℓ and hence can be taken as parameters specifying the potential.

5.2 Relations between Born approximations at different energies

In the previous section we expressed $V_\ell(k^2)$ in terms of $V_m(q^2)$ for a given m and all values of q^2. In practice we do not know the values of $V_m(q^2)$ for large q^2, in fact this is a region where potential theory is not applicable. Therefore it is useful to express $V_\ell(k^2)$ in terms of $V_m(q^2)$ at a given energy q^2 and for all m, since these fall off rapidly with increasing m and we need not consider large q^2. We show in this section that $V_\ell(k^2)$ for all ℓ and $k^2 \leqslant q^2$ can be deduced from all the $V_m(q^2)$ at energy q^2.

Using (5.1) and setting

$$\mu' = 1 - (1-\mu)k^2/q^2 \tag{5.31}$$

we have

$$\begin{aligned} \sum_{\ell=0}^{\infty}(2\ell+1)V_\ell(q^2)P_\ell(\mu') &= f^B(-2q^2(1-\mu')) \\ &= f^B(-2k^2(1-\mu)) \\ &= \sum_{\ell=0}^{\infty}(2\ell+1)V_\ell(k^2)P_\ell(\mu) \end{aligned} \tag{5.32}$$

The orthogonality of the Legendre polynomials then yields

$$V_\ell(k^2) = \sum_{m=\ell}^{\infty} R_{\ell,m}(k^2,q^2)\,V_m(q^2) \tag{5.33}$$

where

$$\begin{aligned} R_{\ell,m}(k^2,q^2) &= (m+\tfrac{1}{2})\int_{-1}^{1}P_\ell(\mu)\,P_m(\mu')\,d\mu \\ &= \frac{(2m+1)}{(m+\ell+1)}\left(\frac{k^2}{q^2}\right)^{\ell} P_{m-\ell}^{(2\ell+1,-1)}\left(1-\frac{2k^2}{q^2}\right) \\ &= (-1)^{\ell}\sum_{n=0}^{m-\ell}\left(-\frac{k^2}{q^2}\right)^{n+\ell}\frac{(2m+1)(\ell+m+n)!}{n!\,(m-\ell-n)!\,(2\ell+n+1)!} \end{aligned} \tag{5.34}$$

The proof of (5.34) is given in Appendix 1 but an independent derivation of this result and (5.33) are given in Section 5.3. The justification for interchanging the order of integration and summation in obtaining (5.33) from (5.32) is that (5.33) is absolutely convergent for any potential such that $\int_0^\infty |V(r)| r^2 dr < \infty$. This can be seen since, from (5.34), $|R_{\ell,m}| \leq (2m+1)$ and hence

$$\begin{aligned}
\sum_{m=\ell}^{\infty} \left| R_{\ell,m}(k^2, q^2)\, V_m(q^2) \right| &\leq \sum_{m=\ell}^{\infty} (2m+1) \left| V_m(q^2) \right| \\
&\leq \sum_{m=0}^{\infty} (2m+1) \int_0^\infty j_m^2(kr)\, |V(r)|\, r^2 dr \\
&\leq \int_0^\infty |V(r)|\, r^2 dr
\end{aligned}$$

using the properties of the $j_m(x)$ described in Chapter 1.

From the asymptotic behaviour of the Jacobi polynomials (Szegö (1937) ; 8.21.18) we see that the $R_{\ell,m}$ fall off as $m^{-\frac{1}{2}}$ as $m \to \infty$ for $k^2 < q^2$, thereby ensuring the rapid convergence of (5.33). For $k^2 > q^2$, the $R_{\ell,m}$ diverge as $m^{-\frac{1}{2}} \exp(\gamma m)$ where $\cosh\gamma = -1 + 2k^2/q^2$ (Szegö (1937) ; 8.21.7) and so the series (5.33) will diverge unless the $V_m(q^2)$ fall off faster than $\exp(-\gamma m)$. From the asymptotic behaviour of the Legendre functions of the second kind

$$Q_\ell(x) \sim \sqrt{\tfrac{\pi}{2}}\, (\ell \sinh\eta)^{-\frac{1}{2}} \exp\left[-(\ell+\tfrac{1}{2})\eta\right] \tag{5.35}$$

as $\ell \to \infty$, with $\cosh\eta = x$, we see that for any short-range potential of range α the $V_\ell(q^2)$ will have the asymptotic form

$$V_\ell(q^2) \sim \exp(-\ell\eta) \tag{5.36}$$

with $\cosh\eta = 1 + 1/(2\alpha^2 q^2)$, and hence that (5.33) will converge provided

$$k^2 < q^2 + (2\alpha)^{-2} \tag{5.37}$$

Therefore it is only for potentials which have a very short range that (5.33) will converge for any significant distance outside the region $q^2 \geqslant k^2$. In Chapter 1 we showed that the $V_\ell(k^2)$ for a potential of range α^{-1} were analytic in the strip $|\mathrm{Im}\, k| < \frac{\alpha}{2}$. Therefore for any such potential we can, in principle, obtain a continuation of $V_\ell(q^2)$ to arbitrarily large k^2 by repeated application of (5.33) for $q^2 < k^2 < q^2 + (2\alpha)^{-2}$. In practice this breaks down since any errors introduced at one step of the process are increased in the subsequent stages and the results rapidly become unreliable.

Note that for the special case of $q^2 = k^2$, (5.33) reduces to $V_\ell(k^2) = V_\ell(k^2)$, since $R_{\ell,m}(k^2,k^2) = \delta_{\ell,m}$.

The consistency of the repeated application of (5.33) is quaranteed by the sum rule

$$\sum_{n=\ell}^{m} R_{\ell,n}(k^2,p^2)\, R_{n,m}(p^2,q^2) = R_{\ell,m}(k^2,q^2) \tag{5.38}$$

since then, from (5.33), we have

$$\begin{aligned} V_\ell(k^2) &= \sum_{n=\ell}^{\infty} R_{\ell,n}(k^2,p^2) \sum_{m=n}^{\infty} R_{n,m}(p^2,q^2)\, V_m(q^2) \\ &= \sum_{m=\ell}^{\infty} V_m(q^2) \sum_{n=\ell}^{m} R_{\ell,n}(k^2,p^2) R_{n,m}(p^2,q^2) \\ &= \sum_{m=\ell}^{\infty} R_{\ell,m}(k^2,q^2)\, V_m(q^2) \end{aligned}$$

The proof of the sum rule (5.38) is given in Appendix 1.

5.3 The Low-energy expansion

As discussed in Chapter 1, the partial-wave amplitudes for any short-range potential of range α^{-1} are analytic in the complex k^2 plane for $|\mathcal{I}m\, k| < \frac{\alpha}{2}$. As this includes the region $k^2 < \alpha^2/4$ we expect that the series expansion of $V_\ell(k^2)$ about $k^2 = 0$ given by

$$V_\ell(k^2) = \sum_{m=\ell}^{\infty} (-1)^{m-\ell} \binom{2m+1}{m-\ell} \beta_m\, k^{2m} \tag{5.39}$$

will converge for at least $k^2 < \alpha^2/4$.

A truncated form of this series was derived in (5.30) and since the β_m do not depend on ℓ, they can be taken as parameters specifying the potential.

To verify that the β_m are independent of ℓ we assume that (5.39) is true for $\ell = 0,1,\cdots\cdots,r$ and consider

$$V_{r+1}(k^2) = \sum_{m=r+1}^{\infty} (-1)^{m-r-1} \binom{2m+1}{m-r-1} \hat{\beta}_m k^{2m} \tag{5.40}$$

Using (5.12) and equating coefficients of k^{2n} ($n \geqslant r+1$) we have

$$(-1)^{n-r-1}\binom{2n+1}{n-r-1}\hat{\beta}_n(n+r+2) + (-1)^{n-r}\binom{2n+1}{n-r}\beta_n(n-r) = 0 \tag{5.41}$$

which yields $\hat{\beta}_n = \beta_n$ thus proving the independence of the β_m on ℓ.

Using the identity (Abramowitz and Stegun (1968) ; 22.3.2)

$$\begin{aligned} P_{m-\ell}^{(2\ell+1,-1)}(-1) &= (m+\ell+1)\sum_{n=0}^{m-\ell}(-1)^n \frac{(m+\ell+n)!}{(m-\ell-n)!\,n!\,(2\ell+n+1)!} \\ &= \delta_{m,\ell} \end{aligned} \tag{5.42}$$

it is possible to invert (5.39) to give

$$(2\ell+1)\beta_\ell k^{2\ell} = \sum_{m=\ell}^{\infty}(2m+1)\binom{m+\ell}{m-\ell}V_m(k^2) \tag{5.43}$$

since then, from (5.39), we have

$$\begin{aligned} V_\ell(k^2) &= \sum_{m=\ell}^{\infty}(-1)^{m-\ell}\binom{2m+1}{m-\ell}\sum_{n=m}^{\infty}\frac{(2n+1)}{(2m+1)}\binom{n+m}{n-m}V_n(k^2) \\ &= \sum_{n=\ell}^{\infty}(2n+1)\,V_n(k^2)\sum_{m=0}^{n-\ell}(-1)^m\frac{(n+m+\ell)!}{m!\,(n-m-\ell)!\,(2\ell+m+1)!} \\ &= V_\ell(k^2) \end{aligned}$$

In practice it is expected that for $m \geqslant M$ the $V_m(k^2)$ will be well approximated by $\delta_\ell(k^2)/k$ and so it is useful to eliminate the β_m for $m \geqslant M$ in (5.39). Using (5.43) we have

$$\begin{aligned} V_\ell(k^2) = &\sum_{m=\ell}^{M-1}(-1)^{m-\ell}\binom{2m+1}{m-\ell}\beta_m k^{2m} \\ &+ \sum_{m=M}^{\infty}(-1)^{m-\ell}\binom{2m+1}{m-\ell}\sum_{n=m}^{\infty}\frac{(2n+1)}{(2m+1)}\binom{n+m}{n-m}V_n(k^2) \end{aligned}$$

and so

$$\begin{aligned} V_\ell(k^2) = &\sum_{m=\ell}^{M-1}(-1)^{m-\ell}\binom{2m+1}{m-\ell}\beta_m k^{2m} \\ &+ \sum_{m=M}^{\infty}(-1)^{M-\ell}\frac{2M(2m+1)}{(m+\ell+1)(m-\ell)}\binom{m+M}{m-M}\binom{2M-1}{M+\ell}V_m(k^2) \end{aligned} \tag{5.44}$$

for $\ell < M$ after using the identity

$$\sum_{m=M}^{n}(-1)^m \frac{(n+m)!}{(m-\ell)!\,(m+\ell+1)!\,(n-m)!} = (-1)^M \frac{(n+M)!}{(n-\ell)\,(n+\ell+1)\,(n-M)!\,(M+\ell)!\,(M-\ell-1)!} \tag{5.45}$$

for $\ell < M \leqslant n$, which can easily be proved by induction on M.

Therefore (5.44) provides an M-parameter representation of the potential and the parameters $\beta_0, \beta_1, \cdots\cdots, \beta_{M-1}$ can be fitted from knowledge of $\delta_0(k^2)$, $\delta_1(k^2), \cdots\cdots, \delta_{M-1}(k^2)$.

The series expansion of $V_\ell(k^2)$ now allows us to obtain an expression for the Born approximation to the scattering amplitude in terms of the β_n.

From (5.1) and (5.39)

$$\begin{aligned} f^B(-2k^2(1-\mu)) &= \sum_{\ell=0}^{\infty}(2\ell+1)\,V_\ell(k^2)\,P_\ell(\mu) \\ &= \sum_{m=0}^{\infty}(-1)^m(2m+1)!\,\beta_m k^{2m}\sum_{\ell=0}^{m}(-1)^\ell \frac{(2\ell+1)\,P_\ell(\mu)}{(m-\ell)!\,(m+\ell+1)!} \\ &= \sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)!\,\beta_m k^{2m}}{2^{m+1}\,(m!)^2}\sum_{\ell=0}^{m}(2\ell+1)P_\ell(\mu)\int_{-1}^{1}(1-x)^m P_\ell(x)\,dx \end{aligned} \tag{5.46}$$

where we have used (Gradshteyn (1965) ;7.127)

$$\int_{-1}^{1}(1-\mu)^m P_\ell(\mu)\,d\mu = \frac{(-1)^\ell\, 2^{m+1}\,(m!)^2}{(m+\ell+1)!\,(m-\ell)!} \qquad (0 \leqslant \ell \leqslant m) \tag{5.47}$$

The summation over ℓ and the integral can now be evaluated (Abramowitz and Stegun (1968) ; 8.9.1 , 8.9.2 and Gradshteyn (1965) ; 7.224.4) giving

$$\begin{aligned} f^B(-2k^2(1-\mu)) &= \sum_{m=0}^{\infty}(-1)^m 2^{-m}(2m+1)!\,(m!)^{-2}\beta_m k^{2m}(1-\mu)^m \\ &= \sum_{m=0}^{\infty}(2m+1)\,4^{-m}\binom{2m}{m}\beta_m\left[-2k^2(1-\mu)\right]^m \end{aligned} \tag{5.48}$$

Therefore as V(r) can in principle be derived from

$$r\,V(r) = -\pi^{-1}\int_{-\infty}^{0}\sin(r\sqrt{-t})\,f^B(t)\,dt \tag{5.49}$$

it follows that the parameters β_m specify the potential uniquely, since the analytic continuation of $f^B(t)$ outside the region of convergence of (5.48) will be unique for real $t < 0$.

Although the series (5.39) and (5.48) do not always converge, we show that for any short-range potential the series (5.43) and (5.44) converge for all k^2. To prove this convergence we use (5.30) which was derived directly from the recurrence relation (5.12) and is valid for any potential such that $\int_0^\infty |V(r)| r^2 dr < \infty$.

Introducing the function

$$I_{\ell,p}(k^2) = k^{2\ell}\int_0^{k^2} q^{-2\ell-2}\, V_{\ell+p}(q^2)\, dq^2 \qquad (p \geq \ell) \qquad (5.50)$$

we can rewrite (5.30) in the form

$$V_\ell(k^2) = (-1)^{M-\ell} V_M(k^2) + \sum_{m=\ell}^{M-1} (-1)^{m-\ell}\binom{2m+1}{m-\ell}\beta_m k^{2m}$$
$$- \sum_{m=\ell}^{M-1} \frac{(-1)^{m-\ell}\,(M+m+1)!}{(m-\ell)!\,(M-m-1)!\,(\ell+m+1)!}\, I_{m,M-m}(k^2) \qquad (5.51)$$

where we have used (5.18) for the Jacobi polynomials.

From (5.12) we see that

$$(\ell+p+2)\, I_{\ell,p+1}(k^2) = k^{2\ell}\int_0^{k^2} q^{-2\ell-2}\Big[(\ell+p)V_{\ell+p}(q^2) - q^2 V'_{\ell+p+1}(q^2)$$
$$- q^2 V'_{\ell+p}(q^2)\Big]\, dq^2$$
$$= (\ell+p)\, I_{\ell,p}(k^2) - V_{\ell+p+1}(k^2) - \ell\, I_{\ell,p+1}(k^2)$$
$$- V_{\ell+p}(k^2) - \ell\, I_{\ell,p}(k^2)$$

and so

$$p\, I_{\ell,p} = V_{\ell+p} + V_{\ell+p+1} + (2\ell+p+2)\, I_{\ell,p+1} \qquad (5.52)$$

which can be generalised to give

$$p\, I_{\ell,p} = V_{\ell+p} + \sum_{m=\ell+p+1}^{N-1} \frac{p!\,(m+\ell)!\,(2m+1)}{(m-\ell)!\,(2\ell+p+1)!}\, V_m$$
$$+ \frac{(\ell+N)!\,p!}{(2\ell+p+1)!\,(N-\ell-1)!}\Big[V_N + (\ell+N+1)\, I_{\ell,N-\ell}\Big] \qquad (5.53)$$

for $p \geq 1$ and $N > \ell + p + 1$.

Inserting (5.53) in (5.51) we obtain

$$V_\ell(k^2) = (-1)^{M-\ell} V_M(k^2) - \sum_{m=\ell}^{M-1} \frac{(-1)^{m-\ell}(M+m+1)!}{(m-\ell)!\,(M-m)!\,(\ell+m+1)!} V_M(k^2)$$

$$+ \sum_{m=\ell}^{M-1} (-1)^{m-\ell} \binom{2m+1}{m-\ell} \beta_m k^{2m} + R_N$$

$$+ \sum_{p=M+1}^{N-1} (2p+1)\, V_p(k^2) \sum_{m=\ell}^{M-1} \frac{(-1)^{m+\ell+1}(p+m)!}{(m-\ell)!\,(\ell+m+1)!\,(p-m)!} \tag{5.54}$$

where

$$R_N = \sum_{p=\ell}^{M-1} (-1)^{p-\ell+1} \binom{2p+1}{p-\ell} \binom{p+N}{N-p-1} \left[V_N(k^2) + (N+p+1) k^{2p} \int_0^{k^2} q^{-2p-2} V_N(q^2)\, dq^2 \right] \tag{5.55}$$

for $\ell < M < N$.

Therefore using

$$(-1)^{M-\ell} P_{M-\ell}^{(2\ell+1,0)}(-1) = \sum_{m=\ell}^{M} (-1)^{M-m} \frac{(m+M+1)!}{(m+\ell+1)!\,(m-\ell)!\,(M-m)!} = 1 \tag{5.56}$$

we have

$$V_\ell(k^2) = (-1)^{M-\ell} \frac{(2M+1)!\; V_M(k^2)}{(\ell+M+1)!\,(M-\ell)!} + \sum_{m=\ell}^{M-1} (-1)^{m-\ell} \binom{2m+1}{m-\ell} \beta_m k^{2m}$$

$$+ \sum_{p=M+1}^{N-1} (2p+1) V_p(k^2) \sum_{m=\ell}^{M-1} \frac{(-1)^{m+\ell+1}(p+m)!}{(m-\ell)!\,(\ell+m+1)!\,(p-m)!} + R_N$$

and hence

$$V_\ell(k^2) = \sum_{m=\ell}^{M-1} (-1)^{m-\ell} \binom{2m+1}{m-\ell} \beta_m k^{2m} + R_N$$

$$+ \sum_{m=M}^{N-1} (-1)^{M-\ell} \frac{2M\,(2m+1)}{(m-\ell)(m+\ell+1)} \binom{m+M}{m-M} \binom{2M-1}{M+\ell} V_m(k^2) \qquad (\ell < M < N) \tag{5.57}$$

by virtue of the identity

$$\sum_{p=\ell}^{M-1} (-1)^{M-p-1} \frac{(m+p)!}{(p+\ell+1)!\,(p-\ell)!\,(m-p)!}$$

$$= \frac{(m+M)!}{(m+\ell+1)(m-\ell)(m-M)!\,(M+\ell)!\,(M-\ell-1)!} \tag{5.58}$$

for $m \geq M \geq \ell + 1$, which can be proved by induction on M. It should be noted that (5.57) is also true for M = N provided we make the convention of setting the last term to zero.

With the aid of (5.55) and (5.57) we can now investigate the convergence of the series in (5.43) and (5.44). In Chapter 1 it was shown that for any short-range potential the $V_N(k^2)$ fall off faster than any inverse power of N and therefore for such a potential we see from (5.55) that $R_N \to 0$ as $N \to \infty$.

In the limit $N \to \infty$ and $R_N \to 0$ we see that (5.57) reduces to (5.44) and on setting $M = \ell + 1$ in (5.57) we have

$$V_\ell(k^2) = \beta_\ell k^{2\ell} - \sum_{m=\ell+1}^{N-1} \frac{(2m+1)\,(m+\ell)!}{(2\ell+1)!\,(m-\ell)!} V_m(k^2) + R_N$$

that is

$$(2\ell+1)\beta_\ell k^{2\ell} = \sum_{m=\ell}^{N-1} (2m+1)\binom{m+\ell}{m-\ell} V_m(k^2) - R_N \tag{5.59}$$

which reduces to (5.43) in this limit.

Therefore for any short-range potential the series (5.43) and (5.44) will converge for all k^2.

In any practical application of these formulae, the phase shifts at any energy will only be known up to $\delta_N(k^2)$. Therefore using (5.57) it is possible to derive bounds on the errors in $V_\ell(k^2)$ by obtaining bounds on R_N.

We can now obtain an expression for $V_\ell(k^2)$ in terms of $V_m(q^2)$ with $m \geq \ell$, that is, recover (5.33). From (5.39) and (5.43) we have

$$\begin{aligned} V_\ell(k^2) &= \sum_{m=\ell}^{\infty} (-1)^{m-\ell}\binom{2m+1}{m-\ell}\left(\frac{k^2}{q^2}\right)^m \sum_{n=m}^{\infty} \frac{(2n+1)}{(2m+1)}\binom{n+m}{n-m} V_n(q^2) \\ &= \sum_{n=\ell}^{\infty} (2n+1) V_n(q^2) \sum_{m=\ell}^{n} \frac{(-1)^{m-\ell}\,(n+m)!}{(m-\ell)!\,(m+\ell+1)!\,(n-m)!}\left(\frac{k^2}{q^2}\right)^m \\ &= \sum_{n=\ell}^{\infty} R_{\ell,n}(k^2,q^2)\, V_n(q^2) \end{aligned} \tag{5.60}$$

using the definition of $R_{\ell,m}$ in (5.34).

As discussed in Section 5.2, this series will certainly converge for $k^2 \leq q^2$ and for short-range potentials it will also converge for a small distance outside of this region.

To illustrate the convergence of (5.60) we consider the Yukawa potential

$V(r) = -\exp(-\mu r)/r$ for the two cases $\mu = 1$ and $\mu = 7$, that is, potentials of range 1 and 1/7 respectively. In Table 5.1 calculated values of $V_\ell(k^2)$ are given for $k^2 = 0.5$, 1.0, 1.5, 1.9, 2.1 and 2.2 when the input $V_\ell(q^2)$ are those of the Yukawa of range 1 at energy $q^2 = 2$. Table 5.2 contains similar results for the Yukawa of range 1/7 but with the $V_\ell(k^2)$ being calculated at higher energies k^2. As can be seen from these tables, good results are obtained in the region $k^2 > q^2$ although as k^2 approaches $q^2 + \mu^2/4$ the series (5.60) oscillates about the correct value as the number of terms included in the summation is increased. For $k^2 > q^2 + \mu^2/4$ the series diverges rapidly and no meaningful results can be obtained.

5.4 The High-energy expansion

Since the low-energy expansion yields an expression for $V_\ell(k^2)$ in terms of $V_m(q^2)$ which, in general, only converges for $k^2 \leqslant q^2$ we investigate methods of determining $V_\ell(k^2)$ for $k^2 > q^2$. The reason for (5.60) only converging in the region $k^2 \leqslant q^2$ stems from the fact that $V_m(q^2)$ only determines $f^\beta(t)$ for $-4q^2 \leqslant t < 0$. In order to determine $V_\ell(k^2)$ for $k^2 > q^2$ we must analytically continue $f^\beta(t)$ to $t = -4k^2$. This is, in principle, a unique procedure and a natural method of continuation is to sum the series (5.48) by Padé approximants.

We set

$$f^\beta(t) = P_{L-1}(t)/Q_L(t) + O(t^{2L}) \tag{5.61}$$

where P_{L-1} and Q_L are polynomials whose coefficients are determined by the β's. In general this is equivalent to approximating the potential by a sum of Yukawas. As an example we take L = 1 and then from (5.48) we have

$$f^\beta(t) \simeq \beta_0^2/(\beta_0 - \tfrac{3}{2}\beta_1 t) \tag{5.62}$$

which corresponds to the potential $V(r) = \beta_0\mu^2\exp(-\mu r)/r$ with

$$\mu^2 = \tfrac{2}{3}\beta_0/\beta_1 \tag{5.63}$$

Therefore

$$V_\ell(k^2) = -\left(\frac{\beta_0 \mu^2}{2k^2}\right) Q_\ell\left(1+\frac{\mu^2}{2k^2}\right) \tag{5.64}$$

where

$$\beta_0 = \sum_{m=0}^{\infty} (2m+1)\, V_m(q^2) \tag{5.65}$$

and

$$6\beta_1 q^2 = \sum_{m=1}^{\infty} (2m+1)(m+1)m \; V_m(q^2) \tag{5.66}$$

The weakness of this approach is that (5.64) is not even exact at $q^2 = k^2$ since the constants β_n are only related to the threshold behaviour of the $V_n(q^2)$.

Therefore, instead of using the Padé approximant approach, we use a method analogous to the low-energy expansion but based on the high-energy expansion

$$V_\ell(k^2) = \sum_{i=1}^{\infty} k^{-2i}\left(\alpha_{i,\ell} \log k^2 + \gamma_{i,\ell}\right) \tag{5.67}$$

As shown in Section 1.1, this expansion is valid for any Yukawa potential and hence is also valid for any potential of the form

$$V(r) = \sum_j \sum_{i \geqslant -1} A_{ij}\, r^i \exp(-\mu_j r) \tag{5.68}$$

The coefficients $\alpha_{i,\ell}$ and $\gamma_{i,\ell}$ are not independent as can be seen by substituting (5.67) into (5.12). We have

$$\sum_{i=1}^{\infty} k^{-2i}\Big[(\ell-i+2)\,\gamma_{i,\ell+1} - (\ell+i)\,\gamma_{i,\ell} + \alpha_{i,\ell+1} + \alpha_{i,\ell} + \log k^2\, \alpha_{i,\ell+1}(\ell-i+2) - \log k^2\, \alpha_{i,\ell}(\ell+i)\Big] = 0 \tag{5.69}$$

which, to be true for all k^2, requires

$$\alpha_{i,\ell} = \frac{(\ell-i+2)}{(\ell+i)}\,\alpha_{i,\ell+1} \tag{5.70}$$

and

$$\gamma_{i,\ell} = \frac{(\ell-i+2)}{(\ell+i)}\gamma_{i,\ell+1} + \frac{(2\ell+2)}{(\ell+i)^2}\alpha_{i,\ell+1} \tag{5.71}$$

From (5.70) we see directly that

$$\alpha_{i,\ell} = 0 \qquad (i > \ell+1) \tag{5.72}$$

and by repeated application of (5.70) we obtain

$$\alpha_{i,\ell} = \binom{\ell+i-1}{\ell-i+1}\alpha_i \qquad (i \leq \ell+1) \tag{5.73}$$

where $\alpha_i = \alpha_{i,i-1}$.

Similarly, from (5.71), we obtain

$$\gamma_{\ell+2,\ell} = (2\ell+2)^{-1}\alpha_{\ell+2} \tag{5.74}$$

$$\gamma_{i,\ell} = (-1)^{i-\ell}\alpha_i\left[(\ell+i)\binom{2i-2}{i-\ell-2}\right]^{-1} \qquad (i > \ell+1) \tag{5.75}$$

and

$$\gamma_{i,\ell} = \binom{\ell+i-1}{\ell-i+1}\left[\gamma_i - 2\alpha_i\sum_{p=0}^{\ell-i}\frac{(\ell-p)}{(\ell-p+i-1)(\ell-p-i+1)}\right] \quad (i \leq \ell+1) \tag{5.76}$$

where $\gamma_i = \gamma_{i,i-1}$.

Therefore, inserting (5.70) through (5.76) into (5.67), we finally obtain

$$V_\ell(k^2) = \sum_{i=1}^{\ell+1} k^{-2i}\binom{\ell+i-1}{\ell-i+1}\left[\alpha_i \log k^2 + \gamma_i - 2\alpha_i\sum_{p=0}^{\ell-i}\frac{(\ell-p)}{(\ell-p+i-1)(\ell-p-i+1)}\right]$$
$$+ \sum_{i=\ell+2}^{\infty}(-1)^{i-\ell}k^{-2i}\alpha_i\left[(\ell+i)\binom{2i-2}{i-\ell-2}\right]^{-1} \tag{5.77}$$

where α_i and γ_i are constants which parameterise the potential at high energies just as the β_i do at low energies.

Unfortunately we have not been able to invert (5.77) explicitly to obtain expressions, analogous to (5.43), giving α_i and γ_i in terms of the $V_m(q^2)$ for a given q^2. However, in practice, $V_m(q^2)$ will be set zero for $m > N$ and hence the series in (5.77) can be truncated to give a finite set of linear equations for α_i and γ_i in terms of $V_m(q^2)$. In order to obtain a consistent approximation it is necessary to take N odd, giving an even number of equations, otherwise difficulties arise with the $\log q^2$ terms.

As an example we set N = 1, that is $V_m(q^2) = 0$ for $m > 1$, and we have

$$\begin{aligned} V_0(q^2) &= q^{-2}(\alpha_1 \log q^2 + \gamma_1) \\ V_1(q^2) &= q^{-2}(\alpha_1 \log q^2 + \gamma_1 - 2\alpha_1) \end{aligned} \tag{5.78}$$

which gives

$$\alpha_1 = \tfrac{1}{2} q^2 [V_0(q^2) - V_1(q^2)]$$

and

$$\gamma_1 = \tfrac{1}{2} q^2 [2V_0(q^2) - \{V_0(q^2) - V_1(q^2)\} \log q^2] \tag{5.79}$$

Substituting (5.79) into (5.78) with $q^2 \to k^2$ then gives

$$V_0(k^2) = \tfrac{1}{2}[V_0(q^2) - V_1(q^2)] \tfrac{q^2}{k^2} \log\left(\tfrac{k^2}{q^2}\right) + \tfrac{q^2}{k^2} V_0(q^2)$$

and

$$V_1(k^2) = V_0(k^2) - \tfrac{q^2}{k^2}[V_0(q^2) - V_1(q^2)] \tag{5.80}$$

These expressions for $V_0(k^2)$ and $V_1(k^2)$ are exact at $k^2 = q^2$ and are of the correct order as $k^2 \to \infty$.

Taking N = 1,3,5,$\cdots\cdots$ we obtain a sequence of approximations which converge rapidly even if q^2 is such that the original series (5.67) diverges. For each value of N, (5.77) can be truncated at $2i = N + 1$ and $\ell = N$, to form a system of N + 1 linear equations which can easily be solved for any particular set of $V_\ell(q^2)$ to give the α_i and γ_i. These constants can then be used to calculate estimates to $V_m(k^2)$ for $m \leqslant \ell$ and $k^2 > q^2$.

Therefore by using this high-energy expansion we obtain an approximate continuation of $V_\ell(k^2)$ into the region $k^2 > q^2$ which, when used in conjunction with the low-energy expansion, allows $V_\ell(k^2)$ for any value of k^2 to be determined from all the $V_m(q^2)$ at an arbitrary energy q^2.

In Tables 5.3 and 5.4 we illustrate the convergence of the high-energy expansion for the Yukawa potential $V(r) = -\exp(-r)/r$ and the exponential potential $V(r) = -\exp(-r)$ respectively. The results were obtained by truncating the series in (5.77) at $2i = N + 1$ and $\ell = N$ and as can be seen the estimated $V_\ell(k^2)$ rapidly converge with increasing N. We have also investigated the convergence of (5.77) for potentials which are not of the form (5.68), for example the δ-function potential $V(r) = -\delta(r-1)$, and have found that the truncated series no longer converges.

In order to determine the effect of errors in the input $V_\ell(q^2)$ we once more consider the Yukawa potential $V(r) = -\exp(-r)/r$. We have found that a small error in any $V_m(q^2)$ for $m = M$ (say) has little effect on $V_\ell(k^2)$ for $\ell > M$, that is, the relative error in the $V_\ell(k^2)$ is less than that in $V_M(q^2)$. For $\ell < M$ the relative errors are generally less than those in $V_M(q^2)$ although this is not always the case, and for $\ell = M$ the errors are almost always increased. This behaviour of the $V_\ell(k^2)$ can be seen in Table 5.5 where relative errors of 10% have been introduced into $V_M(q^2)$ at energy $q^2 = 1$ for $M = 0,1,2$ and 3.

	Estimated $V_\ell(k^2)$		Exact $V_\ell(k^2)$	
ℓ	k^2=0.5	k^2=1.0	k^2=0.5	k^2=1.0
0	5.49333	4.02374	5.49306	4.02359
1	0.98594	1.03526	0.98612	1.03539
2	0.21180	0.31793	0.21184	0.31783
3	0.04911	0.10432	0.04871	0.10433
4	0.01122	0.03535	0.01161	0.03548
ℓ	k^2=1.5	k^2=1.9	k^2=1.5	k^2=1.9
0	3.24327	2.83130	3.24318	2.83127
1	0.99083	0.94472	0.99091	0.94476
2	0.36031	0.37447	0.36023	0.37443
3	0.13983	0.15841	0.13991	0.15844
4	0.05632	0.06945	0.05627	0.06942
ℓ	k^2=2.1	k^2=2.2	k^2=2.1	k^2=2.2
0	2.66991	2.59212	2.66751	2.59362
1	0.92147	0.90919	0.92168	0.91035
2	0.37747	0.37836	0.37794	0.37906
3	0.16524	0.16775	0.16542	0.16845
4	0.07455	0.07723	0.07495	0.07749

Table 5.1 : Estimated values of the Born approximations $V_\ell(k^2)$ obtained from those at energy $q^2 = 2$ by using (5.60) truncated at $n = 10$, for the Yukawa potential $V(r) = -10\exp(-r)/r$.

	Estimated $V_\ell(k^2)$		Exact $V_\ell(k^2)$	
ℓ	k^2=4.0	k^2=6.0	k^2=4.0	k^2=6.0
0	1.76604	1.66099	1.76604	1.66099
1	0.08306	0.11006	0.08306	0.11007
2	0.00469	0.00874	0.00469	0.00875
3	0.00028	0.00074	0.00028	0.00074
4	0.00002	0.00007	0.00002	0.00007
ℓ	k^2=8.0	k^2=10.0	k^2=8.0	k^2=10.0
0	1.57079	1.49305	1.57072	1.49204
1	0.13096	0.14753	0.13103	0.14754
2	0.01316	0.01744	0.01311	0.01749
3	0.00138	0.00223	0.00140	0.00222
4	0.00016	0.00030	0.00016	0.00029
ℓ	k^2=12.0	k^2=14.0	k^2=12.0	k^2=14.0
0	1.42240	1.36003	1.42269	1.36096
1	0.16054	0.17203	0.16068	0.17122
2	0.02186	0.02531	0.02175	0.02581
3	0.00322	0.00441	0.00315	0.00417
4	0.00043	0.00067	0.00047	0.00070

Table 5.2 : Estimated values of the Born approximations $V_\ell(k^2)$ obtained from those at energy $q^2 = 2$ by using (5.60) truncated at $n = 10$, for the Yukawa potential $V(r) = -10^2\exp(-7r)/r$.

k^2	ℓ	Calculated $V_\ell(k^2)$ N=3	N=5	N=7	N=9	Exact $V_\ell(k^2)$
3.0	0	2.07466	2.12990	2.13295	2.13621	2.13746
	1	0.84788	0.83354	0.82903	0.82764	0.82703
	2	0.38647	0.37983	0.37871	0.37856	0.37858
	3	0.18129	0.18358	0.18442	0.18468	0.18477
	4		0.09263	0.09321	0.09330	0.09331
5.0	0	1.45576	1.50277	1.51639	1.52046	1.52226
	1	0.68357	0.67843	0.67599	0.67503	0.67449
	2	0.36319	0.35479	0.35256	0.35198	0.35177
	3	0.19869	0.19562	0.19522	0.19522	0.19526
	4		0.11130	0.11180	0.11197	0.11205
7.0	0	1.13987	1.18335	1.19655	1.20067	1.20261
	1	0.57597	0.57601	0.57510	0.57459	0.57422
	2	0.33125	0.32457	0.32248	0.32183	0.32155
	3	0.19724	0.19266	0.19165	0.19143	0.19139
	4		0.11761	0.11757	0.11763	0.11768
9.0	0	0.94490	0.98471	0.99712	1.00110	1.00303
	1	0.50061	0.50354	0.50361	0.50342	0.50320
	2	0.30265	0.29782	0.29609	0.29551	0.29522
	3	0.19016	0.18553	0.18432	0.18400	0.18390
	4		0.11873	0.11832	0.11828	0.11829

Table 5.3 : Estimated values of the Born approximations $V_\ell(k^2)$ obtained from those at energy $q^2 = 1$ by using (5.77) truncated at $2i = N + 1$ and $\ell = N$, for the potential $V(r) = -10\exp(-r)/r$.

k^2	Calculated $V_\ell(k^2)$					Exact $V_\ell(k^2)$
	ℓ	N=3	N=5	N=7	N=9	
3.0	0	1.65025	1.58642	1.55653	1.54488	1.53846
	1	1.04303	1.06305	1.07428	1.07926	1.08239
	2	0.69928	0.70830	0.71102	0.71154	0.71143
	3	0.45816	0.45491	0.45283	0.45192	0.45142
	4		0.28207	0.28065	0.28032	0.28029
5.0	0	1.07545	1.01022	0.97634	0.96174	0.95238
	1	0.72334	0.73072	0.73685	0.74030	0.74317
	2	0.52743	0.53905	0.54456	0.54666	0.54769
	3	0.38652	0.39065	0.39161	0.39164	0.39129
	4		0.27655	0.27530	0.27469	0.27432
7.0	0	0.80793	0.74745	0.71459	0.69979	0.68966
	1	0.56079	0.56097	0.56327	0.56513	0.56712
	2	0.42527	0.43458	0.43980	0.44212	0.44356
	3	0.32766	0.33394	0.33645	0.33722	0.33744
	4		0.25259	0.25266	0.25245	0.25218
9.0	0	0.65137	0.59593	0.56499	0.55071	0.54054
	1	0.46123	0.45735	0.45720	0.45791	0.45912
	2	0.35811	0.36489	0.36920	0.37131	0.37281
	3	0.28405	0.29045	0.29345	0.29458	0.29511
	4		0.22867	0.22967	0.22985	0.22977

Table 5.4 : Estimated values of the Born approximations $V_\ell(k^2)$ obtained from those at energy $q^2 = 1$ by using (5.77) truncated at $2i = N + 1$ and $\ell = N$, for the exponential potential $V(r) = -10\exp(-r)$.

k^2	Calculated $V_\ell(k^2)$					Exact $V_\ell(k^2)$
	ℓ	M=0	M=1	M=2	M=3	
3.0	0	2.48467	1.87226	2.35225	2.00831	2.13746
	1	0.85757	0.92758	0.74613	0.87896	0.82703
	2	0.37288	0.43353	0.36087	0.38718	0.27858
	3	0.18209	0.19349	0.20302	0.17546	0.18477
	4	0.09389	0.08837	0.10981	0.08909	0.09331
5.0	0	1.81076	1.23680	1.76896	1.36731	1.52226
	1	0.72281	0.72022	0.62832	0.70743	0.67449
	2	0.35298	0.40931	0.31305	0.37517	0.35177
	3	0.19087	0.22276	0.18979	0.19740	0.19526
	4	0.11004	0.11796	0.12296	0.10613	0.11205
7.0	0	1.44953	0.93080	1.44390	1.04829	1.20261
	1	0.62912	0.58698	0.55527	0.59010	0.57422
	2	0.32921	0.36845	0.28402	0.34569	0.32155
	3	0.18858	0.22387	0.17438	0.20094	0.19139
	4	0.11469	0.13209	0.11835	0.11666	0.11768
9.0	0	1.21974	0.74966	1.23139	0.85550	1.00303
	1	0.56012	0.49626	0.50209	0.50735	0.50320
	2	0.30743	0.33114	0.26363	0.31636	0.29522
	3	0.18331	0.21565	0.16289	0.19656	0.18390
	4	0.11548	0.13687	0.11206	0.12135	0.11829

Table 5.5 : Estimated values of the Born approximations $V_\ell(k^2)$ obtained from those at energy $q^2 = 1$ by using (5.77) truncated at $\ell = 2i - 1 = 9$, for the potential $V(r) = -10\exp(-r)/r$, when relative errors of 10% have been introduced into the $V_M(k^2)$.

CHAPTER 6

THE OFF-SHELL BORN APPROXIMATION IN TERMS OF THE ON-SHELL

In Chapter 5 we described methods of determining the on-shell Born approximations $V_\ell(k^2)$ at any energy k^2, either in terms of all the $V_m(q^2)$ at one energy q^2 or in terms of the values of one of them at all energies. In this chapter we show that the off-shell Born approximations $V_\ell(p_1^2, p_2^2)$ can be obtained from all the $V_m(k^2)$ at a particular energy k^2 and hence, using the results of Chapter 5, that $V_\ell(p_1^2, p_2^2)$ can be derived from either all the $V_m(q^2)$ at any energy q^2, or one particular $V_m(q^2)$ at all energies.

6.1 The off-shell Born approximations

The Born approximations to the off-shell partial-wave amplitudes are given in terms of the scattering amplitude as

$$f^B\left(-(\underset{\sim}{p}_1-\underset{\sim}{p}_2)^2\right) = \sum_{\ell=0}^{\infty}(2\ell+1)\, V_\ell(p_1^2, p_2^2)\, P_\ell(\hat{\underset{\sim}{p}}_1\cdot\hat{\underset{\sim}{p}}_2) \tag{6.1}$$

and in terms of the underlying local potential by

$$V_\ell(p_1^2, p_2^2) = -\int_0^\infty j_\ell(p_1 r)\, j_\ell(p_2 r)\, V(r)\, r^2\, dr \tag{6.2}$$

which is well defined provided

$$\int_0^\infty |V(r)|\, r^2\, dr < \infty \tag{6.3}$$

For the on-shell amplitudes we have used the notation

$$V_\ell(k^2) = V_\ell(k^2, k^2) \tag{6.4}$$

and these are given in terms of the on-shell scattering amplitude by

$$f^B\left(-2k^2(1-\mu)\right) = \sum_{\ell=0}^{\infty}(2\ell+1)\, V_\ell(k^2)\, P_\ell(\mu) \tag{6.5}$$

with $\mu = \hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{k}}'$.

Given any particular values for p_1 and p_2, we can choose k^2 such that

$$(p_1 - p_2)^2 = 2k^2(1-\mu) \tag{6.6}$$

with $-1 \leqslant \mu \leqslant 1$ provided only that $p_1 + p_2 \leqslant 2k$. Therefore, from (6.1) and (6.5), we have

$$\sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(\rho_1^2, \rho_2^2)\, P_\ell(\hat{p}_1 \cdot \hat{p}_2) = \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell(k^2)\, P_\ell(\mu) \tag{6.7}$$

and hence

$$V_\ell(\rho_1^2, \rho_2^2) = \sum_{m=\ell}^{\infty} I_{\ell,m}(\rho_1^2, \rho_2^2, k^2)\, V_m(k^2) \tag{6.8}$$

with

$$I_{\ell,m}(\rho_1^2, \rho_2^2, k^2) = (m+\tfrac{1}{2}) \int_{-1}^{1} P_\ell(\mu)\, P_m\left(1 - \frac{\rho_1^2+\rho_2^2-2\rho_1\rho_2\mu}{2k^2}\right) d\mu \tag{6.9}$$

That the summation in (6.8) is from $m = \ell$ and not $m = 0$ can be seen directly from (6.9), since $I_{\ell,m} = 0$ for $m < \ell$.

For general values of k^2, the integral in (6.9) is not readily expressible in a closed form, but it is possible to calculate $I_{\ell,m}$ for arbitrary ℓ and m as described below.

Setting

$$I_{\ell,m}(a,b) = (m+\tfrac{1}{2}) \int_{-1}^{1} P_\ell(\mu)\, P_m(a+b\mu)\, d\mu \tag{6.10}$$

we see, from Appendix 2, that

$$\frac{(2\ell+1)}{(2m+3)} I_{\ell,m}(a,b) + b\, I_{\ell+1,m}(a,b) = \frac{(2\ell+1)}{(2m-1)} I_{\ell,m-1}(a,b) + b\, I_{\ell-1,m}(a,b) \tag{6.11}$$

for $m > \ell \geqslant 1$, with

$$I_{\ell,\ell}(a,b) = b^\ell$$

$$I_{\ell,\ell+1}(a,b) = (2\ell+3)\, a\, b^\ell \tag{6.12}$$

and

$$2b\, I_{0,m}(a,b) = P_{m+1}(a+b) - P_{m+1}(a-b) + P_{m-1}(a-b) - P_{m-1}(a+b)$$

Therefore, by repeated application of the recurrence relation (6.11) and use of the 'initial values' (6.12), we can obtain $I_{\ell,m}$ for any values of ℓ and m, and hence determine the off-shell amplitudes from (6.8).

Although this approach gives good results for small ℓ (see Section 6.2), it tends to produce large errors in the $I_{\ell,m}$ for large ℓ (and m) and so we look at another method for calculating (6.8).

Since k and μ can be chosen arbitrarily, subject only to condition (6.6), we can certainly select the value $k = \frac{1}{2}(p_1 + p_2)$. With this choice of k, (6.9) becomes

$$I_{\ell,m}\left(p_1^2, p_2^2, \left(\frac{p_1+p_2}{2}\right)^2\right) = (m+\tfrac{1}{2})\int_{-1}^{1} P_\ell(\mu)\, P_m(-1+\alpha+\alpha\mu)\, d\mu \qquad (6.13)$$

where

$$0 \leq \alpha = 4p_1p_2/(p_1+p_2)^2 \leq 1 \qquad (6.14)$$

Hence

$$\begin{aligned} I_{\ell,m}\left(p_1^2, p_2^2, \left(\frac{p_1+p_2}{2}\right)^2\right) &= (-1)^{m-\ell}(m+\tfrac{1}{2})\int_{-1}^{1} P_\ell(\mu)\, P_m(1-\alpha+\alpha\mu)\, d\mu \\ &= (-1)^{m-\ell} R_{\ell,m}(\alpha q^2, q^2) \\ &= (-1)^{m-\ell}\left(\frac{2m+1}{m+\ell+1}\right)\alpha^\ell P_{m-\ell}^{(2\ell+1,-1)}(1-2\alpha) \end{aligned} \qquad (6.15)$$

by virtue of (5.34).

Because of the bounds (6.14) on α, the convergence of (6.8) is guaranteed for all p_1 and p_2 provided the underlying potential belongs to the class in (6.3). Moreover, we see from (6.15), that each term of (6.8) has the threshold behaviour $(p_1 p_2)^\ell$ as p_1 or $p_2 \to 0$, as is required for $V_\ell(p_1^2, p_2^2)$ for any short-range potential, by (6.2).

Since the Jacobi polynomials in (6.15) have already been required to relate the Born approximations at different energies, as described in Chapter 5, very little extra work is involved in going off-shell using (6.8) and (6.15).

Therefore, given all the $V_\ell(q^2)$ for some energy q^2, we can determine

$V(p_1^2, p_2^2)$ for any p_1^2 and p_2^2 such that $p_1 + p_2 \leqslant 2q$ by using (6.8), the recurrence relation (6.11) and the 'initial values' (6.12). For $p_1 + p_2 > 2q$ we need to know $V_\ell(k^2)$ for $k^2 > q^2$ (where $k^2 > (p_1 + p_2)^2/4$). These can be calculated for short-range potentials by the methods described in Section 5.2, and for the more restricted class of potentials (5.68) by using the high-energy expansion in Section 5.4. Therefore, provided the potential belongs to the class (5.68), we can determine $V_\ell(p_1^2, p_2^2)$ from the $V_\ell(q^2)$ for $p_1 + p_2 > 2q$ by first using the high energy expansion to obtain $V_\ell((p_1 + p_2)^2/4)$ and then making use of (6.8) and (6.15).

Having described the general method for determining the off-shell amplitudes we now consider two cases of particular interest. The first case is when p_1 and p_2 do not differ too greatly from k, that is, we are close to the energy shell. Since $p_1 \simeq p_2 \simeq k$, we see from (6.14) that $\alpha \simeq 1$ and hence from (6.15) and the definition of the Jacobi polynomials we have

$$I_{\ell,m} \simeq (2m+1)(\alpha-1) \qquad (m > \ell)$$

and (6.16)

$$I_{\ell,\ell} \simeq 1 + \ell(\alpha - 1)$$

Therefore, to first order in $(1-\alpha)$, we have

$$V_\ell(p_1^2, p_2^2) \simeq V_\ell(k^2) - (1-\alpha)\Big[\ell V_\ell(k^2) + \sum_{m=\ell+1}^{\infty}(2m+1)V_m(k^2)\Big] \tag{6.17}$$

where $k = \frac{1}{2}(p_1 + p_2)$.

Using (5.43) for the case of the s wave, we can rewrite (6.17) in the form

$$V_\ell(p_1^2, p_2^2) \simeq V_\ell(k^2) - (1-\alpha)\Big[\ell V_\ell(k^2) + \beta_0 - \sum_{m=0}^{\ell}(2m+1)V_m(k^2)\Big] \tag{6.18}$$

with $\beta_0 = V_0(0)$. Therefore, for the s wave, we have

$$V_0(p_1^2, p_2^2) = V_0(k^2) - (1-\alpha)\big[V_0(0) - V_0(k^2)\big] + O\big((1-\alpha)^2\big) \tag{6.19}$$

which only involves V_0.

As an example of this approximation we can calculate $V_0(1.5,0.5)$ from the phase shifts to the potential $V(r) = -\exp(-r)/r$ at energy $k^2 = 0.933$. Using (6.19) we obtain $V_0(1.5,0.5) = 0.375$ which compares favourably with the correct value of 0.380.

The second case of interest is when we are far off-shell, for instance when $p_1 >> p_2$. From (6.14) we see that this implies $\alpha \simeq 0$ and hence from (6.15)

$$I_{\ell,m} \simeq (-1)^{m-\ell} \frac{(2m+1)}{(m+\ell+1)} \alpha^{\ell} P_{m-\ell}^{(2\ell+1,-1)}(0)$$

$$\simeq (-1)^{m-\ell} \frac{(2m+1)}{(m+\ell+1)} \binom{m+\ell+1}{m-\ell} \left(\frac{4p_2}{p_1}\right)^{\ell} \tag{6.20}$$

Therefore, for the s wave, we obtain

$$V_0(p_1^2, p_2^2) \simeq \sum_{m=0}^{\infty} (-1)^m (2m+1) V_m(k^2) \tag{6.21}$$

where $k = p_1/2$. Note that (6.21) is independent of p_2 and is in fact exact for $p_2 = 0$. For example, we obtain $V_0(4,1) = 0.2000$ using (6.21) for the Yukawa potential $V(r) = -\exp(-r)/r$, whereas the exact value is 0.2012.

6.2 Numerical Results

We now give some results for $V_\ell(p_1^2,p_2^2)$ when the recurrence relation (6.11) is used to evaluate (6.10) and compare these results with those obtained when (6.15) is used in place of (6.10), i.e. when k is chosen to take the value $k = \frac{1}{2}(p_1 + p_2)$.

From (6.11) we see that in the process of calculating $I_{\ell,m}$ it is necessary to calculate nearly all the $I_{\ell',m'}$ for $m' < m$ and $\ell' < m'$. Not only is this a time consuming process but the recurrence relations (6.11) are unstable in the sense that any errors produced in $I_{\ell,m}$ are amplified when $I_{\ell,m+2}$ is calculated. The instability does not become noticable until $m > 10$ and, provided the summation in (6.8) is truncated at $m = 9$, good results are obtained.

In Table 6.1 $V_\ell(p_1^2,p_2^2)$ is given for $p_1 = 1,4$, $p_2 = 1,2,3,4$ and $\ell = 0,1,$

2,3 when the $V_\ell(k^2)$ are taken to be those of the potential $V(r) = -\exp(-r)/r$ at energy $k^2 = 4.0$. Table 6.2 contains similar results when the $V_\ell(k^2)$ are taken to be those of the δ-function potential $V(r) = -\delta(r-1)$. The results in both of these tables were obtained by truncating the summation in (6.8) at m = 9 and as can be seen, at least 3 figure accuracy is achieved for all the values. The high accuracy of the results in Table 6.2 is due to the rapid decay of $V_\ell(k^2)$ with increasing ℓ for the δ-function potential (in fact $V_q(4.0) \simeq 10^{-7}$ compared with 10^{-3} for the Yukawa potential).

As a comparison of the use of (6.11) against (6.15) we give results in Table 6.3 for $V_\ell(p_1^2,p_2^2)$ with $p_1 = 1,4$, $p_2 = 1,2,3,4$ and $\ell = 0,1,2$ when the $V_\ell(k^2)$ are taken as those of the potential $V(r) = -\exp(-r)/r$ at energy $k^2 = (p_1 + p_2)^2/4$. From these results we see that (6.15),in general, gives at least one extra figure accuracy over (6.11). In fact, the use of (6.15) for the δ-function potential gives 7 figure accuracy when only the first 9 terms of the summation are included.

Therefore the use of (6.15) leads to more accurate results than does (6.11) and hence is the preferred method of calculating $V_\ell(p_1^2,p_2^2)$, although it does mean that a new set of $V_\ell(k^2)$ have to be obtained whenever p_1 or p_2 are altered.

		$V_\ell(p_1^2, p_2^2)$			
p_1	ℓ	$p_2 = 1.0$	$p_2 = 2.0$	$p_2 = 3.0$	$p_2 = 4.0$
1.0	0	0.40234	0.20053	0.10189	0.05982
	1	0.10365	0.05173	0.02045	0.00936
	2	0.03158	0.01535	0.00475	0.00178
	3	0.01011	0.00507	0.00110	0.00031
1.0 Exact	0	0.40236	0.20118	0.10198	0.05972
	1	0.10354	0.05177	0.02030	0.00937
	2	0.03178	0.01589	0.00483	0.00176
	3	0.01043	0.00522	0.00123	0.00035
4.0	0		0.06255	0.06705	0.06525
	1		0.01960	0.03085	0.03609
	2		0.00726	0.01681	0.02308
	3		0.00293	0.00970	0.01564
4.0 Exact	0		0.06255	0.05706	0.06522
	1		0.01960	0.03098	0.03601
	2		0.00730	0.01681	0.02310
	3		0.00290	0.00971	0.01569

Table 6.1 : $V_\ell(p_1^2, p_2^2)$ calculated using the recurrence relation (6.11), the $V_\ell(k^2)$ being those of the Yukawa potential $V(r) = -\exp(-r)/r$ at energy $k^2 = 4.0$. The exact result for the Yukawa is $V_\ell(p_1^2, p_2^2) = (2p_1 p_2)^{-1} Q_\ell((1+p_1^2+p_2^2)/2p_1 p_2)$ and these values are given in rows 2 and 4.

		$V_\ell(p_1^2,p_2^2)$			
p_1	ℓ	p_2=1.0	p_2=2.0	p_2=3.0	p_2=4.0
1.0	0	0.70804	0.38256	0.03959	-0.15920
	1	0.09071	0.13112	0.10411	0.03497
	2	0.00385	0.01230	0.01852	0.01715
	3	0.00008	0.00055	0.00014	0.00206
1.0 Exact	0	0.70807	0.38257	0.03958	-0.15921
	1	0.09070	0.13113	0.10411	0.03497
	2	0.00385	0.01231	0.01853	0.01714
	3	0.00008	0.00055	0.00014	0.00206
4.0	0		-0.08602	-0.00890	0.03580
	1		0.05056	0.04013	0.01348
	2		0.05483	0.08251	0.07633
	3		0.01392	0.03485	0.05255
4.0 Exact	0		-0.08602	-0.00890	0.03580
	1		0.05055	0.04014	0.01348
	2		0.05483	0.08251	0.07633
	3		0.01392	0.03486	0.05255

Table 6.2 : $V_\ell(p_1^2,p_2^2)$ calculated using the recurrence relation (6.11), the $V_\ell(k^2)$ being those of the potential $V(r) = -\delta(r-1)$ at energy $k^2 = 4.0$. The exact result for this potential is $V_\ell(p_1^2,p_2^2) = j_\ell(p_1)j_\ell(p_2)$ and these are given in rows 2 and 4.

		$V_\ell(p_1^2,p_2^2)$			
p_1		$p_2=1.0$	$p_2=2.0$	$p_2=3.0$	$p_2=4.0$
1.0	0	0.40236	0.20119	0.10194	0.05974
	1	0.10354	0.05178	0.02029	0.00936
	2	0.03178	0.01590	0.00484	0.00175
1.0 Exact	0	0.40236	0.20118	0.10198	0.05972
	1	0.10354	0.05177	0.02030	0.00937
	2	0.03178	0.01589	0.00483	0.00176
4.0	0		0.06255	0.06706	0.06522
	1		0.01959	0.03099	0.03601
	2		0.00729	0.01682	0.02310
4.0 Exact	0		0.06255	0.06706	0.06522
	1		0.01960	0.03098	0.03601
	2		0.00730	0.01681	0.02310

Table 6.3 : $V_\ell(p_1^2,p_2^2)$ calculated from $V_\ell(k^2)$ with $k = \frac{1}{2}(p_1 + p_2)$ using (6.15), the $V_\ell(k^2)$ being those of the Yukawa potential $V(r) = -\exp(-r)/r$. The exact result for the Yukawa is $V_\ell(p_1^2,p_2^2) = (2p_1p_2)^{-1}Q_\ell((1+p_1^2+\bar{p}_2^2)/2p_1p_2)$ and these values are given in rows 2 and 4.

APPENDIX 1

In this appendix we prove that

$$R_{\ell,m}(k^2,q^2) = P_{\ell,m}\left(\frac{k^2}{q^2}\right) \tag{A1.1}$$

where

$$R_{\ell,m}(k^2,q^2) = \left(m+\tfrac{1}{2}\right)\int_{-1}^{1} P_\ell(\mu)\, P_m(\mu')\, d\mu \tag{A1.2}$$

with $\mu' = 1-(1-\mu)k^2/q^2$, and

$$P_{\ell,m}\left(\frac{k^2}{q^2}\right) = \left(\frac{2m+1}{m+\ell+1}\right)\left(\frac{k^2}{q^2}\right)^{\ell} P_{m-\ell}^{(2\ell+1,-1)}\left(1-\frac{2k^2}{q^2}\right) \tag{A1.3}$$

We also show that the $R_{\ell,m}$ satisfy the sum rule

$$\sum_{n=\ell}^{m} R_{\ell,n}(k^2,p^2)\, R_{n,m}(p^2,q^2) = R_{\ell,m}(k^2,q^2) \tag{A1.4}$$

From the results in Appendix 2 we know that $R_{\ell,m}(k^2,q^2)$ satisfies the recurrence relation

$$\left(\frac{2\ell+1}{2m+3}\right)R_{\ell,m+1} + \frac{k^2}{q^2}R_{\ell+1,m} = \left(\frac{2\ell+1}{2m-1}\right)R_{\ell,m-1} + \frac{k^2}{q^2}R_{\ell-1,m} \qquad (m>\ell\geqslant 1) \tag{A1.5}$$

and we show that the function $P_{\ell,m}$ satisfies the same recurrence relation.

The Jacobi polynomials of different order are related by (Abramowitz and Stegun (1968) ; 22.7.15 and 22.7.18)

$$\left(n+\frac{\alpha+1}{2}\right)(1-x)\, P_n^{(\alpha+1,-1)}(x) = (n+\alpha+1)\, P_n^{(\alpha,-1)}(x) - (n+1)\, P_{n+1}^{(\alpha,-1)}(x) \tag{A1.6}$$

and

$$(2n+\alpha-1)\, P_n^{(\alpha-1,-1)}(x) = (n+\alpha-1)\, P_n^{(\alpha,-1)}(x) - (n-1)\, P_{n-1}^{(\alpha,-1)}(x) \tag{A1.7}$$

where $P_n^{(\alpha,-1)}(x) = 0$ for $n < 0$.

Repeated use of (A1.6) and (A1.7) gives

$$\frac{k^2}{q^2}P_{\ell+1,m} = \frac{(m+\ell+1)(m+\ell)}{2m(2m-1)}P_{\ell,m-1} + \frac{(m-\ell)(m-\ell+1)}{(2m+2)(2m+3)}P_{\ell,m+1} + \frac{(m-\ell)(m+\ell+1)}{2m\,(m+1)}P_{\ell,m} \tag{A1.8}$$

and

$$\frac{(m+\ell+1)}{(m+1)}P_{\ell,m} = \frac{(m-\ell-1)}{(2m-1)}P_{\ell,m-1} - \frac{k^2}{q^2}\frac{2m}{(m-\ell)}P_{\ell-1,m} + \frac{m(m+\ell+1)(m+\ell+2)}{(m-\ell)(m-1)(2m+3)}P_{\ell,m+1} \tag{A1.9}$$

respectively, and so

$$\frac{k^2}{q^2} P_{\ell+1,m} = \frac{k^2}{q^2} P_{\ell-1,m} + \frac{(2\ell+1)}{(2m-1)} P_{\ell,m-1} - \frac{(2\ell+1)}{(2m+3)} P_{\ell,m+1} \tag{A1.10}$$

which is the same as (A1.5).

Having shown that $R_{\ell,m}$ and $P_{\ell,m}$ satisfy the same recurrence relation, we must now verify that (A1.1) is valid for the cases :-

(i) $\ell = m$

(ii) $\ell = m - 1$

and (iii) $\ell = 0$

since all other values of ℓ and m can be obtained from these by using (A1.5) and (A1.10).

Again using the results in Appendix 2, we see that

$$R_{\ell,\ell}(k^2, q^2) = \left(\frac{k^2}{q^2}\right)^{\ell} \tag{A1.11}$$

$$R_{\ell,\ell+1}(k^2, q^2) = (2\ell+3)\left(1 - \frac{k^2}{q^2}\right)\left(\frac{k^2}{q^2}\right)^{\ell} \tag{A1.12}$$

and

$$R_{0,m}(k^2, q^2) = \frac{q^2}{2k^2}\left[P_{m-1}\left(1-\frac{2k^2}{q^2}\right) - P_{m+1}\left(1-\frac{2k^2}{q^2}\right)\right] \tag{A1.13}$$

from (A2.10), (A2.11) and (A2.7) respectively.

Also, from the properties of the Jacobi polynomials (see, for example, Abramowitz and Stegun (1968) ; Chapter 22) we have

$$\begin{aligned} P_{\ell,\ell}\left(\frac{k^2}{q^2}\right) &= \left(\frac{k^2}{q^2}\right)^{\ell} P_0^{(2\ell+1,-1)}\left(1 - \frac{2k^2}{q^2}\right) \\ &= \left(\frac{k^2}{q^2}\right)^{\ell} \end{aligned} \tag{A1.14}$$

$$\begin{aligned} P_{\ell,\ell+1}\left(\frac{k^2}{q^2}\right) &= \frac{(2\ell+3)}{(2\ell+2)}\left(\frac{k^2}{q^2}\right)^{\ell} P_1^{(2\ell+1,-1)}\left(1 - \frac{2k^2}{q^2}\right) \\ &= (2\ell+3)\left(1 - \frac{k^2}{q^2}\right)\left(\frac{k^2}{q^2}\right)^{\ell} \end{aligned} \tag{A1.15}$$

and

$$\begin{aligned} P_{0,m}\left(\frac{k^2}{q^2}\right) &= \frac{(2m+1)}{(m+1)} P_m^{(1,-1)}\left(1 - \frac{2k^2}{q^2}\right) \\ &= \frac{q^2}{2k^2}\left[P_{m-1}\left(1-\frac{2k^2}{q^2}\right) - P_{m+1}\left(1 - \frac{2k^2}{q^2}\right)\right] \end{aligned} \tag{A1.16}$$

thereby completing the proof of (A1.1).

Note that the recurrence relation in (A1.5) is also valid for $m = \ell$ since $R_{\ell,m} = 0$ for $\ell > m$.

The proof of the sum rule (A1.4) is obtained by expanding each $R_{\ell,m}(k^2,q^2)$ as a series in k^2/q^2, and we see that (A1.4) becomes

$$\sum_{n=\ell}^{m}\sum_{i=0}^{n-\ell}\sum_{j=0}^{m-n}(-1)^{i+j}\left(\frac{k^2}{p^2}\right)^{i+\ell}\left(\frac{p^2}{q^2}\right)^{j+n}\frac{(2n+1)(2m+1)(n+i+\ell)!\,(m+j+n)!}{i!\,j!\,(i+2\ell+1)!\,(n-i-\ell)!\,(j+2n+1)!\,(m-j-n)!}$$
$$= \sum_{i=0}^{m-\ell}(-1)^{i}\left(\frac{k^2}{q^2}\right)^{i+\ell}\frac{(2m+1)\,(m+i+\ell)!}{i!\,(i+2\ell+1)!\,(m-i-\ell)!} \qquad \text{(A1.17)}$$

which we assert is true for all values of k^2, q^2 and p^2. Interchanging the order of the n and i summations we see that the limits become $i = 0$ to $m-\ell$ and $n = \ell+i$ to m. Therefore, for (A1.17) to be true for all k^2, we require

$$\sum_{n=\ell+i}^{m}\sum_{j=0}^{m-n}(-1)^{j}\left(\frac{q^2}{p^2}\right)^{\ell+i}\left(\frac{p^2}{q^2}\right)^{j+n}\frac{(2n+1)\,(\ell+n+i)!\,(m+j+n)!}{j!\,(n-\ell-i)!\,(j+2n+1)!\,(m-j-n)!}$$
$$= \frac{(m+i+\ell)!}{(m-i-\ell)!} \qquad \text{(A1.18)}$$

for all q^2 and p^2 and $0 \leq i \leq m-\ell$.

Setting $r = j + n$ we then have

$$\sum_{r=i+\ell}^{m}\sum_{j=0}^{r-i-\ell}(-1)^{j}\left(\frac{p^2}{q^2}\right)^{r-i-\ell}\frac{(2r-2j+1)\,(\ell+r-j+i)!\,(m+r)!}{j!\,(r-j-\ell-i)!\,(2r-j+1)!\,(m-r)!}$$
$$= \frac{(m+i+\ell)!}{(m-i-\ell)!} \qquad (0 \leq i \leq m-\ell) \qquad \text{(A1.19)}$$

for all q^2 and p^2, and so

$$\sum_{j=0}^{r-i-\ell}(-1)^{j}\frac{(2r-2j+1)\,(\ell+r-j+i)!\,(m+r)!}{j!\,(r-j-\ell-i)!\,(2r-j+1)!\,(m-r)!} = \frac{(m+i+\ell)!}{(m-i-\ell)!}\,\delta_{r,i+\ell} \qquad \text{(A1.20)}$$

for $0 \leq i \leq m-\ell$ and $i+\ell \leq r \leq m$.

Therefore we require

$$\sum_{j=0}^{r-i-\ell}(-1)^{j}\frac{(2r-2j+1)(\ell+r-j+i)!}{j!\,(r-j-\ell-i)!\,(2r-j+1)!} = \delta_{r,i+\ell} \qquad \text{(A1.21)}$$

for $r \geq i+\ell$, which after setting $N = r - i - \ell$, $a = 2i + 2\ell$ and $j = N - j$ becomes

$$\sum_{j=0}^{N}(-1)^{j}\frac{(2j+a+1)\,(j+a)!}{j!\,(N-j)!\,(N+a+j+1)!} = \delta_{N,0} \qquad \text{(A1.22)}$$

for $N \geq 0$, $a \geq 0$ and even.

Therefore, for (A1.4) to be valid for all k^2 ,q^2 and p^2 we require (A1.22) to be valid for all $N \geqslant 0$. For $N = 0$ the result is trivial and for $N > 0$ we have

$$\sum_{j=0}^{N} (-1)^j \frac{(2j+a+1)\,(j+a)!}{j!\,(N-j)!\,(N+a+j+1)!} = \sum_{j=0}^{N} (-1)^j \frac{(j+a+1)!}{j!\,(N-j)!\,(N+a+j+1)!} - \sum_{j=0}^{N-1} (-1)^j \frac{(j+a+1)!}{j!\,(N-j-1)!\,(N+a+j+2)!}$$

$$= \frac{(1+a)!}{(2N+a+1)!}\left[P_N^{(N+a+1,-2N)}(-1) - P_{N-1}^{(N+a+2,-2N)}(-1) \right]$$

$$= (-1)^N \frac{(1+a)!}{(2N+a+1)!}\left[\binom{-N}{N} + \binom{-N-1}{N-1} \right] = 0$$

Therefore the sum rule (A1.4) is true for all values of k^2 ,q^2 and p^2 .

APPENDIX 2

In this appendix we prove that the function

$$J_{\ell,m}(a,b) = \int_{-1}^{1} P_\ell(\mu)\, P_m(a+b\mu)\, d\mu \tag{A2.1}$$

satisfies the recurrence relation

$$(2\ell+1)\, J_{\ell,m+1} + (2m+1)b\, J_{\ell+1,m} = (2\ell+1)\, J_{\ell,m-1} + (2m+1)b\, J_{\ell-1,m} \tag{A2.2}$$

for $m > \ell \geqslant 1$.

Since (Abramowitz and Stegun (1968) ; 8.5.3 and 8.5.4)

$$\int_{-1}^{\mu} P_\ell(x)\, dx = (2\ell+1)^{-1}\left[P_{\ell+1}(\mu) - P_{\ell-1}(\mu)\right] \tag{A2.3}$$

for $\ell \neq 0$, we have

$$(2\ell+1)^{-1}\left[P_{\ell+1}(a+b\mu) - P_{\ell-1}(a+b\mu)\right] = b\int_{\frac{-1-a}{b}}^{\mu} P_\ell(a+bx)\, dx \tag{A2.4}$$

and hence, for m > 0,

$$\begin{aligned} J_{\ell,m+1} - J_{\ell,m-1} &= (2m+1)b\int_{-1}^{1} P_\ell(\mu) \int_{\frac{-1-a}{b}}^{\mu} P_m(a+bx)\, dx\, d\mu \\ &= (2m+1)b\int_{-1}^{1} P_\ell(x)\, dx \int_{\frac{-1-a}{b}}^{1} P_m(a+bx)\, dx \\ &\quad - (2m+1)b\int_{-1}^{1} P_m(a+b\mu)\int_{-1}^{\mu} P_\ell(x)\, dx\, d\mu \\ &= -(2m+1)b\int_{-1}^{1} P_m(a+b\mu)\int_{-1}^{\mu} P_\ell(x)\, dx\, d\mu \end{aligned} \tag{A2.5}$$

since $\int_{-1}^{1} P_\ell(x)\, dx = 0$ for $\ell > 0$.

Therefore

$$\begin{aligned} J_{\ell,m+1} - J_{\ell,m-1} &= -\frac{(2m+1)b}{(2\ell+1)}\int_{-1}^{1} P_m(a+b\mu)\left[P_{\ell+1}(\mu) - P_{\ell-1}(\mu)\right] d\mu \\ &= -\frac{(2m+1)b}{(2\ell+1)}\left[J_{\ell+1,m} - J_{\ell-1,m}\right] \end{aligned} \tag{A2.6}$$

which is the same as (A2.2) thereby proving the recurrence relation.

Also, from (A2.4), we readily see that

$$J_{0,m}(a,b) = b^{-1}(2m+1)^{-1}\Big[P_{m+1}(a+b) - P_{m-1}(a+b) - P_{m+1}(a-b) + P_{m-1}(a-b) \Big] \tag{A2.7}$$

Using (Abramowitz and Stegun (1968) ; 8.14.15)

$$\int_{-1}^{1} P_\ell(x)\, x^{\ell+2n}\, dx = \begin{cases} 2^{\ell+1} \dfrac{(\ell+2n)!\,(\ell+n)!}{n!\,(2n+2\ell+1)!} & (n \geqslant 0) \\ 0 & (n<0) \end{cases} \tag{A2.8}$$

and (Abramowitz and Stegun (1968) ;22.3.8)

$$P_\ell(x) = 2^{-\ell} \sum_{n=0}^{[\frac{\ell}{2}]} (-1)^n \binom{\ell}{n} \binom{2\ell-2n}{\ell} x^{\ell-2n} \tag{A2.9}$$

we also see that

$$\begin{aligned} J_{\ell,\ell}(a,b) &= \int_{-1}^{1} P_\ell(x)\, 2^{-\ell} \binom{2\ell}{\ell} b^\ell x^\ell\, dx \\ &= 2(2\ell+1)^{-1} b^\ell \end{aligned} \tag{A2.10}$$

and

$$\begin{aligned} J_{\ell,\ell+1}(a,b) &= \int_{-1}^{1} P_\ell(x)\, 2^{-\ell-1} \binom{2\ell+2}{\ell+1} (\ell+1)\, a b^\ell x^\ell\, dx \\ &= 2ab^\ell \end{aligned} \tag{A2.11}$$

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