THE UNIVERSITY OF HULL

Spectral theory of Herglotz functions and their compositions, and the Schrödinger equation

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Yiannis Takis Christodoulides, BSc(Hons)

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Abstract

In this thesis we generalize the theory of value distribution associated to a Herglotz function. Compositions of Herglotz functions are studied, and some results regarding the integral representation of a composed Herglotz function are obtained. Properties of spectral measures corresponding to Herglotz functions are derived, and an application to the Schrödinger equation is given.

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Chapter 1

Introduction

A main theme of this thesis will be the spectral theory of Herglotz functions, that is functions analytic and with positive imaginary part in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : Im z > 0\}$. To look into the spectral properties of a given Herglotz function, we need to consider the associated measures. A general Herglotz function $F(z)$ has the representation

$$F(z) = a_F + b_F z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\rho(t), \tag{1.1}$$

where a_F and b_F are real constants ($b_F > 0$), the 'spectral function' $\rho(t)$ is non-decreasing and right-continuous and the convergence condition

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\rho(t) < \infty \tag{1.2}$$

is satisfied. For given $F(z)$, the values of a_F and b_F are determined by

$$a_F = ReF(i), \quad b_F = \lim_{s\to\infty} \frac{ImF(is)}{s} \tag{1.3}$$

($s \to \infty$ through real values), and the function $\rho(t)$ is determined up to an additive constant. Hence ρ determines a σ-finite measure $\mu = d\rho$, where $\mu((a,b]) = \rho(b) - \rho(a)$ for finite intervals $(a,b]$, and μ extends to a measure on Borel subsets of $\mathbb{R}$. Measures satisfying (1.2) are called Herglotz measures since they may be used to define a Herglotz function, and we refer to μ as the Herglotz measure corresponding to F.

The spectral measure μ may be decomposed into its absolutely continuous and singular parts, $\mu = \mu_{ac} + \mu_s$ through an analysis of the boundary behaviour of F as z approaches the real axis. In this thesis we will be interested mostly in the absolutely continuous measure for a given Herglotz function. We will denote by $F_+(\lambda)$ the boundary value of F as z approaches the real axis; that is

$$F_+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + i\varepsilon),$$

for each $\lambda \in \mathbb{R}$ for which this limit exists finitely. The theory of boundary values of analytic functions tells us that $F_+(\lambda)$ is defined for (Lebesgue) almost all λ. The density function of μ_{ac} is $\frac{1}{\pi} Im F_+(\lambda)$. In the case of a Herglotz function with purely singular measure, the boundary value is real almost everywhere, and hence the Herglotz function defines a real valued boundary value function $f(\lambda) = F_+(\lambda)$. Although one might think that the two cases are independent, any Herglotz function with complex boundary values may be obtained as a limit of a sequence of Herglotz functions F_n with real boundary values almost everywhere. We shall show in Chapter 5 that, if μ_n are the measures corresponding to the functions F_n, we have in that case $\mu_n \to \mu$. Hence even in looking at Herglotz functions with absolutely continuous measure, the case of singular measure will also be important. In fact, in the spectral analysis of Herglotz functions one needs not one but many Herglotz functions, that is a whole family of them.

Thus, given a Herglotz function F, we generate a corresponding one-parameter family of Herglotz functions F_y $(y \in \mathbb{R})$ defined by

$$F_y(z) = \frac{1}{y - F(z)}.$$

We turn now to the idea of value distribution for real valued functions. By value distribution of a real valued, Lebesgue measurable function f, we shall refer to a corresponding mapping $\mathcal{M}$ assigning a non-negative extended real number, which we shall denote by $\mathcal{M}(A, S; f)$, to each pair of Borel subsets A and S. In the case of a Herglotz function F with purely singular measure, we define (and denote) the value distribution corresponding to the real valued function $f(\lambda)$ by

$$\mathcal{M}(A, S; F) = \int_S \mu_y(A)dy, \tag{1.4}$$

where μ_y are the measures corresponding to the Herglotz functions F_y. We refer to the integral in (1.5) as the value distribution associated with the Herglotz function F. Moreover, we have in this case

$$\int_S \mu_y(A)dy = |A \cap F_+^{-1}(S)|, \tag{1.5}$$

where $|.|$ denotes Lebesgue measure, so that $\mathcal{M}(A, S; F)$ is the Lebesgue measure of the points $\lambda \in A$ for which $F_+(\lambda) \in S$. Note that the value distribution formula (1.4) may be regarded as a formula for some kind of average (over Lebesgue measure) of the spectral measure. For an analysis of the value distribution in (1.4) see [23]. Because an absolutely continuous Herglotz measure can be approximated by a sequence of singular Herglotz measures, we can also associate a value distribution with a general Herglotz function.

A major argument in this thesis will be to generalize this theory in two directions:

(i) We will consider the case in which Lebesgue measure dy in (1.4) is replaced by a measure $d\sigma$ corresponding to a Herglotz function ϕ_s; that is, we will take the average of the spectral measure over other measures.
(ii) We will examine the case in which $|A \cap F_+^{-1}(S)|$ becomes $\nu_s(A \cap F_+^{-1}(S))$, where ν_s is some measure other than Lebesgue measure.

In fact these two directions are not independent because $d\nu_s$ is related to $d\sigma$. We shall find in Chapter 3 that ν_s satisfies the convergence condition (1.2), and thus may be used to define a Herglotz function through the formula (1.1). Furthermore, we will show that there is a close connection between ν_s and the measure corresponding to the composed Herglotz function $\phi_s \circ F$.

In Chapter 5 we will apply the generalized theory of value distribution to the Schrödinger equation

$$-\frac{d^2 f(x)}{dx^2} + V(x)f(x) = zf(x), \qquad z \in \mathbb{C}_+, \tag{1.6}$$

where V is a locally integrable potential giving rise to absolutely continuous spectrum. The crucial link is the Weyl-Titchmarsh m-function of equation (1.6), which is a Herglotz function and defined by the condition that

$$u(.,z) + m(z)v(.,z) \in L^2(0,\infty),$$

where u and v are solutions of (1.6) satisfying specific initial conditions. Our main result, to obtain which we shall need to impose certain conditions, may be regarded as the generalized value distribution for the logarithmic derivative $q(v) = v'/v$ of the solution v of (1.6). In this Chapter we will use some ideas of the geometry of the upper half plane, namely the angle subtended $\theta(z,S)$ at a point $z \in \mathbb{C}_+$ by a Borel set on the real axis, and the distance of separation $\gamma(z_1, z_2)$ of two points z_1, $z_2 \in \mathbb{C}_+$.

Chapter 2

Mathematical background

2.1 Introduction

This Chapter serves as a mathematical background for the rest of the thesis. The subjects to be discussed are: measure theory, analysis, complex analysis, Herglotz functions, analysis of measures, existence and uniqueness theorems for differential equations, and the Weyl-Titchmarsh m-function. The material presented in each section is *drawn* either from *all* the references mentioned in that section and only those, or otherwise, from the references that we state at the end of the section.

2.2 Measure theory

Definition 2.1 *(σ-algebras of sets). Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is a σ-algebra on X if:*

(i) $X \in \mathcal{A}$,

(ii) for each set A that belongs to $\mathcal{A}$ the set A^c belongs to $\mathcal{A}$, where $A^c = X \backslash A$ is the complement of A in X, and

(iii) for each infinite sequence $\{A_i\}$ of sets that belong to $\mathcal{A}$ the set $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{A}$.

Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation and under the formation of countable unions. The most common σ-algebra is the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$, which is generated by the collection of all open subsets of $\mathbb{R}$.

A collection $\mathcal{A}'$ of subsets of X is an *algebra* on X if it satisfies conditions (i) and (ii) in definition (2.1), and also condition (iii) for a *finite* sequence $\{A'_i\}$ of sets that belong to $\mathcal{A}'$.

Definition 2.2 *(Measures). Let X be a set, and let $\mathcal{A}$ be a σ-algebra on X. A measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ that satisfies:*
(i) $\mu(\emptyset) = 0$, and
(ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, for each infinite sequence $\{A_i\}$ of disjoint sets that belong to $\mathcal{A}$. (Since $\mu(A_i)$ is non-negative for each i, the sum $\sum_{i=1}^{\infty} \mu(A_i)$ always exists, either as a real number or as $+\infty$.)
The sets in $\mathcal{A}$ are said to be measurable with respect to μ.

We refer to condition (ii) in definition (2.2) as the *countable additivity* property of measures. A measure μ is called a finite measure if $\mu(X) < +\infty$, and is a σ-finite measure if X is the union of a sequence A_1, A_2,... of sets that belong to $\mathcal{A}$ and satisfy $\mu(A_i) < +\infty$ for each i. If X is a set, $\mathcal{A}$ is a σ-algebra on X, and μ is a measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a measure space. Likewise, if X is a set and $\mathcal{A}$ a σ-algebra on X, then the pair $(X, \mathcal{A})$ is often called a measurable space. The following lemmas give some elementary but useful properties of measures.

Lemma 2.3 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let A and B be subsets of X that belong to $\mathcal{A}$ and satisfy $A \subseteq B$. Then $\mu(A) \leq \mu(B)$. If in addition A satisfies $\mu(A) < +\infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.*

Proof. The sets A and $B \backslash A$ are disjoint and satisfy $B = A \cup (B \backslash A)$; thus the additivity of μ implies that

$$\mu(B) = \mu(A) + \mu(B \backslash A).$$

Since $\mu(B \backslash A) \geq 0$, it follows that $\mu(B) \geq \mu(A)$. In case $\mu(A) < +\infty$, the relation $\mu(B) - \mu(A) = \mu(B \backslash A)$ also follows.

Lemma 2.4 *Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\{A_k\}$ is an arbitrary sequence of sets that belong to $\mathcal{A}$, then*

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Proof. Define a sequence $\{B_k\}$ of subsets of X by letting $B_1 = A_1$ and $B_k = A_k \backslash (\bigcup_{i=1}^{k-1} A_i)$ if $k > 1$. Then each B_k belongs to $\mathcal{A}$ and is a subset of the corresponding A_k, and so satisfies $\mu(B_k) \leq \mu(A_k)$. Since in addition the sets B_k are disjoint and satisfy $\bigcup_k B_k = \bigcup_k A_k$, it follows that

$$\mu\Big(\bigcup_k A_k\Big) = \sum_k \mu(B_k) \leq \sum_k \mu(A_k).$$

Thus the countable additivity of μ implies the *countable subadditivity* of μ.

Lemma 2.5 *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

(i) If $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{A}$, then $\mu(\bigcup_k A_k) = \lim_k \mu(A_k)$.

(ii) If $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{A}$, and if $\mu(A_n) < +\infty$ holds for some n, then $\mu(\bigcap_k A_k) = \lim_k \mu(A_k)$.

Proof. First suppose that $\{A_k\}$ is an increasing sequence of sets that belong to $\mathcal{A}$, and define a sequence $\{B_j\}$ of sets by letting $B_1 = A_1$ and letting $B_j = A_j \backslash A_{j-1}$ if $j > 1$. The sets just constructed are disjoint, belong to $\mathcal{A}$, and satisfy $A_k = \bigcup_{j=1}^{k} B_j$ for each k. It follows that $\bigcup_k A_k = \bigcup_j B_j$ and hence that

$$\mu\Big(\bigcup_k A_k\Big) = \sum_j \mu(B_j) = \lim_k \sum_{j=1}^{k} \mu(B_j)$$
$$= \lim_k \mu\Big(\bigcup_{j=1}^{k} B_j\Big) = \lim_k \mu(A_k).$$

This completes the proof of (i).

Now suppose that $\{A_k\}$ is a decreasing sequence of sets that belong to $\mathcal{A}$, and that $\mu(A_n) < +\infty$ holds for some n. We can assume that $n = 1$. For each k let $C_k = A_1 \backslash A_k$. Then $\{C_k\}$ is an increasing sequence of sets that belong to $\mathcal{A}$, and $\bigcup_k C_k = A_1 \backslash (\bigcap_k A_k)$. It follows from part (i) that $\mu(\bigcup_k C_k) = \lim_k \mu(C_k)$ and hence that

$$\mu\Big(A_1 \backslash \Big(\bigcap_k A_k\Big)\Big) = \lim_k \mu(A_1 \backslash A_k).$$

In view of lemma (2.3) and the assumption that $\mu(A_1) < +\infty$, this implies that $\mu(\bigcap_k A_k) = \lim_k \mu(A_k)$.

Definition 2.6 (*Null sets*). *Let $(X, \mathcal{A}, \mu)$ be a measure space. A subset B of X is said to be μ-null if there is a subset A of X such that $A \in \mathcal{A}$, $B \subset A$, and $\mu(A) = 0$.*

Definition 2.7 (*Simple functions*). *A simple function f is a function which admits a finite number of values.*

Definition 2.8 (*Measurable functions*). *Let $(X, \mathcal{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathcal{A}$. A function $f : A \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable if for each real number t the set $\{x \in A : f(x) < t\}$ belongs to $\mathcal{A}$.*

The following basic result is needed in order to obtain more powerful results like, for example, the monotone convergence theorem.

Lemma 2.9 *Let $(X, \mathcal{A})$ be a measurable space, let A be a subset of X that belongs to $\mathcal{A}$, and let f be a $[0, +\infty]$-valued measurable function on A. Then there is a sequence $\{f_n\}$ of $[0, +\infty)$-valued simple measurable functions on A that satisfy, at each x in A,*

(i) $f_1(x) \leq f_2(x) \leq \ldots$, and

(ii) $f(x) = \lim_n f_n(x)$.

Proof. For each positive integer n and for $k = 1, 2, \ldots, n2^n$ let $A_{n,k} = \{x \in A : (k-1)/2^n \leq f(x) < k/2^n\}$. The measurability of f implies that each $A_{n,k}$ belongs to $\mathcal{A}$. Define a sequence $\{f_n\}$ of functions from A to $\mathbb{R}$ by requiring f_n to have value $(k-1)/2^n$ at each point in $A_{n,k}$ (for $k = 1, \ldots, n2^n$) and to have value n at each point in $A \setminus \bigcup_k A_{n,k}$. The functions so defined are simple and measurable, and it is easy to check that they satisfy (i) and (ii) at each x in A.

We now construct the integral. The construction will take place in three stages. We begin with the simple functions. Let $(X, \mathcal{A})$ be a measurable space. We shall denote by $\mathcal{S}$ the collection of all real-valued simple $\mathcal{A}$-measurable functions on X, and by $\mathcal{S}_+$ the collection of non-negative functions in $\mathcal{S}$.

Let μ be a measure on $(X, \mathcal{A})$. If f belongs to $\mathcal{S}_+$ and is given by $f = \sum_{i=1}^m a_i \chi_{A_i}$, where $a_1, \ldots, a_m$ are non-negative real numbers and $A_1, \ldots, A_m$ are disjoint subsets of X that belong to $\mathcal{A}$, then $\int f d\mu$, the *integral* of f with respect to μ, is defined to be $\sum_{i=1}^m a_i \mu(A_i)$ (note that this sum is either a non-negative real number or $+\infty$). The integral $\int f d\mu$ depends only on f, and not on $a_1, \ldots, a_m$ and $A_1, \ldots, A_m$.

Next we define the integral of an arbitrary $[0, +\infty]$-valued $\mathcal{A}$-measurable function on X. For such a function f let

$$\int f d\mu = \sup\left\{\int g d\mu : g \in \mathcal{S}_+ \text{ and } g \leq f\right\}.$$

Before we proceed to the last stage of the construction of the integral, we give some properties of the integral defined so far.

Lemma 2.10 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let f be a $[0, +\infty]$-valued $\mathcal{A}$-measurable function on X, and let $\{f_n\}$ be a non-decreasing sequence of functions in $\mathcal{S}_+$ for which $f(x) = \lim_n f_n(x)$ holds at each $x \in X$. Then $\int f d\mu = \lim_n \int f_n d\mu$.*

Proof. See [10], Prop. (2.3.3), p.63.

Lemma 2.11 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let f and g be $[0, +\infty]$-valued $\mathcal{A}$-measurable functions on X, and let α be a non-negative real number. Then*

(i) $\int \alpha f d\mu = \alpha \int f d\mu$,

(ii) $\int (f + g) d\mu = \int f d\mu + \int g d\mu$, and

(iii) if $f(x) \leq g(x)$ holds at each x in X, then $\int f d\mu \leq \int g d\mu$.

Proof. Choose non-decreasing sequences $\{f_n\}$ and $\{g_n\}$ of functions in $\mathcal{S}_+$ such that $f = \lim_n f_n$ and $g = \lim_n g_n$ (see lemma (2.9)). Then $\{\alpha f_n\}$ and $\{f_n + g_n\}$ are non-decreasing sequences of functions in $\mathcal{S}_+$ that satisfy $\alpha f = \lim_n \alpha f_n$ and $f + g = \lim_n (f_n + g_n)$, and by using lemma (2.10) we conclude that

$$\int \alpha f d\mu = \lim_n \int \alpha f_n d\mu = \lim_n \alpha \int f_n d\mu = \alpha \int f d\mu$$

and

$$\int (f+g)d\mu = \lim_n \int (f_n + g_n)d\mu = \lim_n \left(\int f_n d\mu + \int g_n d\mu \right)$$
$$= \int f d\mu + \int g d\mu.$$

Thus parts (i) and (ii) are proved. For part (iii), note that if $f \leq g$, then the class of functions h in $\mathcal{S}_+$ that satisfy $h \leq f$ is included in the class of functions h in $\mathcal{S}_+$ that satisfy $h \leq g$; it follows that $\int f d\mu \leq \int g d\mu$.

Finally, let f be an arbitrary $[-\infty, +\infty]$-valued $\mathcal{A}$-measurable function on X. The *positive part* and the *negative part* of f are the extended real-valued functions defined by

$$f^+(x) = \max(f(x), 0), \qquad f^-(x) = -\min(f(x), 0).$$

If $\int f^+ d\mu$ and $\int f^- d\mu$ are both finite, then f is called *integrable* (or *μ-integrable*), and its *integral* $\int f d\mu$ is defined to be

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

The integral of f is said to exist if at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite, and again in this case $\int f d\mu$ is defined to be $\int f^+ d\mu - \int f^- d\mu$.

We proceed to prove the basic limit theorems of integration theory, which are extremely important. We shall use these results repeatedly.

Theorem 2.12 (*The monotone convergence theorem*). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and $f_1, f_2, \ldots$ be $[0, +\infty]$-valued $\mathcal{A}$-measurable functions on X. Suppose that the relations*

(i) $f_1(x) \leq f_2(x) \leq \ldots$, *and*

(ii) $f(x) = \lim_n f_n(x)$

hold at every x in X. Then, we have

$$\int f d\mu = \lim_n \int f_n d\mu. \tag{2.1}$$

Proof. The monotonicity of the integral (part (iii) of lemma (2.11)) implies that

$$\int f_1 d\mu \leq \int f_2 d\mu \leq ... \leq \int f d\mu;$$

hence the sequence $\{\int f_n d\mu\}$ converges (perhaps to $+\infty$), and its limit satisfies $\lim_n \int f_n d\mu \leq \int f d\mu$. We turn to the reverse inequality. For each n choose a non-decreasing sequence $\{g_{n,k}\}_{k=1}^{\infty}$ of simple $[0, +\infty)$-valued measurable functions such that $f_n = \lim_k g_{n,k}$ (lemma (2.9)). For each n define a function h_n by

$$h_n = \max(g_{1,n}, g_{2,n}, ..., g_{n,n}).$$

Then $\{h_n\}$ is a non-decreasing sequence of simple $[0, +\infty)$-valued measurable functions that satisfy $h_n \leq f_n$ and $f = \lim_n h_n$. It follows from these remarks, lemma (2.10), and the monotonicity of the integral that

$$\int f d\mu = \lim_n \int h_n d\mu \leq \lim_n \int f_n d\mu.$$

Hence $\int f d\mu = \lim_n \int f_n d\mu$.

Theorem 2.13 (*Fatou's Lemma*). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of $[0, +\infty]$-valued $\mathcal{A}$-measurable functions on X. Then*

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Proof. For each positive integer n let $g_n = \inf_{k \geq n} f_k$. Each g_n is $\mathcal{A}$-measurable (see [10], Prop. (2.1.4), p.51), and it is easy to check that

$$g_1(x) \leq g_2(x) \leq \ldots$$

and

$$\liminf_n f_n(x) = \lim_n g_n(x)$$

hold at each x in X. Thus it follows from the monotone convergence theorem (2.12) and the inequality $g_n \leq f_n$ that

$$\int \liminf_n f_n d\mu = \lim_n \int g_n d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem 2.14 (*Lebesgue's dominated convergence theorem*). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let g be a $[0, +\infty)$-valued integrable function on X, and let f and $f_1, f_2, \ldots$ be $[-\infty, +\infty]$-valued $\mathcal{A}$-measurable functions on X such that the relations*

(i) $f(x) = \lim_n f_n(x)$, *and*

(ii) $|f_n(x)| \leq g(x), \quad n = 1, 2, \ldots$

hold at every x in X. Then f and $f_1, f_2, \ldots$ are integrable, and

$$\int f d\mu = \lim_n \int f_n d\mu. \tag{2.2}$$

Proof. The integrability of f and of $f_1, f_2, ..$ follows from that of g; (see [10], Prop. (2.3.7), (2.3.8), p.66-67, and part (iii) of our lemma (2.11)). Note that $\{g + f_n\}$ is a sequence of non-negative $\mathcal{A}$-measurable functions such that $(g + f)(x) = \lim_n (g + f_n)(x)$ holds at each x in X, and so Fatou's lemma (Theorem (2.13)) implies that

$$\int (g + f) d\mu \leq \liminf_n \int (g + f_n) d\mu$$

and hence that

$$\int f d\mu \leq \liminf_n \int f_n d\mu.$$

A similar argument, applied to the sequence $\{g - f_n\}$, shows that

$$\int (g - f) d\mu \leq \liminf_n \int (g - f_n) d\mu$$

and hence that

$$\limsup_n \int f_n d\mu \leq \int f d\mu.$$

Consequently, (2.2) is proved.

Definition 2.15 (*Absolutely continuous measures*). *Let $(X, \mathcal{A})$ be a measurable space, and let μ and ν be measures on $(X, \mathcal{A})$. Then ν is absolutely continuous with respect to μ if each set A that belongs to $\mathcal{A}$ and satisfies $\mu(A) = 0$ also satisfies $\nu(A) = 0$. We write $\nu \ll \mu$ to indicate that ν is absolutely continuous with respect to μ.*

The following result, which we shall only state, is very important.

Theorem 2.16 (*Radon-Nikodym theorem*). *Let $(X, \mathcal{A})$ be a measurable space, and let μ and ν be σ-finite measures on $(X, \mathcal{A})$. If ν is absolutely continuous with respect to μ, then there is an $\mathcal{A}$-measurable function $g : X \to [0, +\infty)$ such that $\nu(A) = \int_A g d\mu$ holds for each A in $\mathcal{A}$. The function g is unique up to μ-almost everywhere equality, that is g is unique except on a μ-null set.*

Proof. See [10], Thm. (4.2.2), p.132.

Definition 2.17 (*Singular measures*). *Let $(X, \mathcal{A})$ be a measurable space. A measure μ on $(X, \mathcal{A})$ is concentrated on the $\mathcal{A}$-measurable set E if $\mu(E^c) = 0$.*

Suppose that μ and ν are measures on $(X, \mathcal{A})$. Then μ and ν are said to be mutually singular, *(or singular with respect to each other), if there is an $\mathcal{A}$-measurable set E such that μ is concentrated on E and ν is concentrated on E^c. We write $\mu \perp \nu$ to indicate that μ and ν are mutually singular.*

We conclude this section with another very important result, which states that a finite or σ-finite measure can be split into an absolutely continuous and a singular part.

Theorem 2.18 *(Lebesgue decomposition theorem). Let $(X, \mathcal{A})$ be a measurable space, let μ be a measure on $(X, \mathcal{A})$, and let ν be a finite or a σ-finite measure on $(X, \mathcal{A})$. Then there are unique measures ν_a and ν_s on $(X, \mathcal{A})$ such that*
(i) ν_a is absolutely continuous with respect to μ,
(ii) ν_s is singular with respect to μ, and
(iii) $\nu = \nu_a + \nu_s$.
The decomposition $\nu = \nu_a + \nu_s$ is called the Lebesgue decomposition of ν, while ν_a and ν_s are called the absolutely continuous and singular parts of ν.

Proof. We begin with the case in which ν is a finite measure. Define $\mathcal{N}_\mu$ by

$$\mathcal{N}_\mu = \{B \in \mathcal{A} : \mu(B) = 0\},$$

and choose a sequence $\{B_j\}$ of sets in $\mathcal{N}_\mu$ such that

$$\lim_j \nu(B_j) = \sup\{\nu(B) : B \in \mathcal{N}_\mu\}.$$

Let $N = \bigcup_j B_j$, and define measures ν_a and ν_s on $(X, \mathcal{A})$ by $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$. Of course $\nu = \nu_a + \nu_s$. The countable subadditivity of μ

implies that $\mu(N) = 0$ and hence that ν_s is singular with respect to μ. Since

$$\nu(N) = \sup\{\nu(B) : B \in \mathcal{N}_\mu\},$$

each $\mathcal{A}$-measurable subset B of N^c that satisfies $\mu(B) = 0$ also satisfies $\nu(B) = 0$ (otherwise $N \cup B$ would belong to $\mathcal{N}_\mu$ and satisfy $\nu(N \cup B) > \nu(N)$). The absolute continuity of ν_a follows.

Now suppose that ν is a σ-finite measure, and let $\{D_k\}$ be a partition of X into $\mathcal{A}$-measurable sets that have finite measure under ν. For each k let $\mathcal{A}_k$ be the σ-algebra on D_k that consists of the $\mathcal{A}$-measurable subsets of D_k, and apply the construction above to the restrictions of the measures μ and ν to the spaces $(D_k, \mathcal{A}_k)$. Let $N_1, N_2, \ldots$ be the μ-null subsets of $D_1, D_2, ..$ thus constructed, and let $N = \bigcup_k N_k$. Then the measures ν_a and ν_s defined by $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$ form a Lebesgue decomposition of ν.

We turn to the uniqueness of the Lebesgue decomposition. Let $\nu = \nu_a + \nu_s$ and $\nu = \tilde{\nu}_a + \tilde{\nu}_s$ be Lebesgue decompositions of ν. First suppose that ν is a finite measure. Then

$$\nu_a - \tilde{\nu}_a = \tilde{\nu}_s - \nu_s;$$

since $(\nu_a - \tilde{\nu}_a) \ll \mu$ and $(\tilde{\nu}_s - \nu_s) \perp \mu$, it follows that

$$\nu_a - \tilde{\nu}_a = \tilde{\nu}_s - \nu_s = 0.$$

Thus $\nu_a = \tilde{\nu}_a$ and $\nu_s = \tilde{\nu}_s$. The case where ν is a σ-finite measure can be dealt with by choosing a partition $\{D_k\}$ of X into $\mathcal{A}$-measurable subsets that have finite measure under ν, and applying the preceding argument to the restrictions of ν_a, ν_s, $\tilde{\nu}_a$ and $\tilde{\nu}_s$ to the $\mathcal{A}$-measurable subsets of the sets D_k.

One sometimes goes a step further for a finite measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $C = \{x \in \mathbb{R} : \nu(\{x\}) \neq 0\}$, and note that C is countable (for each positive integer n there are only finitely many points x such that $\nu(\{x\}) \geq \frac{1}{n}$). Let ν_1 be the measure on $\mathcal{B}(\mathbb{R})$ defined by $\nu_1(A) = \nu(A \cap C)$, and let ν_2 and ν_3 be the singular and absolutely continuous (with respect to Lebesgue measure) parts of the measure $A \to \nu(A \cap C^c)$. Then $\nu = \nu_1 + \nu_2 + \nu_3$ is a decomposition of ν into the sum of a discrete measure, a continuous but singular measure, and an absolutely continuous measure. It is easy to check that the measures appearing in this decomposition are unique.

2.3 Analysis

Definition 2.19 (*Functions of bounded variation*). *Let $f(x)$ be a real-valued function which is defined and finite for all x in a closed bounded interval $a \leq x \leq b$. Let*

$$\Gamma = \{x_0, x_1, ..., x_m\}$$

be a partition of $[a, b]$; that is, Γ is a collection of points x_i, $i = 0, 1, ..., m$, satisfying $x_0 = a$, $x_m = b$, and $x_{i-1} < x_i$, for $i = 1, ..., m$. With each partition Γ, we associate the sum

$$S_\Gamma = S_\Gamma[f; a, b] = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|.$$

The variation of f over $[a, b]$ is defined as

$$V = V[f; a, b] = \sup_\Gamma S_\Gamma,$$

where the supremum is taken over all partitions Γ of $[a, b]$. Since $0 \leq S_\Gamma < +\infty$, we have $0 \leq V \leq +\infty$. If $V < +\infty$, f is said to be of bounded variation on $[a, b]$; if $V = +\infty$, f is of unbounded variation on $[a, b]$.

Corollary 2.20 *A function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference of two bounded increasing functions on $[a, b]$.*

Proof. See [28], Cor. (2.7), p.19.

Theorem 2.21 *If f is of bounded variation on $[a, b]$, then f' exists almost everywhere in $[a, b]$, and f' is integrable on $[a, b]$.*

Proof. See [28], Cor. (7.23), p.113.

Definition 2.22 (*Points of density*). *Let $E \subset \mathbb{R}$ be measurable with respect to Lebesgue measure, and let $x \in E$. Then, x is called a point of density of E if and only if*

$$\lim_{h \to 0^+} \frac{|E \cap [x - h, x + h]|}{2h} = 1,$$

where $|.|$ denotes Lebesgue measure.

Theorem 2.23 *Let $E \subset \mathbb{R}$ be measurable with respect to Lebesgue measure. Then, almost every point of E is a point of density of E. Note that, unless otherwise stated, 'almost everywhere' will refer to Lebesgue measure.*

Proof. See [28], Thm. (7.13), p.107.

Definition 2.24 (*Approximately monotonic, constant, and oscillatory functions*). *Let f be a Lebesgue measurable real valued function, finite almost everywhere on $\mathbb{R}$ and let $I_F = \{x \in \mathbb{R} : f(x)$ is finite$\}$. Then, f is said to be approximately right monotonic increasing at $x \in I_F$ provided that*

$$\lim_{h \to 0^+} |\{t \in [x, x + h] \cap I_F : f(t) > f(x)\}|/h = 1,$$

and approximately left monotonic increasing if

$$\lim_{h\to 0+} |\{t \in [x-h, x] \cap I_F : f(t) < f(x)\}|/h = 1,$$

where $|.|$ *denotes Lebesgue measure. Approximately right (left) monotonic decreasing functions are defined similarly. A function which is both approximately right and left monotonic increasing at a point* x *is said to be approximately monotonic increasing at* x.

We say that f *is approximately constant at a point* x_0 *if* x_0 *is a point of density for* $f^{-1}(\{f(x_0)\})$.

We say that f *is approximately oscillatory to the right at a point* x*, if there are sequences of real numbers* h_n, $h'_n > 0$ *approaching zero, such that both*

$$|\{t \in [x, x+h_n] \cap I_F : f(t) < f(x)\}|/h_n$$

and

$$|\{t \in [x, x+h'_n] \cap I_F : f(t) > f(x)\}|/h'_n$$

are arbitrarily close to zero. In a similar way we define f *to be approximately oscillatory to the left at* x*. A function which is approximately oscillatory both to the right and to the left of* x *will be described as approximately oscillatory at* x.

Theorem 2.25 *Let* f *be a measurable and finite almost everywhere function. Then, at almost all* x, f *is either*

(i) approximately monotonic increasing, or

(ii) approximately monotonic decreasing, or

(iii) approximately constant, or

(iv) approximately oscillatory.

Proof. See [13], Thm. 3, p.494.

Definition 2.26 (*Component Interval*). *Let S be an open subset of $\mathbb{R}$. An open interval I (which may be finite or infinite) is called a component interval of S if $I \subset S$ and if there is no open interval $J \neq I$ such that $I \subset J \subset S$.*

Theorem 2.27 *Every point of a nonempty open set S belongs to one and only one component interval of S.*

Proof. See [2], Thm. (3.10), p.51.

Theorem 2.28 (*Representation theorem for open sets on the real line*). *Every non-empty open set S in $\mathbb{R}$ is the union of a countable collection of disjoint component intervals of S.*

Proof. See [2], Thm. (3.11), p.51.

2.4 Complex analysis

Definition 2.29 (*Connected sets, domains*). *A set S is said to be connected if there do not exist disjoint open sets U and V satisfying the following conditions:*
(i) $U \cup V \supset S$,
(ii) $U \cap S \neq \emptyset, \quad V \cap S \neq \emptyset$.
An open connected set is called a domain.

The 'negative' definition for connectedness is sometimes difficult to visualize. But when the connected set is a domain, we have the following useful property.

Lemma 2.30 *Any two points in a domain can be joined by a polygonal line that lies in the domain.*

Proof. See [25], Thm. (2.2), p.21.

Definition 2.31 (*Simple and closed curves, simply connected domains*). *A continuous curve in the complex plane is defined parametrically by*

$$z(t) = x(t) + iy(t), \qquad a \le t \le b, \tag{2.3}$$

where $x(t)$ *and* $y(t)$ *are real valued, continuous functions of the real parameter* t. *For a curve* C *defined by* (2.3), *the point* $z(a)$ *is called the initial point and* $z(b)$ *the terminal point. If the initial and terminal points coincide* ($z(a) = z(b)$), *then* C *is said to be a closed curve. If* $z(t_1) \neq z(t_2)$ *when* $t_1 \neq t_2$, *so that* C *does not intersect itself, the curve is said to be simple. A closed curve that is simple in the interval* $a \le t \le b$ *is said to be a simple closed curve. A domain* $\mathcal{D}$ *is simply connected if all points enclosed by any simple closed curve contained in* $\mathcal{D}$ *are elements of* $\mathcal{D}$.

Definition 2.32 (*Analytic functions*). *A function* $f(z)$ *of the complex variable* z *is said to be analytic at a point* z_0 *if* f *is differentiable in a 'neighborhood' of* z_0. *A function which is analytic at every point of the complex plane is called an entire function.*

Theorem 2.33 (*Cauchy's integral theorem*). *Let* $\mathcal{D}$ *be a simply connected domain, and let* $f(z)$ *be a single-valued analytic function on* $\mathcal{D}$. *Then,*

$$\int_C f(z)dz = 0, \tag{2.4}$$

where C *is any closed curve with finite 'length', contained in* $\mathcal{D}$.

Proof. See [19], Thm. (13.1), p.268.

Definition 2.34 (*Harmonic functions, Cauchy-Riemann equations*). *A function $u(x,y)$ of two real variables is said to be harmonic on a domain G if the partial derivatives*

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2}$$

exist and are continuous on G, and if at every point of G, $u(x,y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2.5}$$

known as Laplace's equation. Let $u(x,y)$ and $v(x,y)$ be two functions harmonic on a domain G, which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.6}$$

at every point of G. Then $u(x,y)$ and $v(x,y)$ are said to be conjugate harmonic functions (on G), and each of the functions $u(x,y)$, $v(x,y)$ is said to be the conjugate harmonic function (or simply the harmonic conjugate) of the other.

There is an intimate connection between harmonic and analytic functions, as shown by the following result.

Lemma 2.35 *A necessary and sufficient condition condition for a function $f(z) = u(x,y) + iv(x,y)$ to be analytic on a domain G is that its real part $u(x,y)$ and imaginary part $v(x,y)$ be conjugate harmonic functions on G.*

Proof. See [20], Thm. (5.1), p.144.

Theorem 2.36 *Given a function $u(z)$, harmonic in a simply connected domain $\mathcal{D}$, there exists an analytic function $f(z)$ such that $Re f(z) = u(z)$ in $\mathcal{D}$.*

Proof. See [25], Thm. (10.1), p.268.

Theorem 2.37 (*Cauchy's theorem on the expansion of an analytic function in a power series*). *Let $f(z)$ be an analytic function on a domain G, let z_0 be an arbitrary (finite) point of G, and let $\Delta = \Delta(z_0)$ be the 'distance' between z_0 and the boundary of G. Then, there exists a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.7}$$

converging to $f(z)$ on the disk $\{z \in G : |z - z_0| < \Delta\}$.

Proof. See [19], Thm. (16.7), p.361.

Theorem 2.38 *Let $u(x, y)$ be a harmonic function on a domain G, with harmonic conjugate $v(x, y)$, let z_0 be an arbitrary (finite) point of G, and let $\Delta = \Delta(z_0)$ be the distance between z_0 and the boundary of G. Then $u(x, y)$ and $v(x, y)$ have expansions of the form*

$$u(x, y) = u(r, \theta) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n, \tag{2.8}$$

$$v(x, y) = v(r, \theta) = \beta_0 + \sum_{n=1}^{\infty} (\beta_n \cos n\theta + \alpha_n \sin n\theta) r^n \tag{2.9}$$

on the disk $|z - z_0| < \Delta$, where $z - z_0 = re^{i\theta}$.

Proof. According to theorem (2.36), there is a function $f(z)$ which is analytic on G and has $u(x, y)$ as its real part. According to theorem (2.37), $f(z)$ has the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.10}$$

on the disk $|z - z_0| < \Delta$. To obtain (2.8), we substitute

$$a_n = \alpha_n + i\beta_n, \quad z - z_0 = re^{i\theta}$$

into (2.10) and then take the real part of the resulting equation

$$f(z) = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n e^{in\theta}. \tag{2.11}$$

To obtain (2.9), we take the imaginary part of (2.11).

Lemma 2.39 *The series*

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \phi)} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n \cos n(\theta - \phi),$$
$$\frac{2\rho r \sin(\theta - \phi)}{\rho^2 + r^2 - 2\rho r \cos(\theta - \phi)} = 2\sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n \sin n(\theta - \phi) \tag{2.12}$$

converge uniformly on every compact (that is, closed and bounded) subset of the disc $|z - z_0| < \rho$.

Proof. See [20], Lem. p.149.

The integral

$$\frac{1}{2\pi} \int_0^{2\pi} f(\rho, \phi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \phi)} d\phi,$$

with the 'kernel'

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \phi)} = Re\left[\frac{\rho e^{i\phi} + (z - z_0)}{\rho e^{i\phi} - (z - z_0)}\right],$$

is called *Poisson's integral.* As we now show, a characteristic feature of harmonic functions is that they can be represented as Poisson's integrals:

Theorem 2.40 *With the same notation as in theorem (2.38), we have*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta-\phi)}d\phi, \tag{2.13}$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} v(\rho,\phi)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta-\phi)}d\phi \tag{2.14}$$

for $r < \rho < \Delta$ *and arbitrary* θ. *Moreover*

$$v(r,\theta) = \beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi)\frac{2\rho r\sin(\theta-\phi)}{\rho^2 + r^2 - 2\rho r\cos(\theta-\phi)}d\phi, \tag{2.15}$$

in terms of $u(r,\theta)$.

Proof. We start from formula (2.8), with r replaced by ρ ($\rho < \Delta$), θ replaced by ϕ, and n replaced by m:

$$u(\rho,\phi) = \alpha_0 + \sum_{m=1}^{\infty}[\alpha_m\cos(m\phi) - \beta_m\sin(m\phi)]\rho^m. \tag{2.16}$$

Using the uniform convergence of (2.16) in ϕ for every $\rho < \Delta$, we multiply (2.16) by $\cos(n\phi)$ and integrate term by term with respect to ϕ between 0 and 2π. The result is

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi)d\phi,$$

$$\tag{2.17}$$

$$\alpha_n = \frac{1}{\pi\rho^n}\int_0^{2\pi} u(\rho,\phi)\cos(n\phi)d\phi \qquad (n \geq 1).$$

Similarly, multiplying (2.16) by $\sin(n\phi)$ and integrating term by term with respect to ϕ between the same limits, we obtain

$$-\beta_n = \frac{1}{\pi\rho^n}\int_0^{2\pi} u(\rho,\phi)\sin(n\phi)d\phi \qquad (n \geq 1). \tag{2.18}$$

Substitution of (2.17) and (2.18) into (2.8) and (2.9) gives

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi)d\phi + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho,\phi)\left(\frac{r}{\rho}\right)^n \cos n(\theta-\phi)d\phi,$$

$$v(r,\theta) = \beta_0 + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho,\phi)\left(\frac{r}{\rho}\right)^n \sin n(\theta-\phi)d\phi. \tag{2.19}$$

Formulas (2.13) and (2.15) now follow at once from lemma (2.39), after multiplying the series (2.12) by

$$\frac{1}{2\pi}u(\rho,\phi),$$

integrating term by term with respect to ϕ from 0 to 2π for fixed r and ρ ($r < \rho$), and comparing the results with (2.19). Moreover, since (2.13) holds for an *arbitrary* harmonic function on G, we can replace $u(r,\theta)$ by $v(r,\theta)$, obtaining (2.14).

Lemma 2.41 (*Schwarz's Lemma*). *Suppose $f(z)$ is analytic for $|z| < R$ with $f(0) = 0$. If $|f(z)| \leq M$ in $|z| < R$,, then*

$$|f(z)| \leq \frac{Mr}{R}, \qquad |z| = r < R,$$

with equality only for $f(z) = (M/R)e^{i\alpha}z$, α real.

Proof. See [25], Lem. p.218.

Theorem 2.42 *Let $f(z)$ be a bounded analytic function in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and O, A, B be points on the boundary of the disk $\{z \in \mathbb{C} : |z| = 1\}$ such that the chords OA and OB are of equal length. If $f(z)$ converges to the number a as z approaches the point O along a simple curve γ, then $f(z)$ converges* *to a also if z approaches O along any other simple curve γ' ending at O and lying entirely within the angular sector AOB.*

Proof. See [8], §307-308, p.39.

The following result regarding boundary values of bounded analytic functions in the unit disc is very important.

Theorem 2.43 (*Fatou's Theorem*). *Let $f(z)$ be a bounded analytic function in the unit disc $|z| < 1$. Then, the points at which*

$$\lim_{r \to 1} f(re^{i\theta})$$

fails to exist constitute a set of Lebesgue measure zero.

Proof. Because of theorem (2.42), we need prove the theorem only for radial approach to the boundary. Also, we may assume without loss of generality that $f(0) = 0$, since this can always be achieved by merely adding a constant.

Let $f(z)$, then, be analytic in $|z| < 1$, and let $|f(z)| < M$ in this disc. We introduce the function

$$F(\rho, \theta) = \int_0^\theta f(\rho e^{i\theta})d\theta = \int_\rho^{\rho e^{i\theta}} \frac{f(z)}{iz}dz. \tag{2.20}$$

The second integral in this relation can be taken along any path that joins the points $z = \rho$ and $z = \rho e^{i\theta}$ and lies entirely within $|z| < 1$, since $\frac{f(z)}{iz}$ is analytic in the disc $|z| < 1$ (we define this function to have the value $\frac{1}{i}f'(0)$ at $z = 0$). Thus $F(\rho, \theta)$ is a single-valued function in $|z| < 1$, whence in particular,

$$F(\rho, -\pi) = F(\rho, \pi) \tag{2.21}$$

follows. Now by (2.20),

$$|F(\rho+\Delta\rho,\theta)-F(\rho,\theta)| \leq \left|\int_{\rho}^{\rho+\Delta\rho}\frac{f(z)}{iz}dz\right| + \left|\int_{\rho e^{i\theta}}^{(\rho+\Delta\rho)e^{i\theta}}\frac{f(z)}{iz}dz\right| < 2M|\Delta\rho|. \tag{2.22}$$

Here we have used the fact that

$$\left|\frac{f(z)}{z}\right| < M$$

holds, by Schwarz's lemma (Lemma (2.41)). On the other hand, if we fix ρ and let θ vary we obtain

$$|F(\rho,\theta+\Delta\theta)-F(\rho,\theta)| < M|\Delta\theta|. \tag{2.23}$$

From (2.22) we infer that

$$F(\theta) = \lim_{\rho\to 1} F(\rho,\theta) \tag{2.24}$$

exists for every θ, and that the convergence is uniform. By (2.23), the function $F(\theta)$ satisfies the relation

$$\left|\frac{F(\theta+\Delta\theta)-F(\theta)}{\Delta\theta}\right| \leq M,$$

which shows that $F(\theta)$ is continuous and which implies that $F(\theta)$ is of bounded variation.

We now set $z = e^{i\theta}$ and apply the Poisson integral to the circle $|z| = \rho < 1$, where $r < \rho$, obtaining

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{i\phi})\frac{\rho^2 - r^2}{\rho^2 - 2r\cos(\phi-\theta) + r^2}d\phi.$$

Upon integrating by parts, we see from (2.21) that the first resulting term (the one which is not an integral) vanishes, so that

$$f(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\rho,\phi)\frac{d}{d\phi}\Big[\frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\phi-\theta) + r^2}\Big]d\phi.$$

The left-hand side of this equation is independent of ρ. Since the integrand converges to a continuous function of ϕ as ρ tends to unity, we may set $\rho = 1$ in the last formula, obtaining

$$f(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\phi)\frac{d}{d\phi}\Big[\frac{1 - r^2}{1 - 2r\cos(\phi-\theta) + r^2}\Big]d\phi. \tag{2.25}$$

An analogous formula holds in the case that $f(z)$ is a constant C; for in this case, we can write

$$C = \frac{1}{2\pi}\int_{-\pi}^{\pi} C\frac{1 - r^2}{1 - 2r\cos(\phi-\theta) + r^2}d\phi,$$

which after an integration by parts yields

$$C = \frac{C(1-r^2)}{1+2r\cos\theta + r^2} - \frac{1}{2\pi}\int_{-\pi}^{\pi} C\phi\frac{d}{d\phi}\Big[\frac{1 - r^2}{1 - 2r\cos(\phi-\theta) + r^2}\Big]d\phi. \tag{2.26}$$

In particular, for $C = 1$ and $\theta = 0$,

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi\frac{d}{d\phi}\Big[\frac{1 - r^2}{1 - 2r\cos\phi + r^2}\Big]d\phi = 1 - \frac{1-r^2}{(1+r)^2} = \frac{2}{1+\frac{1}{r}}. \tag{2.27}$$

Let $e^{i\theta_0}$ be a point of the unit circle at which $F(\phi)$ has a finite derivative $F'(\theta_0)$. We shall show that in this case,

$$\lim_{r\to 1} f(re^{i\theta_0}) = F'(\theta_0) \tag{2.28}$$

holds. Here it suffices to take $\theta_0 = 0$, since we can reduce the general case to the special case $\theta_0 = 0$ by a rotation of the coordinate system. By (2.25) and (2.26),

$$f(r) - F'(0) = -F'(0)\Big(\frac{1-r}{1+r}\Big) - \frac{1}{2\pi}\int_{-\pi}^{\pi} [F(\phi) - \phi F'(0)]\frac{d}{d\phi}\Big[\frac{1 - r^2}{1 - 2r\cos\phi + r^2}\Big]d\phi.$$

Now we select a positive number $\lambda < \pi$ and set

$$H(r,\lambda) = -\frac{1}{2\pi}\int_{-\lambda}^{\lambda}[F(\phi) - \phi F'(0)]\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi + r^2}\right]d\phi \tag{2.29}$$

and

$$J(r,\lambda) = -\frac{1}{2\pi}\left\{\int_{-\pi}^{-\lambda} + \int_{\lambda}^{\pi}\right\} \tag{2.30}$$

(with the same integrand), so that

$$f(r) - F'(0) = -F'(0)\left(\frac{1-r}{1+r}\right) + H(r,\lambda) + J(r,\lambda) \tag{2.31}$$

holds. If we note that (2.20) and (2.24) imply that $F(0) = 0$, and if we set

$$F(\phi) = \phi F'(0) + \phi\eta(\phi), \tag{2.32}$$

and denote the least upper bound of $|\eta(\phi)|$ in the interval from $-\lambda$ to λ by $h(\lambda)$, we find that

$$\lim_{\lambda\to 0} h(\lambda) = 0. \tag{2.33}$$

From (2.29) and (2.32) it follows that

$$H(r,\lambda) = -\frac{1}{2\pi}\int_{-\lambda}^{\lambda}\phi\eta(\phi)\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi + r^2}\right]d\phi,$$

and if we note that

$$-\phi\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi + r^2}\right] = \frac{2r(1-r^2)\phi\sin\phi}{(1-2r\cos\phi + r^2)^2} \geq 0 \tag{2.34}$$

holds, we obtain

$$|H(r,\lambda)| \leq -\frac{h(\lambda)}{2\pi}\int_{-\lambda}^{\lambda}\phi\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi + r^2}\right]d\phi$$

$$< -\frac{h(\lambda)}{2\pi}\int_{-\pi}^{\pi}\phi\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi+r^2}\right]d\phi,$$

and hence finally, by (2.27),

$$|H(r,\lambda)| < \frac{2h(\lambda)}{1+\frac{1}{r}} < h(\lambda). \tag{2.35}$$

In order to obtain a bound for $J(r,\lambda)$, we first observe that by (2.23) and (2.24),

$$|F(\phi)| < \pi M$$

holds in the interval of integration, and that in the same interval, we have

$$\left|\frac{d}{d\phi}\left[\frac{1-r^2}{1-2r\cos\phi+r^2}\right]\right| < \frac{2r(1-r^2)}{(1-2r\cos\lambda+r^2)^2} < \frac{2(1-r^2)}{(1-\cos^2\lambda)^2}.$$

Hence (2.30) yields

$$|J(r,\lambda)| < \frac{2\pi(M+|F'(0)|)}{\sin^4\lambda}(1-r^2). \tag{2.36}$$

Finally, we obtain from (2.31), (2.35) and (2.36) that

$$|f(r)-F'(0)| < h(\lambda)+\left\{\frac{2\pi(M+|F'(0)|)}{\sin^4\lambda}+\frac{|F'(0)|}{(1+r)^2}\right\}(1-r^2). \tag{2.37}$$

Given any positive number ε, then by (2.33) we can choose λ so small that $h(\lambda) < \frac{\varepsilon}{2}$. It then follows from (2.37) that for sufficiently small values of $1-r^2$,

$$|f(r)-F'(0)| < \varepsilon. \tag{2.38}$$

This shows that for every θ_0 where $F(\theta)$ is differentiable, the given function $f(z)$ converges to the value $F'(\theta_0)$ as z approaches $e^{i\theta_0}$ radially.

According to Theorem (2.21), both the real and the imaginary part of $F(\phi)$ are differentiable except at most on a set of measure zero, and this, together with the result stated above, proves Fatou's Theorem.

Definition 2.44 (*Möbius transformations*). *A Möbius transformation $L(z)$ is a transformation of the form*

$$L(z) = \frac{az+b}{cz+d}, \qquad z \in \mathbb{C},$$

where a, b, c, d are arbitrary complex numbers satisfying $ad - bc \neq 0$.

A characteristic feature of Möbius transformations is the following.

Theorem 2.45 *Every Möbius transformation maps a circle or line onto a circle or line. Moreover, if the circle or line K is mapped onto the circle or line K^*, then one of the complementary domains of K is mapped onto one of the complementary domains of K^*, and the other complementary domain of K is mapped onto the other complementary domain of K^*. (For a circle, the two complementary domains are the interior and the exterior of the circle. In the case of a line the complementary domains are two half planes, one on each side of the line).*

Proof. See [19], Thm. (10.4), p.169, and Rem. p.170.

Definition 2.46 (*Symmetric points*). *Two points z_1 and z_2 are symmetric with respect to a given straight line κ if κ is the perpendicular bisector of the line segment joining z_1 and z_2. Two points z_1 and z_2 are symmetric with respect to a given circle K if every straight line or circle passing through z_1 and z_2 is orthogonal to K.*

The following theorem shows that Möbius transformations preserve symmetry.

Theorem 2.47 *Let z_1 and z_2 be any two points symmetric with respect to a given straight line or circle κ, and let $L(z) = w$ be any Möbius transformation. Then* *the points $w_1 = L(z_1)$ and $w_2 = L(z_2)$ are symmetric with respect to the straight line or circle $K = L(\kappa)$.*

Proof. See [19], Thm. (10.8), p.178.

2.5 Herglotz functions

Theorem 2.48 (*Helly's selection theorem*). *Let the real non-decreasing functions $f_n(x)$ and the positive constant k be such that*

$$|f_n(x)| < k, \qquad (n = 0, 1, ...; a \leq x \leq b).$$

Then, there is a set of integers $n_0 < n_1 < ...$ and a non-decreasing bounded function $f(x)$ such that

$$\lim_{\nu \to \infty} f_{n_\nu}(x) = f(x), \qquad (a \leq x \leq b).$$

Proof. See [26], p.165.

Theorem 2.49 *If $W(z,t)$ is a continuous function of two variables: the complex variable z ranging over the open set G, and the real variable t ranging over the interval $[a, b]$, then the function*

$$H(z) = \int_a^b W(z,t)dt$$

is continuous in the set G.

If, in addition, for each $t \in [a, b]$ the function $W(z,t)$ has, at every point $z \in G$, a partial derivative $W_z'(z,t)$ continuous with respect to both variables z and t, then the function $H(z)$ is analytic in G and

$$H'(z) = \int_a^b W_z'(z,t)dt.$$

Proof. See [24], Chapter II, Thm. (3.1), p.107.

Theorem 2.50 *A necessary and sufficient condition that a function $f(z)$, analytic in the circle $|z| < 1$, satisfy the condition $Ref(z) \geq 0$ for $|z| < 1$ is that $f(z)$ be representable as an integral*

$$f(z) = \int_{-\pi}^{+\pi} \frac{e^{i\psi} + z}{e^{i\psi} - z} d\tau(\psi) + ic, \tag{2.39}$$

where $\tau(\psi)$ is a real non-decreasing function, $-\pi \leq \psi \leq +\pi$, with the properties

$$\tau(-\pi) = 0; \quad \tau(\psi + 0) = \tau(\psi), \quad -\pi \leq \psi < +\pi,$$

and c is a real constant, equal to $Imf(0)$. When this representation is possible, it is unique.

Proof. A function $f(z)$ defined by an integral of the type described in the theorem is single-valued, and analytic in the circle $|z| < 1$ by Theorem (2.49). If we put $z = re^{i\theta}$, where $r = |z|$, we see at once that

$$Ref(z) = \int_{-\pi}^{+\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \psi) + r^2} d\tau(\psi) \geq 0$$

for $r < 1$, since the integrand is positive under this condition. Hence the integral representation is sufficient for $f(z)$ to have the indicated properties.

If $f(z)$ is a single-valued function analytic in the circle $|z| < 1$ and if $Ref(z) \geq 0$ for $|z| < 1$, we consider the function $f_n(z) = f(\frac{n-1}{n}z)$. The function $Ref_n(z)$ is harmonic and non-negative in the closed region $|z| \leq 1$ and assumes continuous non-negative boundary values $u_n(\theta) = Ref_n(z)$ on the circumference $z = e^{i\theta}$. By Poisson's integral formula (Theorem (2.40)) we have

$$Ref_n(z) = \int_{-\pi}^{+\pi} \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2} d\tau_n(\psi)$$

where

$$\tau_n(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\psi} u_n(\theta)d\theta$$

is a continuous real monotone-increasing function for $-\pi \leq \psi \leq +\pi$. When $z = 0$, this formula becomes

$$Ref(0) = Ref_n(0) = \int_{-\pi}^{+\pi} d\tau_n(\psi) = \tau_n(\pi) - \tau_n(-\pi) = \tau_n(\pi).$$

Hence the sequence $\{\tau_n(\psi)\}$ satisfies the conditions of Helly's theorem (Theorem (2.48)). We can therefore determine a sequence of integers $\{n(k)\}$ and a real monotone-increasing function $\tau(\psi)$ such that

$$\lim_{k\to\infty} n(k) = \infty, \quad \lim_{k\to\infty} \tau_{n(k)}(\psi) = \tau(\psi), \quad 0 \leq \tau(\psi) \leq Ref(0),$$

for $-\pi \leq \psi \leq +\pi$. If we allow n to tend to infinity through values in the sequence $\{n(k)\}$ in the formula for $Ref_n(z)$ given above, we find by virtue of the relations just noted and of the further relations

$$\lim_{n\to\infty} f_n(z) = f(z), \quad \lim_{n\to\infty} Ref_n(z) = Ref(z), \quad |z| < 1,$$

that

$$Ref(z) = \int_{-\pi}^{+\pi} \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2} d\tau(\psi)$$

for $|z| < 1$. The function $Imf(z)$, harmonic in the open region $|z| < 1$, is now determined by the fact that it is conjugate to $Ref(z)$ and assumes the value $Imf(0)$ for $z = 0$. Hence it is given by the formula

$$Imf(z) = \int_{-\pi}^{+\pi} \frac{2r\sin(\theta-\psi)}{1-2r\cos(\theta-\psi)+r^2} d\tau(\psi) + Imf(0).$$

By combining the expressions for $Ref(z)$ and $Imf(z)$, we find that

$$f(z) = \int_{-\pi}^{+\pi} \frac{e^{i\psi}+z}{e^{i\psi}-z} d\tau(\psi) + iImf(0)$$

for $|z| < 1$. The function $\tau(\psi)$ may not have all the properties demanded by the theorem; but it can be replaced by the function $\tau^*(\psi)$ defined by the equations

$$\tau^*(-\pi) = 0; \quad \tau^*(\pi) = \tau(\pi) - \tau(-\pi);$$

$$\tau^*(\psi) = \tau(\psi+0) - \tau(-\pi+0), \quad -\pi < \psi < +\pi,$$

a function which has the desired properties.

Since we make little use of the uniqueness of the integral representation under discussion, we omit the proof of it.

Theorem (2.50) is the basis for the following result which we shall use repeatedly.

Theorem 2.51 *(Integral representation of Herglotz functions). A Herglotz function $F(z)$ admits the representation*

$$F(z) = a_F + b_F z + \int_{-\infty}^{+\infty} \frac{1+tz}{t-z} d\rho'(t), \tag{2.40}$$

where $b_F \geq 0$ and a_F are two real constants, and $\rho'(t)$ is a non-decreasing function.

Note that the representation (2.40) *can also be written in the form*

$$F(z) = a_F + b_F z + \int_{-\infty}^{\infty} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} d\rho(t), \tag{2.41}$$

where $d\rho(t) = (1+t^2)d\rho'(t)$.

Proof. The transformation given by the equations

$$z = \frac{i-\zeta}{i+\zeta}, \qquad \zeta = i\frac{1-z}{1+z} \tag{2.42}$$

maps the half plane $Im\zeta \geq 0$ on the disk $|z| \leq 1$, in such a manner that, when ζ increases through real values from $-\infty$ to $+\infty$, z lies on the unit circle $|z| = 1$ and the variable $\theta = \arg z$ increases from $-\pi$ to $+\pi$. When z and ζ are connected by these relations, the equation $if(z) = F(\zeta)$ sets up a one-to-one correspondence between the class of all functions $f(z)$ which are single-valued and analytic in the circle $|z| < 1$ and have positive real parts in that region, and the class of all functions $F(\zeta)$ which are single-valued and analytic in the half-plane $Im\zeta > 0$ and have positive imaginary parts there. Similarly, when z and ζ are connected by these relations, we can write

$$-c + i\int_{-\pi}^{+\pi} \frac{e^{i\psi}+z}{e^{i\psi}-z} d\tau(\psi) = a_F + b_F\zeta + \int_{-\infty}^{+\infty} \frac{1+t\zeta}{t-\zeta} d\rho'(t)$$

where

$$\psi = 2\arctan t, \quad t = i\frac{1-e^{i\psi}}{1+e^{i\psi}} = \tan\left(\frac{\psi}{2}\right), \quad \tau(\psi) = \rho'(t)$$

for $-\pi < \psi < +\pi$ and $-\infty < t < +\infty$, $b_F = \tau(\pi) - \tau(\pi - 0)$, and $a_F = -c$; the functions $\tau(\psi)$ and $\rho'(t)$ are real monotone-increasing functions subject to the conditions stated in theorem (2.50) and in the present theorem respectively, and the equation holds in the sense that, if the integral on either side exists, the integral on the other also exists and is equal to it. By combining these facts with the result stated in theorem (2.50), we see that a function $F(\zeta)$ single-valued and analytic in the half plane $Im\zeta > 0$ satisfies the condition $ImF(\zeta) > 0$ for $Im\zeta > 0$ if and only if $F(\zeta)$ is representable by the integral formula

$$F(\zeta) = a_F + b_F\zeta + \int_{-\infty}^{+\infty} \frac{1+t\zeta}{t-\zeta} d\rho'(t)$$

where a_F, b_F and $\rho'(t)$ are subject to the conditions indicated in the statement of the present theorem. This completes the proof.

We see, further, that this representation is unique. When $F(\zeta)$ is a function analytic in the half planes $Im\zeta > 0$, $Im\zeta < 0$, it satisfies the condition $ImF(\zeta) > 0$ for $Im\zeta > 0$ if and only if it is representable in the half plane $Im\zeta > 0$ by the indicated integral formula. This formula is significant in the half plane $Im\zeta < 0$ also; but it represents $F(\zeta)$ in this half plane if and only if $F(\zeta)$ satisfies the functional relation $F(\overline{\zeta}) = \overline{F(\zeta)}$. When the latter relation is valid, the inequality $ImF(\zeta) > 0$, $Im\zeta > 0$ implies the inequality $ImF(\zeta) < 0$, $Im\zeta < 0$.

Corollary 2.52 *Let F be a Herglotz function. Then, the boundary values of F, as z approaches the real axis, exist almost everywhere.*

Proof. If F is bounded, the result follows from Fatou's Theorem (2.43) through the transformation (2.42). In particular, the boundary values of F exist almost everywhere as 'wedgy' limits, that is along any curve γ which approaches the real axis in a non-tangential way.

If F is unbounded, note that

$$G(z) = -\frac{1}{F(z) + i}$$

is a bounded Herglotz function, and the existence of boundary values for G implies the existence of boundary values for F.

Lemma 2.53 *Let F be a Herglotz function with integral representation*

$$F(z) = a_F + b_F z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu(t). \tag{2.43}$$

Then, the constants a_F and b_F are determined by

$$a_F = ReF(i), \qquad b_F = \lim_{s\to\infty} \frac{ImF(is)}{s}.$$

Proof. From (2.43) we have

$$F(i) = a_F + i\Big(b_F + \int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu(t)\Big).$$

Thus, $a_F = ReF(i)$. We also have

$$\frac{ImF(is)}{s} = b_F + \int_{\mathbb{R}} \frac{1}{t^2+s^2}\, d\mu(t). \qquad (2.44)$$

Since

$$\frac{1}{t^2+s^2} \le \frac{1}{1+t^2} \qquad (s \ge 1),$$

which is integrable with respect to μ, we can apply the Lebesgue dominated convergence theorem in (2.44) in the limit $s \to \infty$ to obtain $\lim_{s\to\infty} \frac{ImF(is)}{s} = b_F$.

The following lemma characterizes measures corresponding to Herglotz functions.

Lemma 2.54 *A sufficient condition for the representation of $F(z)$ in* (2.43) *to converge is that*

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu(t) < +\infty. \qquad (2.45)$$

Proof. Note that

$$\frac{1}{t-z} - \frac{t}{1+t^2} = \frac{1+tz}{(t-z)(1+t^2)},$$

where the function $G(t,z) = \frac{1+tz}{t-z}$ is bounded for $t \in \mathbb{R}$ and fixed $z \in \mathbb{C}_+$. [As $t \to \pm\infty$ $G(t,z) \to z$ and so there is an N such that $G(t,z) < 2z$ provided

$t > N$. Also, being a continuous function, G attains its maximum on the interval $t \in [-N, N]$.] Therefore, if condition (2.45) is satisfied, then the representation of $F(z)$ in (2.43) converges absolutely.

2.6 Analysis of measures

Let ρ be a function having the following two properties:

(i) $\rho(\lambda)$ is a non-decreasing function of λ,

(ii) $\rho(\lambda)$ is continuous from the right, i.e.

$$\lim_{\varepsilon \to 0^+} \rho(\lambda + \varepsilon) = \rho(\lambda).$$

Then we can define a measure μ on the Borel algebra by

$$\mu((a, b]) = \rho(b) - \rho(a).$$

We refer to μ as the Borel-Stieltjes measure generated by ρ. For a Borel-Stieltjes measure, a single point λ_0 will have strictly positive measure if and only if the function $\rho(\lambda)$ is discontinuous at λ_0. The measure of the single point λ_0 is given by

$$\mu(\{\lambda_0\}) = \rho(\lambda_0) - \lim_{\varepsilon \to 0^+} \rho(\lambda_0 - \varepsilon).$$

Points having strictly positive measure will be referred to as the discrete points of the measure. For a Borel-Stieltjes measure there are at most countably many discrete points; if there are no discrete points, so that the function $\rho(\lambda)$ is continuous, we refer to a continuous measure. For any continuous Borel-Stieltjes measure the measure of any countable set of points will be zero.

By the Lebesgue decomposition theorem (2.18), given a Borel-Stieltjes measure μ, we can decompose it into its absolutely continuous and singular parts with respect to Lebesgue measure, $\mu = \mu_{ac} + \mu_s$. The singular component μ_s may be decomposed further into its singular continuous and discrete components respectively. Thus μ_{sc} is a singular measure that is also continuous in the sense that single points have zero measure, or equivalently $\mu_{sc}(a, x]$ is a continuous function of x. On the other hand, the discrete component μ_d of the measure μ is concentrated on those points (finite or countable in number) that have strictly positive measure. These are the so-called discrete points of the measure. Writing $\mu_s = \mu_{sc} + \mu_d$, we then have the complete decomposition

$$\mu = \mu_{ac} + \mu_{sc} + \mu_d.$$

The following lemma provides a local comparison of the measure μ with Lebesgue measure, which will be denoted by $|.|$.

Lemma 2.55 *Let μ be a Borel-Stieltjes measure, and let $\mathcal{S}_m$ be a Borel set such that, for each $\lambda \in \mathcal{S}_m$ and for $\delta > 0$ sufficiently small (how small depends on λ),*

$$\mu(\mathcal{I}) \leq m|\mathcal{I}| \tag{2.46}$$

for every subinterval $\mathcal{I}$, containing the point λ, of the interval $[\lambda - \delta, \lambda + \delta]$. Then

$$\mu(\mathcal{S}_m) \leq m|\mathcal{S}_m|. \tag{2.47}$$

Proof. (i) Consider first of all the case in which the set $\mathcal{S}_m$ is closed and bounded. Then, given $\varepsilon > 0$, there is an open set $\mathcal{S}_0$ such that $\mathcal{S}_m \subseteq \mathcal{S}_0$ and

$$|\mathcal{S}_0| \leq |\mathcal{S}_m| + \varepsilon. \tag{2.48}$$

For each $\lambda \in \mathcal{S}_m$, we find an open interval $\mathcal{I}_\lambda$ containing λ such that (2.46) is satisfied for subintervals $\mathcal{I}$, containing λ, of $\mathcal{I}_\lambda$. Without loss of generality, it may be assumed that $\mathcal{I}_\lambda$ is taken sufficiently small that $\mathcal{I}_\lambda \subseteq \mathcal{S}_0$ in each case. The collection of intervals $\{\mathcal{I}_\lambda\}$ constitute a covering of $\mathcal{S}_m$ by open intervals, and by the Heine-Borel theorem (see [2], Thm. (3.29), p.58) the set $\mathcal{S}_m$ will be covered by a *finite* subset of the $\{\mathcal{I}_\lambda\}$. By removing intervals if necessary, this finite covering may be assumed to be *minimal*, in the sense that no further intervals may be removed without uncovering some point of $\mathcal{S}_m$. It is then possible to shrink each interval (if necessary), while retaining the property $\lambda \in \mathcal{I}_\lambda$, in such a way that the intervals do not overlap, and such that $\mathcal{S}_m$ is still completely covered. (The shrunk intervals need no longer be open, however.)

For example, if $\mathcal{I}_{\lambda_1} = (\lambda_1 - \varepsilon_1, \lambda_1 + \varepsilon_1)$ and $\mathcal{I}_{\lambda_2} = (\lambda_2 - \varepsilon_2, \lambda_2 + \varepsilon_2)$ are two overlapping intervals with $\lambda_1 < \lambda_2$ then they could be replaced by $\mathcal{I}_{\lambda_1} = (\lambda_1 - \varepsilon_1, \frac{1}{2}(\lambda_1 + \lambda_2))$ and $\mathcal{I}_{\lambda_2} = (\frac{1}{2}(\lambda_1 + \lambda_2), \lambda_2 + \varepsilon_2)$, which do not overlap, but which together cover the same set of points.

Now, for each of the shrunken intervals, we have

$$\mu(\mathcal{I}_\lambda) \leq m|\mathcal{I}_\lambda|.$$

Adding this inequality over the $\mathcal{I}_\lambda$, we find that the total μ-measure of the finite covering of $\mathcal{S}_m$ cannot exceed m times its Lebesgue measure. Since the finite covering is also contained in $\mathcal{S}_0$, we have

$$\mu(\mathcal{S}_m) \leq m|\mathcal{S}_0|,$$

so that, from (2.48),

$$\mu(\mathcal{S}_m) \leq m(|\mathcal{S}_m| + \varepsilon).$$

But this inequality holds for all $\varepsilon > 0$, and we may deduce (2.47).

(ii) We now let $\mathcal{S}_m$ be an arbitrary Borel set satisfying the hypothesis of the lemma. Since

$$\mu(\mathcal{S}_m) = \lim_{N\to\infty} \mu(\mathcal{S}_m \cap [-N, N])$$

and

$$|\mathcal{S}_m| = \lim_{N\to\infty} |\mathcal{S}_m \cap [-N, N]|,$$

in order to prove (2.47) it will be sufficient to consider sets that are contained in a finite closed interval $[-N, N]$.

We suppose then that $\mathcal{S}_m \subseteq [-N, N]$, and let $\mathcal{S}_m^c$ denote the complement of $\mathcal{S}_m$ with respect to $[-N, N]$ (i.e. $\mathcal{S}_m^c = [-N, N]\backslash\mathcal{S}_m$). Given $\varepsilon > 0$, we cover $\mathcal{S}_m^c$ by an open set $\mathcal{S}_0'$ such that

$$\mu(\mathcal{S}_0') \leq \mu(\mathcal{S}_m^c) + \varepsilon. \tag{2.49}$$

If $(\mathcal{S}_0')^c$ denotes the complement of $\mathcal{S}_0'$ with respect to $[-N, N]$ then $(\mathcal{S}_0')^c \subseteq \mathcal{S}_m$, and is a closed bounded set satisfying the hypothesis of the corollary. Hence by (i) above we have

$$\mu((\mathcal{S}_0')^c) \leq m|(\mathcal{S}_0')^c|. \tag{2.50}$$

Also,

$$\mu[-N, N] = \mu(\mathcal{S}_m) + \mu(\mathcal{S}_m^c) \leq \mu(\mathcal{S}_0') + \mu((\mathcal{S}_0')^c).$$

From (2.49) and (2.50) we now have

$$\mu(\mathcal{S}_m) \leq \mu((\mathcal{S}_0')^c) + \varepsilon \leq m|(\mathcal{S}_0')^c| + \varepsilon.$$

But $(\mathcal{S}_0')^c \subseteq \mathcal{S}_m$, so that

$$\mu(\mathcal{S}_m) \leq m|\mathcal{S}_m| + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, (2.47) follows and the Lemma is proved.

In the following theorem, we shall say that $\frac{\mu(\mathcal{I}_\lambda)}{|\mathcal{I}_\lambda|}$ is bounded, for some fixed λ, if there exists $m > 0$, which may be different for different λ, such that $\mu(\mathcal{I}_\lambda) \leq$

$m|\mathcal{I}_\lambda|$ for all intervals $\mathcal{I}_\lambda$ containing λ such that $|\mathcal{I}_\lambda| < const.$ We shall say that $\lim_{|\mathcal{I}_\lambda|\to 0} \frac{\mu(\mathcal{I}_\lambda)}{|\mathcal{I}_\lambda|} = c_\lambda$, say, if this limit exists and is the same for every sequence of intervals $\mathcal{I}_\lambda^{(n)}$ such that $\lim_{n\to\infty} |\mathcal{I}_\lambda^{(n)}| = 0$. It is easy to see that this will be so if and only if

$$\lim_{\varepsilon\to 0+} \frac{\mu((\lambda, \lambda+\varepsilon])}{\varepsilon} = \lim_{\varepsilon\to 0+} \frac{\mu((\lambda-\varepsilon, \lambda])}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \frac{\mu((\lambda-\varepsilon, \lambda+\varepsilon])}{2\varepsilon} = c_\lambda. \tag{2.51}$$

In other words, for any constant c, the function $\mu((c, \lambda])$ should be differentiable at the point λ. This function is locally of bounded variation and differentiable for almost all values of λ. It follows that $\frac{\mu(\mathcal{I}_\lambda)}{|\mathcal{I}_\lambda|}$ converges to a limit, and hence in particular it is bounded, for almost all λ.

The following Theorem gives a local characterization of the respective supports of the measures μ_{ac} and μ_s.

Theorem 2.56 *For a Borel-Stieltjes measure μ let S be the set of values of λ at which $\mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|$ is bounded. Then*

(i) the absolutely continuous part of μ is given by

$$\mu_{ac} = \mu|_S \qquad (\textit{restriction of } \mu \textit{ to } S);$$

μ_{ac} has density function $f(\lambda)$ given (for almost all λ) by

$$f(\lambda) = \lim_{|\mathcal{I}_\lambda|\to 0} \mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|, \tag{2.52}$$

and any Borel set of points λ for which this limit is zero has μ-measure zero;

(ii) the singular part of μ is given by

$$\mu_s = \mu|_{S^c} \qquad (\textit{restriction of } \mu \textit{ to the complement of } S).$$

Moreover, the measure μ is singular (i.e. $\mu = \mu_s$) if and only if

$$\lim_{|\mathcal{I}_\lambda| \to 0} \mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda| = 0 \tag{2.53}$$

for almost all values of λ.

Remark 2.57 *If F is the Herglotz function corresponding to μ, then an alternative characterization of the supports of μ_{ac} and μ_s is ([23], p.131) the following:*

$$\mu_{ac} = \mu \ \textit{restricted to} \ \{\lambda \in \mathbb{R} : ImF(\lambda + i\varepsilon) \ \textit{is bounded in} \ 0 < \varepsilon < 1\},$$

$$\mu_s = \mu \ \textit{restricted to} \ \{\lambda \in \mathbb{R} : \lim_{\varepsilon \to 0^+} ImF(\lambda + i\varepsilon) = \infty\}.$$

Proof. That $\mathcal{S}$ is a Borel set follows from [22], p.53. We next prove that the restriction of μ to $\mathcal{S}$ is absolutely continuous. The restriction to $\mathcal{S}$ is defined by

$$(\mu|_S)(E) = \mu(E \cap \mathcal{S}),$$

and to demonstrate absolute continuity it will be sufficient to show that, for any subset A of $\mathcal{S}$,

$$|A| = 0 \Rightarrow \mu(A) = 0. \tag{2.54}$$

Suppose then that $|A| = 0$. Consider $\mathcal{S}_m$ as in lemma (2.55). Then we can write $\mathcal{S} = \bigcup_m \mathcal{S}_m$ as m runs over the set of positive integers. Moreover, $|A \cap \mathcal{S}_m| = 0$, where the set $A \cap \mathcal{S}_m$ satisfies the hypothesis of lemma (2.55), so that

$$\mu(A \cap \mathcal{S}_m) \leq m|A \cap \mathcal{S}_m|.$$

It follows that $\mu(A \cap \mathcal{S}_m) = 0$. Hence, we have

$$\mu(A) = \mu(A \cap \mathcal{S}) = \lim_{m \to \infty} \mu(A \cap \mathcal{S}_m) = 0,$$

so that (2.54) is satisfied and $\mu|_{\mathcal{S}}$ is absolutely continuous.

Since $\mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|$ is bounded for almost all λ, it follows that the complement of $\mathcal{S}$ has Lebesgue measure zero. Hence $\mu|_{\mathcal{S}^c}$ is a measure concentrated on a set of Lebesgue measure zero, and is therefore singular with respect to Lebesgue measure. It follows immediately that

$$\mu = \mu_{ac} + \mu_s,$$

with

$$\mu_{ac} = \mu|_{\mathcal{S}} \quad \textit{and} \quad \mu_s = \mu|_{\mathcal{S}^c}.$$

We now consider a Borel set E of points λ for which the limit on the right-hand side of (2.52) is zero. Then Lemma (2.55) may be applied to the set E, with the value m arbitrarily small in (2.46). From (2.47) in the limit $m \to 0$, with $\mathcal{S}_m = E$, we have $\mu(E) = 0$.

The next step in the proof is to show that, for almost all λ,

$$\lim_{|\mathcal{I}_\lambda| \to 0} \mu_s(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda| = 0. \tag{2.55}$$

Let $\mathcal{S}_s$ be the set of values of λ such that $\mu_s(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|$ is bounded. From the first part of the theorem, applied to μ_s, we know that the absolutely continuous part of μ_s is $\mu_s|_{\mathcal{S}_s}$. However, μ_s is purely singular, so that $\mu_s|_{\mathcal{S}_s} = 0$, or, in other words, $\mu_s(\mathcal{S}_s) = 0$.

Consider the set E_m of points λ at which the limit on the left-hand side of (2.55) exceeds m^{-1}. Then for each $\lambda \in E_m$, and for any $\theta > 0$,

$$\mu_s(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda| > (m + \theta)^{-1},$$

for $|\mathcal{I}_\lambda|$ sufficiently small. Inverting this inequality then gives

$$|\mathcal{I}_\lambda| < (m+\theta)\mu_s(\mathcal{I}_\lambda).$$

Lemma (2.55) may now be applied, with the roles of $|.|$ and μ_s exchanged, and we have

$$|E_m| \leq (m+\theta)\mu_s(E_m).$$

However, $E_m \subseteq \mathcal{S}_s$ and $\mu_s(\mathcal{S}_s) = 0$, so that $|E_m| = 0$. This means that the limit on the left-hand side of (2.55), which exists for almost all λ, is less than or equal to m^{-1} for almost all λ, independently of the (positive) value of m. The Lebesgue measure of the set of points at which this limit differs from zero is $|\bigcup_m E_m|$, where m runs over the set of positive integers. Since $|\bigcup_m E_m| = \lim_{m\to\infty} |E_m| = 0$, we have proved (2.55) for almost all λ.

The density function for μ_{ac} is almost everywhere given by

$$f(\lambda) = \frac{d}{d\lambda}\rho_{ac}(\lambda),$$

where ρ_{ac} is the function that generates the measure μ_{ac}. Thus

$$f(\lambda) = \lim_{|\mathcal{I}_\lambda|\to 0} \mu_{ac}(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|,$$

which gives (2.52) on setting $\mu_{ac} = \mu - \mu_s$ and using (2.55). If (2.53) holds then $f(\lambda) = 0$ for almost all λ, in which case $\mu_{ac} = 0$, and the measure μ is purely singular. On the other hand, if μ is singular, so that $\mu = \mu_s$, then (2.55) $\Rightarrow$ (2.53). So (2.53) is a necessary and sufficient condition for a singular measure, and the proof is complete.

The above theorem provides a necessary and sufficient condition for a measure to be singular, and a sufficient condition for the measure to be absolutely continuous. Namely, μ will be purely absolutely continuous provided $\mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|$ is bounded for all values of λ. If, on the other hand, $\mu(\mathcal{I}_\lambda)/|\mathcal{I}_\lambda|$ is bounded for all except a *countable* set of values of λ, one may deduce $\mu_{sc} = 0$, since a singular *continuous* measure, restricted to a discrete set of points, must vanish. In that case μ decomposes into an absolutely continuous and discrete component.

Theorem 2.58 *Let μ be a finite Borel-Stieltjes measure, generated by $\rho(\lambda)$, and define $v(z)$ by*

$$v(z) = \int_{-\infty}^{\infty} \frac{y}{(\lambda - x)^2 + y^2} d\rho(\lambda), \qquad (z = x + iy). \tag{2.56}$$

Then

(i) for almost all x,

$$\lim_{y \to 0^+} v(x + iy) = \pi \frac{d\rho(x)}{dx}, \tag{2.57}$$

(ii) if ϕ is any real continuous function such that $\lim_{x \to \pm\infty} \phi(x) = 0$ *then*

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} v(x + iy)\phi(x) dx = \pi \int_{-\infty}^{\infty} \phi(\lambda) d\rho(\lambda), \tag{2.58}$$

Proof. See [22], lem. (2.8), p.63.

Lemma 2.59 *Under the hypothesis of the above theorem, if $(a, b]$ is any finite interval such that neither a nor b have positive μ-measure then*

$$\mu((a, b]) = \lim_{y \to 0^+} \frac{1}{\pi} \int_a^b v(x + iy) dx. \tag{2.59}$$

Proof. Equation (2.59) follows from (2.58) if we are allowed to take, for ϕ, the characteristic function of the interval $(a, b]$. However, such a choice for ϕ would not be continuous, and the best we can do is to take, for ϕ, a smooth function that satisfies $0 \leq \phi(x) \leq 1$, and (for small positive ε)

$$\phi(x) = \begin{cases} 1 & (a \le x \le b), \\ 0 & (x \le a - \varepsilon), \\ 0 & (x \ge b + \varepsilon). \end{cases}$$

The contributions to the right-hand side of (2.58) from integration over the intervals $(a-\varepsilon, a]$ and $(b, b+\varepsilon]$ respectively vanish in the limit as $\varepsilon \to 0$, and to prove (2.59) it remains only to show that, as ε and y *both* tend to zero,

$$\lim \int_{a-\varepsilon}^{a} v(x+iy)dx = \lim \int_{b}^{b+\varepsilon} v(x+iy)dx = 0.$$

But this is again a consequence of applying (2.58) respectively to functions $\phi = \phi_a$ and ϕ_b satisfying $0 \le \phi(x) \le 1$ and

$$\phi_a(x) = \begin{cases} 1 & (a - \varepsilon \le a), \\ 0 & (x \le a - 2\varepsilon), \\ 0 & (x \ge a + 2\varepsilon). \end{cases}$$

$$\phi_b(x) = \begin{cases} 1 & (b \le x \le b + \varepsilon), \\ 0 & (x \le b - 2\varepsilon), \\ 0 & (x \ge b + 2\varepsilon). \end{cases}$$

We have, for example,

$$\begin{aligned} \lim \int_{a-\varepsilon}^{a} v(x+iy)dx &\le \lim \int_{-\infty}^{\infty} v(x+iy)\phi_a(x)dx \\ &= \pi \lim \int_{-\infty}^{\infty} \phi_a(\lambda)d\rho(\lambda) \\ &\le \pi \lim_{\varepsilon \to 0^+} \int_{a-2\varepsilon}^{a+2\varepsilon} d\rho(\lambda) = 0, \end{aligned}$$

since a does not belong to the discrete set of points of the measure μ. Similarly $\lim \int_b^{b+\varepsilon} v(x+iy)dx = 0$. These two results allow us to replace ϕ in (2.58) by the characteristic function of $(a, b]$, and (2.59) follows.

Remark 2.60 *If F is a Herglotz function with μ as corresponding measure and having the representation* (1.1) *in Chapter* 1, *then we find from this representation that*

$$v(z) = ImF(x+iy) - b_F y.$$

Noting that the term $b_F y$ will not contribute to the integral in (2.59), (*this follows by an application of the Lebesgue dominated convergence theorem*), *and writing $z = \lambda + i\varepsilon$, we can rewrite* (2.59) *as*

$$\mu((a,b]) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b ImF(\lambda + i\varepsilon)d\lambda. \tag{2.60}$$

Equation (2.60) *characterizes the Herglotz measure of an interval $(a,b]$ whose endpoints are not discrete points of the measure, in terms of the corresponding Herglotz function.*

Reference: [22].

2.7 Existence and uniqueness theorems for differential equations

At the end of Chapter 5 we will be dealing with differential equations. In this section, we give an existence and uniqueness theorem for differential equations.

Definition 2.61 *Suppose that a function $f(x,y)$ is defined in an open set D in the (x,y) plane and suppose that there exists $k > 0$ such that if (x,y_1), $(x,y_2) \in D$, then*

$$|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2|.$$

Then we say that f satisfies the Lipschitz condition with respect to y in D, and k is a Lipschitz constant for f.

Theorem 2.62 *Let D be an open set in the x,y plane. Let $(x_0, y_0) \in D$ and let a, b be positive constants such that the set*

$$R = \{(x,y) : |x - x_0| \leq a,\ |y - y_0| \leq b\}$$

is contained in D. Suppose that the function f is defined and continuous on D and satisfies a Lipschitz condition with respect to y on R. Let

$$M = \max_{(x,y)\in R} |f(x,y)|, \qquad A = \min\{a, b/M\}.$$

Then the differential equation

$$\frac{dy}{dx} = f(x,y), \tag{2.61}$$

has a unique solution $y(x)$ on $(x_0 - A, x_0 + A)$ such that $y(x_0) = y_0$. This solution $y(x)$ is such that

$$|y(x) - y_0| \leq MA$$

for all $x \in (x_0 - A, x_0 + A)$.

Proof. We shall prove this theorem by the method of successive approximations which goes back to the days of Isaac Newton.

We first note that a necessary and sufficient condition that the function $y(x)$, continuous in x and satisfying the condition $y(x_0) = y_0$, is a solution of (2.61) on the interval $(x_0 - r, x_0 + r)$ $(r > 0)$ is that $y(x)$ satisfies the equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2.62}$$

If $y(x)$ satisfies (2.62), then since f and y are continuous, we can differentiate (2.62) and the result is (2.61). If $y(x)$ satisfies equation (2.61), then, taking the definite integral from x_0 to x where $x \in (x_0 - r, x_0 + r)$ on both sides of the equation, we obtain

$$\int_{x_0}^{x} \frac{dy}{ds} ds = y(x) - y(x_0) = \int_{x_0}^{x} f(s, y) ds$$

which gives the result.

The remainder of the proof of the theorem consists in showing that the sequence

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0) dt,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t)) dt,$$

$$\cdots\cdots\cdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t)) dt$$

converges on $[x_0 - A, x_0 + A]$ to a function which is a solution on $(x_0 - A, x_0 + A)$ of (2.62), and then showing that this solution is unique. We will show that the sequence converges on $[x_0, x_0 + A]$ and that the limit function is a solution on $[x_0, x_0 + A]$ of (2.62). A similar argument applies to the interval $[x_0 - A, x_0]$.

We first show that for each m, the function $y_m(x)$ is defined and continuous on $[x_0, x_0 + A]$, and moreover if $x \in [x_0, x_0 + A]$ then

$$|y_m(x) - y_0| \leq M|x - x_0|. \tag{2.63}$$

If $m = 0$, the statement is clearly true. If the statement is true for $m = q$, then for $x \in [x_0, x_0 + A]$, $|y_q(x) - y_0| \le MA \le b$. Therefore $f(x, y_q(x))$ is defined for $x \in [x_0, x_0 + A]$. Since $f(x, y_q(x))$ is a continuous function of x, then

$$y_{q+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_q(t))dt$$

is defined and continuous. Also,

$$|y_{q+1}(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_q(t))dt \right| \le M(x - x_0).$$

Next we show that the sequence $\{y_m(t)\}$ converges uniformly on $[x_0, x_0 + A]$ to a continuous function $y(x)$.

We will use the Weierstrass M-test (see [19], Thm. (15.2), p.323) to prove that the series

$$y_0(x) + \sum_{n=0}^{\infty} [y_{n+1}(x) - y_n(x)] \tag{2.64}$$

converges uniformly on $[x_0, x_0 + A]$. For $x \in [x_0, x_0 + A]$, let

$$\Delta_n(x) = |y_{n+1}(x) - y_n(x)|.$$

Then, for each n, we have

$$\begin{aligned} \Delta_n(x) &= \left| \int_{x_0}^{x} [f(t, y_n(t)) - f(t, y_{n-1}(t))]dt \right| \\ &\le \int_{x_0}^{x} |f(t, y_n(t)) - f(t, y_{n-1}(t))|dt \\ &\le k \int_{x_0}^{x} |y_n(t) - y_{n-1}(t)|dt \\ &= k \int_{x_0}^{x} \Delta_{n-1}(t)dt \end{aligned}$$

where k is a Lipschitz constant for f on D.

Now we obtain an estimate for $\Delta_n(x)$ by induction. We have seen that,

$$\Delta_0(x) = |y_1(t) - y_0(t)| \leq M|x - x_0|, \quad for\ x \in [x_0, x_0 + A].$$

Assume inductively that if $x \in [x_0, x_0 + A]$, then

$$\Delta_n(x) \leq \frac{M}{k} \frac{k^{n+1}(x - x_0)^{n+1}}{(n+1)!}.$$

In that case,

$$\Delta_{n+1}(x) \leq k \int_{x_0}^{x} \Delta_n(t)dt \leq k \frac{M}{k} \frac{k^{n+1}}{(n+1)!} \int_{x_0}^{x} (t - x_0)^{n+1} dt$$

$$= \frac{M}{k} \frac{k^{n+2}}{(n+1)!} \frac{1}{n+2} (x - x_0)^{n+2},$$

which verifies the inductive hypothesis. Thus, if $x \in [x_0, x_0 + A]$, we have

$$\sum_{n=0}^{\infty} \Delta_n(x) \leq \frac{M}{k} \sum_{n=0}^{\infty} \frac{k^{n+1}(x - x_0)^{n+1}}{(n+1)!}$$

$$\leq \frac{M}{k} \sum_{n=0}^{\infty} \frac{k^{n+1} A^{n+1}}{(n+1)!} = \frac{M}{k}(e^{kA} - 1).$$

Thus the uniform convergence of (2.64) follows from the Weierstrass M-test, by using the convergence of the exponential series.

Next, we show that the function $y(x)$ defined by the series (2.64) is a solution of (2.62) such that $y(x_0) = y_0$. First we show that for $x \in [x_0, x_0 + A]$, $|y(x) - y_0| \leq b$, and hence that for all $x \in [x_0, x_0 + A]$, $f(x, y(x))$ is defined. If $x \in [x_0, x_0 + A]$ and if $\varepsilon > 0$, then if m is sufficiently large, we have

$$|y(x) - y_0| \leq |y(x) - y_m(x)| + |y_m(x) - y_0| < \varepsilon + M(x - x_0).$$

Therefore,

$$|y(x) - y_0| \le M(x - x_0) \le MA \le b.$$

By the Lipschitz condition on f, we have: for $\varepsilon > 0$,

$$\left| \int_{x_0}^{x} [f(t, y(t)) - f(t, y_m(t))]dt \right| \le \int_{x_0}^{x} |f(t, y(t)) - f(t, y_m(t))|dt$$

$$\le k \int_{x_0}^{x} |y(t) - y_m(t)|dt \le k\varepsilon(x - x_0) \le k\varepsilon A,$$

provided m is sufficiently large. Therefore

$$\lim_{m\to\infty} \int_{x_0}^{x} f(t, y_m(t))dt = \int_{x_0}^{x} f(t, y(t))dt.$$

Taking the limit in m on both sides of the equation

$$y_{m+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_m(t))dt,$$

we obtain

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt,$$

so that (2.62) is satisfied.

Finally, we show that the solution $y(x)$ of (2.62), which satisfies the initial condition $y(x_0) = y_0$, is the only solution of (2.61) which satisfies this initial condition.

Suppose there exist solutions $y(x)$ and $Y(x)$ of (2.62) on an interval $(x_0 - r, x_0 + r)$, where r is a positive number, such that $y(x_0) = Y(x_0) = y_0$. By induction, we obtain an estimate on $|y(x) - Y(x)|$ for $x \in [x_0, x_0 + r - \delta]$ where $0 < \delta < r$. A similar estimate can be obtained for $x \in [x_0 - r + \delta, x_0]$. Since $y(x), Y(x)$ are continuous on $[x_0, x_0 + r - \delta]$ for fixed δ there exists $B > 0$ such that, if $x \in [x_0, x_0 + r - \delta]$, then $|y(x) - Y(x)| \le B$. But

$$|y(x) - Y(x)| \le \int_{x_0}^{x} |f(t, y(t)) - f(t, Y(t))|dt$$

$$\le k \int_{x_0}^{x} |y(t) - Y(t)|dt. \tag{2.65}$$

Therefore

$$|y(x) - Y(x)| \le kB(x - x_0).$$

Assume inductively that

$$|y(x) - Y(x)| \le \frac{k^m}{m!} B(x - x_0)^m,$$

for m a positive integer. Then by (2.65),

$$|y(x) - Y(x)| \le \frac{k^{m+1}}{(m+1)!} B(x - x_0)^{m+1},$$

which is the $(m+2)th$ term in the (convergent) series for $Be^{k(x-x_0)}$. Therefore for any $\varepsilon > 0$ we have $|y(x) - Y(x)| < \varepsilon$; hence $y(x) = Y(x)$ for $x \in [x_0, x_0 + r - \delta]$. Since δ is arbitrarily small, $y(x) = Y(x)$ for $x \in (x_0, x_0 + r)$. This completes the proof of Theorem (2.62).

We emphasize that the uniqueness result, although straightforward to prove, is crucially important both in later development of the theory and in applications of ordinary differential equations.

Extension of the Method of Successive Approximation to a system of Equations of the First Order.

Consider the system of equations

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_m),$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_m),$$

. . . .

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2, ..., y_m).$$

Then, under conditions which will be stated, there exists a unique set of continuous solutions of this system of equations which assume values y_1^0, y_2^0,..., y_m^0 when $x = x_0$. A bare outline of the proof will be given; the method follows on the lines of our preceding discussion.

The functions $f_1, f_2, ..., f_m$ are supposed to be single-valued and continuous in their $m + 1$ variables when these variables are restricted to lie in the domain D defined by

$$|x - x_0| \leq a, \quad |y_1 - y_1^0| \leq b_1, ..., |y_m - y_m^0| \leq b_m.$$

Let the greatest of the upper bounds of $f_1, f_2, ..., f_m$ in this domain be M; if A is the least of a, $\frac{b_1}{M}, ..., \frac{b_m}{M}$, let x be further restricted, if necessary, by the condition $|x - x_0| < A$. The Lipschitz condition to be imposed is

$$|f_r(x, Y_1, Y_2, ..., Y_m) - f_r(x, y_1, y_2, ..., y_m)|$$

$$< k_1|Y_1 - y_1| + k_2|Y_2 - y_2| + ... + k_m|Y_m - y_m|,$$

for $r = 1, 2, ..., m$. Now define the functions $y_1^n(x)$, $y_2^n(x)$,...,$y_m^n(x)$ by the relations

$$y_r^n(x) = y_r^0 + \int_{x_0}^{x} f_r(t, y_1^{n-1}(t), y_2^{n-1}(t), ..., y_m^{n-1}(t))dt.$$

Then, it can be proved by induction that

$$|y_r^n(x) - y_r^{n-1}(x)| < \frac{M(k_1 + k_2 + ... + k_m)^{n-1}}{n!}|x - x_0|^n,$$

and the existence, continuity, and uniqueness of solutions follow as in previous arguments.

Extension of the Method of Successive Approximation to a differential Equation of Order m.

Since the differential equation of order m

$$\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{m-1}y}{dx^{m-1}}\right)$$

is equivalent to the set of m equations of the first order

$$\frac{dy}{dx} = y_1, \quad \frac{dy_1}{dx} = y_2, \ldots, \frac{dy_{m-2}}{dx} = y_{m-1},$$

$$\frac{dy_{m-1}}{dx} = f(x, y, y_1, \ldots, y_{m-1}),$$

it follows that if f is continuous and satisfies a Lipschitz condition in a domain D, the equation admits of a unique continuous solution which, together with its first m-1 derivatives, which are also continuous, will assume an arbitrary set of initial conditions for the initial value $x = x_0$.

As a final remark for this section, we note that the method of successive approximations can be extended, with slight changes, to differential equations which contain a complex parameter z.

References: [11], and [17].

2.8 The Weyl-Titchmarsh m-function

Consider the Schrödinger equation

$$-\frac{d^2 f(x)}{dx^2} + V(x)f(x) = zf(x), \qquad 0 \leq x < +\infty, \tag{2.66}$$

where the potential $V(x)$ is real valued and integrable over compact intervals of $\mathbb{R}$, and z is a complex parameter. Let u and v be two independent solutions of (2.66) determined by the initial conditions

$$\left.\begin{array}{l} u(0,.)=1 \\ u'(0,.)=0 \end{array}\right\} \qquad \left.\begin{array}{l} v(0,.)=0 \\ v'(0,.)=1 \end{array}\right\} \tag{2.67}$$

Then, the functions $u(x,z)$ and $v(x,z)$, together with their derivatives $u'(x,z)$ and $v'(x,z)$ are entire functions of z.

We note that

$$u(x,\bar{z}) = \overline{u(x,z)}, \quad v(x,\bar{z}) = \overline{v(x,z)} \tag{2.68}$$

which may be verified as follows. Taking the complex conjugate of

$$-\frac{d^2u(x,z)}{dx^2} + V(x)u(x,z) = zu(x,z)$$

we obtain

$$-\overline{\frac{d^2u(x,z)}{dx^2}} + V(x)\overline{u(x,z)} = \bar{z}\overline{u(x,z)}.$$

Hence $\overline{u(x,z)}$ is a solution of the differential equation

$$-\frac{d^2f(x)}{dx^2} + V(x)f(x) = \bar{z}f(x)$$

satisfying the initial conditions

$$f(0,\bar{z}) = 1, \quad f'(0,\bar{z}) = 0.$$

On the other hand, according to its definition, $f = u(x,\bar{z})$ is also a solution of the above equation satisfying the same initial conditions. Therefore, the uniqueness of the solution implies

$$u(x,\bar{z}) = \overline{u(x,z)}.$$

The corresponding property of v may be verified in the same manner.

Lemma 2.63 *Suppose that $y_1(x, z_1)$ and $y_2(x, z_2)$ are solutions of (2.66) with corresponding complex parameters z_1 and z_2 respectively. Then, we have Green's formula*

$$(z_2 - z_1) \int_0^x y_1 y_2 dx = W_0(y_1, y_2) - W_x(y_1, y_2), \tag{2.69}$$

where $W_x(y_1, y_2)$ is the Wronskian of y_1 and y_2, defined by

$$W_x(y_1, y_2) = y_1(x) y_2'(x) - y_1'(x) y_2(x) \tag{2.70}$$

and $W_0(y_1, y_2) = W(y_1, y_2)|_{x=0}$.

Proof. The identity

$$(z_2 - z_1) y_1(x, z_1) y_2(x, z_2) = -\frac{d}{dx}[y_1(x, z_1) y_2'(x, z_2) - y_1'(x, z_1) y_2(x, z_2)]$$

can easily be verified. Equation (2.69) follows on integrating both sides of the above equation on the interval $(0, x)$.

Now let $y = y(x, z)$, $\overline{y} = y(x, \overline{z})$ be solutions of (2.66) such that either $y = u$ or $y = v$. Suppose that at $x = N$ we have

$$y \cos\beta + y' \sin\beta = 0, \tag{2.71}$$

$$\overline{y} \cos\beta + \overline{y}' \sin\beta = 0, \tag{2.72}$$

where β is a real number. By Green's formula (2.69) we have

$$(\overline{z} - z) \int_0^N y(x, z) y(x, \overline{z}) dx = W_0(y, \overline{y}) - W_N(y, \overline{y}). \tag{2.73}$$

By the conditions (2.67), $W_0(y, \overline{y}) = 0$. Also, if we multiply (2.71) by $\overline{y}' \cos\beta$, (2.72) by $y' \cos\beta$ and subtract the two obtained equations we obtain

$$(y\overline{y}' - y'\overline{y})\cos^2\beta = 0. \tag{2.74}$$

In a similar way, if we multiply (2.71) by $\overline{y}\sin\beta$, (2.72) by $y\sin\beta$ and subtract the two obtained equations, we have

$$(y\overline{y}' - y'\overline{y})\sin^2\beta = 0. \tag{2.75}$$

From equations (2.74) and (2.75) it follows that $W_N(y,\overline{y}) = 0$. Furthermore, since $y(x,\overline{z}) = \overline{y(x,z)}$ the integrand in (2.73) is just $|y(x,z)|^2$. Thus, we have

$$(\overline{z} - z)\int_0^N |y(x,z)|^2 dx = 0.$$

Since the conditions (2.67) exclude the trivial solution $y(x,z) = 0$, we must have $z = \overline{z}$, that is z is real. This proves the following:

Theorem 2.64 *Let N be a positive number and β a real number. Then every zero of the entire function*

$$u(N,z)\cos\beta + u'(N,z)\sin\beta \tag{2.76}$$

is real. The same is true for $v(N,z)\cos\beta + v'(N,z)\sin\beta$.

For any number p, the expression $u(x,z) + pv(x,z)$ satisfies the equation (2.66). We now choose p so that $u(x,z)+pv(x,z)$ satisfies the boundary condition

$$[u(N,z) + pv(N,z)]\cos\beta + [u'(N,z) + pv'(N,z)]\sin\beta = 0 \tag{2.77}$$

at the point $x = N$ and we denote p by $\ell_N(z)$. Then $\ell_N(z)$ must satisfy

$$\ell_N(z) = -\frac{u(N,z)\cos\beta + u'(N,z)\sin\beta}{v(N,z)\cos\beta + v'(N,z)\sin\beta}. \tag{2.78}$$

Since $u(N,z)$, $v(N,z)$, $u'(N,z)$ and $v'(N,z)$ are all entire functions of z, $\ell_N(z)$ is a meromorphic function of z, that is, a ratio of two entire functions. Furthermore, by Theorem (2.64), all the poles of $\ell_N(z)$ lie on the real axis. We write

$$\ell_N(z,\zeta) = -\frac{u(N,z)\zeta + u'(N,z)}{v(N,z)\zeta + v'(N,z)}, \tag{2.79}$$

where $\zeta = \cot\beta$ is real (ζ infinite corresponding to $\sin\beta = 0$ and $\ell_N = -\frac{u}{v}$). If N and z are fixed, and ζ varies, (2.79) may be written as

$$\ell = \frac{a\zeta + b}{c\zeta + d}, \tag{2.80}$$

where a, b, c and d are fixed.

This is a Möbius transformation, which we defined in section 3. Since

$$|bc - ad| = |u(N,z)v'(N,z) - u'(N,z)v(N,z)| \neq 0,$$

the transformation (2.80) is a one-to-one mapping which transforms circles into circles; straight lines being considered as circles with infinite radii. Therefore, if $Imz \neq 0$, then $\ell_N(z,\zeta)$ varies on a circle $C_N(z)$, with a finite radius, in the ℓ plane, as ζ varies over real values. The circle $C_N(z)$ is the image of the real line with respect to the Möbius transformation.

The centre and the radius of the circle $C_N(z)$ may be determined as follows. The centre of the circle is the symmetric point of the point at infinity with respect to the circle. Thus if we set

$$\ell_N(z,\zeta') = \infty,$$

$$\ell_N(z,\zeta'') = the\ centre\ of\ C_N(z),$$

then ζ'' must be the symmetric point of ζ' with respect to the real axis of the ζ plane, namely, $\zeta' = \overline{\zeta}''$. On the other hand,

$$\ell_N\left(z,-\frac{v'(N,z)}{v(N,z)}\right)=\infty, \tag{2.81}$$

as can be seen from (2.79). Therefore, the centre of the circle $C_N(z)$ is given by

$$\ell_N\left(z,-\frac{\overline{v'(N,z)}}{\overline{v(N,z)}}\right)=-\frac{W_N(u,\overline{v})}{W_N(v,\overline{v})}. \tag{2.82}$$

The radius $r_N(z)$ of the circle $C_N(z)$ is equal to the distance between the centre of $C_N(z)$ and the point $\ell_N(z,0)$ on the circle $C_N(z)$. Hence

$$r_N(z)=\left|\frac{u'(N,z)}{v'(N,z)}-\frac{W_N(u,\overline{v})}{W_N(v,\overline{v})}\right|$$

$$=\left|\frac{u'v\overline{v}'-u'\overline{v}v'-v'u\overline{v}'+v'\overline{v}u'}{v'W_N(v,\overline{v})}\right|$$

$$=\left|\frac{-\overline{v}'(uv'-u'v)}{v'W_N(v,\overline{v})}\right|=\left|\frac{W_N(u,v)}{W_N(v,\overline{v})}\right|.$$

On the other hand, by virtue of (2.67),

$$W_N(u,v)=W_0(u,v)=1.$$

Further, by virtue of (2.67), (2.68) and by making use of (2.69), we have

$$2Imz\int_0^N|v(x,z)|^2dx=2Imz\int_0^N v(x,z)v(x,\overline{z})dx \tag{2.83}$$

$$=iW_0[v(x,z),v(x,\overline{z})]-iW_N[v(x,z),v(x,\overline{z})]$$

$$=-iW_N[v(x,z),v(x,\overline{z})].$$

Therefore, we obtain

$$r_N(z)=\frac{1}{2|Imz|\int_0^N|v(x,z)|^2dx}. \tag{2.84}$$

We shall now prove the following:

Theorem 2.65 *If $Im z > 0$, then the lower half plane of the ζ plane is mapped onto the interior of the circle $C_N(z)$ by the transformation* (2.79).

Proof. Since the circle $C_N(z)$ is the image of the real axis of the ζ plane by the transformation (2.79), the lower half plane of the ζ plane is mapped onto either the interior or the exterior of the circle $C_N(z)$, and further, the point $\frac{-v'(N,z)}{v(N,z)}$ of the ζ plane is mapped onto the point at infinity of the ℓ plane.

On the other hand, by making use of (2.68) and (2.83), we obtain

$$Im\left[-\frac{v'(N,z)}{v(N,z)}\right] = \frac{i}{2}\left[\frac{v'(N,z)}{v(N,z)} - \frac{\overline{v'(N,z)}}{\overline{v(N,z)}}\right] \tag{2.85}$$

$$= \frac{-i}{2}\frac{W_N(v,\overline{v})}{|v(N,z)|^2} = \frac{Im z \int_0^N |v(x,z)|^2 dx}{|v(N,z)|^2} > 0.$$

This means that $\frac{-v'(N,z)}{v(N,z)}$ belongs to the upper half plane of the ζ plane. Hence the upper half of the ζ plane is mapped onto the exterior of $C_N(z)$, and so the lower half plane of the ζ plane is mapped onto the interior of $C_N(z)$. This completes the proof.

Since $W_0(u,v) = 1$, the transformation (2.79) has a unique inverse which is given by

$$\zeta = -\frac{v'(N,z)\ell_N + u'(N,z)}{v(N,z)\ell_N + u(N,z)} \tag{2.86}$$

In view of Theorem (2.65), if $Im z > 0$, ℓ belongs to the interior of the circle $C_N(z)$ if and only if $Im\zeta < 0$, namely, $i(\zeta - \overline{\zeta}) > 0$. From (2.86) it follows that

$$i(\zeta - \overline{\zeta}) = i\left[-\frac{v'(N,z)\ell + u'(N,z)}{v(N,z)\ell + u(N,z)} + \frac{\overline{v'(N,z)}\overline{\ell} + \overline{u'(N,z)}}{\overline{v(N,z)}\overline{\ell} + \overline{u(N,z)}}\right]$$

$$= \frac{iW_N(u+\ell v, \overline{u} + \overline{\ell}\overline{v})}{|v(N,z)\ell + u(N,z)|^2}.$$

Therefore, $Im\zeta < 0$ if and only if

$$iW_N(u+\ell v, \overline{u}+\overline{\ell}\overline{v}) > 0.$$

By Green's formula (2.69), we have

$$2Imz \int_0^N |u+\ell v|^2 dx = i[W_0(u+\ell v, \overline{u}+\overline{\ell}\overline{v}) - W_N(u+\ell v, \overline{u}+\overline{\ell}\overline{v})].$$

We obtain further by (2.67)

$$W_0(u+\ell v, \overline{u}+\overline{\ell}\overline{v})$$

$$= W_0(u,\overline{u}) + W_0(v,\overline{u})\ell + W_0(u,\overline{v})\overline{\ell} + W_0(v,\overline{v})|\ell|^2$$

$$= -\ell + \overline{\ell} = -2iIm\ell.$$

Consequently, we obtain the following:

Theorem 2.66 *If $Imz > 0$, then ℓ belongs to the interior of the circle $C_N(z)$ if and only if*

$$\int_0^N |u(x,z)+\ell v(x,z)|^2 dx < \frac{(Im\ell)}{(Imz)} \tag{2.87}$$

and ℓ lies on the circle $C_N(z)$ if and only if

$$\int_0^N |u(x,z)+\ell v(x,z)|^2 dx = \frac{(Im\ell)}{(Imz)} \tag{2.88}$$

Remark 2.67 *It is easy to see that Theorem* (2.66) *also holds when $Imz < 0$.*

If ℓ belongs to the interior of the circle $C_N(z)$ and $0 < N' < N$, then

$$\int_0^{N'} |u+\ell v|^2 dx \leq \int_0^N |u+\ell v|^2 dx < \frac{(Im\ell)}{(Imz)}$$

Hence, from Theorem (2.66), we have the following:

Theorem 2.68 *If $Im z \neq 0$, and $0 < N' < N$, then we have*

$$\overline{C_N(z)} \subseteq \overline{C_{N'}(z)}$$

where $\overline{C_N(z)}$ is the set composed of the circle $C_N(z)$ and its interior.

Theorem (2.68) implies that, if $Im z \neq 0$, then the set

$$\bigcap_{N>0} \overline{C_N(z)} = C_\infty(z) = m(z) \tag{2.89}$$

is either a point or a closed disc with a non-zero finite radius.

The function $m(z)$ is the Weyl-Titchmarsh m-function. According as $m(z)$ is a point or a disc, the singular boundary point $x = \infty$ is said to be in the *limit point case* or the *limit circle case.*

According to this definition, the classification would appear to depend upon both $V(x)$ and z. However, it depends only on $V(x)$, as is shown in the following Theorem:

Theorem 2.69 *(i) If for some z_0, $Im z_0 \neq 0$, the point $x = \infty$ is in the limit circle case, then, for every z, every solution $f(x)$ of the equation*

$$-\frac{d^2 f(x)}{dx^2} + V(x) f(x) = z f(x), \qquad 0 \leq x < \infty \tag{2.90}$$

satisfies

$$\int_0^\infty |f(x,z)|^2 dx < \infty. \tag{2.91}$$

(ii) If for some z_0, $Im z_0 \neq 0$, every solution of the equation

$$-\frac{d^2 f(x)}{dx^2} + V(x) f(x) = z_0 f(x), \qquad 0 \leq x < \infty \tag{2.92}$$

satisfies (2.91) *with $z = z_0$, then the point $x = \infty$ is in the limit circle case for this z_0.*

Remark 2.70 *According to Theorem* (2.69), *the classification is independent of* z. *Thus, the point* $x = \infty$ *is in the limit circle case if and only if, for every* z *every solution* $f(x)$ *of* (2.90) *satisfies*

$$\int_0^\infty |f(x)|^2 dx < \infty.$$

The point $x = \infty$ *is in the limit point case if and only if, for every* z, *there exists at least one solution of* (2.90) *such that*

$$\int_0^\infty |f(x)|^2 dx = \infty.$$

Remark 2.71 *Even if the point* $x = \infty$ *is in the limit point case, there exists, for every* z, $Imz \neq 0$, *at least one solution of* (2.90) *such that*

$$\int_0^\infty |f(x)|^2 dx < \infty.$$

In fact, from Theorem (2.66), *it follows that*

$$\int_0^\infty |u(x,z) + \ell v(x,z)|^2 dx \leq \frac{(Im\ell)}{(Imz)} < \infty,$$

where $\ell = C_\infty(z) = m(z)$.

Proof of (ii). By assumption, we have

$$\int_0^\infty |v(x,z_0)|^2 dx < \infty.$$

Hence, by virtue of (2.84), the radius $r_N(z_0)$ of the circle $C_N(z_0)$ remains positive as $N \to \infty$. Thus the proof is completed.

Proof of (i). See [29], Thm. (43.4), p.166.

Theorem 2.72 *In the limit-point case the limit point $m(z)$ is an analytic function of z for $Im z > 0$ (and $Im z < 0$). Also, $Imm(z) > 0$ for $Im z > 0$.*

Proof. From (2.82), (2.83) and (2.84) it follows that the center and the radius of the circle $C_N(z)$ are continuous functions of z, for $Im z \neq 0$. Therefore, since $\overline{C_{N'}(z)} \subseteq \overline{C_N(z)}$ for $N' \leq N$ (Theorem (2.68)), if z is restricted to any bounded domain Λ of the z plane which does not meet the real axis, the circles $\ell_N(z)$ are uniformly bounded as $N \to \infty$. It can then be shown (see [9], Thm. (2.3), p.229) that $m(z)$ is the uniform limit of a sequence of analytic functions, and is thus analytic. Moreover, since $m(z)$ is inside $C_N(z)$, it follows from (2.87) that $Im\, m(z) > 0$ for $Im z > 0$.

We have treated the case of the Schrödinger operator defined on the positive real line. The results obtained can be shown to hold for the negative line in a similar way. In particular, we have shown that the function $m(z)$ is an example of a Herglotz function. The analysis of Herglotz functions which arise from differential operators provides an important tool of Spectral Theory.

Chapter 3

Generalized Value distribution of Herglotz functions.

3.1 Introduction

We begin this chapter by introducing the notion of value distribution for a real-valued, measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$. We then explain what we mean by the value distribution associated to a Herglotz function F, and indicate how we can generalize this idea. Finally, we show how this generalized value distribution may be related to the composition of pairs of Herglotz functions.

3.2 Value distribution

Given a (Lebesgue) measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, and any two Borel sets A and S, we are interested in quantities such as

$$\mathcal{M}(A, S) = |\{\lambda \in A : f(\lambda) \in S\}| = |A \cap f^{-1}(S)|.$$

Here $|.|$ stands for Lebesgue measure, and $\mathcal{M}(A,S)$ is the measure of the points $\lambda \in A$ for which $f(\lambda) \in S$. The mapping $\mathcal{M} : (A,S) \to \mathcal{M}(A,S)$, which assigns an extended real non-negative number to pairs of Borel subsets of $\mathbb{R}$, has the properties

(i) $A \to \mathcal{M}(A,S)$ defines a measure on Borel subsets of $\mathbb{R}$, for fixed S;

$S \to \mathcal{M}(A,S)$ defines a measure on Borel subsets of $\mathbb{R}$, for fixed A;

(ii) $\mathcal{M}(A,\mathbb{R}) = |A|$, hence in particular the measure $A \to \mathcal{M}(A,S)$ is absolutely continuous with respect to Lebesgue measure;

In addition, we shall assume that

(iii) the measure $S \to \mathcal{M}(A,S)$ is absolutely continuous with respect to Lebesgue measure.

Definition 3.1 *Any mapping $(A,S) \to \mathcal{M}(A,S)$, where A, S are Borel subsets of $\mathbb{R}$, and satisfying properties (i) – (iii) above, is called a value distribution function.*

Properties (ii) and (iii) imply that any value distribution function $\mathcal{M}$ may be represented in terms of measures $\{\mu_y\}$ $(y \in \mathbb{R})$ as

$$\mathcal{M}(A,S) = \int_S \mu_y(A)dy. \tag{3.1}$$

In considering the notion of value distribution for Herglotz functions, the family of measures in the integral in (3.1) will arise from the Herglotz integral representation, and these measures will be Herglotz measures. The precise construction will be given in the next section.

3.3 The one parameter family of Herglotz functions F_y

Given a Herglotz function F, we may construct a one parameter family of Herglotz functions F_y, defined by

$$F_y(z) = \frac{1}{y - F(z)}, \qquad y \in \mathbb{R}. \tag{3.2}$$

Suppose that the functions F_y have representations

$$F_y(z) = a_y + b_y z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu_y(t), \tag{3.3}$$

so that $d\mu_y(t)$ are the measures corresponding to the Herglotz functions F_y. We refer to the integral in (3.1), where the measures μ_y are now the measures corresponding to the Herglotz functions F_y, as the value distribution associated with the Herglotz function F. Moreover, we have the following result.

Lemma 3.2 *Suppose that F has real boundary values almost everywhere, so that μ is purely singular. Then, for any Borel sets A, $S \subseteq \mathbb{R}$ we have*

$$\int_S \mu_y(A) dy = |A \cap F_+^{-1}(S)|, \tag{3.4}$$

where F_+ denotes the boundary value of F as z approaches the real axis, and $|.|$ denotes Lebesgue measure.

Proof. Note that the functions F_y will also have real boundary values almost everywhere, and so the measures μ_y will be purely singular as well. Denote by $Supp\,\mu_y$ the support of μ_y, and note that $Supp\,\mu_y = \{\lambda \in \mathbb{R} : F_+(\lambda) = y\}$ (see remark (2.57)). Hence, for $y \in S$ we have $Supp\,\mu_y \subseteq F_+^{-1}(S)$ and we can write

$$\int_S \mu_y(A)dy = \int_S \mu_y(A \cap F_+^{-1}(S))dy.$$

However, for $y \notin S$ we have $Supp\, \mu_y \bigcap F_+^{-1}(S) = \emptyset$, and so

$$\int_S \mu_y(A \cap F_+^{-1}(S))dy = \int_{\mathbf{R}} \mu_y(A \cap F_+^{-1}(S))dy.$$

We now use the remarkable identity (see [23]) that

$$\int_{\mathbf{R}} \mu_y(A)dy = |A| \tag{3.5}$$

for any set $A \subseteq \mathbb{R}$, to obtain

$$\int_S \mu_y(A)dy = |A \cap F_+^{-1}(S)|,$$

and the lemma is proved.

Corollary 3.3 *With the same notation as in lemma* (3.2), *suppose that F has real boundary values almost everywhere on A. Then, equation* (3.4) *again holds.*

Proof. Let μ_y^{ac}, μ_y^s denote the absolutely continuous and singular components of μ_y respectively. The assumption that F has real boundary values almost everywhere on A implies that the set $Supp\, \mu_y^{ac} \bigcap A$ has zero Lebesgue measure, and on this set μ_y^{ac} vanishes. Thus, $\mu_y(A) = \mu_y^s(A)$ and corollary (3.3) now follows by our previous argument.

It will be important for later calculations to know how the constants b_y in (3.3) depend on y. The next lemma provides an answer to this question.

Lemma 3.4 *For a given Herglotz function $F(z)$, the non-negative constants b_y appearing in* (3.3) *are zero, except possibly for a single value of y. Moreover,* *it is possible for b_y to be strictly positive for some value $y = y_0$; for any given $y_0 \in \mathbb{R}$, the condition $b_{y_0} > 0$ is equivalent to the condition that the point $t = 0$ is a discrete point of the measure $dg(t)$, where $dg(t)$ is the measure corresponding to the Herglotz function $G(z)$ defined by*

$$G(z) = \frac{1}{y_0 - F(-\frac{1}{z})}. \tag{3.6}$$

Proof. By lemma (2.53), the constants b_y are determined by

$$b_y = \lim_{s\to\infty} \frac{1}{s} Im\, F_y(is) = \lim_{s\to\infty} \frac{1}{s} Im\Big[\frac{1}{y - F(is)}\Big].$$

Suppose that $F(is) = \alpha(s) + i\beta(s)$, with α, β real. Then,

$$b_y = \lim_{s\to\infty} \frac{1}{s} \frac{\beta(s)}{[y - \alpha(s)]^2 + [\beta(s)]^2}.$$

If $b_y > 0$, then $\alpha(s) \to y$ and $\beta(s) \to 0$ as $s \to \infty$, so that $F(is) \to y$ as $s \to \infty$. That is, $F(is) \to y$ as $s \to \infty$ is a necessary condition for $b_y > 0$, where y is any real number. This means that there can only be at most one value of y such that $b_y > 0$, since $F(is)$ can not tend to two different limits as $s \to \infty$.

Consider now the Herglotz function $G(z)$ defined in (3.6), and in particular, the limit

$$\ell = \lim_{w\to 0^+} w\, Im G(iw), \qquad w \in \mathbb{R}.$$

Writing $w = \frac{1}{s}$, we have

$$\ell = \lim_{s\to\infty} \frac{1}{s} Im\Big[\frac{1}{y_0 - F(is)}\Big].$$

Therefore, $b_{y_0} = \ell$, and thus $b_{y_0} > 0$ is equivalent to $\ell > 0$.

Suppose that G admits the representation

$$G(z) = a_G + b_G z + \int_{\mathbf{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} dg(t).$$

Then,

$$\ell = \lim_{w \to 0^+} w \, Im G(iw) = \lim_{w \to 0^+} \int_{\mathbf{R}} \frac{w^2}{t^2+w^2} dg(t). \tag{3.7}$$

The integrand in the above equation has 0 as its pointwise limit, as $w \to 0^+$, except in the case when $t = 0$, in which case the limit is 1. Also, note that

$$\frac{w^2}{t^2+w^2} = \frac{1}{\frac{t^2}{w^2}+1} \leq \frac{1}{t^2+1}, \qquad for\, 0 < w \leq 1,$$

where

$$\int_{\mathbf{R}} \frac{1}{1+t^2} dg(t) < \infty,$$

since the measure $dg(t)$ defines a Herglotz function. Hence, we can apply the Lebesgue dominated convergence theorem in (3.7), to deduce that

$$\lim_{w \to 0^+} \int_{\mathbf{R}} \frac{w^2}{t^2+w^2} dg(t) = \int_{\mathbf{R}} \lim_{w \to 0^+} \frac{w^2}{t^2+w^2} dg(t) = g(\{0\}).$$

So, $b_{y_0} > 0$ is equivalent to $t = 0$ being a discrete point of the $dg(t)$ measure.

Now we show how to construct a family of Herglotz functions $\tilde{F}_y(z)$, such that, for a given (real) value of y, the constant $\tilde{b}_y$ appearing in the representation of the functions $\tilde{F}_y(z)$ is strictly positive. We first take a Herglotz function $\tilde{G}(z)$, whose corresponding measure $d\tilde{g}(t)$ has a discrete point at $t = 0$. A Herglotz measure has a discrete point at $t = t_0$, provided that the non-decreasing function which gives rise to this measure has a discontinuity at the point $t = t_0$. Suppose for example that we take

$$\tilde{\rho}(t) = \begin{cases} 0 & t < 0, \\ t+1 & t \geq 0, \end{cases}$$

so that $d\tilde{g}(t)$ is the zero measure on the negative real line, and is equal to Lebesgue measure on the positive real line. Now let the Herglotz function $\tilde{F}(z)$ be defined by

$$\tilde{G}(z) = \frac{1}{y_1 - \tilde{F}(-\frac{1}{z})}, \qquad y_1 \in \mathbb{R},$$

which implies on inverting the equation that

$$\tilde{F}(z) = y_1 - \frac{1}{\tilde{G}(-\frac{1}{z})}.$$

Next, define a family of Herglotz functions $\tilde{F}_y(z)$ by

$$\tilde{F}_y(z) = \frac{1}{y - \tilde{F}(z)}, \qquad y \in \mathbb{R}.$$

Then, it follows by our previous analysis that the constant $\tilde{b}_{y_1}$, appearing in the representation of the Herglotz function $\tilde{F}_{y_1}(z)$, is strictly positive. Moreover, we then have $b_y = 0$ for any $y \neq y_1$.

3.4 Generalized Value distribution of Herglotz functions, and its dependence on composition of Herglotz functions

We now extend the idea of the value distribution of a Herglotz function, which was defined in the previous section. Given a Herglotz function $F(z)$, and a Borel subset S of $\mathbb{R}$, we define the integral-measures ν and ν_S by

$$\nu(X) = \int_{\mathbf{R}} \mu_y(X) d\sigma(y) \tag{3.8}$$

and

$$\nu_S(X) = \int_S \mu_y(X) d\sigma(y) \tag{3.9}$$

respectively, for any Borel subset X of $\mathbb{R}$. Here, the measures μ_y are the measures corresponding to the Herglotz function F_y, which were defined in (3.2), and $d\sigma$ is any Herglotz measure. In the remainder of the section, we will obtain some results which describe the measures ν and ν_S.

Lemma 3.5 *Suppose that the measure $d\sigma$ is absolutely continuous. Then, the measures ν and ν_S are absolutely continuous.*

Proof. We will use the identity (3.5). Assuming $|A| = 0$ we then have ([10], Cor. (2.3.11), p.68) $\mu_y(A) = 0$ almost everywhere, and since by assumption the measure $d\sigma$ is absolutely continuous it follows ([10], Prop. (2.3.8), p.67) that $\nu(A) = \nu_S(A) = 0$. This proves the absolute continuity of ν and ν_S.

Lemma 3.6 *Suppose that F has real boundary values almost everywhere, so that the measure μ is purely singular. Then, we have*

$$\nu_S(A) = \nu_S(A \cap F_+^{-1}(S)) = \nu(A \cap F_+^{-1}(S)). \tag{3.10}$$

Proof. The result follows from the proof of lemma (3.2) and the definitions of ν and ν_S from which we have

$$\nu_S(A) = \int_S \mu_y(A) d\sigma(y) = \int_S \mu_y(A \cap F_+^{-1}(S)) d\sigma(y) = \nu_S(A \cap F_+^{-1}(S))$$

$$= \int_{\mathbb{R}} \mu_y(A \cap F_+^{-1}(S)) d\sigma(y) = \nu(A \cap F_+^{-1}(S)).$$

Thus ν_S agrees with ν on $F_+^{-1}(S)$, though in general they are different.

Lemma 3.7 *The measure ν_S is a Herglotz measure.*

Proof. We shall first of all verify the identity

$$\int_{\mathbb{R}} h(t)d\nu_S(t) = \int_S \left\{ \int_{\mathbb{R}} h(t)d\mu_y(t) \right\} d\sigma(y), \tag{3.11}$$

where h is any measurable function for which these integrals are absolutely convergent. (In fact, it is enough that either the left-hand side or the right-hand side is absolutely convergent). Suppose first that h is the characteristic function of a measurable set $E \subset \mathbb{R}$. Then,

$$\int_{\mathbb{R}} \chi_E(t)d\nu_S(t) = \nu_S(E) = \int_S \mu_y(E)d\sigma(y)$$

$$= \int_S \left\{ \int_E d\mu_y(t) \right\} d\sigma(y) = \int_S \left\{ \int_{\mathbb{R}} \chi_E(t)d\mu_y(t) \right\} d\sigma(y).$$

Hence the identity holds in the case $h(t) = \chi_E(t)$. Next, suppose that h is a simple function, $h(t) = \sum_{i=1}^n a_i \chi_{A_i}(t)$. Then, we have

$$\int_{\mathbb{R}} \sum_{i=1}^n a_i \chi_{A_i}(t)d\nu_S(t) = \sum_{i=1}^n a_i \int_{\mathbb{R}} \chi_{A_i}(t)d\nu_S(t) =$$

$$= \sum_{i=1}^n a_i \int_S \left\{ \int_{\mathbb{R}} \chi_{A_i}(t)d\mu_y(t) \right\} d\sigma(y) = \int_S \left\{ \sum_{i=1}^n a_i \int_{\mathbb{R}} \chi_{A_i}(t)d\mu_y(t) \right\} d\sigma(y) =$$

$$= \int_S \left\{ \int_{\mathbb{R}} \sum_{i=1}^n a_i \chi_{A_i}(t)d\mu_y(t) \right\} d\sigma(y),$$

and hence (3.11) is proved in this case also.

Now suppose that h is a non-negative measurable function. Thus, by lemma (2.9) there is a sequence $\{f_n\}$ of simple functions such that

(1) $f_1(t) \leq f_2(t) \leq ...$, and

(2) $\lim_{n \to \infty} f_n(t) = h(t)$.

Then, we have

$$\int_{\mathbb{R}} h(t)d\nu_s(t) = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(t)d\nu_s(t) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)d\nu_s(t),$$

by an application of the monotone convergence theorem. By the previous part of this proof, and another application of the monotone convergence theorem, we see that the above expression is equal to

$$\lim_{n\to\infty} \int_S \left\{ \int_{\mathbb{R}} f_n(t)d\mu_y(t) \right\} d\sigma(y) = \int_S \left\{ \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)d\mu_y(t) \right\} d\sigma(y).$$

Finally, by another application of the monotone convergence theorem, we see that this integral is equal to

$$\int_S \left\{ \int_{\mathbb{R}} \lim_{n\to\infty} f_n(t)d\mu_y(t) \right\} d\sigma(y) = \int_S \left\{ \int_{\mathbb{R}} h(t)d\mu_y(t) \right\} d\sigma(y),$$

which shows that (3.11) holds in this case as well.

Suppose now that h is any measurable function. Then, $h = h^+ - h^-$, where h^+, h^- are respectively the positive and negative parts of h, and h^+ and h^- are non-negative, measurable functions. Hence, (3.11) follows from the second part of this proof. Note that the integrals involving h^+ and h^- are finite, since $|h| = h^+ + h^-$, and by our assumption $|h|$ is integrable.

We now consider the function $p(t) = 1/(1+t^2)$, which is continuous and hence measurable. We will show that, with $h(t) = p(t)$, the double integral on the right hand side of (3.11) is finite. This will imply that the integral on the left hand side of (3.11) is also finite, (having the same value since the two sides are equal), which implies that ν_s is a Herglotz measure.

From the representation of the Herglotz functions F_y in (3.3), we obtain

$$Im F_y(i) = b_y + \int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_y(t).$$

Therefore, we have

$$\int_S \left\{ \int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_y(t) \right\} d\sigma(y) = \int_S \{Im F_y(i) - b_y\} d\sigma(y). \tag{3.12}$$

Let us consider

$$\int_S Im F_y(z) d\sigma(y) = \int_S Im \left[\frac{1}{y - F(z)} \right] d\sigma(y).$$

Suppose that, for any fixed z, $F(z) = A_z + iB_z$, with $B_z > 0$. Then,

$$\int_S Im F_y(z) d\sigma(y) = \int_S \frac{B_z}{(y - A_z)^2 + B_z^2} d\sigma(y).$$

Consider now the function

$$H(y) = \frac{B_z(1+y^2)}{(y - A_z)^2 + B_z^2}.$$

This function tends to B_z as $y \to \pm\infty$. Hence, there exists a constant $y_0 > 0$ such that, if $|y| > y_0$, then $H(y) < B_z + 1$. Since the function H is continuous on the closed interval $[-y_0, y_0]$, it attains its maximum. Let $c = \max H(y)$ on the interval $[-y_0, y_0]$, and let $K_z = \max\{B_z + 1, c\}$. Note that $K_z = K(z)$ will depend on the value of $z \in \mathbb{C}_+$. We then have

$$\frac{B_z}{(y - A_z)^2 + B_z^2} \leq K_z \frac{1}{1+y^2},$$

and since $1/(1+y^2)$ is σ-integrable, it follows that

$$\int_S Im F_y(z) d\sigma(y) < +\infty.$$

By noting that $Im F_y(z) - b_y \leq Im F_y(z)$, since b_y is non-negative, and setting $z = i$, we obtain from (3.11) and (3.12)

$$\int_S \{Im F_y(i) - b_y\} d\sigma(y) = \int_{\mathbb{R}} \frac{1}{1+t^2} d\nu_S(t) < +\infty,$$

which proves from (3.12) that $d\nu_S(t)$ is a Herglotz measure, for arbitrary Borel sets S.

Now let $H_S(z)$ and $\phi(z)$ be Herglotz functions corresponding to the Herglotz measures $d\nu_S(t)$ and $d\sigma(t)$, respectively, with the following representations:

$$H_S(z) = a_H + b_H z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} d\nu_S(t), \tag{3.13}$$

$$\phi(z) = a_\phi + b_\phi z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t). \tag{3.14}$$

Let also $\phi_S(z)$ be the Herglotz function having the same representation as that of $\phi(z)$ in (3.14), except that, now integration takes place over the set S instead of $\mathbb{R}$, that is

$$\phi_S(z) = a_\phi + b_\phi z + \int_S \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\sigma(t). \tag{3.15}$$

Moreover, let the composed Herglotz function $(\phi_S \circ F)(z)$ have the following representation:

$$(\phi_S \circ F)(z) = a_{(\phi_S \circ F)} + b_{(\phi_S \circ F)} z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu_{(\phi_S \circ F)}(t). \tag{3.16}$$

The following lemma shows how the measure ν_S is related to the measure $\mu_{(\phi_S \circ F)}$.

Lemma 3.8 *For any Borel subset B of $\mathbb{R}$, we have*

$$\nu_S(B) = \mu_{(\phi_S \circ F)}(B) - b_\phi \mu(B),$$

where $\mu_{(\phi_S \circ F)}$ is the measure corresponding to the composed Herglotz function $(\phi_S \circ F)$, μ is the measure corresponding to the Herglotz function F, and the *constant b_ϕ appears in the representation of the Herglotz function ϕ_S in* (3.15) (*and also in* (3.14)). *Note that here we do not use absolute continuity of the measure σ.*

Proof. From the representation of $H_S(z)$ in (3.13) we have

$$Im H_S(z) = b_H Im z + \int_{\mathbf{R}} Im\left[\frac{1}{t-z}\right] d\nu_S(t). \tag{3.17}$$

The function $Im[1/(t-z)]$ is a continuous, and hence measurable function, and thus we have from equation (3.11)

$$\int_{\mathbf{R}} Im\left[\frac{1}{t-z}\right] d\nu_S(t) = \int_S \left\{ \int_{\mathbf{R}} Im\left[\frac{1}{t-z}\right] d\mu_y(t) \right\} d\sigma(y). \tag{3.18}$$

The representation of the functions $F_y(z)$ in (3.3) leads to

$$Im F_y(z) = b_y Im z + \int_{\mathbf{R}} Im\left[\frac{1}{t-z}\right] d\mu_y(t). \tag{3.19}$$

Substituting (3.19) into (3.18) , and then (3.18) into (3.17) we obtain

$$Im H_S(z) = b_H Im z + \int_S \{Im F_y(z) - b_y Im z\} d\sigma(y)$$

$$= b_H Im z + \int_S Im\left[\frac{1}{y - F(z)}\right] d\sigma(y) - \int_S b_y Im z \, d\sigma(y). \tag{3.20}$$

In (3.20), $\int_S b_y Im z \, d\sigma(y)$ may be identified with the integral $\int_S f(y) d\sigma(y)$, where

$$f(y) = b_y Im z = \begin{cases} b_{y^*} Im z & y = y^*, \\ 0 & y \neq y^*, \end{cases}$$

where y^* is the point for which $b_y > 0$, if this point exists. Hence,

$$\int_S b_y Im z \, d\sigma(y) = \begin{cases} b_{y^*} (Im z) \sigma(\{y^*\}) & y^* \in S, \\ 0 & y^* \notin S. \end{cases} \tag{3.21}$$

Also, from (3.15) we have

$$Im\,\phi_S(z) = b_\phi\, Im\, z + \int_S Im\left[\frac{1}{t-z}\right] d\sigma(t). \tag{3.22}$$

With (3.21) and (3.22), we obtain from (3.20)

$$Im\,H_S(z) = b_H\, Im\, z + Im\,\phi_S(F(z)) - b_\phi\, Im\,F(z) - b_{y^*}(Im\,z)\sigma(\{y^*\}\cap S), \tag{3.23}$$

where y^* is the point for which $b_y > 0$, whenever such a point exists.

Suppose that the points a and b are not discrete points of the measure ν_S. By using lemma (2.59) (also remark (2.60)) and equation (3.23) we then have

$$\nu_S((a,b]) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b \Big\{ b_H\, Im(\lambda+i\varepsilon) + Im\,\phi_S(F(\lambda+i\varepsilon)) -$$

$$-b_\phi\, Im\,F(\lambda+i\varepsilon) - b_{y^*} Im(\lambda+i\varepsilon)\sigma(\{y^*\}\cap S)\Big\} d\lambda =$$

$$= b_H \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im(\lambda+i\varepsilon)d\lambda + \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im\,\phi_S(F(\lambda+i\varepsilon))d\lambda -$$

$$b_\phi \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im\,F(\lambda+i\varepsilon)d\lambda - b_{y^*}\sigma(\{y^*\}\cap S)\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im(\lambda+i\varepsilon)d\lambda. \tag{3.24}$$

However,

$$\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im(\lambda+i\varepsilon)d\lambda = 0.$$

Hence, we have from (3.24)

$$\nu_S((a,b]) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im\,\phi_S(F(\lambda+i\varepsilon))d\lambda - b_\phi \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b Im\,F(\lambda+i\varepsilon)d\lambda.$$

Suppose that the points a and b are not discrete points of any of the measures ν_S, $\mu_{(\phi_S\circ F)}$ or μ. (The set of discrete points of each of these measures is a countable set of points. This is because the functions that generate Herglotz measures are non-decreasing, and thus can only have a countable number of discontinuities.) By using again the characterization of Herglotz measures of intervals $(a, b]$, we now have

$$\nu_S((a,b]) = \mu_{(\phi_S \circ F)}((a,b]) - b_\phi \mu((a,b]), \tag{3.25}$$

We shall now show that (3.25) holds, for arbitrary points a and b. As noted above, the set of discrete points of the measures ν_S, $\mu_{(\phi_S \circ F)}$, and μ is countable. Take points a and b. Then, given any $\varepsilon > 0$, there are points in the intervals $(a-\varepsilon, a)$ and $(b, b+\varepsilon)$ respectively, which are not discrete points of either of the measures ν_S, $\mu_{(\phi_S \circ F)}$, or μ. This is because the intervals $(a-\varepsilon, a)$ and $(b, b+\varepsilon)$ are both uncountable, having positive Lebesgue measure ε. Hence, we can construct two sequences of such points, $\{c_i\}_{i \in \mathbb{N}}$ and $\{d_i\}_{i \in \mathbb{N}}$, with $c_i \to a_-$ and $d_i \to b_+$. We then have, on taking the limit of $\nu_S((c_i, d_i])$

$$\nu_S([a,b]) = \mu_{(\phi_S \circ F)}([a,b]) - b_\phi \mu([a,b]).$$

By the same argument, we have

$$\nu_S(\{x\}) = \mu_{(\phi_S \circ F)}(\{x\}) - b_\phi \mu(\{x\}), \qquad \forall x \in \mathbb{R}.$$

Combining these two equations, we obtain

$$\nu_S((a,b]) = \mu_{(\phi_S \circ F)}((a,b]) - b_\phi \mu((a,b]), \tag{3.26}$$

for all points a and b of $\mathbb{R}$.

Equation (3.26) implies that ν_S and $(\mu_{(\phi_S \circ F)} - b_\phi \mu)$ are measures defined on the algebra of countable unions of intervals of the form $(a, b]$. The fact that ν_S is a Herglotz measure, implies that $\nu_S((-N, N])$ is finite for any integer N.

Since $\mathbb{R} = \bigcup_{N\in\mathbb{N}}(-N, N]$, it follows that ν_S, and also $(\mu_{(\phi_S \circ F)} - b_\phi \mu)$, are σ-finite measures. Hence, there is a unique extension of these measures to the collection of Lebesgue measurable sets, called the corresponding Lebesgue-Stieltjes measure, and restricting this measure to the Borel sets, we have a unique extension to all Borel sets. Therefore, with this extended measure we have shown that

$$\nu_S(B) = \mu_{(\phi_S \circ F)}(B) - b_\phi \mu(B), \tag{3.27}$$

for all Borel sets B, and lemma (3.8) is proved.

In view of the significance of the composition $(\phi_S \circ F)$ in (3.27) for the measure ν_S, we shall give further results for compositions of Herglotz functions in Chapter 4.

Chapter 4

Compositions of Herglotz functions

4.1 Introduction

The composition of two Herglotz functions is a Herglotz function. Given two Herglotz functions, the question arises how the Herglotz representation for the composed function will depend on the Herglotz representation for the two given Herglotz functions. We shall answer this question:

(a) with respect to the coefficient $b_F z$ in the Herglotz representation formula (1.1); how does the coefficient b_F in this term depend on the representation of the two given functions?

(b) with respect to the measure $d\mu(t)$ in the Herglotz representation formula (1.1); in the case that $d\mu(t)$ is absolutely continuous we know ([23], p.131) that the density function h is given by $h(\lambda) = \frac{1}{\pi} Im \lim_{\varepsilon \to 0^+} F(\lambda + i\varepsilon)$. So the question arises how does the boundary value for the composed function depend on the boundary values of the two given functions. In fact, we will answer this question in more generality, and we will make no additional assumptions about the measures.

4.2 The term linear in z in the representation of a composed Herglotz function

The following result provides a criterion for a Herglotz measure to be finite.

Lemma 4.1 *Let a function $c(s)$ be defined for $s \in \mathbb{R}$ by*

$$c(s) = \int_{\mathbb{R}} \frac{1}{s^2 + t^2} dg(t),$$

where $dg(t)$ is an arbitrary Herglotz measure. Then, $c(s)s^2 \to a < +\infty$ as $s \to \infty$, if and only if the measure $dg(t)$ is finite. If $G(z)$ is a Herglotz function associated with the measure $dg(t)$, the above condition is the same as $sImG(is) - b_G s^2 \to a < +\infty$ as $s \to \infty$, where b_G is the constant appearing in the Herglotz representation of $G(z)$. Note also that this implies $c(s)s^2 \to +\infty$ as $s \to \infty$ if and only if the measure $dg(t)$ is infinite.

Proof. Suppose that $c(s)s^2 \to a < +\infty$, as $s \to +\infty$. We will show that the measure $dg(t)$ is finite. Assume on the contrary that $dg(t)$ is infinite. Then, there is an $N_0 \in \mathbb{N}$ such that $g([-N_0, N_0]) > a$, and we have

$$a = \lim_{s\to\infty} \int_{\mathbb{R}} \frac{s^2}{s^2 + t^2} dg(t) \geq \lim_{s\to\infty} \int_{-N_0}^{N_0} \frac{s^2}{s^2 + t^2} dg(t)$$

$$= \int_{-N_0}^{N_0} \lim_{s\to\infty} \frac{s^2}{s^2 + t^2} dg(t)$$

by an application of the Lebesgue dominated convergence theorem, since for $t \in [-N_0, N_0]$, we have

$$\frac{s^2}{s^2+t^2} \le 1 \le (1+N_0^2)\frac{1}{1+t^2}, \qquad \forall\, s \in \mathbb{R}.$$

Note that for $t \in [-N_0, N_0]$, $\lim_{s\to\infty} s^2/(s^2+t^2) = 1$, and hence we obtain

$$a = g([-N_0, N_0]) > a,$$

which is a contradiction.

Now suppose conversely that $dg(t)$ is finite. Then,

$$\lim_{s\to\infty} c(s)s^2 = \lim_{s\to\infty} \int_{\mathbb{R}} \frac{s^2}{s^2+t^2}\,dg(t) = g((-\infty,\infty)) < +\infty,$$

by the Lebesgue dominated convergence theorem (since in this case $\frac{s^2}{s^2+t^2} \le 1$ which is integrable with respect to the measure $dg(t)$).

Remark 4.2 *We note in passing that, if $c(s)s^2 \to 0$ as $s \to \infty$, then $dg(t)$ is the zero measure. To see this, note that*

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\,dg(t) \le \int_{\mathbb{R}} \frac{s^2}{s^2+t^2}\,dg(t), \qquad \forall\, s \ge 1,$$

and $\lim_{s\to\infty} c(s)s^2 = 0$ would imply $\int_{\mathbb{R}} \frac{1}{1+t^2}\,dg(t) = 0$. Hence, for any interval $[-N, N]$ we would have

$$0 = \int_{-N}^{N} \frac{1}{1+t^2}\,dg(t) \ge \frac{1}{(1+N^2)}\,g([-N, N]),$$

so that $g([-N, N]) = 0 \quad \forall\, N \in \mathbb{N}$, and thus

$$g((-\infty, +\infty)) = \lim_{N\to\infty} g([-N, N]) = 0.$$

Theorem 4.3 *Let $F(z)$ and $G(z)$ be two arbitrary Herglotz functions with the following respective integral representations:*

$$F(z) = a_F + b_F z + \int_{\mathbf{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu(t), \tag{4.1}$$

$$G(z) = a_G + b_G z + \int_{\mathbf{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} dg(t). \tag{4.2}$$

Consider the composed Herglotz function $(F \circ G)(z)$, *and let* $b_{F\circ G} z$ *be the term linear in* z *appearing in its representation. Then, if* $b_G \neq 0$, *we have* $b_{F\circ G} = b_F b_G$. *If* $b_G = 0$ *and the measure* $dg(t)$ *is infinite, then* $b_{F\circ G} = 0$, *(so that again* $b_{F\circ G} = b_F b_G$ *holds), and if* $b_G = 0$ *and* $dg(t)$ *is finite, then* $b_{F\circ G} = \frac{1}{a}\mu(\{t_0\})$, *where* $a = g((-\infty, \infty)) < +\infty$, *and*

$$t_0 = a_G - \int_{\mathbf{R}} \frac{t}{1+t^2} dg(t).$$

Note that if $dg(t)$ *is the zero measure, then* $b_G \neq 0$.

Proof. The constant $b_{F\circ G}$ is given (lemma (2.53)) by $b_{F\circ G} = \lim_{s\to\infty} \frac{1}{s} \mathrm{Im}\, F(G(is))$. From (4.2) we have

$$G(is) = a_G + b_G is + \int_{\mathbf{R}} \left\{ \frac{t+is}{s^2+t^2} - \frac{t}{1+t^2} \right\} dg(t)$$

$$= a_G + \int_{\mathbf{R}} \frac{t(1-s^2)}{(s^2+t^2)(1+t^2)} dg(t) + i\left\{ b_G s + s \int_{\mathbf{R}} \frac{1}{s^2+t^2} dg(t) \right\}.$$

Let the functions $A(s)$ and $c(s)$ be defined by

$$A(s) = a_G + \int_{\mathbf{R}} \frac{t(1-s^2)}{(s^2+t^2)(1+t^2)} dg(t), \qquad c(s) = \int_{\mathbf{R}} \frac{1}{s^2+t^2} dg(t). \tag{4.3}$$

Here,

$$\lim_{s\to\infty} c(s) = 0. \tag{4.4}$$

This follows by an application of the Lebesgue dominated convergence theorem, since

$$\frac{1}{s^2+t^2} \leq \frac{1}{1+t^2}, \qquad for\ s \geq 1,$$

where the function $1/(1+t^2)$ is integrable with respect to $dg(t)$. So, we have

$$G(is) = A(s) + i[b_G s + c(s)s], \tag{4.5}$$

and from (4.1) we obtain

$$F(G(is)) = a_F + b_F\{A(s) + i[b_G s + c(s)s]\} +$$

$$+ \int_{\mathbb{R}} \left\{ \frac{1}{t - A(s) - i[b_G s + c(s)s]} - \frac{t}{t^2+1} \right\} d\mu(t)$$

$$= a_F + b_F A(s) + i b_F b_G s + i b_F c(s) s +$$

$$+ \int_{\mathbb{R}} \left\{ \frac{t - A(s) + i[b_G s + c(s)s]}{[t-A(s)]^2 + [b_G s + c(s)s]^2} - \frac{t}{t^2+1} \right\} d\mu(t),$$

implying that

$$\frac{1}{s} Im F(G(is)) = b_F b_G + b_F c(s) + b_G \int_{\mathbb{R}} \frac{1}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} d\mu(t) +$$

$$+ \int_{\mathbb{R}} \frac{c(s)}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} d\mu(t).$$

Equation (4.4) implies that $b_F c(s) \to 0$ as $s \to \infty$. Thus, we have

$$b_{F \circ G} = \lim_{s\to\infty} \frac{1}{s} Im F(G(is)) = b_F b_G +$$

$$\lim_{s\to\infty} \left\{ b_G \int_{\mathbb{R}} \frac{1}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} d\mu(t) + \int_{\mathbb{R}} \frac{c(s)}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} d\mu(t) \right\} \tag{4.6}$$

[Note that if $dg(t)$ is the zero measure, (in which case $b_G \neq 0$), then $A(s) = a_G$ and $c(s) = 0$, and thus

$$b_{F\circ G} = b_F b_G + \lim_{s\to\infty} b_G \int_{\mathbb{R}} \frac{1}{(t-a_G)^2 + s^2 b_G^2} d\mu(t).$$

It is straightforward to show that

$$\frac{1}{(t-a_G)^2 + s^2 b_G^2} \leq const. \frac{1}{1+t^2}, \qquad \forall\, t \in \mathbb{R}, s \geq 1,$$

which is integrable with respect to $d\mu(t)$, and hence an application of the Lebesgue dominated convergence theorem gives $b_{F\circ G} = b_F b_G$].

Suppose first that $b_G = 0$. Then, from (4.6) we obtain

$$\begin{aligned} b_{F\circ G} &= \lim_{s\to\infty} \int_{\mathbb{R}} \frac{c(s)}{[t-A(s)]^2 + s^2[c(s)]^2} d\mu(t) \\ &= \lim_{s\to\infty} \int_{\mathbb{R}} \frac{1}{\frac{1}{c(s)}[t-A(s)]^2 + s^2 c(s)} d\mu(t). \end{aligned} \tag{4.7}$$

We now distinguish the two subcases. Suppose first that the measure $dg(t)$ is finite. Note that

$$\left|\frac{t(1-s^2)}{(s^2+t^2)(1+t^2)}\right| \leq \frac{|t|s^2}{(s^2+t^2)(1+t^2)} \leq 1, \qquad for\; s \geq 1,$$

and since $dg(t)$ is finite, an application of the Lebesgue dominated convergence theorem gives

$$\begin{aligned} \lim_{s\to\infty} A(s) &= \lim_{s\to\infty} \left\{ a_G + \int_{\mathbb{R}} \frac{t(1-s^2)}{(s^2+t^2)(1+t^2)} dg(t) \right\} \\ &= a_G - \int_{\mathbb{R}} \frac{t}{1+t^2} dg(t) = t_0, \end{aligned} \tag{4.8}$$

where $|t_0| < +\infty$. Note also that by the proof of Lemma (4.1) we have $c(s)s^2 \to a = g((-\infty,\infty))$, as $s \to \infty$.

We claim that

$$\frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2+c(s)s^2} \le const., \tag{4.9}$$

for s sufficiently large and for all real values of t. There is an $S_0 > 1$ such that, if $s > S_0$ then $c(s)s^2 \ge \frac{a}{2}$ and $|A(s)| \le |t_0| + 1$. Fix N such that $N \ge 4(|t_0|+1)$. For $t \in [-N, N]$ and $s > S_0$ we have

$$\frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2+c(s)s^2} \le \frac{t^2+1}{c(s)s^2} \le \frac{2}{a}(N^2+1),$$

and for $t \in [-N, N]^c$ and $s > S_0$ we have

$$\frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2+c(s)s^2} \le \frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2}$$

$$= \frac{1+\frac{1}{t^2}}{\frac{1}{c(s)}\left[1+\frac{A^2(s)}{t^2}-2\frac{A(s)}{t}\right]} \le \frac{1+1/16(1+|t_0|)^2}{\frac{1}{c(s)}\left[1-2\frac{A(s)}{t}\right]}$$

$$\le 2c(s)\left(1+\frac{1}{16(1+|t_0|)^2}\right) \le 2\left(1+\frac{1}{16(1+|t_0|)^2}\right)\int_{\mathbf{R}} \frac{s^2}{s^2+t^2}dg(t)$$

$$\le 2\left(1+\frac{1}{16(1+|t_0|)^2}\right)g((-\infty,+\infty)) = 2\left(1+\frac{1}{16(1+|t_0|)^2}\right)a < \infty.$$

Thus, if $s > S_0$, then

$$\frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2+c(s)s^2} \le \max\left\{\frac{2(N^2+1)}{a},\ 2\left(1+\frac{1}{16(1+|t_0|)^2}\right)a\right\},$$

and (4.9) is verified. Therefore, by an application of the Lebesgue dominated convergence theorem in (4.7), (since μ is a Herglotz measure and so the function $\frac{1}{1+t^2}$ is integrable with respect to μ), we obtain

$$b_{F\circ G} = \int_{\mathbf{R}} \lim_{s\to\infty}\left[\frac{1}{\frac{1}{c(s)}[t-A(s)]^2+c(s)s^2}\right]d\mu(t). \tag{4.10}$$

Since $c(s) \to 0$ as $s \to \infty$, the limit in the integrand in (4.10) is zero, provided $t \neq t_0$. Thus, if the point t_0 is not a discrete point of the measure $d\mu(t)$, then $b_{F\circ G} = 0$. If the point $t = t_0$ is a discrete point of the measure $d\mu(t)$, then

$$b_{F\circ G} = \mu(\{t_0\}) \lim_{s\to\infty} \left[\frac{1}{\frac{1}{c(s)}[t_0 - A(s)]^2 + c(s)s^2}\right]. \tag{4.11}$$

Consider the term $\lim_{s\to\infty} \frac{1}{c(s)}[t_0 - A(s)]^2$. Note that

$$\begin{aligned} t_0 - A(s) &= a_G - \int_{\mathbf{R}} \frac{t}{1+t^2}\,dg(t) - a_G - \int_{\mathbf{R}} \frac{t(1-s^2)}{(s^2+t^2)(1+t^2)}\,dg(t) \\ &= \int_{\mathbf{R}} \frac{-t}{s^2+t^2}\,dg(t), \end{aligned}$$

and also that

$$\begin{aligned} \int_{\mathbf{R}} \frac{|t|}{s^2+t^2}\,dg(t) &= \int_{\mathbf{R}} \frac{1}{\sqrt{s^2+t^2}} \frac{|t|}{\sqrt{s^2+t^2}}\,dg(t) \\ &\le \left\{ \int_{\mathbf{R}} \frac{1}{s^2+t^2}\,dg(t)\right\}^{\frac{1}{2}} \left\{ \int_{\mathbf{R}} \frac{t^2}{s^2+t^2}\,dg(t)\right\}^{\frac{1}{2}} \qquad (4.12) \\ &= \sqrt{c(s)} \left\{ \int_{\mathbf{R}} \frac{t^2}{s^2+t^2}\,dg(t)\right\}^{\frac{1}{2}}, \end{aligned}$$

by the Schwarz inequality, since the integrals in (4.12) are finite (the integrands being bounded functions with $dg(t)$ finite). Hence

$$\lim_{s\to\infty} \frac{1}{c(s)}[t_0 - A(s)]^2 \le \lim_{s\to\infty} \int_{\mathbf{R}} \frac{t^2}{s^2+t^2}\,dg(t) = 0,$$

by an application of the Lebesgue dominated convergence theorem. Thus, from (4.11) we obtain $b_{F\circ G} = \mu(\{t_0\}) \lim_{s\to\infty} \frac{1}{c(s)s^2} = \frac{1}{a}\mu(\{t_0\})$, where $a = g((-\infty,\infty))$. This completes the case $b_G = 0$ in the subcase when $dg(t)$ is finite.

We note here a useful inequality for $A(s)$. For $s \ge 1$ we have

$$|A(s) - a_G| \leq \int_{\mathbf{R}} \frac{s^2|t|}{(s^2+t^2)(1+t^2)}\, dg(t)$$

$$= s \int_{\mathbf{R}} \frac{1}{\sqrt{s^2+t^2}} \frac{s|t|}{\sqrt{s^2+t^2}(1+t^2)}\, dg(t)$$

$$\leq s\Big\{ \int_{\mathbf{R}} \frac{1}{s^2+t^2}\, dg(t)\Big\}^{\frac{1}{2}} \Big\{ \int_{\mathbf{R}} \frac{s^2t^2}{(s^2+t^2)(1+t^2)^2}\, dg(t)\Big\}^{\frac{1}{2}} \tag{4.13}$$

$$= s\sqrt{c(s)}\Big\{ \int_{\mathbf{R}} \frac{s^2t^2}{(s^2+t^2)(1+t^2)^2}\, dg(t)\Big\}^{\frac{1}{2}} = \alpha s\sqrt{c(s)}, \tag{4.14}$$

where $\alpha = \alpha(s) = \Big\{ \int_{\mathbf{R}} \frac{s^2t^2}{(s^2+t^2)(1+t^2)^2}\, dg(t)\Big\}^{\frac{1}{2}} < +\infty$. We have again used the Schwarz inequality, since the integrals in (4.13) are finite (for $s \geq 1$ each of the integrands is less than or equal to $const./(1+t^2)$, which is integrable with respect to the Herglotz measure $dg(t)$). We shall be using this estimate later.

Now consider the second subcase, for which $dg(t)$ is infinite with again $b_G = 0$. This implies $c(s)s^2 \to +\infty$ as $s \to +\infty$, by remark (4.2). We claim that

$$\frac{t^2+1}{\frac{1}{c(s)}[t-A(s)]^2 + c(s)s^2}$$

$$= \frac{t^2}{\frac{1}{c(s)}[t-A(s)]^2 + c(s)s^2} + \frac{1}{\frac{1}{c(s)}[t-A(s)]^2 + c(s)s^2} \leq const., \tag{4.15}$$

for s sufficiently large and for all real values of t. In this case, $A(s)$ may not tend to a finite limit as $s \to \infty$, and so our previous treatment does not apply.

Since

$$\frac{1}{(t-a_G)^2+1} \leq const. \frac{1}{1+t^2},$$

it will be sufficient to show

$$\frac{(t-a_G)^2+1}{\frac{1}{c(s)}[t-A(s)]^2 + c(s)s^2} \leq const.$$

But with $t' = t - a_G$, this is the same as

$$\frac{(t')^2 + 1}{\frac{1}{c(s)}[t' - (A(s) - a_G)]^2 + c(s)s^2} \leq const.,$$

where now $A(s) - a_G$ does not have the constant term. So without loss of generality we can deal with the case $a_G = 0$.

We proceed instead as follows. There is an $S_1 > 0$ such that if $s > S_1$ then $c(s)s^2 \geq 1$. Thus, for $s > S_1$ we have

$$\frac{1}{\frac{1}{c(s)}[t - A(s)]^2 + c(s)s^2} \leq \frac{1}{c(s)s^2} \leq 1,$$

and so in order to verify (4.15), it is sufficient to show that

$$\frac{t^2}{\frac{1}{c(s)}[t - A(s)]^2 + c(s)s^2} \leq const.,$$

or equivalently,

$$\frac{\frac{1}{c(s)}[t - A(s)]^2 + c(s)s^2}{t^2} \geq k_1 > 0, \tag{4.16}$$

for s sufficiently large and for all real values of t, where k_1 is a (positive) constant. Let us solve the quadratic

$$\frac{1}{c(s)}[t - A(s)]^2 + c(s)s^2 - K_1 t^2 = 0, \qquad K_1 > 0. \tag{4.17}$$

Rewriting this we obtain

$$\left(\frac{1}{c(s)} - K_1\right)t^2 - \frac{2A(s)}{c(s)}t + \frac{A^2(s)}{c(s)} + c(s)s^2 = 0.$$

Using the standard formula for the roots of the quadratic, we find that the expression inside the square root, except from a factor 4, is

$$-s^2 + \frac{K_1}{c(s)}[A(s)]^2 + K_1 c(s)s^2,$$

and by (4.14) in the case $a_G = 0$ we have

$$-s^2 + \frac{K_1}{c(s)}[A(s)]^2 + K_1 c(s) s^2 \leq -s^2 + K_1 \alpha^2 s^2 + K_1 c(s) s^2$$

$$= s^2(K_1\alpha^2 + K_1 c(s) - 1).$$

An application of the Lebesgue dominated convergence theorem gives

$$\lim_{s\to\infty} \alpha^2 = \lim_{s\to\infty} \int_{\mathbb{R}} \frac{s^2t^2}{(s^2+t^2)(1+t^2)^2}\, dg(t)$$

$$= \int_{\mathbb{R}} \frac{t^2}{(1+t^2)^2}\, dg(t) = \tilde{\alpha} < \infty,$$

and so there is an $S_2 > 0$ such that, if $s > S_2$ then $\alpha^2 < 2\tilde{\alpha}$. Set $K_1 = 1/4\tilde{\alpha}$. Since $c(s) \to 0$ as $s \to \infty$, there is also an $S_3 > 0$ such that, if $s > S_3$ then $K_1 c(s) < 1/4$. Let $S = \max\{S_2, S_3\}$. Then, if $s > S$ we have

$$K_1\alpha^2 + K_1 c(s) - 1 < 0.$$

This shows that the quadratic in (4.17) has no real root, for $s > S$. Note also that, when $t = 0$ the value of this quadratic is

$$\frac{A^2(s)}{c(s)} + c(s)s^2 \geq c(s)s^2,$$

where $c(s)s^2 \to \infty$ as $s \to \infty$. Hence, the quadratic is positive, for $s > S$ and $\forall\, t \in \mathbb{R}$. Therefore, the condition in (4.16) is satisfied, with $k_1 = 1/4\tilde{\alpha}$. If, furthermore, $s > \max\{S, S_1\}$, then the bound in (4.15) is verified and extends to the case $a_G \neq 0$ (though with a different constant), and we can apply the Lebesgue dominated convergence theorem in (4.7) to deduce that $b_{F\circ G} = 0$, since the denominator is greater than or equal to $c(s)s^2$ which diverges in the limit $s \to \infty$. This completes the case $b_G = 0$.

Now suppose that $b_G > 0$. As before, we will apply the Lebesgue dominated convergence theorem in (4.6). This will imply that the limits of both integrals in (4.6) are zero, since the denominator of both integrands is bounded below by $b_G^2 s^2$, which tends to infinity as $s \to \infty$ and $c(s) \to 0$. It will be sufficient, in order to use an argument based on the Lebesgue dominated convergence theorem, to verify that

$$\frac{(t^2+1)}{[t-A(s)]^2 + s^2[b_G + c(s)]^2}$$

$$= \frac{t^2}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} + \frac{1}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} \leq const., \qquad (4.18)$$

for s sufficiently large and for all real values of t. Again, by our previous argument, we can assume $a_G = 0$. There is an $S_1 > 0$ such that, if $s > S_1$ then $b_G^2 s^2 \geq 1$. Thus, if $s > S_1$ we have

$$\frac{1}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} \leq \frac{1}{s^2[b_G + c(s)]^2} \leq \frac{1}{s^2 b_G^2} \leq 1.$$

So, in order to prove (4.18) it is sufficient to show that

$$\frac{t^2}{[t-A(s)]^2 + s^2[b_G + c(s)]^2} \leq const.,$$

or equivalently that

$$\frac{[t-A(s)]^2 + s^2[b_G + c(s)]^2}{t^2} \geq k_2 > 0, \qquad (4.19)$$

for s sufficiently large and for all real values of t, where k_2 is a (positive) constant. Let us solve the quadratic

$$[t-A(s)]^2 + s^2[b_G + c(s)]^2 - K_2 t^2 = 0, \qquad K_2 > 0.$$

Rewriting this we have

$$(1-K_2)t^2 - 2A(s)t + A^2(s) + s^2[b_G + c(s)]^2 = 0. \qquad (4.20)$$

By the standard formula for the roots of this quadratic, we find in this case that the expression inside the square root, except from a factor 4, is

$$s^2[b_G + c(s)]^2(K_2 - 1) + K_2[A(s)]^2$$

and by (4.14) we have, with $a_G = 0$,

$$s^2[b_G + c(s)]^2(K_2 - 1) + K_2[A(s)]^2 \leq s^2[b_G + c(s)]^2(K_2 - 1) + K_2\alpha^2 s^2 c(s)$$

$$= s^2\Big\{[b_G + c(s)]^2(K_2 - 1) + K_2\alpha^2 c(s)\Big\}.$$

Fix K_2 in the interval $0 < K_2 < 1$. Since $\alpha^2 \to \tilde{\alpha} < \infty$ as $s \to \infty$, and $c(s) \to 0$ as $s \to \infty$, it follows that $K_2\alpha^2 c(s) \to 0$ as $s \to \infty$. Thus, there exists an $S > 0$ such that, if $s > S$ then $K_2\alpha^2 c(s) < (1 - K_2)b_G^2$. Then, for $s > S$ the expression inside the square root is negative. This shows that for $s > S$ the quadratic in (4.20) has no real root. Note also that when $t = 0$, the value of this quadratic is

$$A^2(s) + s^2[b_G + c(s)]^2 \geq s^2 b_G^2 > 0,$$

for $s > S$. Therefore, this quadratic is positive for $s > S$ and $\forall\, t \in \mathbb{R}$, and so the condition in (4.19) is satisfied with fixed k_2 in the interval $0 < k_2 < 1$. If, furthermore, $s > \max\{S, S_1\}$, then the condition in (4.18) is satisfied. Hence, we can apply the Lebesgue dominated convergence theorem in (4.6) to obtain $b_{F\circ G} = b_F b_G$. This completes the proof of the theorem.

4.3 Boundary values of composed Herglotz functions

Theorem 4.4 *Let $\phi(z)$ and $F(z)$ be two arbitrary Herglotz functions, and denote by I_F, $\tilde{I}_F$ the sets $I_F = \{\lambda \in \mathbb{R} : F_+(\lambda)$ exists and $F_+(\lambda) \in \mathbb{R}\}$ and $\tilde{I}_F = \{\lambda \in \mathbb{R} : F_+(\lambda)$ exists and $Im F_+(\lambda) > 0\}$ respectively, where $F_+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + i\varepsilon)$ is the boundary value of F at the point λ. (The boundary value $\phi_+(\lambda)$ of ϕ at the point λ is defined similarly). Then, at almost all points $\lambda \in \mathbb{R}$ we have*

$$\lim_{\varepsilon \to 0^+} (\phi \circ F)(\lambda + i\varepsilon) = (\phi \circ F)_+(\lambda) = \begin{cases} \phi_+(F_+(\lambda)) & \lambda \in I_F, \\ \phi(F_+(\lambda)) & \lambda \in \tilde{I}_F. \end{cases} \tag{4.21}$$

Proof. It is straightforward to see that, if $\lambda \in \tilde{I}_F$ so that $F_+(\lambda)$ is a complex number with strictly positive imaginary part, then $(\phi \circ F)_+(\lambda) = \lim_{\varepsilon \to 0^+} \phi(F(\lambda + i\varepsilon)) = \phi(F_+(\lambda))$, since ϕ is analytic and thus continuous in the upper half-plane. As a result, it remains only to consider points $\lambda \in I_F$.

By a wedge-shaped area with vertex the point $\lambda \in \mathbb{R}$ we shall mean a set of the form $\{z \in \mathbb{C}_+ : \alpha < Arg(z - \lambda) < \beta,\ 0 < \alpha < \beta < \pi\}$. By the 'wedgy' limit of ϕ at the point $\lambda \in \mathbb{R}$ we shall mean the limit as z approaches λ along a simple curve ending at λ and lying entirely in a wedge-shaped area with vertex the point λ. By corollary (2.52) we know that if the limit of ϕ at λ exists along a simple curve, then it also exists along any other simple curve ending at λ and contained in a wedge-shaped area with vertex the point λ (and the two limits are equal). We will denote by $\phi_w(\lambda)$ the wedgy limit of ϕ at $\lambda \in \mathbb{R}$.

Let A be the set

$$A = \{\lambda \in I_F : \phi_+(F_+(\lambda)) \text{ and } \phi_w(F_+(\lambda)) \text{ exist}\},$$

and note that $|I_F \backslash A| = 0$. [The fact that $\phi(z)$ is a Herglotz function implies that the set of points $\lambda \in \mathbb{R}$ for which $\phi_+(\lambda)$ does not exist has zero Lebesgue measure. By corollary (3.3) with $|S| = 0$ and $A = I_F$ we have

$$|\{\lambda \in I_F : F_+(\lambda) \in S\}| = 0.] \tag{4.22}$$

We consider first the case when F has real boundary values almost everywhere, so that the complement of I_F has zero Lebesgue measure. In this case we have $|A^c| = 0$. We define the function f by $f(\lambda) = F_+(\lambda)$, so that f is almost everywhere the real boundary value of F. The function f is real-valued, measurable, and finite almost everywhere, and thus it follows from Theorem (2.25) that at almost all $\lambda \in \mathbb{R}$, (and at almost all $\lambda \in A$), f is either approximately monotonic increasing, approximately monotonic decreasing, approximately constant, or approximately oscillatory (see definition (2.24)).

None of the points in I_F, (and so none of the points in A), is a point of approximate constancy for f: If f was approximately constant at a point $\lambda_0 \in I_F$, then, from the definition of approximate constancy we would have $|\{\lambda \in I_F : F_+(\lambda) = f(\lambda_0)\}| > 0$, and (4.22) would be violated with $S = \{f(\lambda_0)\}$. [In fact, f can not be approximately constant at points $\lambda \in \tilde{I}_F$ either, because F has real boundary values almost everywhere.] Therefore, at almost all points in A, f is either approximately monotonic increasing, or approximately monotonic decreasing, or approximately oscillatory.

We shall show that for points $\lambda \in A$ at which f is either approximately monotonic increasing, or approximately monotonic decreasing, or approximately oscillatory, we can find sequences $z_n = \lambda + i\varepsilon_n$ with $\varepsilon_n \to 0^+$ such that the points $F(\lambda + i\varepsilon_n)$ lie in a wedge-shaped area with vertex the point $f(\lambda)$.

Define now the log function for complex argument in the upper half plane by

$$\log(re^{i\theta}) = \log r + i\theta \qquad for\ r > 0,\ 0 \leq \theta \leq \pi.$$

Then, we have

$$Arg\,(F(\lambda+i\varepsilon)-f(\lambda)) = Im\log(F(\lambda+i\varepsilon)-f(\lambda)).$$

Now $z \to \log(F(z)-f(\lambda))$, with $\lambda \in A$ fixed, defines a Herglotz function $H(z)$ with $0 < Im H(z) < \pi$. Any Herglotz function H with bounded imaginary part has a representation with absolutely continuous measure $\xi(t)dt$, where the density function $\xi(t)$ is given by $\lim_{\varepsilon\to 0^+} \frac{1}{\pi} Im H(t+i\varepsilon)$. In this case, we find

$$H(z) = \log|F(i)-f(\lambda)| + \int_{\mathbb{R}} \left\{\frac{1}{t-z} - \frac{t}{1+t^2}\right\}\xi(t)dt,$$

where for almost all t we have

$$\xi(t) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} Arg(F(t+i\varepsilon)-f(\lambda)) = \begin{cases} 1 & f(t) < f(\lambda), \\ 0 & otherwise. \end{cases} \tag{4.23}$$

Hence $\xi(t)$ is almost everywhere the characteristic function of the set $\{t \in I_F : f(t) < f(\lambda)\}$. Thus, we have

$$Arg(F(\lambda+i\varepsilon)-f(\lambda)) = Im H(\lambda+i\varepsilon) = \int_{\mathbb{R}} \frac{\varepsilon\xi(t)dt}{(t-\lambda)^2+\varepsilon^2}. \tag{4.24}$$

which gives the angle between the real axis and the direction from $f(\lambda)$ to $F(\lambda + i\varepsilon)$, for fixed $\varepsilon > 0$ and $\lambda \in \mathbb{R}$. The limit as $\varepsilon \to 0^+$ of the integral in (4.24) may be equated (see [22], Lem (2.4), p.44) with

$$\lim_{h\to 0^+} \pi \int_{\lambda-h}^{\lambda+h} \frac{\xi(t)}{2h} dt, \tag{4.25}$$

in the sense that if either limit exists then both exist and they are equal. We then have

$$\lim_{\varepsilon\to 0^+} Arg\,(F(\lambda+i\varepsilon)-f(\lambda))$$

$$= \frac{\pi}{2} \lim_{h\to 0+} |\{t \in [\lambda - h, \lambda + h] \cap I_F : f(t) < f(\lambda)\}|/h. \tag{4.26}$$

Suppose that λ is any point of A at which f is either approximately monotonic increasing, or approximately monotonic decreasing. In this case, the limit in (4.26) equals 1, so that the limiting angle $\lim_{\varepsilon\to 0+}\text{Arg}\,(F(\lambda+i\varepsilon)-f(\lambda))$ is $\pi/2$. Thus, for ε sufficiently small, the points $F(\lambda + i\varepsilon)$ will make an angle arbitrary close to $\pi/2$ relative to the point $f(\lambda)$ on the real axis. Therefore, the points $F(\lambda + i\varepsilon)$ will all lie, for small enough values of ε, in any given wedge-shaped area with vertex at the point $f(\lambda)$. Since $\lambda \in A$, the limit $\lim_{\varepsilon\to 0+} \phi(F_+(\lambda) + i\varepsilon)$ exists as does the wedgy limit, and it follows that

$$(\phi \circ F)_+(\lambda) = \lim_{\varepsilon\to 0+} (\phi \circ F)(\lambda + i\varepsilon)$$

$$= \lim_{\varepsilon\to 0+} \phi(F(\lambda + i\varepsilon)) = \lim_{\varepsilon\to 0+} \phi(F_+(\lambda) + i\varepsilon) = \phi_+(F_+(\lambda)).$$

Now take any point $\lambda \in A$ at which f is approximately oscillatory. Then, for any ε_0 with $0 < \varepsilon_0 < \frac{\pi}{4}$ there are sequences h_n and h'_n of positive numbers with $h_n \to 0$, $h'_n \to 0$, (without loss of generality we may assume $h_1 > h'_1 > h_2 > h'_2 > \ldots$), such that for all $n \in \mathrm{N}$ we have

$$|\{t \in [\lambda - h_n, \lambda + h_n] \cap I_F : f(t) < f(\lambda)\}| > 2h_n(1 - \varepsilon_0), \tag{4.27}$$

$$|\{t \in [\lambda - h'_n, \lambda + h'_n] \cap I_F : f(t) > f(\lambda)\}| > 2h'_n(1 - \varepsilon_0). \tag{4.28}$$

Note that $ImF(\lambda+ih_n)$, $ImF(\lambda+ih'_n) > 0$ for all n, and consider first the angle given by

$$Arg\,(F(\lambda + ih_1) - f(\lambda)) = \int_{-\infty}^{\infty} \frac{h_1\xi(t)}{(t-\lambda)^2 + h_1^2}\,dt$$

$$\geq \int_{\lambda-h_1}^{\lambda+h_1} \frac{h_1\xi(t)}{(t-\lambda)^2 + h_1^2}\,dt. \tag{4.29}$$

Note that the integrand in (4.29) is bounded above by $1/h_1$ and that

$$\int_{\lambda-h_1}^{\lambda+h_1} \frac{h_1}{(t-\lambda)^2+h_1^2}\,dt = \frac{\pi}{2}.$$

From (4.27) we have $|\{t \in [\lambda - h_1, \lambda + h_1] \cap I_F : \xi(t) = 0\}| < 2h_1\varepsilon_0$, and hence we obtain

$$\frac{\pi}{2} - 2h_1\varepsilon_0\frac{1}{h_1} \leq Arg\,(F(\lambda + ih_1) - f(\lambda)) < \pi.$$

Next consider

$$Arg\,(F(\lambda + ih_1') - f(\lambda)) = \int_{-\infty}^{\infty} \frac{h_1'\xi(t)}{(t-\lambda)^2 + (h_1')^2}\,dt$$

$$= \int_{\mathbb{R}/[\lambda-h_1',\lambda+h_1']} \frac{h_1'\xi(t)}{(t-\lambda)^2+(h_1')^2}\,dt + \int_{\lambda-h_1'}^{\lambda+h_1'} \frac{h_1'\xi(t)}{(t-\lambda)^2+(h_1')^2}\,dt. \tag{4.30}$$

Note that

$$\int_{\mathbb{R}/[\lambda-h_1',\lambda+h_1']} \frac{h_1'}{(t-\lambda)^2+(h_1')^2}\,dt = \int_{\lambda-h_1'}^{\lambda+h_1'} \frac{h_1'}{(t-\lambda)^2+(h_1')^2}\,dt = \frac{\pi}{2},$$

and from (4.28) we have $|\{t \in [\lambda - h_1', \lambda + h_1'] \cap I_F : \xi(t) = 1\}| < 2h_1'\varepsilon_0$. Thus, (4.30) implies that

$$0 < Arg\,(F(\lambda + ih_1') - f(\lambda)) < \frac{\pi}{2} + 2h_1'\varepsilon_0\frac{1}{h_1'}.$$

Continuing in this way, we can construct a strictly-decreasing sequence ε_j with $\varepsilon_j \to 0$, by setting $\varepsilon_1 = h_1$, $\varepsilon_2 = h_1'$, $\varepsilon_3 = h_2, \ldots$ (ε_j taking the values of h_n and h_n' successively), such that

$$\frac{\pi}{2} - 2\varepsilon_0 < Arg(F(\lambda + i\varepsilon_j) - f(\lambda)) < \pi, \tag{4.31}$$

if j is odd, and

$$0 < Arg(F(\lambda + i\varepsilon_j) - f(\lambda)) < \frac{\pi}{2} + 2\varepsilon_0, \tag{4.32}$$

if j is even.

We now construct a sequence of positive numbers ε_j' with $\varepsilon_j' \to 0$, such that

$$\frac{\pi}{2} - 2\varepsilon_0 < Arg\,(F(\lambda + i\varepsilon_j') - f(\lambda)) < \frac{\pi}{2} + 2\varepsilon_0, \tag{4.33}$$

for all j, as follows. If (4.32) is also satisfied with $j = 1$, (as well as (4.31)), we set $\varepsilon_1' = \varepsilon_1$. If that is not the case but (4.31) is satisfied with $j = 2$, (as well as (4.32)), we neglect ε_1 and set $\varepsilon_1' = \varepsilon_2$. If we have none of these two cases, then there must exist a positive number $\varepsilon_{1,2}$ with $\varepsilon_2 < \varepsilon_{1,2} < \varepsilon_1$, such that both (4.31) and (4.32) are satisfied with $\varepsilon_j = \varepsilon_{1,2}$. Such a number must exist because of the continuity of F. In this case we set $\varepsilon_1' = \varepsilon_{1,2}$. Proceeding in this way, we generate a sequence ε_j' with $\varepsilon_j' \to 0$ such that (4.33) is satisfied, implying that the points $F(\lambda + i\varepsilon_j')$ lie in a wedge with vertex at $f(\lambda)$. (The angle of the wedge is $4\varepsilon_0$, where $\varepsilon_0 > 0$ was arbitrary, and the angle between either side of the wedge and the perpendicular at $f(\lambda)$ is $2\varepsilon_0$.) Since $\lambda \in A$, the limits $\lim_{\varepsilon\to 0^+} \phi(F_+(\lambda) + i\varepsilon)$ and $\phi_w(F_+(\lambda))$ exist, and we have

$$(\phi \circ F)_+(\lambda) = \lim_{\varepsilon\to 0^+} \phi(F(\lambda + i\varepsilon))$$

$$= \lim_{j\to\infty} \phi(F(\lambda + i\varepsilon_j')) = \lim_{\varepsilon\to\infty} \phi(F_+(\lambda) + i\varepsilon) = \phi_+(F_+(\lambda)).$$

This completes the case when F has real boundary values almost everywhere.

Now consider the general case when F takes boundary values with strictly positive imaginary part on a set of positive Lebesgue measure. As noted at the beginning of this proof, for points $\lambda \in \tilde{I}_F$ we have $(\phi \circ F)_+(\lambda) = \phi(F_+(\lambda))$. Thus, we only have to consider points $\lambda \in I_F$.

Define a function $\tilde{f}$ by

$$\tilde{f}(\lambda) = \begin{cases} F_+(\lambda) & \lambda \in I_F, \\ 0 & \lambda \in \tilde{I}_F. \end{cases}$$

The function $\tilde{f}$ is real-valued, measurable, and finite almost everywhere, and therefore at almost all λ, $\tilde{f}$ is either approximately monotonic increasing, or approximately monotonic decreasing, or approximately constant, or approximately oscillatory (Theorem (2.25)).

Since I_F and $\tilde{I}_F$ are measurable, almost all points of I_F are points of density of I_F, and almost all points of $\tilde{I}_F$ are points of density of $\tilde{I}_F$ (Theorem (2.23)). Points of density of $\tilde{I}_F$ are points of approximate constancy of $\tilde{f}$, (with value 0), and hence at almost all points $\lambda \in \tilde{I}_F$, $\tilde{f}$ is approximately constant. On the other hand, only points $\lambda \in I_F$ with $F_+(\lambda) = 0$ can be points of approximate constancy of $\tilde{f}$, (because of (4.22)), and the set of these points has zero Lebesgue measure (again from (4.22)).

Denote by A_d the set of points $\lambda \in A$ which are density points of A, and note that the complement of A_d in I_F has zero Lebesgue measure. With $\lambda \in A_d$ we have in this case

$$\xi(t) = \begin{cases} 1 & t \in I_F \textit{ and } \tilde{f}(t) < \tilde{f}(\lambda), \\ 0 & t \in I_F \textit{ and } \tilde{f}(t) \geq \tilde{f}(\lambda), \end{cases} \tag{4.34}$$

and $0 < \xi(t) < 1$ for $t \in \tilde{I}_F$. Hence, (4.26) in this case becomes inequality, with the left hand side being greater than or equal to the right hand side. However, for $\lambda \in A_d$ and in the limit $h \to 0^+$ we have

$$\lim_{h \to 0^+} |\{t \in [\lambda - h, \lambda + h] \cap I_F\}|/h = 2.$$

Hence, for points $\lambda \in A_d$ at which $\tilde{f}$ is either approximately monotonic increasing or approximately monotonic decreasing the limit in (4.26) equals 1, so that the limiting angle $\lim_{\varepsilon \to 0^+} Arg\,(F(\lambda + i\varepsilon) - \tilde{f}(\lambda))$ is again $\pi/2$. Therefore, by the same argument used in the first part of this proof, it follows that $(\phi \circ F)_+(\lambda) = \phi_+(F_+(\lambda))$.

For points $\lambda \in A_d$ at which $\tilde{f}$ is approximately oscillatory, we can apply our treatment for the corresponding case in the first part of the proof, with slight changes. There exist sequences h_n and h'_n of positive real numbers converging to zero, which satisfy (4.27) and (4.28) respectively, as before. Inequality (4.29) holds, where now $|\{t \in [\lambda - h_1, \lambda + h_1] : \xi(t) \neq 1\}| < 2h_1\varepsilon_0$, and so again we have

$$\frac{\pi}{2} - 2\varepsilon_0 \leq Arg(F(\lambda + ih_1) - \tilde{f}(\lambda)) < \pi.$$

Also, (4.30) holds, where now $|\{t \in [\lambda - h'_1, \lambda + h'_1] : \xi(t) \neq 0\}| < 2h'_1\varepsilon_0$. Hence, we have

$$0 < Arg(F(\lambda + ih'_1) - \tilde{f}(\lambda)) \leq \frac{\pi}{2} + 2\varepsilon_0.$$

Hence the result follows in this case as well by the same argument that we used in the corresponding case in the first part of the proof, and the proof of the theorem is completed.

Corollary 4.5 *Suppose that the measure $d\sigma$ corresponding to the Herglotz function F is absolutely continuous. Suppose also that the constant b_ϕ appearing in the representation of the Herglotz function ϕ_s in* (3.15) *in Chapter* 3 *is zero. Then, the measure $\mu_{(\phi_s \circ F)}$ corresponding to the composed Herglotz function $(\phi_s \circ F)$ is absolutely continuous, and moreover, (with the same notation as in Theorem* (4.4)*), the density function $h(\lambda)$ of $\mu_{(\phi_s \circ F)}$ is given almost everywhere* *by*

$$h(\lambda) = \begin{cases} \frac{1}{\pi} Im\, (\phi_s)_+(F_+(\lambda)) & \lambda \in I_F, \\ \frac{1}{\pi} Im\, \phi_s(F_+(\lambda)) & \lambda \in \tilde{I}_F. \end{cases}$$

Proof. From lemma (3.8) we have

$$\nu_s(B) = \mu_{(\phi_s \circ F)}(B)$$

for any Borel set B, where the measure ν_s was defined in (3.9). By lemma (3.5), the absolute continuity of σ implies the absolute continuity of ν_s. Hence $\mu_{(\phi_s \circ F)}$ is also absolutely continuous, with density function $h = h(\lambda)$. By definition, $h(\lambda) = \frac{1}{\pi} Im\, (\phi_s \circ F)_+(\lambda)$, and Corollary (4.5) now follows from Theorem (4.4).

Chapter 5

Averaged Herglotz measures and the Schrödinger equation.

5.1 Introduction

In this Chapter we consider some convergent sequences of Herglotz functions and derive some consequences for spectral theory. In section 5.2 we examine uniformly convergent sequences of Herglotz functions and their limiting measures. In section 5.3 we present a remarkable estimate regarding convergence in the limit $\delta \to 0^+$ of the value distribution associated to a Herglotz function $F^\delta(z)$, which has been obtained from a given Herglotz function $F(z)$ by translation through a small increment $i\delta$ parallel to the imaginary axis. This estimate will be used in section 5.4, in applications to the Schrödinger equation.

5.2 Herglotz functions and uniform convergence

Here we consider sequences of Herglotz functions $F_n(z)$, with corresponding measures $d\mu_n(t)$, having the respective integral representations

$$F_n(z) = a_n + b_n z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\mu_n(t). \tag{5.1}$$

We suppose that the functions $F_n(z)$ converge uniformly in the limit $n \to \infty$ to the Herglotz function $F(z)$, on compact subsets of the upper half-plane. We are interested in the behaviour of the family of measures $d\mu_n(t)$ in this limit. Moreover, we define a corresponding family of Herglotz functions $F_y^n(z)$ $(y \in \mathbb{R})$ by

$$F_y^n(z) = \frac{1}{y - F_n(z)},$$

having measures μ_y^n and the integral representations

$$F_y^n(z) = a_y^n + b_y^n z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} d\mu_y^n(t). \tag{5.2}$$

We shall use the following result later.

Lemma 5.1 *Let F_n be a sequence of Herglotz functions such that $F_n(i) \to q$ as $n \to \infty$, with $Im q > 0$. Then, for any fixed N and any $y \in \mathbb{R}$ $\mu_y^n([-N, N])$ is bounded, and there exists a constant $c > 0$ independent of y such that*

$$\mu_y^n([-N, N]) \leq c\frac{1}{1+y^2}, \qquad n \in \mathbb{N}.$$

Proof. Since $F_n(i) \to q$ as $n \to \infty$ with $Imq > 0$, there exists a compact set D of the upper half plane such that for all n, the points $F_n(i) = A_n + iB_n$ lie in D. Let K_D be a constant such that $|z| \leq K_D$ for all $z \in D$, and $\delta_D = \inf_{z \in D} Im z > 0$.

[Such a δ_D exists: otherwise there exists a sequence of complex numbers z_n in D such that $\lim_{n\to\infty} Im z_n = 0$. Also, since D is a compact set, there exists a subsequence z_{n_k} of z_n such that $z_{n_k} \to z_0$, with $z_0 \in D$. Since $Im z$ is a continuous function, we have $Im z_{n_k} \to Im z_0 = 0$, which is a contradiction.] For any $y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\mu_y^n([-N,N]) = \int_{-N}^{N} d\mu_y^n(t) \leq (1+N^2)\int_{-N}^{N} \frac{1}{1+t^2} d\mu_y^n(t)$$

$$\leq (1+N^2)\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_y^n(t) = (1+N^2)(Im F_y^n(i) - b_y^n)$$

$$\leq (1+N^2) Im F_y^n(i),$$

since $b_y^n \geq 0$ for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Note that

$$Im F_y^n(i) = Im\left[\frac{1}{y - F_n(i)}\right] = \frac{B_n}{(y-A_n)^2 + B_n^2},$$

where $0 < \delta_D \leq B_n \leq K_D$ and $|A_n| \leq K_D$. It is straightforward to show that $Im F_y^n(i)$ is bounded, and thus that $\mu_y^n([-N,N])$ is bounded for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover, it is also straightforward to show that $Im F_y^n(i) \leq c_1 \frac{1}{1+y^2}$, where c_1 is a constant. The second assertion of the Lemma follows with $c = (1+N^2)c_1$, and the proof is completed.

Lemma 5.2 *Let $F(z)$ be a Herglotz function with representation* (1.1) *and Herglotz measure $d\mu(t)$. If the point a is not a discrete point of the measure $d\mu(t)$, then,*

$$\lim_{\delta\to 0+} \int_{\mathbb{R}} \frac{\delta^2}{(t-a)^2 + \delta^2} d\mu(t) = 0.$$

Proof. Note that

$$\frac{\delta^2}{(t-a)^2+\delta^2} = \frac{1}{\frac{(t-a)^2}{\delta^2}+1} \leq \frac{1}{(t-a)^2+1}, \qquad \forall\, t \in \mathbb{R},$$

for $0 < \delta < 1$. Moreover,

$$\lim_{\delta \to 0^+} \frac{\delta^2}{(t-a)^2+\delta^2} = \begin{cases} 0 & t \neq a, \\ 1 & t = a. \end{cases}$$

Hence, by an application of the Lebesgue dominated convergence theorem we obtain

$$\lim_{\delta \to 0^+} \int_{\mathbf{R}} \frac{\delta^2}{(t-a)^2+\delta^2}\, d\mu(t) = \mu(\{a\}) = 0,$$

since a is not a discrete point of $d\mu(t)$.

Lemma 5.3 *Let $F_n(z)$ be a sequence of Herglotz functions, given by equation* (5.1), *converging uniformly to the Herglotz function $F(z)$ on compact subsets of the upper half-plane. Let $F(z)$ have the Herglotz representation* (1.1), *with Herglotz measure $d\mu(t)$. Suppose that the points a and b, $a < b$, are not discrete points of the measure $d\mu(t)$. Let $\varepsilon > 0$ be given. Then, there exists a positive number δ_0 with $0 < \delta_0 < \frac{b-a}{2}$, and an $N_0 \in \mathbf{N}$ depending on δ_0 and ε, such that if $n > N_0$ then $\mu_n(J_0) < \varepsilon$, where $J_0 = [a-\delta_0, a+\delta_0] \cup [b-\delta_0, b+\delta_0]$.*

Proof. From (5.1) we have

$$\delta\, Im F_n(a+i\delta) = b_n \delta^2 + \int_{\mathbf{R}} \frac{\delta^2}{(t-a)^2+\delta^2}\, d\mu_n(t),$$

and from the representation of $F(z)$ in (1.1)

$$\delta\, Im F(a+i\delta) = b_F \delta^2 + \int_{\mathbf{R}} \frac{\delta^2}{(t-a)^2+\delta^2}\, d\mu(t).$$

Similar expressions hold for $\delta Im F_n(b+i\delta)$ and $\delta Im F(b+i\delta)$. It follows from Lemma (5.2) that

$$\lim_{\delta\to 0^+} \delta Im F(a+i\delta) = \lim_{\delta\to 0^+} \delta Im F(b+i\delta) = 0. \tag{5.3}$$

Let $\varepsilon > 0$ be given. In view of (5.3), we can choose $\delta_0 > 0$ such that

$$\delta_0 Im[F(a+i\delta_0) + F(b+i\delta_0)] < \varepsilon/8. \tag{5.4}$$

Since the functions $F_n(z)$ converge to $F(z)$ at the points $z = a+i\delta_0$ and $z = b+i\delta_0$, there is an $N_0 \in \mathrm{N}$ such that, if $n > N_0$ then

$$\delta_0 |Im\{F_n(a+i\delta_0) - F(a+i\delta_0) + F_n(b+i\delta_0) - F(b+i\delta_0)\}| < \varepsilon/8. \tag{5.5}$$

It follows from (5.4) and (5.5) that, for $n > N_0$ we have

$$\delta_0 Im[F_n(a+i\delta_0) + F_n(b+i\delta_0)]$$

$$= 2b_n\delta_0^2 + \int_{\mathrm{R}} \left[\frac{\delta_0^2}{(t-a)^2+\delta_0^2} + \frac{\delta_0^2}{(t-b)^2+\delta_0^2}\right] d\mu_n(t) < \frac{\varepsilon}{4}. \tag{5.6}$$

Since $b_n\delta_0^2 \geq 0$, (5.6) implies

$$\int_{\mathrm{R}} \left[\frac{\delta_0^2}{(t-a)^2+\delta_0^2} + \frac{\delta_0^2}{(t-b)^2+\delta_0^2}\right] d\mu_n(t) < \frac{\varepsilon}{4},$$

so that

$$\int_{\mathrm{R}} \frac{\delta_0^2}{(t-a)^2+\delta_0^2} d\mu_n(t) < \frac{\varepsilon}{4} \quad \textit{and} \quad \int_{\mathrm{R}} \frac{\delta_0^2}{(t-b)^2+\delta_0^2} d\mu_n(t) < \frac{\varepsilon}{4}.$$

In particular, specializing to the intervals $[a-\delta_0, a+\delta_0]$, $[b-\delta_0, b+\delta_0]$, we have

$$\int_{a-\delta_0}^{a+\delta_0} \frac{\delta_0^2}{(t-a)^2+\delta_0^2} d\mu_n(t) < \frac{\varepsilon}{4} \quad \textit{and} \quad \int_{b-\delta_0}^{b+\delta_0} \frac{\delta_0^2}{(t-b)^2+\delta_0^2} d\mu_n(t) < \frac{\varepsilon}{4}.$$

For $t \in [a-\delta_0, a+\delta_0]$ we have $(t-a)^2 \le \delta_0^2$, so that

$$\frac{\delta_0^2}{(t-a)^2+\delta_0^2} \ge \frac{1}{2},$$

and hence,

$$\int_{a-\delta_0}^{a+\delta_0} \frac{1}{2}\, d\mu_n(t) \le \int_{a-\delta_0}^{a+\delta_0} \frac{\delta_0^2}{(t-a)^2+\delta_0^2}\, d\mu_n(t) < \frac{\varepsilon}{4}.$$

Therefore, for $n > N_0$ we have

$$\mu_n([a-\delta_0, a+\delta_0]) < \varepsilon/2.$$

Similarly, for $n > N_0$ we have

$$\mu_n([b-\delta_0, b+\delta_0]) < \varepsilon/2.$$

Hence $\mu_n(J) < \varepsilon$, provided $n > N_0$, as stated in the Lemma.

Remark 5.4 *On examination of the proof of Lemma* (5.3) *the reader will find that, a sufficient condition is that the functions $F_n(z)$ converge to $F(z)$ at the points $z = a + i\delta_0$ and $z = b + i\delta_0$.*

Lemma 5.5 *With the same notation as in the statement of lemma* (5.2)*, suppose that the points a and b $(a < b)$ are not discrete points of the measure μ, and let $\varepsilon > 0$ be given. Then $\delta > 0$ can be chosen such that*

$$\left| \frac{1}{\pi} \int_0^c \left\{ \int_J \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2} \right] d\mu(t) \right\} ds \right| < \varepsilon, \tag{5.7}$$

where $c > 0$ is arbitrary, $J = (a-\delta, a+\delta) \cup (b-\delta, b+\delta)$, and δ is taken to lie in the interval $0 < \delta < \frac{b-a}{2}$.

Proof. Let $c > 0$ be a constant, and define a function $K(c,t)$ by

$$K(c,t) = \int_0^c \left[\frac{|t-a|}{(t-a)^2+s^2} + \frac{|t-b|}{(t-b)^2+s^2}\right] ds,$$

so that

$$K(c,t) = \begin{cases} \tan^{-1}\frac{c}{|b-a|} & t = a \text{ or } t = b, \\ \\ \tan^{-1}\frac{c}{|t-a|} + \tan^{-1}\frac{c}{|t-b|} & t \neq a,\ t \neq b, \end{cases}$$

and hence $K(c,t) \leq \pi$. Since $\mu(\{a\}) = \mu(\{b\}) = 0$, it follows that there is a $\delta_1 > 0$ such that $\mu((a-\delta_1, a+\delta_1]) < \frac{\varepsilon}{2}$, and a $\delta_2 > 0$ such that $\mu((b-\delta_2, b+\delta_2]) < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2, \frac{b-a}{2}\}$, and $J = [a-\delta, a+\delta] \cup [b-\delta, b+\delta]$, so that $\mu(J) < \varepsilon$. Note that the double integral

$$\int_J \left\{ \int_0^c \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] ds \right\} d\mu(t)$$

is absolutely convergent, so that we can change the order of integration in (5.7), and obtain

$$\left| \frac{1}{\pi} \int_0^c \left\{ \int_J \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu(t) \right\} ds \right|$$

$$\leq \frac{1}{\pi} \int_J K(c,t) d\mu(t) \leq \mu(J) < \varepsilon,$$

by our choice of δ in the definition of J, and (5.7) is proved.

We are now ready to prove the first theorem of this section regarding the limiting behaviour of a family of Herglotz measures $d\mu_n(t)$, where the corresponding functions $F_n(z)$ converge uniformly to a Herglotz function $F(z)$, on compact subsets of the upper half-plane.

Theorem 5.6 *Let $F_n(z)$ be a sequence of Herglotz functions with corresponding measures μ_n. Suppose that the points a and b $(a < b)$ are not discrete points* *of any of the measures μ_n ($n \in \mathbb{N}$) or μ, the measure corresponding to the Herglotz function $F(z)$, and that the functions $F_n(z)$ converge uniformly to $F(z)$ on compact subsets of the upper half-plane. Then, we have*

$$\mu_n((a,b]) \to \mu((a,b]). \tag{5.8}$$

Proof. We start with the standard result regarding Herglotz measures of intervals $(a,b]$, whose endpoints are not discrete points of the measure (lemma (2.59) and remark (2.60)). With this result, we have

$$|\mu_n((a,b]) - \mu((a,b])|$$

$$= \left| \lim_{w\to 0+} \frac{1}{\pi}\int_a^b Im\, F_n(\lambda + iw)d\lambda - \lim_{w\to 0+} \frac{1}{\pi}\int_a^b Im\, F(\lambda+iw)d\lambda \right|. \tag{5.9}$$

The problem is to control the behaviour of the functions $F_n(z)$ and $F(z)$ close to the real axis, without imposing additional restrictions on the behaviour of these functions. We achieve this by using ideas from complex contour integration, which allow us to rely on properties of the functions $F_n(z)$ and $F(z)$ in the upper half-plane, where, by assumption, they satisfy convergence conditions. We shall also need to control the behaviour of integrals near the endpoints a and b of the interval $(a,b]$, on a contour perpendicular to the real axis. We proceed as follows.

Given any $\varepsilon > 0$, first set

$$\varepsilon_0 = \frac{\varepsilon\pi}{6M(b-a)},$$

where the constant M is defined to be

$$M = \frac{2}{s_0} Im F(is_0) + 1,$$

for any $s_0 > 0$. The role of ε_0 and the constant M will become clear shortly.

Let $z = \lambda + iw$, for some fixed w, $0 < w < \varepsilon_0$, so that $dz = d\lambda$. Also let $A = a + iw$, $B = b + iw$. We thus have

$$|\mu_n((a,b]) - \mu((a,b])| = \left| \lim_{w \to 0^+} \frac{1}{\pi} Im \int_A^B F_n(z)dz - \lim_{w \to 0^+} \frac{1}{\pi} Im \int_A^B F(z)dz \right|. \quad (5.10)$$

Now let $C = b + i\varepsilon_0$, $D = a + i\varepsilon_0$, and consider the contour $ABCD$. Since the functions $F_n(z)$ and $F(z)$ are analytic in the upper half complex plane, it follows by Cauchy's theorem (Theorem (2.33)) that

$$\int_{ABCD} F_n(z)dz = \int_{ABCD} F(z)dz = 0,$$

so that

$$\int_A^B F_n(z)dz = \int_A^D F_n(z)dz + \int_D^C F_n(z)dz + \int_C^B F_n(z)dz.$$

A similar expression holds for $F(z)$. On the contour AD, let $z = a + is$, for $w \le s \le \varepsilon_0$, so that $dz = ids$. On the contour DC let $z = s + i\varepsilon_0$, for $a \le s \le b$ so that $dz = ds$, and on the contour CB let $z = b + is$, for $w \le s \le \varepsilon_0$, which implies $dz = ids$. Then,

$$\int_A^B F_n(z)dz = \int_w^{\varepsilon_0} iF_n(a + is)ds + \int_a^b F_n(s + i\varepsilon_0)ds + \int_{\varepsilon_0}^w iF_n(b + is)ds.$$

Therefore,

$$Im \int_A^B F_n(z)dz = Re \int_w^{\varepsilon_0} F_n(a+is)ds + Im \int_a^b F_n(s+i\varepsilon_0)ds - Re \int_w^{\varepsilon_0} F_n(b+is)ds$$

$$= Re \int_w^{\varepsilon_0} [F_n(a + is) - F_n(b + is)]ds + Im \int_a^b F_n(s + i\varepsilon_0)ds.$$

From equation (5.1), it follows that

$$Re\,[F_n(a+is)-F_n(b+is)]$$

$$= (a-b)b_n + \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu_n(t), \tag{5.11}$$

and thus we have

$$Im \int_A^B F_n(z)dz = \int_w^{\varepsilon_0} (a-b)b_n ds + Im \int_a^b F_n(s+i\varepsilon_0)ds +$$

$$+ \int_w^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu_n(t) \right\} ds. \tag{5.12}$$

In a similar way, we obtain

$$Im \int_A^B F(z)dz = \int_w^{\varepsilon_0} (a-b)b_F ds + Im \int_a^b F(s+i\varepsilon_0)ds +$$

$$+ \int_w^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu(t) \right\} ds. \tag{5.13}$$

Hence, from equations (5.10), (5.12) and (5.13) we have

$$|\mu_n((a,b]) - \mu((a,b])|$$

$$\leq \left| \lim_{w\to 0+} \frac{1}{\pi} \int_w^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu_n(t) \right\} ds - \right.$$

$$\left. - \lim_{w\to 0+} \frac{1}{\pi} \int_w^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu(t) \right\} ds \right| +$$

$$+ \left| \lim_{w\to 0+} \frac{1}{\pi} \int_w^{\varepsilon_0} (a-b)(b_n-b_F)ds \right| + \left| \frac{1}{\pi} \int_a^b Im\,[F_n(s+i\varepsilon_0) - F(s+i\varepsilon_0)]ds \right|. \tag{5.14}$$

The last term of the above inequality is the result of integrating along the contour CD, and is a straightforward estimate. The other terms are the result of integrating along the contours BC and AD. The term involving the constants b_n and b_F is also not difficult to be dealt with, and we begin with this term.

From the representations of the functions $F_n(z)$ and $F(z)$ in (5.1) and (1.1) respectively, we have

$$\frac{1}{s} Im F_n(is) = b_n + \int_{\mathbb{R}} \frac{1}{t^2+s^2} d\mu_n(t),$$

and

$$\frac{1}{s} Im F(is) = b_F + \int_{\mathbb{R}} \frac{1}{t^2+s^2} d\mu(t).$$

Thus,

$$b_F < \frac{1}{s} Im F(is), \qquad \forall s \in \mathbb{R},$$

$$b_n < \frac{1}{s} Im F_n(is), \qquad \forall s \in \mathbb{R}.$$

In particular, $b_F < \frac{1}{s_0} Im F(is_0)$, and $b_n < \frac{1}{s_0} Im F_n(is_0)$. Since $F_n \to F$, as $n \to \infty$, at $z = is_0$, there is a $N_1 \in \mathbb{N}$, such that if $n > N_1$, then $\frac{1}{s_0} |Im[F_n(is_0) - F(is_0)]| < \frac{1}{2}$. So we obtain $b_n < \frac{1}{s_0} Im F(is_0) + \frac{1}{2}$. Hence,

$$|b_n - b_F| \leq b_n + b_F < \frac{2}{s_0} Im F(is_0) + \frac{1}{2} < M.$$

From this result, with an application of the Lebesgue dominated convergence theorem, we have

$$\left| \lim_{w \to 0^+} \frac{1}{\pi} \int_w^{\varepsilon_0} (a-b)(b_n - b_F) ds \right| \leq \lim_{w \to 0^+} \frac{1}{\pi} \int_w^{\varepsilon_0} (b-a)|b_n - b_F| ds$$

$$< \frac{1}{\pi} \int_0^{\varepsilon_0} (b-a) M ds = \frac{1}{\pi}(b-a) M \varepsilon_0 = \frac{\varepsilon}{6}. \tag{5.15}$$

Since, by assumption, $F_n(z) \to F(z)$ uniformly as $n \to \infty$, on the horizontal contour joining the points $a + i\varepsilon_0$ and $b + i\varepsilon_0$, there is an $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|F_n(s + i\varepsilon_0) - F(s + i\varepsilon_0)| < \frac{\varepsilon\pi}{6(b-a)}$, $a \leq s \leq b$. Hence, we also have

$$\left|\frac{1}{\pi}\int_a^b Im[F_n(s+i\varepsilon_0)-F(s+i\varepsilon_0)]ds\right| \leq \frac{1}{\pi}\int_a^b |Im[F_n(s+i\varepsilon_0)-F(s+i\varepsilon_0)]|ds$$

$$\leq \frac{1}{\pi}\int_a^b |F_n(s+i\varepsilon_0)-F(s+i\varepsilon_0)|ds < \frac{\varepsilon}{6}. \tag{5.16}$$

We now consider the double integrals appearing in (5.14). The idea here is to split the real line, over which the integration with respect to $d\mu_n(t)$ and $d\mu(t)$ respectively is taken, into two subsets. The first subset, which is the union of two intervals surrounding the points a and b, has arbitrary small measure in each case. On the second subset, which is the complement of the first, we can obtain appropriate bounds for the integrand. The detailed argument is as folows .

By lemma (5.3), there is a $\delta_1 > 0$, a corresponding set $J_1 = [a-\delta_1, a+\delta_1] \cup [b-\delta_1, b+\delta_1]$, and an $N_3 \in \mathbb{N}$ such that, if $n > N_3$, then

$$\mu_n(J_1) < \frac{\varepsilon}{18}.$$

Note that the integral

$$\int_w^{\varepsilon_0}\left\{\int_{J_1}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds$$

is absolutely convergent, so that we can change the order of integration, and by an application of the Lebesgue dominated convergence theorem we obtain

$$\left|\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{J_1}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds\right|$$

$$=\left|\frac{1}{\pi}\int_{J_1}\left\{\int_0^{\varepsilon_0}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]ds\right\}d\mu_n(t)\right|$$

$$\leq \frac{1}{\pi}\int_{J_1} K(\varepsilon_0,t)d\mu_n(t) \leq \mu_n(J_1) < \frac{\varepsilon}{18},$$

provided that $n > N_3$. The function $K(c,t)$, $t \in \mathbb{R}$ and any $c > 0$ fixed, is defined in the proof of lemma (5.5).

Also, by lemma (5.5), there is a $\delta_2 > 0$ and a corresponding set $J_2 = [a-\delta_2, a+\delta_2] \cup [b-\delta_2, b+\delta_2]$ such that the following inequality holds:

$$\left|\frac{1}{\pi}\int_0^{\varepsilon_0}\left\{\int_{J_2}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right| < \frac{\varepsilon}{18}.$$

By a change in order of integration, and an application of the Lebesgue dominated convergence theorem we now have

$$\left|\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{J_2}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right|$$

$$=\left|\frac{1}{\pi}\int_{J_2}\left\{\int_0^{\varepsilon_0}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]ds\right\}d\mu(t)\right| < \frac{\varepsilon}{18}.$$

We can choose $\delta = \min\{\delta_1, \delta_2\}$, and set $J = [a-\delta, a+\delta] \cup [b-\delta, b+\delta]$, which then gives

$$\left|\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds\right|+$$

$$+\left|\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right| < \frac{\varepsilon}{9}. \tag{5.17}$$

From the inequalities (5.15), (5.16), and (5.17) we may deduce from (5.14) that

$$|\mu_n((a,b]) - \mu((a,b])|$$

$$< \frac{4\varepsilon}{9} + \left|\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds-\right.$$

$$\left.-\lim_{w\to 0+}\frac{1}{\pi}\int_w^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right|. \tag{5.18}$$

It remains to estimate the integrand for $t \in \mathbb{R}/J$, and on this set we have

$$\left|\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right| \le c_J\frac{1}{1+t^2},$$

for some constant $c_J > 0$.

[To verify this bound, note that

$$\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}$$

$$= \frac{(t-a)[(t-b)^2+s^2] - (t-b)[(t-a)^2+s^2]}{[(t-a)^2+s^2][(t-b)^2+s^2]}$$

$$= \frac{(a-b)[t^2-(a+b)t+ab-s^2]}{[(t-a)^2+s^2][(t-b)^2+s^2]} = L_1(t),$$

after some algebraic manipulation. Since $t \in \mathbb{R}/J$, the denominator of $L_1(t)$ is bounded below by a positive constant. Consider now the function

$$L(t) = L_1(t)(1+t^2).$$

Note that $L(t) \to 1$ as $t \to \pm\infty$, and it is straightforward to show that $L(t)$ is bounded for all $s \geq 0$.].

The fact that $d\mu_n(t)$ and $d\mu(t)$ are Herglotz measures now implies that the integrals

$$c_1 = \int_{\mathbb{R}/J} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu_n(t),$$

and

$$c_2 = \int_{\mathbb{R}/J} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2}\right] d\mu(t),$$

converge absolutely. Note that

$$|c_1| \leq c_J \int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_n(t) = c_J(Im F_n(i) - b_n) \leq c_J Im F_n(i), \tag{5.19}$$

and thus c_1 is bounded uniformly in n, since $F_n(i) \to F(i)$. Hence, it follows by the Lebesgue dominated convergence theorem that

$$\lim_{w\to 0+}\frac{1}{\pi}\int_{w}^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds=$$

$$=\frac{1}{\pi}\int_{0}^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds, \qquad (5.20)$$

and similarly for the second double integral. This shows that the integration with respect to Lebesgue measure ds in equation (5.18) must be performed on the interval $[0,\varepsilon_0]$.

Suppose that $|c_1|\le K$, for all $n\in\mathbb{N}$, for some constant $K>0$. We now let $c'=\max\{K,|c_2|\}$, $\varepsilon_1=\frac{\varepsilon\pi}{9c'}$, and set $\varepsilon'=\frac{1}{2}\min\{\varepsilon_1,\varepsilon_0\}$. We then have

$$\left|\frac{1}{\pi}\int_{0}^{\varepsilon'}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds-\right.$$

$$\left.-\frac{1}{\pi}\int_{0}^{\varepsilon'}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right|\le\frac{\varepsilon}{9}. \qquad (5.21)$$

From (5.18), (5.20), and (5.21) we have

$$|\mu_n((a,b])-\mu((a,b])|$$

$$<\frac{5\varepsilon}{9}+\left|\frac{1}{\pi}\int_{\varepsilon'}^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds-\right.$$

$$\left.-\frac{1}{\pi}\int_{\varepsilon'}^{\varepsilon_0}\left\{\int_{\mathbb{R}/J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right|. \qquad (5.22)$$

But our previous arguments show that

$$\left|\frac{1}{\pi}\int_{\varepsilon'}^{\varepsilon_0}\left\{\int_{J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu_n(t)\right\}ds\right|<\frac{\varepsilon}{18},$$

provided $n>N_3$, and

$$\left|\frac{1}{\pi}\int_{\varepsilon'}^{\varepsilon_0}\left\{\int_{J}\left[\frac{(t-a)}{(t-a)^2+s^2}-\frac{(t-b)}{(t-b)^2+s^2}\right]d\mu(t)\right\}ds\right|<\frac{\varepsilon}{18}.$$

Therefore, from (5.22) we find

$$|\mu_n((a,b]) - \mu((a,b])|$$

$$< \frac{2\varepsilon}{3} + \left| \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2} \right] d\mu_n(t) \right\} ds - \right.$$

$$\left. - \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} \left\{ \int_{\mathbb{R}} \left[\frac{(t-a)}{(t-a)^2+s^2} - \frac{(t-b)}{(t-b)^2+s^2} \right] d\mu(t) \right\} ds \right|. \tag{5.23}$$

By using equation (5.11) and a similar expression for $F(z)$ which can be obtained from the representation of $F(z)$ in (1.1), and substituting in (5.23), we obtain

$$|\mu_n((a,b]) - \mu((a,b])|$$

$$\leq \frac{2\varepsilon}{3} + \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |Re[F_n(a+is) - F(a+is)]| ds +$$

$$+ \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |Re[F(b+is) - F_n(b+is)]| ds + \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} (b-a)|b_F - b_n| ds$$

$$\leq \frac{2\varepsilon}{3} + \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |F_n(a+is) - F(a+is)| ds +$$

$$+ \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |F_n(b+is) - F(b+is)| ds + \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} (b-a)|b_F - b_n| ds. \tag{5.24}$$

Since $F_n(z) \to F(z)$ uniformly, on compact subsets of the upper half-plane, as $n \to \infty$, we can find an $N_4 \in \mathbb{N}$ such that if $n > N_4$, then

$$\frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |F_n(a+is) - F(a+is)| ds + \frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} |F_n(b+is) - F(b+is)| ds < \frac{\varepsilon}{6}.$$

Also, from (5.15) we have

$$\frac{1}{\pi} \int_{\varepsilon'}^{\varepsilon_0} (b-a)|b_F - b_n| ds \leq \frac{1}{\pi} \int_{0}^{\varepsilon_0} (b-a)|b_F - b_n| ds < \frac{\varepsilon}{6}.$$

Let $N = \max\{N_1, N_2, N_3, N_4\}$. Then, if $n > N$, we have from (5.24)

$$|\mu_n((a,b]) - \mu((a,b])| < \varepsilon,$$

so that Theorem (5.6) is proved.

Corollary 5.7 *Suppose, as in Theorem (5.6), that the points a and b $(a < b)$ are not discrete points of any of the measures $d\mu_n(t)$ $(n \in \mathbb{N})$ or $d\mu(t)$, and moreover, that the Herglotz functions $F_n(z)$ converge to $F(z)$ as $n \to \infty$ at a point $z = is_0$, for any $s_0 \geq 1$, and uniformly on the Π-shaped contour consisting of the following parts: the horizontal contour joining the points $a+i\varepsilon_0$ and $b+i\varepsilon_0$, and the vertical contours $\{z = a + is : \varepsilon' \leq s \leq \varepsilon_0\}$, $\{z = b + is : \varepsilon' \leq s \leq \varepsilon_0\}$, where the positive constants ε_0, ε' were defined in the proof of Theorem (5.6). Then,*

$$\mu_n((a,b]) \to \mu((a,b]).$$

Proof. On examination of the proof of Theorem (5.6) it is found that the Theorem holds provided the convergence conditions stated in Corollary (5.7) are satisfied. Note that (5.19) is modified here as follows:

$$|c_1| \leq c_J \int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_n(t) \leq c_J \int_{\mathbb{R}} \frac{1}{t^2+s_0^2} d\mu_n(t) \qquad (\textit{since } s_0 \geq 1)$$

$$= \frac{c_J}{s_0}(Im\, F_n(is_0) - b_n s_0) \leq \frac{c_J}{s_0} Im\, F_n(is_0),$$

and thus c_1 is uniformly bounded for all n, since by assumption $F_n(is_0) \to F(is_0)$.

A further consequence of uniform convergence of $F_n(z)$ to $F(z)$ on compact subsets of the upper half-plane is that we also have convergence of the corresponding value distributions. We shall prove this result in theorem (5.9), but first we prove a lemma which will be useful in later estimates.

Lemma 5.8 *Suppose that the measure $d\sigma(y)$ is absolutely continuous with respect to Lebesgue measure, and let $h_\sigma(y)$ be the density function of $d\sigma(y)$. For $n \in \mathbb{N}$, let a family of sets X_n be defined by $X_n = \{y \in \mathbb{R} : h_\sigma(y) > n\}$. Then,*

$$\lim_{n\to\infty} \int_{X_n} \frac{1}{1+y^2} d\sigma(y) = 0. \tag{5.25}$$

Proof. Let $\chi_n(y)$ be the characteristic function of the set X_n, and note that

$$\frac{\chi_n(y)}{1+y^2} \le \frac{1}{1+y^2}, \qquad y \in \mathbb{R},\ n \in \mathbb{N},$$

where $1/(1+y^2)$ is integrable with respect to $d\sigma(t)$. Hence, by an application of the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\chi_n(y)}{1+y^2} d\sigma(y) = \int_{\mathbb{R}} \lim_{n\to\infty} \frac{\chi_n(y)}{1+y^2} d\sigma(y).$$

By the Radon-Nikodym theorem, the density function $h_\sigma(y)$ is finite Lebesgue almost everywhere, and thus $\lim_{n\to\infty} \chi_n(y) = 0$ almost everywhere. The lemma now follows from the fact that $d\sigma(y)$ is absolutely continuous with respect to Lebesgue measure.

The above lemma provides a way of estimating and controlling $h_\sigma(y)$, the density function of the measure $d\sigma(y)$, by means of a splitting of the real line in two sets, which we call respectively S_0 and S_1. On S_1 $h_\sigma(y)$ takes large values, although the contribution of this set to the integral in (5.25) will be arbitrary small. On the set S_0 $h_\sigma(y)$ is bounded, and as a result we will be able to bound the measure $d\sigma(y)$ on this set, by means of Lebesgue measure. More precisely, we let the sets S_0 and S_1 be defined by

$$S_0 = \{y \in S : h_\sigma(y) \le C\}, \quad S_1 = \{y \in S : h_\sigma(y) > C\}, \tag{5.26}$$

where $C > 0$ is a constant. The larger the constant C, the smaller the contribution of the set S_1 to the integral in (5.25). This technique will provide us with a useful tool for estimation in our generalization, in the case when Lebesgue measure is replaced by a Herglotz measure $d\sigma$ which is absolutely continuous with respect to Lebesgue measure.

Theorem 5.9 *Let $F_n(z)$ be a family of Herglotz functions with corresponding measures μ_n, such that $F_n(z) \to F(z)$ uniformly, as $n \to \infty$, on compact subsets of the upper half-plane. Suppose that the measure $d\sigma$ is absolutely continuous with respect to Lebesgue measure. Then, for any Borel set S, and any bounded Borel set B, we have*

$$\lim_{n\to\infty} \int_S \mu_y^n(B)d\sigma(y) = \int_S \mu_y(B)d\sigma(y), \tag{5.27}$$

where the measures μ_y^n appear in (5.2), *and the measures μ_y in* (3.3) *in Chapter* 3.

Proof. Suppose $B \subseteq [-N, N]$, and consider the family $\mathcal{A}$ of all Lebesgue measurable subsets A of $[-N, N]$ which satisfy equation (5.27). $\mathcal{A}$ is non-empty; to see this, first note that by lemma (5.1) there exists a constant $K_1 > 0$ such that

$$\mu_y^n([-N-1, N+1]) \leq K_1 \frac{1}{1+y^2}. \tag{5.28}$$

Thus, for any subinterval $(a, b]$ of $[-N, N]$, an application of the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \int_S \mu_y^n((a, b])d\sigma(y) = \int_S \lim_{n\to\infty} \mu_y^n((a, b])d\sigma(y).$$

Since $F_y^n(z) \to F_y(z)$ uniformly, as $n \to \infty$, on compact subsets of the upper half-plane, it follows by Theorem (5.6) that $\mu_y^n((a,b]) \to \mu_y((a,b])$ as $n \to \infty$, provided that the endpoints a and b are not discrete points of any of the measures μ_y^n or μ_y. Hence, the above equation implies that such an interval $(a,b]$ satisfies (5.27).

We will prove that every Borel subset of $[-N,N]$ satisfies (5.27). First, however, we need to show that $\mathcal{A}$ is closed under countable unions of disjoint sets. Let, thus, $\{A_k\}$ be a sequence of disjoint sets in $\mathcal{A}$. We need to show that

$$\lim_{n\to\infty} \int_S \mu_y^n\Big(\bigcup_{k=1}^{\infty} A_k\Big) d\sigma(y) = \int_S \mu_y\Big(\bigcup_{k=1}^{\infty} A_k\Big) d\sigma(y). \tag{5.29}$$

We will split integration over S, to integration over the two disjoint sets S_0 and S_1, whose union is S. There also exists a constant $K_2 > 0$ such that

$$\mu_y([-N-1, N+1]) \le K_2 \frac{1}{1+y^2}, \tag{5.30}$$

for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. (This follows in a very similar way as the result in lemma (5.1)).

Let $\varepsilon > 0$ be given. Then, (5.28), (5.30), and lemma (5.8) enable us to choose the constant C in the definition of the sets S_0 and S_1 in (5.26), such that we have both

$$\int_{S_1} \mu_y^n([-N-1, N+1]) d\sigma(y) < \frac{\varepsilon}{6}, \tag{5.31}$$

and

$$\int_{S_1} \mu_y([-N-1, N+1]) d\sigma(y) < \frac{\varepsilon}{6}. \tag{5.32}$$

Therefore,

$$\int_{S_1} \mu_y^n\Big(\bigcup_k A_k\Big)d\sigma(y) \le \int_{S_1} \mu_y^n([-N-1,N+1])d\sigma(y) < \frac{\varepsilon}{6}, \tag{5.33}$$

and

$$\int_{S_1} \mu_y\Big(\bigcup_k A_k\Big)d\sigma(y) \le \int_{S_1} \mu_y([-N-1,N+1])d\sigma(y) < \frac{\varepsilon}{6}. \tag{5.34}$$

Also, for each k

$$\int_{S_0} \mu_y^n(A_k)d\sigma(y) \le C\int_{S_0} \mu_y^n(A_k)dy$$

$$\le C\int_{\mathbb{R}} \mu_y^n(A_k)dy = C|A_k|,$$

where $|.|$ stands for Lebesgue measure. Since

$$\sum_k C|A_k| \le C|[-N,N]| = 2NC < +\infty,$$

it follows by a discrete version of the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\sum_k \int_{S_0} \mu_y^n(A_k)d\sigma(y) = \sum_k \lim_{n\to\infty}\int_{S_0} \mu_y^n(A_k)d\sigma(y).$$

Therefore, we get

$$\lim_{n\to\infty}\int_{S_0} \mu_y^n\Big(\bigcup_k A_k\Big)d\sigma(y) = \lim_{n\to\infty}\sum_k \int_{S_0} \mu_y^n(A_k)d\sigma(y)$$

$$= \sum_k \lim_{n\to\infty}\int_{S_0} \mu_y^n(A_k)d\sigma(y) = \sum_k \int_{S_0} \mu_y(A_k)d\sigma(y)$$

$$= \int_{S_0} \mu_y\Big(\bigcup_k A_k\Big)d\sigma(y).$$

Hence, there is an $N_1 \in \mathbb{N}$ such that, if $n > N_1$ then

$$\left|\int_{S_0} \mu_y^n\Big(\bigcup_k A_k\Big)d\sigma(y) - \int_{S_0} \mu_y\Big(\bigcup_k A_k\Big)d\sigma(y)\right| < \frac{2\varepsilon}{3}. \tag{5.35}$$

Combining (5.35), (5.33) and (5.34), we see that for $n > N_1$ we have

$$\left|\int_S \mu_y^n\Big(\bigcup_k A_k\Big)d\sigma(y) - \int_S \mu_y\Big(\bigcup_k A_k\Big)d\sigma(y)\right| < \varepsilon,$$

which proves (5.29) and shows that $\mathcal{A}$ is closed under countable unions of disjoint sets.

Now take any measurable subset B of $[-N, N]$. We shall show that B satisfies (5.27) as well. There is an open (and hence measurable) set G such that $G \supset B$ and $|G| < |B| + \frac{\varepsilon}{6C}$. (We may assume that $G \subset [-N-1, N+1]$). Suppose $G = B \cup B_0$ where B and B_0 are disjoint, so that $B_0 = G \backslash B$ and thus B_0 is measurable. Then, $|G| = |B| + |B_0|$, which implies that $|B_0| \leq \frac{\varepsilon}{6C}$. We have $\mu_y^n(G) = \mu_y^n(B) + \mu_y^n(B_0)$, and since (5.28) implies that $\mu_y^n(B_0)$ is bounded, we can subtract $\mu_y^n(B_0)$ from both sides of the equation to obtain $\mu_y^n(B) = \mu_y^n(G) - \mu_y^n(B_0)$. Similarly, we have $\mu_y(B) = \mu_y(G) - \mu_y(B_0)$. Therefore,

$$\begin{aligned}
&\left|\int_S \mu_y^n(B)d\sigma(y) - \int_S \mu_y(B)d\sigma(y)\right| \\
&= \left|\int_S \{\mu_y^n(G) - \mu_y^n(B_0)\}d\sigma(y) - \int_S \{\mu_y(G) - \mu_y(B_0)\}d\sigma(y)\right| \\
&\leq \left|\int_S \mu_y^n(G)d\sigma(y) - \int_S \mu_y(G)d\sigma(y)\right| + \\
&\quad + \int_S \mu_y^n(B_0)d\sigma(y) + \int_S \mu_y(B_0)d\sigma(y). \qquad (5.36)
\end{aligned}$$

As before, we split integration over the set S to integration over the disjoint sets S_0 and S_1. From (5.31) we have

$$\int_{S_1} \mu_y^n(B_0)d\sigma(y) < \int_{S_1} \mu_y^n([-N-1, N+1])d\sigma(y) < \frac{\varepsilon}{6}. \qquad (5.37)$$

Also,

$$\int_{S_0} \mu_y^n(B_0)d\sigma(y) \leq C\int_{S_0} \mu_y^n(B_0)dy$$
$$\leq C\int_{\mathbb{R}} \mu_y^n(B_0)d(y) = C|B_0| \leq C\frac{\varepsilon}{6C} = \frac{\varepsilon}{6}. \tag{5.38}$$

Combining (5.37) and (5.38), we obtain

$$\int_S \mu_y^n(B_0)d\sigma(y) < \frac{\varepsilon}{3}. \tag{5.39}$$

Similarly, we have

$$\int_S \mu_y(B_0)d\sigma(y) < \frac{\varepsilon}{3}. \tag{5.40}$$

Since it is an open set, G is the union of a countable collection of disjoint component intervals of $[-N-1, N+1]$. Thus, by the first part of this proof, (where now, we consider disjoint measurable subsets of $[-N-1, N+1]$, rather than $[-N, N]$, satisfying (5.27)), it follows that there is an $N_2 \in \mathbb{N}$ such that if $n > N_2$ then

$$\left|\int_S \mu_y^n(G)d\sigma(y) - \int_S \mu_y(G)d\sigma(y)\right| < \frac{\varepsilon}{3}. \tag{5.41}$$

Combining (5.39), (5.40) and (5.41), we see from (5.36) that, if $n > N_2$ then

$$\left|\int_S \mu_y^n(B)d\sigma(y) - \int_S \mu_y(B)d\sigma(y)\right| < \varepsilon,$$

which completes the proof.

5.3 An estimate for the Value distribution of a translated Herglotz function $F^\delta(z)$.

In this section we will obtain a remarkably precise estimate of the rate of convergence in the limit as $\delta \to 0^+$. Given a Herglotz function $F(z)$, define first a

Herglotz function $F^\delta(z)$, obtained from $F(z)$ by translation through an increment $i\delta$, thus

$$F^\delta(z) = F(z + i\delta), \qquad \delta > 0.$$

We can also define a family of translated Herglotz functions $F_y^\delta(z)$, by

$$F_y^\delta(z) = \frac{1}{y - F^\delta(z)}, \qquad \delta > 0,\ y \in \mathbb{R}.$$

Lemma 5.10 *We have $F^\delta(z) \to F(z)$ uniformly as $\delta \to 0^+$, on compact subsets of the upper half plane.*

Proof. Let D be any compact subset of the upper half plane. From the representation of F in (1.1) in Chapter 1 we have

$$|F^\delta(z) - F(z)|$$

$$= \Big| a_F + b_F(z + i\delta) + \int_{\mathbb{R}} \left\{ \frac{1}{t - z - i\delta} - \frac{t}{1+t^2} \right\} d\mu(t) -$$

$$- a_F - b_F z - \int_{\mathbb{R}} \left\{ \frac{1}{t - z} - \frac{t}{1+t^2} \right\} d\mu(t) \Big|$$

$$= \left| b_F i\delta + \int_{\mathbb{R}} \left\{ \frac{1}{t - z - i\delta} - \frac{1}{t - z} \right\} d\mu(t) \right| = \left| b_F i\delta + \int_{\mathbb{R}} \frac{i\delta}{(t - z - i\delta)(t - z)} d\mu(t) \right|$$

$$\leq b_F \delta + \delta \int_{\mathbb{R}} \frac{1}{|t - z - i\delta||t - z|} d\mu(t). \qquad (5.42)$$

Since $|t - z - i\delta||t - z| \geq |t - z|^2$, we have

$$\frac{1}{|t - z - i\delta||t - z|} \leq \frac{1}{|t - z|^2} = \frac{1}{(t - Rez)^2 + (Imz)^2}.$$

There are constants K_D, $\varepsilon_D > 0$ such that $|Rez| \leq K_D$ and $0 < \varepsilon_D \leq Imz \leq K_D$ for all $z \in D$. It is straightforward to show that

$$\frac{1}{(t - Rez)^2 + (Imz)^2} \leq c\frac{1}{1+t^2}$$

for some constant $c > 0$, which shows that the integral in (5.42) is finite. Hence, taking the limit as $\delta \to 0^+$ we see that $|F^\delta(z) - F(z)| \to 0$ uniformly for all $z \in D$. Since D was an arbitrary compact subset of the upper half plane, the result follows.

Lemma 5.11 *We have $F_y^\delta(z) \to F_y(z)$ uniformly as $\delta \to 0^+$, on compact subsets of the upper half complex plane, where the functions F_y were defined in* (3.2) *in Chapter* 3.

Proof. By definition,

$$|F_y^\delta(z) - F_y(z)| = \left|\frac{1}{y - F(z+i\delta)} - \frac{1}{y - F(z)}\right|$$

$$= \frac{|F(z+i\delta) - F(z)|}{|y - F(z+i\delta)||y - F(z)|}.$$

Let D be any compact subset of the upper half plane. There is an $\varepsilon'_D > 0$ such that $ImF(z) \geq \varepsilon'_D$ for all $z \in D$. Thus,

$$|y - F(z)| \geq ImF(z) \geq \varepsilon'_D, \qquad \forall\, z \in D.$$

Since $F^\delta(z) \to F(z)$ uniformly as $\delta \to 0^+$ on D, there is a $\delta_0 > 0$ such that if $\delta < \delta_0$ then $\left|ImF(z+i\delta) - ImF(z)\right| < \frac{\varepsilon'_D}{2}$. Hence, if $\delta < \delta_0$, then $|ImF(z+i\delta)| > \frac{\varepsilon'_D}{2}$, and so $|y - F(z+i\delta)| \geq ImF(z+i\delta) > \frac{\varepsilon'_D}{2}$. Thus, if $\delta < \delta_0$ then $|y-F(z+i\delta)||y-F(z)| > \frac{(\varepsilon'_D)^2}{2}$, and $1/|y-F(z+i\delta)||y-F(z)| < 2/(\varepsilon'_D)^2$. Given any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that if $\delta < \delta_1$ then $|F(z+i\delta)-F(z)| < \frac{(\varepsilon'_D)^2\varepsilon}{2}$. Hence, if $\delta < \min\{\delta_0, \delta_1\}$ we have $|F_y^\delta(z) - F_y(z)| < \varepsilon$, for all $z \in D$. Since D and ε were arbitrary the lemma follows.

We introduce now the idea of angle subtended $\theta(z, S)$ at a point $z \in \mathbb{C}_+$ by a Borel set S on the real line, defined by

$$\theta(z, S) = \int_S Im\left[\frac{1}{t-z}\right] dt. \tag{5.43}$$

We shall refer to θ as the 'standard' angle subtended. Moreover, for $z \in \mathbb{C}_+$ we define $\omega(., S; F)$ by

$$\omega(z, S; F) = \frac{1}{\pi}\theta(F(z), S), \tag{5.44}$$

so that $\omega(\lambda, S; F) = \lim_{\delta \to 0^+} \omega(\lambda + i\delta, S; F)$ for $\lambda \in \mathbb{R}$. For almost all $\lambda \in \mathbb{R}$, we have

$$\omega(\lambda, S; F) = \begin{cases} 1 & F_+(\lambda) \in \mathbb{R} \ \mathit{and} \ F_+(\lambda) \in S, \\ 0 & F_+(\lambda) \in \mathbb{R} \ \mathit{and} \ F_+(\lambda) \notin S, \\ \frac{1}{\pi}\theta(F_+(\lambda), S) & Im F_+(\lambda) > 0, \end{cases}$$

where $F_+(\lambda) = \lim_{\varepsilon \to 0^+} F(\lambda + i\varepsilon)$ is the boundary value of F at λ. Hence, in the particular case that $F_+(\lambda)$ is almost everywhere real, $\omega(., S; F)$ is (almost everywhere) the characteristic function of $F_+^{-1}(S)$. In the case of the translated Herglotz function F^δ we set $\omega(\lambda, S; F^\delta) = \omega^\delta(\lambda, S; F) = \frac{1}{\pi}\theta(F(\lambda + i\delta), S)$.

Because in Chapter 3 we considered the average of the spectral measure over other measures than Lebesgue measure, we will need to generalize the notion of angle subtended. Thus, for any point $z \in \mathbb{C}_+$ and any Borel set S on the real line, we define a 'generalized angle subtended' $\theta_\sigma(z, S)$ by

$$\theta_\sigma(z, S) = \int_S Im\left[\frac{1}{t-z}\right] d\sigma(t), \tag{5.45}$$

where the measure $d\sigma$ corresponds to the Herglotz function ϕ. Correspondingly, we define $\omega_\sigma(., S; F)$ by

$$\omega_\sigma(z, S; F) = \frac{1}{\pi}\theta_\sigma(F(z), S), \tag{5.46}$$

so that $\omega_\sigma(\lambda, S; F) = \lim_{\delta\to 0^+} \omega_\sigma(\lambda+i\delta, S; F)$, and we set $\omega_\sigma(\lambda, S; F^\delta) = \omega_\sigma^\delta(\lambda, S; F)$ $= \frac{1}{\pi}\theta_\sigma(F(\lambda + i\delta), S)$.

The following lemma implies that if the measure $d\sigma$ is absolutely continuous, then for fixed $z \in \mathbb{C}_+$ and any Borel set S the generalized angle subtended $\theta_\sigma(z, S)$ is bounded.

Lemma 5.12 *Suppose that the measure $d\sigma(t)$ is absolutely continuous with respect to Lebesgue measure, let S be any Borel set, and z be a point of the upper half complex plane. Then, given any $\varepsilon > 0$, we can choose a constant $C(\varepsilon) > 0$ depending on ε, such that*

$$\theta_\sigma(z, S) \le C\theta(z, S) + \varepsilon.$$

Proof. By definition,

$$\theta_\sigma(z, S) = \int_S Im\left[\frac{1}{t-z}\right] d\sigma(t)$$
$$= \int_{S_0} Im\left[\frac{1}{t-z}\right] d\sigma(t) + \int_{S_1} Im\left[\frac{1}{t-z}\right] d\sigma(t),$$

where the sets S_0 and S_1 are defined in (5.26). Since $Im\, 1/(t-z) \le const./(1+t^2)$, given any $\varepsilon > 0$, we can choose a constant $C(\varepsilon) > 0$, such that $\int_{S_1} Im\, 1/(t-z)\, d\sigma(t) < \varepsilon$. Also, we have

$$\int_{S_0} Im\left[\frac{1}{t-z}\right] d\sigma(t) \le C\int_{S_0} Im\left[\frac{1}{t-z}\right] dt \le C\int_S Im\left[\frac{1}{t-z}\right] dt = C\theta(z, S).$$

Hence, we have

$$\theta_\sigma(z, S) \le C\theta(z, S) + \varepsilon.$$

Lemma 5.13 *Let μ_y^δ be a family of measures corresponding to the Herglotz functions F_y^δ, for $\delta > 0$, $y \in \mathbb{R}$, and $d\sigma$ an arbitrary Herglotz measure. Let A be any bounded Borel set, and S any Borel set. Then, we have*

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}|A|(b_{(\phi_S \circ F)} - b_\phi b_F) + \frac{1}{\pi}\int_{\mathbb{R}} \theta(t+i\delta, A)d\nu_S(t), \tag{5.47}$$

where b_ϕ is the constant appearing in the representations of the Herglotz functions ϕ_S and ϕ in (3.15) *and* (3.14) *respectively in Chapter* 3, *b_F is the constant appearing in the representation of the Herglotz function F in* (1.1) *in Chapter* 1, *$b_{(\phi_S \circ F)}$ is the constant appearing in the representation of the composed Herglotz function $(\phi_S \circ F)$ in* (3.16) *in Chapter* 3, *the measure ν_S was defined in* (3.9) *in Chapter 3, and $\theta(z, S)$ was defined in* (5.43). *Moreover, if the measure $d\sigma$ is absolutely continuous, then*

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\int_{\mathbb{R}} \theta(t+i\delta, A)d\nu_S(t). \tag{5.48}$$

Proof. Fix $\delta > 0$. Then, the Herglotz function $F^\delta(z)$ has boundary values with strictly positive imaginary part, as do the functions $F_y^\delta(z)$. The measures μ_y^δ are thus absolutely continuous, having density functions $\frac{1}{\pi} Im\, F_y(\lambda + i\delta)$, and hence we have

$$\begin{aligned}\int_S \mu_y^\delta(A)d\sigma(y) &= \int_S \left\{\frac{1}{\pi}\int_A Im\, F_y(\lambda+i\delta)d\lambda\right\}d\sigma(y) \\ &= \int_S \left\{\frac{1}{\pi}\int_A Im\left[\frac{1}{y - F(\lambda+i\delta)}\right]d\lambda\right\}d\sigma(y).\end{aligned} \tag{5.49}$$

Since λ is confined in the bounded set A, we have $Im\left[\frac{1}{y-F(\lambda+i\delta)}\right] \leq const. \frac{1}{1+y^2}$. Hence, the integral above is absolutely convergent, and we can change the order of integration to obtain

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\int_A \Big\{ \int_S Im\Big[\frac{1}{y - F(\lambda + i\delta)}\Big] d\sigma(y)\Big\} d\lambda. \tag{5.50}$$

From the integral representation of $\phi_S(z)$ in (3.15) in Chapter 3 we have

$$Im\,\phi_S(F(\lambda + i\delta)) - b_\phi Im F(\lambda + i\delta) = \int_S Im\Big[\frac{1}{t - F(\lambda + i\delta)}\Big] d\sigma(t). \tag{5.51}$$

Substituting (5.51) into (5.50) we have

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\int_A \{Im\,\phi_S(F(\lambda + i\delta)) - b_\phi Im F(\lambda + i\delta)\} d\lambda. \tag{5.52}$$

From the representation of the composed Herglotz function $(\phi_S \circ F)(z)$ in (3.16) in Chapter 3, we have

$$Im(\phi_S \circ F)(\lambda + i\delta) = b_{(\phi_S \circ F)}\delta + \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2 + \delta^2} d\mu_{(\phi_S \circ F)}(t). \tag{5.53}$$

Also, from the representation of $F(z)$ in (1.1) in Chapter 1 we have

$$Im F(\lambda + i\delta) = b_F\delta + \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2 + \delta^2} d\mu(t). \tag{5.54}$$

Substituting (5.53) and (5.54) into (5.52) we obtain

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\int_A \delta(b_{(\phi_S \circ F)} - b_\phi b_F)d\lambda +$$

$$+\frac{1}{\pi}\int_A \Big\{ \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2 + \delta^2} d\mu_{(\phi_S \circ F)}(t) - b_\phi \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2 + \delta^2} d\mu(t)\Big\} d\lambda.$$

In lemma (3.8) in Chapter 3, we obtained the relation

$$\nu_S(B) = \mu_{(\phi_S \circ F)}(B) - b_\phi \mu(B), \tag{5.55}$$

for any Borel set B. Thus, we have

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\delta|A|(b_{(\phi_S\circ F)} - b_F b_\phi) +$$

$$+\frac{1}{\pi}\int_A \Big\{ \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2+\delta^2} d(\mu_{(\phi_S\circ F)}(t) - b_\phi \mu(t)) \Big\} d\lambda$$

$$= \frac{1}{\pi}\delta|A|(b_{(\phi_S\circ F)} - b_F b_\phi) + \frac{1}{\pi}\int_A \Big\{ \int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2+\delta^2} d\nu_S(t) \Big\} d\lambda,$$

since all the above integrals are convergent. The above double integral is also absolutely convergent, since, from the representation of the Herglotz function $H_S(z)$ in (3.13) in Chapter 3 we have

$$\int_{\mathbb{R}} \frac{\delta}{(t-\lambda)^2+\delta^2} d\nu_S(t) = Im H_S(\lambda + i\delta) - b_H \delta \leq Im H_S(\lambda + i\delta),$$

which is uniformly bounded for λ in the bounded set A. Hence, by changing the order of integration we obtain

$$\int_S \mu_y^\delta(A)d\sigma(y) = \frac{1}{\pi}\delta|A|(b_{(\phi_S\circ F)} - b_F b_\phi) + \frac{1}{\pi}\int_{\mathbb{R}} \Big\{ \int_A \frac{\delta}{(t-\lambda)^2+\delta^2} d\lambda \Big\} d\nu_S(t)$$

$$= \frac{1}{\pi}\delta|A|(b_{(\phi_S\circ F)} - b_F b_\phi) + \frac{1}{\pi}\int_{\mathbb{R}} \theta(t+i\delta, A) d\nu_S(t),$$

from the definition of the angle subtended $\theta(z, S)$. If, furthermore, the measure $d\sigma(y)$ is absolutely continuous, then the term $\frac{1}{\pi}\delta|A|(b_{(\phi_S\circ F)} - b_F b_\phi)$ vanishes by Theorem (4.3) in Chapter 4, and the lemma is proved.

Now let the Herglotz functions $\phi_0(z)$, $\phi_1(z)$ and the composed Herglotz functions $(\phi_0 \circ F)(z)$, $(\phi_1 \circ F)(z)$ have the following respective representations:

$$\phi_0(z) = a_\phi + b_\phi z + \int_{S_0} \Big\{ \frac{1}{t-z} - \frac{t}{t^2+1} \Big\} d\sigma(t), \tag{5.56}$$

$$\phi_1(z) = \int_{S_1} \Big\{ \frac{1}{t-z} - \frac{t}{t^2+1} \Big\} d\sigma(t), \tag{5.57}$$

$$(\phi_0 \circ F)(z) = a_0 + b_0 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\nu_0(t), \tag{5.58}$$

$$(\phi_1 \circ F)(z) = a_1 + b_1 z + \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} d\nu_1(t), \tag{5.59}$$

for any Borel set S, where the sets S_0 and S_1 were defined in (5.26).

Lemma 5.14 *Let F, F_1, and F_2 be arbitrary Herglotz functions which satisfy $F(z) = F_1(z) + F_2(z)$ for all z, and which have corresponding measures μ, μ_1, and μ_2 respectively. Then, for any Borel set B we have*

$$\mu(B) = \mu_1(B) + \mu_2(B).$$

Proof. By using the characterization of any Herglotz measure for intervals $(a, b]$, where the points a and b are not discrete points of the measure, we have

$$\mu((a,b]) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b Im F(\lambda + i\varepsilon) d\lambda$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b Im(F_1(\lambda + i\varepsilon) + F_2(\lambda + i\varepsilon)) d\lambda = \mu_1((a,b]) + \mu_2((a,b]),$$

provided that the endpoints a and b are not discrete points of any of the measures μ, μ_1 or μ_2. Now the lemma follows by the same argument which we used in the proof of lemma (3.8) in Chapter 3.

Corollary 5.15 *For any Borel set B we have*

$$\mu_{(\phi_S \circ F)}(B) = \nu_0(B) + \nu_1(B),$$

and also

$$\nu_S(B) = \nu_0(B) + \nu_1(B) - b_\phi \mu(B).$$

Here, the measures $\mu_{(\phi_S \circ F)}$, ν_0, ν_1, ν_S, and μ, correspond to the Herglotz functions $(\phi_S \circ F)$, $(\phi_0 \circ F)$, $(\phi_1 \circ F)$, H_S, and F respectively.

Proof. The first assertion of corollary (5.15) follows from lemma (5.14) and the observation that for all z, we have

$$(\phi_S \circ F)(z) = (\phi_0 \circ F)(z) + (\phi_1 \circ F)(z).$$

Having obtained the first one, the second assertion follows from equation (5.55).

Lemma 5.16 *Suppose that the measure $d\sigma$ is absolutely continuous with respect to Lebesgue measure. Then, with the same notation as in corollary* (5.15) *we have*

$$0 \leq \nu_0(B) - b_\phi \mu(B) \leq C|B|,$$

where the constant $C > 0$ appears in (5.26), *and* $|.|$ *denotes Lebesgue measure.*

Proof. Suppose that the points a and b are not discrete points of either of the measures ν_0 or μ. Then,

$$\nu_0((a,b]) - b_\phi \mu((a,b])$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b \{Im(\phi_0 \circ F)(\lambda + i\varepsilon) - b_\phi Im F(\lambda + i\varepsilon)\} d\lambda.$$

From (5.56) we have

$$Im(\phi_0 \circ F)(\lambda + i\varepsilon) = b_\phi Im F(\lambda + i\varepsilon) + \int_{S_0} Im\left[\frac{1}{t - F(\lambda + i\varepsilon)}\right] d\sigma(t),$$

and hence

$$Im(\phi_0 \circ F)(\lambda + i\varepsilon) - b_\phi Im F(\lambda + i\varepsilon) = \int_{S_0} Im\left[\frac{1}{t - F(\lambda + i\varepsilon)}\right] d\sigma(t)$$

$$\leq C \int_{\mathbf{R}} Im \left\{\frac{1}{t - F(\lambda + i\varepsilon)} - \frac{t}{t^2 + 1}\right\} dt = C\pi,$$

since

$$Im \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{t^2+1} \right\} dt = \pi, \qquad \forall\, z \in \mathbb{C}_+.$$

Thus,

$$\nu_0((a,b]) - b_\phi \mu((a,b]) \le \frac{1}{\pi} \int_a^b C\pi d\lambda = C|(a,b]|.$$

Note that since $Im \int_{S_0} 1/[t - F(z)]d\sigma(t) \ge 0 \ \ \forall\, z \in \mathbb{C}_+$, $\nu_0((a,b]) - b_\phi\mu((a,b])$ is non-negative.

Now take any points c and d. We can construct sequences $\{c_i\}$ and $\{d_i\}$ which are not discrete points of either of the measures ν_0 or μ, such that $c_i \to c_-$ and $d_i \to d_+$. Hence,

$$\begin{aligned}
\nu_0((c,d]) - b_\phi\mu((c,d]) &= \nu_0\Big(\lim_{i\to\infty}(c_i,d_i]\Big) - b_\phi\mu\Big(\lim_{i\to\infty}(c_i,d_i]\Big) \\
&= \lim_{i\to\infty}\Big(\nu_0((c_i,d_i]) - b_\phi\mu((c_i,d_i])\Big) \le \lim_{i\to\infty} C|(c_i,d_i]| \\
&= C|\lim_{i\to\infty}(c_i,d_i]| = C|(c,d]|.
\end{aligned}$$

Note also that since $\nu_0((c_i,d_i]) - b_\phi\mu((c_i,d_i]) \ge 0, \ \forall\, i$, it follows that $\nu_0((c,d]) - b_\phi\mu((c,d])$ is non-negative, for any points c, $d \in \mathbb{R}$. By the same argument, it follows that $\nu_0(\{x\}) - b_\phi\mu(\{x\}) = 0, \ \forall\, x \in \mathbb{R}$. Hence, for any open interval (c,d) we have $\nu_0((c,d)) - b_\phi\mu((c,d)) \le C|(c,d)|$.

Let B be any open Borel set. Then, $B = \bigcup_k A_k$, where A_k are disjoint open intervals, and thus we have

$$\begin{aligned}
\nu_0(B) - b_\phi\mu(B) &= \nu_0\Big(\bigcup_k A_k\Big) - b_\phi\mu\Big(\bigcup_k A_k\Big) \\
&= \sum_k \{\nu_0(A_k) - b_\phi\mu(A_k)\} \le \sum_k C|A_k| = C|B|.
\end{aligned}$$

Observe that $\nu_0(B) - b_\phi\mu(B) \ge 0$, because $\nu_0(A_k) - b_\phi\mu(A_k)$ is non-negative, for all k.

Now let B be any bounded Borel set. For any $\varepsilon > 0$, there is an open set G containing B such that $|G| < |B| + \frac{\varepsilon}{C}$. Suppose $G = B \cup B_1$, where the sets B and B_1 are disjoint, so that $|B_1| < \frac{\varepsilon}{C}$. Then,

$$\nu_0(G) - b_\phi\mu(G) = \nu_0(B \cup B_1) - b_\phi\mu(B \cup B_1)$$

$$= \nu_0(B) + \nu_0(B_1) - b_\phi\mu(B) - b_\phi\mu(B_1).$$

We can assume that G is bounded, which implies that $\nu_0(B_1)$ and $b_\phi\mu(B_1)$ are bounded, and thus obtain

$$\nu_0(B) - b_\phi\mu(B) = \nu_0(G) - b_\phi\mu(G) - \{\nu_0(B_1) + b_\phi\mu(B_1)\}$$

$$\leq \nu_0(G) - b_\phi\mu(G) \leq C|G| < C(|B| + \varepsilon/C) = C|B| + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we can infer that $0 \leq \nu_0(B) - b_\phi\mu(B) \leq C|B|$.

Finally, let B be any Borel set. Then, we have

$$\nu_0(B) - b_\phi\mu(B) = \nu_0\Big(\bigcup_N B \cap [-N, N]\Big) - b_\phi\mu\Big(\bigcup_N B \cap [-N, N]\Big)$$

$$= \lim_{N\to\infty} \big\{\nu_0(B \cap [-N, N]) - b_\phi\mu(B \cap [-N, N])\big\}$$

$$\leq \lim_{N\to\infty} C|B \cap [-N, N]| = C|B|,$$

by our previous argument. This completes the proof.

Lemma 5.17 *Take any points a and b with $a < b$, and fix $\delta > 0$. Then, for $t \in [a-1, b+1]^c$ we have*

$$\theta(t + i\delta, [a, b]) \leq \delta a_1 \frac{1}{1 + t^2},$$

where $\theta(z, S)$ was defined in (5.43), and $a_1 > 0$ is a constant depending on a and b but not on δ.

Proof. For $t \in [a-1, b+1]^c$ we have

$$\theta(t+i\delta,[a,b]) = \tan^{-1}\left(\frac{t-a}{\delta}\right) - \tan^{-1}\left(\frac{t-b}{\delta}\right)$$

$$= \tan^{-1}\left(\frac{\delta(b-a)}{\delta^2+(t-a)(t-b)}\right),$$

and note that the expression inside the brackets is positive. We now use the fact that $\frac{1}{x}\tan^{-1}x \le 1,\ x > 0$. [For $x > 0$ we have

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt < \int_0^x dt = x,$$

and by L' Hopitals rule we have

$$\lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{1}{1+x^2} = 1.] \tag{5.60}$$

Thus, it follows that for $t \in [a-1, b+1]^c$ we have

$$\theta(t+i\delta,[a,b]) \le \frac{\delta(b-a)}{\delta^2+(t-a)(t-b)}.$$

However, it is straightforward to show that for $t \in [a-1, b+1]^c$ we have

$$\frac{b-a}{\delta^2+(t-a)(t-b)} \le \frac{b-a}{(t-a)(t-b)} \le a_1\frac{1}{1+t^2},$$

for some constant $a_1 > 0$ independent of δ, and hence the lemma is proved.

Lemma 5.18 *Let A be any bounded set, contained in the interval $[a,b]$. Let a_1 be chosen as in lemma (5.17), and $a_2 = \max\{1+(a-1)^2, 1+(b+1)^2\}$. Suppose that the measure $d\sigma(t)$ is absolutely continuous with respect to Lebesgue measure. Let $\varepsilon > 0$ be given, and take δ with $0 < \delta < 1$. Define the set S_1 as in (5.26). Then, the constant $C = C(\varepsilon)$ can be chosen such that*

$$(a_1/\pi + a_2)\int_{S_1} Im\Big[\frac{1}{t-F(i)}\Big]d\sigma(t) < \varepsilon. \tag{5.61}$$

In that case, with the same notation as in corollary (5.15), *we then have*

$$(i)\ \frac{1}{\pi}\int_{A^c}\theta(t+i\delta, A)d\nu_1(t) < \varepsilon \quad \text{and} \quad (ii)\ \frac{1}{\pi}\int_{A}\theta(t+i\delta, A^c)d\nu_1(t) < \varepsilon.$$

Proof. We will first obtain inequality (i). We have

$$\frac{1}{\pi}\int_{A^c}\theta(t+i\delta, A)d\nu_1(t)$$

$$= \frac{1}{\pi}\int_{A^c\cap[a-1,b+1]}\theta(t+i\delta, A)d\nu_1(t) + \frac{1}{\pi}\int_{A^c\cap[a-1,b+1]^c}\theta(t+i\delta, A)d\nu_1(t).$$

Since $A\subseteq[a,b]$, we have $\theta(t+i\delta,A)\le\theta(t+i\delta,[a,b])$, and it follows from lemma (5.17) that

$$\frac{1}{\pi}\int_{A^c\cap[a-1,b+1]^c}\theta(t+i\delta, A)d\nu_1(t)$$

$$\le \frac{1}{\pi}\delta a_1\int_{A^c\cap[a-1,b+1]^c}\frac{1}{1+t^2}d\nu_1(t) \le \frac{1}{\pi}\delta a_1\int_{\mathbb{R}}\frac{1}{1+t^2}d\nu_1(t). \tag{5.62}$$

Moreover,

$$\frac{1}{\pi}\int_{A^c\cap[a-1,b+1]}\theta(t+i\delta, A)d\nu_1(t) \le \int_{\mathbb{R}}\chi_{[a-1,b+1]}(t)d\nu_1(t)$$

$$\le a_2\int_{\mathbb{R}}\frac{1}{1+t^2}d\nu_1(t), \tag{5.63}$$

where $a_2 = \max\{1+(a-1)^2,\ 1+(b+1)^2\}$. From (5.59) we have

$$Im(\phi_1\circ F)(i) = b_1 + \int_{\mathbb{R}}\frac{1}{1+t^2}d\nu_1(t) \tag{5.64}$$

and from (5.57)

$$Im\,\phi_1(F(i)) = \int_{S_1} Im\Big[\frac{1}{t-F(i)}\Big]d\sigma(t). \tag{5.65}$$

From (5.62) and (5.63) we obtain

$$\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d\nu_1(t)\leq(\delta a_1/\pi+a_2)\int_{\mathbb{R}}\frac{1}{1+t^2}d\nu_1(t). \tag{5.66}$$

By substituting (5.64) and (5.65) into (5.66) we obtain

$$\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d\nu_1(t)\leq(\delta a_1/\pi+a_2)\Big[\int_{S_1}Im\Big[\frac{1}{t-F(i)}\Big]d\sigma(t)-b_1\Big]$$

$$\leq(\delta a_1/\pi+a_2)\int_{S_1}Im\Big[\frac{1}{t-F(i)}\Big]d\sigma(t).$$

Since $Im\,1/[t-F(i)]\leq const./(1+t^2)$, the constant C can now be chosen, by lemma (5.8), such that (5.61) holds, and inequality (i) in the statement of the lemma follows.

To see that inequality (ii) holds, note that

$$\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d\nu_1(t)\leq\int_{\mathbb{R}}\chi_A(t)d\nu_1(t)\leq a_2\int_{\mathbb{R}}\frac{1}{1+t^2}d\nu_1(t).$$

Therefore, by our previous argument we have

$$\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d\nu_1(t)\leq a_2\int_{S_1}Im\Big[\frac{1}{t-F(i)}\Big]d\sigma(t),$$

and by our choice of C inequality (ii) follows as well.

Lemma 5.19 *With the same notation as in corollary* (5.15), *and the same assumptions as in lemma* (5.18) *we have*

$$\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d(\nu_0(t)-b_\phi\mu(t))\leq CE_A(\delta),$$

and

$$\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d(\nu_0(t)-b_\phi\mu(t))\leq CE_A(\delta),$$

where $E_A(\delta)$ *is defined by*

$$E_A(\delta) = \frac{1}{\pi}\int_{A^c} \theta(t+i\delta, A)dt = \frac{1}{\pi}\int_A \theta(t+i\delta, A^c)dt. \tag{5.67}$$

If, in particular, $A = [a, b]$, *then*

$$E_A(\delta) = \frac{2}{\pi}(b-a)\tan^{-1}\frac{\delta}{(b-a)} + \frac{\delta}{\pi}\ln[(b-a)^2+\delta^2] - \frac{2}{\pi}\delta\ln\delta. \tag{5.68}$$

Proof. By lemma (5.16), $(\nu_0 - b_\phi\mu)(B) \le C|B|$, for any Borel set B. Therefore,

$$\frac{1}{\pi}\int_{A^c} \theta(t+i\delta, A)d(\nu_0(t) - b_\phi\mu(t)) \le \frac{C}{\pi}\int_{A^c}\theta(t+i\delta, A)dt,$$

and

$$\frac{1}{\pi}\int_A \theta(t+i\delta, A^c)d(\nu_0(t) - b_\phi\mu(t)) \le \frac{C}{\pi}\int_A \theta(t+i\delta, A^c)dt.$$

Note that

$$\begin{aligned}
E_A(\delta) &= \frac{1}{\pi}\int_{A^c}\theta(t+i\delta, A)dt = \frac{1}{\pi}\int_{\mathbf{R}}\theta(t+i\delta, A)dt - \frac{1}{\pi}\int_A \theta(t+i\delta, A)dt \\
&= \frac{1}{\pi}\int_{\mathbf{R}}\theta(t+i\delta, A)dt - \frac{1}{\pi}\int_A \{\pi - \theta(t+i\delta, A^c)\}dt \\
&= \frac{1}{\pi}\int_{\mathbf{R}}\left\{\int_A \frac{\delta}{(t-\lambda)^2+\delta^2}d\lambda\right\}dt - |A| + \frac{1}{\pi}\int_A \theta(t+i\delta, A^c)dt \\
&= \frac{1}{\pi}\int_A \theta(t+i\delta, A^c)dt,
\end{aligned}$$

by a change in order of integration in the double integral. If $A = [a, b]$, then

$$\begin{aligned}
E_A(\delta) &= \frac{1}{\pi}\int_A \theta(t+i\delta, A^c)dt = \frac{1}{\pi}\int_a^b\left\{\pi - \int_a^b \frac{\delta}{(\lambda-t)^2+\delta^2}dt\right\}d\lambda \\
&= (b-a) - \frac{1}{\pi}\int_a^b\left\{\int_a^b \frac{\delta}{(\lambda-t)^2+\delta^2}dt\right\}d\lambda.
\end{aligned}$$

The above equation is equal to the expression in (5.68) (see [5]).

Lemma 5.20 *Define the measures μ_y^δ as in lemma* (5.13), *and $E_A(\delta)$ by* (5.67). *With the same assumptions as in lemma* (5.18), *and for any given $\varepsilon > 0$ we then have*

$$\left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right| \leq C E_A(\delta) + \varepsilon,$$

where $C = C(\varepsilon)$ is a constant depending on ε.

Proof. By lemma (5.13) we have

$$\int_S \mu_y^\delta(A) d\sigma(y) = \frac{1}{\pi} \int_{\mathbf{R}} \theta(t + i\delta, A) d\nu_s(t). \tag{5.69}$$

Also, from (3.11) in Chapter 3 we have

$$\int_S \mu_y(A) d\sigma(y) = \int_{\mathbf{R}} \chi_A(t) d\nu_s(t).$$

Therefore,

$$\begin{aligned}
&\left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right| \\
&= \left| \int_{\mathbf{R}} \left\{ \frac{1}{\pi} \theta(t + i\delta, A) - \chi_A(t) \right\} d\nu_s(t) \right| \\
&= \left| \int_A \left\{ \frac{1}{\pi} \theta(t + i\delta, A) - 1 \right\} d\nu_s(t) + \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_s(t) \right| \\
&= \left| \frac{1}{\pi} \int_A -\theta(t + i\delta, A^c) d\nu_s(t) + \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_s(t) \right|.
\end{aligned}$$

In the above expression, the integral on the left is negative and the integral on the right is positive. Hence, an upper bound for this expression is

$$\sup \left\{ \frac{1}{\pi} \int_A \theta(t + i\delta, A^c) d\nu_s(t), \quad \frac{1}{\pi} \int_{A^c} \theta(t + i\delta, A) d\nu_s(t) \right\}.$$

From corollary (5.15) we have $\nu_s(B) = \nu_0(B) + \nu_1(B) - b_\phi \mu(B)$, for any Borel set B, and thus

$$\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d\nu_S(t)=\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d\nu_1(t)+$$

$$+\frac{1}{\pi}\int_{A^c}\theta(t+i\delta,A)d(\nu_0(t)-b_\phi\mu(t)),$$

and

$$\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d\nu_S(t)=\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d\nu_1(t)+$$

$$+\frac{1}{\pi}\int_{A}\theta(t+i\delta,A^c)d(\nu_0(t)-b_\phi\mu(t)).$$

Lemma (5.20) now follows from lemmas (5.18) and (5.19).

Corollary 5.21 *With the same assumptions as those in lemma* (5.18) *we have*

$$\lim_{\delta\to 0}\left|\int_S \mu_y^\delta(A)d\sigma(y)-\int_S \mu_y(A)d\sigma(y)\right|=0,$$

with the convergence being uniform over all Borel sets S and over all Herglotz functions F such that $F(i)$ belongs to some compact subset of the upper half plane.

Proof. The function $E_A(\delta)$ defined in (5.67) is a non-decreasing function of δ (see [5]), and $\lim_{\delta\to 0^+}E_A(\delta)=0$ by an application of the Lebesgue dominated convergence theorem. (The specific expression for $E_A(\delta)$ in (5.68) in the case when $A=[a,b]$ illustrates the convergence of $E_A(\delta)$ to zero in the limit $\delta\to 0^+$.) Corollary (5.21) now follows from lemma (5.20), since $\varepsilon>0$ was arbitrary. The requirement for $F(i)$ to belong to a compact subset of the upper half-plane emerges from lemma (5.18). If this condition is satisfied we then have $Im\ 1/[t-F(i)]\le 1/(1+t^2)$, and by using lemma (5.8) we can choose the constant C such that (5.61) in the statement of lemma (5.18) is satisfied.

Theorem 5.22 *Let $E_A(\delta)$ be defined by* (5.67), *K be a compact subset of $\mathbb{C}_+$, S be any Borel set, and take δ with $0 < \delta < 1$. With the same assumptions as in lemma* (5.18), *and given $\varepsilon > 0$, the constant $C = C(\varepsilon)$ can be chosen such that*

$$\left| \int_A \omega_\sigma^\delta(\lambda, S; F) d\lambda - \int_A \omega_\sigma(\lambda, S; F) d\lambda \right| \leq C E_A(\delta) + \varepsilon, \tag{5.70}$$

for any Herglotz function $F(z)$ such that $F(i) \in K$.

Proof. We will show that

$$\int_S \mu_y^\delta(A) d\sigma(y) = \int_A \omega_\sigma^\delta(\lambda, S; F) d\lambda,$$

and

$$\int_S \mu_y(A) d\sigma(y) = \int_A \omega_\sigma(\lambda, S; F) d\lambda.$$

The above equations are the generalization of results to be found in the work of Breimesser and Pearson in [5] and [6], in the case when Lebesgue measure is replaced by a Herglotz measure $d\sigma$ which is absolutely continuous with respect to Lebesgue measure.

We have from (5.50)

$$\int_S \mu_y^\delta(A) d\sigma(y) = \int_A \left\{ \frac{1}{\pi} \int_S Im\left[\frac{1}{y - F(\lambda + i\delta)}\right] d\sigma(y) \right\} d\lambda$$

$$= \int_A \frac{1}{\pi} \theta_\sigma(F(\lambda + i\delta), S) d\lambda = \int_A \omega_\sigma(\lambda, S; F^\delta) d\lambda$$

$$= \int_A \omega_\sigma^\delta(\lambda, S; F) d\lambda.$$

From the definition of the measure ν_s in (3.9) in Chapter 3 we have

$$\nu_s(A) = \int_S \mu_y(A) d\sigma(y).$$

Here ν_S, according to (5.55), is the measure associated with the Herglotz function $(\phi_S \circ F) - b_\phi F$. From the representations of the Herglotz functions ϕ_S in (3.15) in Chapter 3, and F in (1.1) in Chapter 1, we have

$$\phi_S(F(z)) - b_\phi F(z) = a_\phi + \int_S \left\{ \frac{1}{t - F(z)} - \frac{t}{1+t^2} \right\} d\sigma(t).$$

Let h_{ν_S} be the density function of ν_S. Then, h_{ν_S} is determined almost everywhere by the equation $h_{\nu_S}(\lambda) = \lim_{\delta\to 0^+} \frac{1}{\pi} Im\,[\phi_S(F) - b_\phi F](\lambda + i\delta)$, and is thus given by

$$\begin{aligned} h_{\nu_S}(\lambda) &= \lim_{\delta\to 0^+} \frac{1}{\pi} \int_S Im \left[\frac{1}{t - F(\lambda + i\delta)} \right] d\sigma(t) \\ &= \lim_{\delta\to 0^+} \frac{1}{\pi} \theta_\sigma(F(\lambda + i\delta), S) = \omega_\sigma(\lambda, S). \end{aligned} \tag{5.71}$$

So,

$$\int_S \mu_y(A) d\sigma(y) = \int_A \omega_\sigma(\lambda, S) d\lambda.$$

Therefore,

$$\begin{aligned} &\left| \int_A \omega_\sigma^\delta(\lambda, S; F) d\lambda - \int_A \omega_\sigma(\lambda, S; F) d\lambda \right| \\ &= \left| \int_S \mu_y^\delta(A) d\sigma(y) - \int_S \mu_y(A) d\sigma(y) \right|, \end{aligned}$$

and Theorem (5.22) now follows from lemma (5.20).

Corollary 5.23 *Let h_{ν_S} be the density function of ν_S, and denote by $F_+(\lambda)$ the boundary value of F at $\lambda \in \mathbb{R}$, defined by $F_+(\lambda) = \lim_{\varepsilon\to 0^+} F(\lambda + i\varepsilon)$. Then, at almost all points λ we have*

$$h_{\nu_S}(\lambda) = \begin{cases} \frac{1}{\pi}\theta_\sigma(F_+(\lambda), S) & F_+(\lambda) \in \mathbb{C}_+, \\ \lim_{\delta\to 0^+} \frac{1}{\pi}\theta_\sigma(F_+(\lambda) + i\delta, S) & F_+(\lambda) \in \mathbb{R}. \end{cases} \tag{5.72}$$

Proof. If $F_+(\lambda) \in \mathbb{C}_+$, then for sufficiently small values of δ the points $F(\lambda + i\delta)$ lie in a compact subset of $\mathbb{C}_+$ and we have $Im\left[\frac{1}{t-F(\lambda+i\delta)}\right] \leq const.\frac{1}{1+t^2}$. Thus, the first assertion of the corollary follows from an application of the Lebesgue dominated convergence theorem in (5.71).

Otherwise, if $F_+(\lambda) \in \mathbb{R}$, note that by Theorem (4.4) in Chapter 4 we have $\lim_{\delta\to0^+} \phi_S(F(\lambda+i\delta)) = \lim_{\delta\to0^+} \phi_S(F_+(\lambda)+i\delta)$, so that

$$\lim_{\delta\to0^+}\left[\phi_S \circ F - b_\phi F\right](\lambda+i\delta)$$

$$= \lim_{\delta\to0^+}\left[a_\phi + b_\phi(F_+(\lambda)+i\delta) + \int_S \left\{\frac{1}{t-(F_+(\lambda)+i\delta)} - \frac{t}{1+t^2}\right\} d\sigma(t)\right] - b_\phi F_+(\lambda).$$

Hence,

$$h_{\nu_S}(\lambda) = \frac{1}{\pi} Im \lim_{\delta\to0^+}\left[\phi_S \circ F - b_\phi F\right](\lambda+i\delta)$$

$$= \frac{1}{\pi}\lim_{\delta\to0^+}\int_S Im\left[\frac{1}{t-(F_+(\lambda)+i\delta)}\right] d\sigma(t) = \frac{1}{\pi}\lim_{\delta\to0^+}\theta_\sigma(F_+(\lambda)+i\delta, S),$$

as the corollary states.

5.4 Application to the Schrödinger equation

In this section we will show how the theory of value distribution of Herglotz functions can be applied to solutions of the Schrödinger equation on the half-line, at real spectral parameter λ, through the Weyl-Titchmarsh m-function and its boundary values. We shall assume that there is an absolutely continuous component to the spectrum of the Schrödinger operator, and take λ to belong to an essential support of the absolutely continuous component of the spectral measure. We will see that applications of the theory of value distribution to the

Schrödinger equation are closely linked to geometrical properties of the upper half complex plane. We begin by establishing the notation.

Let a potential function $V(x)$, defined for $0 \le x \le \infty$, be given, with V real-valued and integrable over bounded subintervals of $[0, \infty)$. We make no special assumptions regarding the behaviour of $V(x)$ in the limit as $x \to \infty$. We associate with V the differential expression $\tau = -\frac{d^2}{dx^2} + V$. Then τ may be used to define the self-adjoint operator $T = -\frac{d^2}{dx^2} + V$, acting in $L^2(0, \infty)$ and subject to Dirichlet boundary conditions at $x = 0$. We are assuming here that the differential expression τ belongs to the limit-point case at infinity (see section 7 of Chapter 2), in which case no boundary condition at $x = +\infty$ is required to define T as a self-adjoint operator. The alternative assumption, that τ belongs to the limit-circle case, is known to lead to purely discrete spectrum for T (see [9]). Since we are primarily concerned here with the absolutely continuous part of the spectrum, we need not allow for the possibility of limit-circle at infinity.

We define $u(x, \lambda)$, $v(x, \lambda)$, in the case of real spectral parameter λ, and $u(x, z)$, $v(x, z)$, where the spectral parameter z is complex with $z \in \mathbb{C}_+$, to be solutions of the Schrödinger equation

$$-\frac{d^2 f(x, \lambda)}{dx^2} + V(x) f(x, \lambda) = \lambda f(x, \lambda), \tag{5.73}$$

$$-\frac{d^2 f(x, z)}{dx^2} + V(x) f(x, z) = z f(x, z), \tag{5.74}$$

respectively, in each case on the half-line $0 \le x \le \infty$, which satisfy for $\lambda \in \mathbb{R}$

$$\left.\begin{aligned} u(0, \lambda) &= 1 \\ u'(0, \lambda) &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} v(0, \lambda) &= 0 \\ v'(0, \lambda) &= 1 \end{aligned}\right\} \tag{5.75}$$

and the corresponding initial conditions for $u(x, z)$ and $v(x, z)$, for $z \in \mathbb{C}_+$. For all $z \in \mathbb{C}_+$, the Weyl Titchmarsh m-function is defined (assuming limit-point case at infinity) by the condition that

$$u(.,z) + m(z)v(.,z) \in L^2(0,\infty). \tag{5.76}$$

Then m is a Herglotz function. An alternative characterization of m is through the observation that $m(z) = f'(0,z)/f(0,z)$ for any (non-trivial) solution of equation (5.74), such that $f(.,z) \in L^2(0,\infty)$.

In addition, we will be interested in the m-function related to the differential expression $\tau = -\frac{d^2}{dx^2}+V$ where $V(x)$ is defined on the truncated interval $N \leq x \leq \infty$, for any $N > 0$. Taking for simplicity the case of Dirichlet boundary condition at $x = N$, we may define the self-adjoint operator $T^N = -\frac{d^2}{dx^2} + V$ acting in $L^2(N,\infty)$, subject to boundary condition $f(N) = 0$. Correspondingly, solutions $u^N(.,z)$, $v^N(.,z)$ of equation (5.74) with Im $z > 0$, may be defined subject to initial conditions

$$\left.\begin{array}{r} u^N(N,z) = 1 \\ (u^N)'(N,z) = 0 \end{array}\right\} \qquad \left.\begin{array}{r} v^N(N,z) = 0 \\ (v^N)'(N,z) = 1 \end{array}\right\} \tag{5.77}$$

and the m-function $m^N(.)$ with Dirichlet boundary condition at $x = N$ is determined by the condition that

$$u^N(.,z) + m^N(z)v^N(.,z) \in L^2(N,\infty) \qquad (z \in \mathbb{C}_+).$$

Note that $m^N(.)$ is the standard m-function for the Dirichlet Schrödinger operator $-\frac{d^2}{dx^2} + V(x+N)$ acting in $L^2(0,\infty)$. An alternative characterization of $m^N(.)$ is through the observation that $m^N(z) = \frac{f'(N,z)}{f(N,z)}$ for any (non-trivial) square-integrable solution $f(.,z)$ of equation (5.74). Since $u(.,z) + m(z)v(.,z)$ is just such a square-integrable solution, we can write explicitly

$$m^N(z) = \frac{u'(N,z) + m(z)v'(N,z)}{u(N,z) + m(z)v(N,z)}. \tag{5.78}$$

Before we proceed to our main result, we introduce the notion of 'distance of separation' $\gamma(z_1, z_2)$ of two points z_1, $z_2 \in \mathbb{C}_+$ defined by

$$\gamma(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{Im z_1}\sqrt{Im z_2}}. \tag{5.79}$$

Lemma 5.24 *Let $K \subset \mathbb{C}_+$ be a compact set, $c > 0$ be a constant, and $E = \{z' \in \mathbb{C}_+ : \gamma(z', z) < c, \ \forall\, z \in K\}$. Then, there exists a compact set $K_1 \subset \mathbb{C}_+$ such that $E \subseteq K_1$.*

Proof. Let z'_n be any sequence of points in E. Then, by assumption we have

$$\gamma(z'_n, z) = \frac{|z'_n - z|}{\sqrt{Im z'_n}\sqrt{Im z}} < c \tag{5.80}$$

for all n and $z \in K$. We note first that $Im z'_n$ is bounded. Otherwise, if $Im z'_n \to \infty$ as $n \to \infty$ then (5.80) would not hold, because $\gamma(z'_n, z)$ is greater than or equal to

$$\frac{|Im z'_n - Im z|}{\sqrt{Im z'_n}\sqrt{Im z}} \tag{5.81}$$

which tends to $+\infty$ as $n \to \infty$ in this case (since z is restricted in the compact set K), which is a contradiction. Similarly, if $|Re z'_n| \to \infty$ as $n \to \infty$ then (5.80) would again not hold, because $\gamma(z_1, z_2)$ is also greater than or equal to

$$\frac{|Re z'_n - Re z|}{\sqrt{Im z'_n}\sqrt{Im z}}$$

which tends to $+\infty$ as $n \to \infty$ in this case (since z is restricted in the compact set K and $Im z'_n$ is bounded). Hence $|z'_n|$ is bounded for all n. Moreover, $Im z'_n$ is bounded below by a positive constant; otherwise, if $Im z'_n \to 0^+$ as $n \to \infty$, then (5.80) would not hold, as can be seen from (5.81). Therefore, there exists a compact set $K_1 \subset \mathbb{C}_+$ such that $z'_n \in K_1$ for all n, and the lemma follows.

The following lemma provides an estimate of the generalized angle subtended θ_σ in terms of the distance of separation γ.

Lemma 5.25 *Let z_1 and z_2 be two points in the upper half complex plane, and let S be any Borel set. We then have, for any Herglotz measure $d\sigma$,*

$$|\theta_\sigma(z_1, S) - \theta_\sigma(z_2, S)| \leq \gamma(z_1, z_2)\sqrt{\theta_\sigma(z_1, S)}\sqrt{\theta_\sigma(z_2, S)}.$$

Proof. By definition of θ_σ, we have

$$|\theta_\sigma(z_1, S) - \theta_\sigma(z_2, S)| = \left|\int_S Im\left[\frac{1}{t - z_1}\right]d\sigma(t) - \int_S Im\left[\frac{1}{t - z_2}\right]d\sigma(t)\right| \tag{5.82}$$

$$= \left|Im \int_S \frac{(z_1 - z_2)}{(t - z_1)(t - z_2)}d\sigma(t)\right| \leq \left|\int_S \frac{(z_1 - z_2)}{(t - z_1)(t - z_2)}d\sigma(t)\right|$$

$$\leq \int_S \frac{|z_1 - z_2|}{|t - z_1||t - z_2|}d\sigma(t).$$

From the definition of the distance of separation γ in (5.79) we obtain $|z_1 - z_2| = \gamma(z_1, z_2)\sqrt{Im\, z_1}\sqrt{Im\, z_2}$. Thus, we have

$$|\theta_\sigma(z_1, S) - \theta_\sigma(z_2, S)| \leq \gamma(z_1, z_2) \int_S \frac{\sqrt{Im\, z_1}}{|t - z_1|}\frac{\sqrt{Im\, z_2}}{|t - z_2|}d\sigma(t)$$

$$\leq \gamma(z_1, z_2)\left\{\int_S \frac{Im\, z_1}{|t - z_1|^2}d\sigma(t)\right\}^{\frac{1}{2}}\left\{\int_S \frac{Im\, z_2}{|t - z_2|^2}d\sigma(t)\right\}^{\frac{1}{2}} \tag{5.83}$$

$$= \gamma(z_1, z_2)\sqrt{\theta_\sigma(z_1, S)}\sqrt{\theta_\sigma(z_2, S)}, \tag{5.84}$$

where we have used the Schwarz inequality.

Theorem 5.26 *Suppose that the measure $d\sigma$ is absolutely continuous, with density function h_σ, and let A be a bounded Borel subset of an essential support* *of the absolutely continuous part μ_{ac} of the spectral measure μ for the Dirichlet Schrödinger operator $T = -\frac{d^2}{dx^2} + V$, acting in $L^2(0,\infty)$; moreover, we make the following assumptions:*

(i) For any fixed $z \in \mathbb{C}_+$, there exists a compact subset K_z of $\mathbb{C}_+$ such that for all N sufficiently large we have $-\frac{v'(N,z)}{v(N,z)} \in K_z$.

(ii) There is a compact subset K_1 of $\mathbb{C}_+$ such that for all $\lambda \in A$ and N sufficiently large we have $m_+^N(\lambda) \in K_1$, and also for all N sufficiently large we have $-\frac{v'(N,i)}{v(N,i)} \in K_1$.

(iii) For $z = \lambda + i\delta$, $\lambda \in A$ and any $\delta > 0$ fixed, we have $K_z \subseteq K_1$.

Then, for any Borel subset S of $\mathbb{R}$ we have

$$\lim_{N\to\infty} \left| \tilde{\nu}_{-S}^N(A) - \frac{1}{\pi}\int_A \theta_{\sigma_r}(m_+^N(\lambda), S)d\lambda \right| = 0, \tag{5.85}$$

where the measure $\tilde{\nu}_S^N$ is defined by $\tilde{\nu}_S^N(X) = \int_S \tilde{\mu}_y^N(X)d\sigma(y)$ for any set X, the measures $\tilde{\mu}_y^N$ correspond to the family of Herglotz functions

$$\tilde{F}_y^N(z) = \frac{1}{y + \frac{v'(N,z)}{v(N,z)}}, \qquad y \in \mathbb{R},$$

the set $-S$ is defined by $-S = \{\lambda \in \mathbb{R} : -\lambda \in S\}$, and the measure $d\sigma_r$ has density function h_{σ_r}, given by $h_{\sigma_r}(t) = h_\sigma(-t)$. (Thus, h_{σ_r} is the reflection of h_σ in the vertical axis).

Remark 5.27 *From the proof of Theorem (2.65) in Chapter 2 we see that $-\frac{v'(N,z)}{v(N,z)}$ is a Herglotz function.*

Proof. The proof is based on the proof of Theorem 1 in [6], and other results therein. Let $m(z)$ denote the Weyl-Titchmarsh m−function for the differential operator T. An essential support for μ_{ac} is the set of all $\lambda \in \mathbb{R}$ at which the boundary value $m_+(\lambda)$ of $m(z)$ exists with strictly positive imaginary part. Hence we may assume without loss of generality that $\operatorname{Im} m_+(\lambda) > 0$ for all $\lambda \in A$.

We construct a finite partition $A = A_0 \cup A_1 \cup ... \cup A_n$ of A into $(n+1)$ disjoint sets, with $|A_0| < p|A|$, where p is a positive number, in the following way: Points $\lambda \in A$ at which $|\lambda|$ or $|m_+(\lambda)|$ are large, or at which $\operatorname{Im} m_+(\lambda)$ is small, are put into the set A_0. This leaves the set $A \backslash A_0$. The range of $m_+(\lambda)$, as λ runs over $A \backslash A_0$, is contained in a compact subset D of $\mathbb{C}_+$, and a partition $D = D_1 \cup D_2 \cup ... \cup D_n$ of D into disjoint subsets can be found such that, for all $j = 1, 2, ..., n$, we have $z_1, z_2 \in D_j \Rightarrow \gamma(z_1, z_2) \leq p$, where p is a positive number. We then take $A_j = (A \backslash A_0) \cap m_+^{-1}(D_j)$, and complex numbers $m^{(j)} = m_+(\lambda_j) \in \mathbb{C}_+$ for fixed $\lambda_j \in A_j$, and hence we have

$$\gamma(m_+(\lambda), m^{(j)}) \leq p \quad (all\ \lambda \in A_j,\ j = 1, ..., n) \tag{5.86}$$

Let K be a compact subset of $\mathbb{C}_+$ to be determined later. There exists a positive number δ_0 such that for arbitrary Borel set S and for $j = 1, ..., n$ we have

$$\left| \int_{A_j} \omega_\sigma^{\delta_0}(\lambda, S; F) d\lambda - \int_{A_j} \omega_\sigma(\lambda, S; F) d\lambda \right| \leq p|A_j|, \tag{5.87}$$

for any Herglotz function $F(z)$ such that $F(i) \in K$. That such δ_0 exists follows from Theorem (5.22). A priori we can define δ separately for each value of j; thus δ is a function of j. However, by taking the minimum value δ_0 of $\delta(j)$ as j runs from 1 to n, we may assume that δ_0 is independent of j.

From (5.78) we obtain an expression for the boundary values of the m-function $m^N(.)$, namely

$$m_+^N(\lambda) = \frac{u'(N,\lambda) + m_+(\lambda) v'(N,\lambda)}{u(N,\lambda) + m_+(\lambda) v(N,\lambda)}. \tag{5.88}$$

Hence, for fixed N and λ the mapping from $m_+(\lambda)$ to $m_+^N(\lambda)$ is a Möbius transformation with real coefficients and discriminant $uv' - vu' = 1$. From lemma 2 of [6] which asserts the invariance of $\gamma(.,.)$ under such Möbius transformations, we see that (5.86) implies

$$\gamma\Big(m_+^N(\lambda), \frac{u'(N,\lambda) + m^{(j)}v'(N,\lambda)}{u(N,\lambda) + m^{(j)}v(N,\lambda)}\Big) \le p \quad \textit{all } \lambda \in A_j,\ j = 1, ..., n. \tag{5.89}$$

Since, by assumption, $m_+^N(\lambda) \in K_1$ for all $\lambda \in A$ and N sufficiently large, it follows by (5.89) and lemma (5.24) that there exists a compact subset K_2 of $\mathbb{C}_+$ such that

$$\frac{u'(N,\lambda) + m^{(j)}v'(N,\lambda)}{u(N,\lambda) + m^{(j)}v(N,\lambda)} \in K_2, \qquad \textit{all } \lambda \in A_j,\ j = 1, ..., n, \textit{ and } N \textit{ suff. large.}$$

Let $d\sigma_r$ be an absolutely continuous measure with density function h_{σ_r}, where $h_{\sigma_r}(t) = h_\sigma(-t)$. By lemma (5.25), applied to the measure $d\sigma_r$, and lemma (5.12) we have

$$\Big|\theta_{\sigma_r}(m_+^N(\lambda), S) - \theta_{\sigma_r}\Big(\frac{u'(N,\lambda) + m^{(j)}v'(N,\lambda)}{u(N,\lambda) + m^{(j)}v(N,\lambda)}, S\Big)\Big| \le pC_{1,2}, \tag{5.90}$$

where the constant $C_{1,2}$ depends on the compact sets K_1 and K_2. Then, integration with respect to λ over A_j leads to the bound

$$\Big|\frac{1}{\pi}\int_{A_j} \theta_{\sigma_r}(m_+^N(\lambda), S)d\lambda - \frac{1}{\pi}\int_{A_j} \theta_{\sigma_r}\Big(\frac{u'(N,\lambda) + m^{(j)}v'(N,\lambda)}{u(N,\lambda) + m^{(j)}v(N,\lambda)}, S\Big)d\lambda\Big| \le \frac{1}{\pi}p|A_j|C_{1,2}, \tag{5.91}$$

valid for all $\lambda \in A_j$, $j = 1, ..., n$, and $N > 0$.

Now, for $j = 1, ..., n$ define the subset $A_j^{\delta_0}$ of $\mathbb{C}_+$, consisting of all $z \in \mathbb{C}_+$ of the form $z = \lambda + i\delta_0$, for $\lambda \in A_j$. Thus $A_j^{\delta_0}$ is the translation of A_j by distance δ_0 above the real axis. Since A_j is bounded, $A_j^{\delta_0}$ is contained in a compact subset of $\mathbb{C}_+$. Hence the corollary to lemma 3 of [6] implies that

$$\gamma\Big(- \frac{v'(N,z)}{v(N,z)}, - \frac{u'(N,z) + \overline{m}^{(j)} v'(N,z)}{u(N,z) + \overline{m}^{(j)} v(N,z)} \Big) \to 0 \tag{5.92}$$

uniformly in $\overline{m}^{(j)}$ and for all $z \in A_j^{\delta_0}$, $j = 1, ..., n$, as $N \to \infty$. We may again obtain an estimate of the generalized angle subtended, by using lemmas (5.25) and (5.12). There is an N_0 such that if $N > N_0$, and with $z = \lambda + i\delta_0$, we have

$$\Big| \int_{A_j} \frac{1}{\pi} \theta_\sigma \Big(- \frac{v'(N, \lambda + i\delta_0)}{v(N, \lambda + i\delta_0)}, -S \Big) d\lambda -$$

$$- \int_{A_j} \frac{1}{\pi} \theta_\sigma \Big(- \frac{u'(N, \lambda + i\delta_0) + \overline{m}^{(j)} v'(N, \lambda + i\delta_0)}{u(N, \lambda + i\delta_0) + \overline{m}^{(j)} v(N, \lambda + i\delta_0)}, -S \Big) d\lambda \Big| \leq \frac{1}{\pi} p |A_j|, \tag{5.93}$$

for all $\lambda \in A_j$, $j = 1, ..., n$, where $-S = \{\lambda \in \mathbb{R} : -\lambda \in S\}$.

Each of the two integrals in equation (5.93) is of the form $\int_{A_j} \omega_\sigma^{\delta_0}(\lambda, -S; F) d\lambda$ for some F; namely

$$F = F_1 = - \frac{v'(N, .)}{v(N, .)},$$

$$F = F_2 = - \frac{u'(N, .) + \overline{m}^{(j)} v'(N, .)}{u(N, .) + \overline{m}^{(j)} v(N, .)}.$$

By assumption, $-\frac{v'(N,i)}{v(N,i)} \in K_1$ for all N sufficiently large. By the corollary to lemma 3 in [6], there exists an N_1 such that (5.92) holds for $N > N_1$ and with $z = i$. Thus, by lemma (5.24) there exists a compact subset K_4 of $\mathbb{C}_+$ such that

$$- \frac{u'(N,i) + \overline{m}^{(j)} v'(N,i)}{u(N,i) + \overline{m}^{(j)} v(N,i)} \in K_4, \qquad N > N_1.$$

Now let K, the compact subset of $\mathbb{C}_+$ on which the choice of δ_0 in (5.87) depends on, be given by $K = K_1 \cup K_4$. Then, by using (5.87) in each case, we may compare the difference between the two integrals in (5.93) with the corresponding difference in the limit as $\delta \to 0^+$. Let $\tilde{F}_y^N$ be the family of Herglotz functions

$\tilde{F}_y^N(z) = 1/\left[y + \frac{v'(N,z)}{v(N,z)}\right]$, $y \in \mathbb{R}$, $N > 0$, with corresponding measures $\tilde{\mu}_y^N$. Then, we have

$$\lim_{\delta\to 0^+} \frac{1}{\pi}\int_{A_j} \theta_\sigma\Big(-\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)}, -S\Big) d\lambda = \lim_{\delta\to 0^+}\int_{A_j}\left\{\frac{1}{\pi}\int_{-S} Im\left[\frac{1}{y+\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)}}\right] d\sigma(y)\right\} d\lambda$$

$$= \lim_{\delta\to 0^+}\int_{-S}\left\{\frac{1}{\pi}\int_{A_j} Im\left[\frac{1}{y+\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)}}\right] d\lambda\right\} d\sigma(y)$$

$$= \lim_{\delta\to 0^+}\int_{-S}\left\{\frac{1}{\pi}\int_{A_j} Im\, \tilde{F}_y^N(\lambda+i\delta) d\lambda\right\} d\sigma(y)$$

$$= \lim_{\delta\to 0^+}\int_{-S} \tilde{\mu}_y^{N,\delta}(A_j) d\sigma(y) = \int_{-S} \tilde{\mu}_y^N(A_j) d\sigma(y) = \tilde{\nu}_{-S}^N(A_j). \tag{5.94}$$

We were able to change the order of integration, since, for λ in the bounded set $\bigcup_{j=1}^n A_j$, we have the estimate, valid for any fixed $\delta > 0$

$$Im\left[\frac{1}{y+\frac{v'(N,\lambda+i\delta)}{v(N,\lambda+i\delta)}}\right] \le const. \frac{1}{1+y^2}.$$

Moreover, equation (5.94) follows from Theorem (5.9), since $-\frac{v'(N,z+i\delta)}{v(N,z+i\delta)} \to -\frac{v'(N,z)}{v(N,z)}$ uniformly on compact subsets of $\mathbb{C}_+$, as $\delta \to 0^+$. Therefore, we arrive at the bound

$$\left|\tilde{\nu}_{-S}^N(A_j) - \frac{1}{\pi}\int_{A_j}\theta_\sigma\Big(-\frac{u'(N,\lambda)+\overline{m}^{(j)}v'(N,\lambda)}{u(N,\lambda)+\overline{m}^{(j)}v(N,\lambda)}, -S\Big) d\lambda\right| \le p|A_j|(2+1/\pi). \tag{5.95}$$

Since $\theta_\sigma(z,S) = \theta_{\sigma_r}(-\overline{z}, -S)$, we obtain from (5.95)

$$\left|\tilde{\nu}_{-S}^N(A_j) - \frac{1}{\pi}\int_{A_j}\theta_{\sigma_r}\Big(\frac{u'(N,\lambda)+m^{(j)}v'(N,\lambda)}{u(N,\lambda)+m^{(j)}v(N,\lambda)}, S\Big) d\lambda\right| \le p|A_j|(2+1/\pi), \tag{5.96}$$

which holds for all $\lambda \in A_j$, $j = 1,\dots,n$ and $N \ge \max\{N_0, N_1\}$. Combining inequalities (5.96) and (5.91) now yields

$$\left|\tilde{\nu}_{-S}^N(A_j) - \frac{1}{\pi}\int_{A_j}\theta_{\sigma_r}(m_+^N(\lambda), S) d\lambda\right| \le \frac{1}{\pi} p|A_j|C_{1,2} + p|A_j|(2+1/\pi), \tag{5.97}$$

for all $\lambda \in A_j$, $j = 1, ..., n$ and $N \geq \max\{N_0, N_1\}$. We now have for all $N \geq \max\{N_0, N_1\}$

$$\left|\tilde{\nu}^N_{-s}(A) - \frac{1}{\pi}\int_A \theta_{\sigma_r}(m^N_+(\lambda), S)d\lambda\right|$$

$$\leq \sum_{j=0}^{n}\left|\tilde{\nu}^N_{-s}(A_j) - \frac{1}{\pi}\int_{A_j} \theta_{\sigma_r}(m^N_+(\lambda), S)d\lambda\right|$$

$$\leq \tilde{\nu}^N_{-s}(A_0) + \frac{1}{\pi}\int_{A_0} \theta_{\sigma_r}(m^N_+(\lambda), S)d\lambda + \frac{1}{\pi}p(1 + 2\pi + C_{1,2})\sum_{j=1}^{n}|A_j|.$$

We have

$$\tilde{\nu}^N_{-s}(A_0) = \int_{-S} \tilde{\mu}^N_y(A_0)d\sigma(y)$$

$$= \int_{-S_0} \tilde{\mu}^N_y(A_0)d\sigma(y) + \int_{-S_1} \tilde{\mu}^N_y(A_0)d\sigma(y),$$

where $-S_0 = \{y \in -S : h_\sigma(y) \leq C_0\}$, for some constant C_0, and $-S_1 = -S\backslash(-S_0)$. Suppose that $A_0 \subseteq [-N_0, N_0]$. From the proof of lemma (5.1) we have $\tilde{\mu}^N_y(A_0) \leq (1+N_0^2)Im\tilde{F}^N_y(i)$. By assumption $-\frac{v'(N,i)}{v(N,i)} \in K_1$ for all $N > 0$, which implies that $Im\tilde{F}^N_y(i) \leq const./(1 + y^2)$. Let $\varepsilon > 0$ be given. Then, we can choose C_0 such that $\int_{-S_1} \tilde{\mu}^N_y(A_0)d\sigma(y) < \varepsilon$, for all $N > 0$. Therefore, we have

$$\tilde{\nu}^N_{-s}(A_0) \leq C_0 \int_{\mathbf{R}} \tilde{\mu}^N_y(A_0)dy + \varepsilon = C_0|A_0| + \varepsilon. \tag{5.98}$$

Also, we have

$$\frac{1}{\pi}\int_{A_0} \theta_{\sigma_r}(m^N_+(\lambda), S)d\lambda \leq \frac{1}{\pi}|A_0|C_{1,2}.$$

Noting that A_0 was chosen such that $|A_0| \leq p|A|$, we now have, for $N \geq \max\{N_0, N_1\}$

$$\left|\tilde{\nu}^N_{-s}(A) - \frac{1}{\pi}\int_A \theta_{\sigma_r}(m^N_+(\lambda), S)d\lambda\right|$$

$$\leq p|A|C_0 + \varepsilon + \frac{1}{\pi}p|A|C_{1,2} + \frac{1}{\pi}p|A|(1 + 2\pi + C_{1,2})$$

$$= \frac{1}{\pi}p|A|(1 + 2\pi + C_0\pi + 2C_{1,2}) + \varepsilon.$$

Since $p > 0$ and $\varepsilon > 0$ were arbitrary, we have verified equation (5.85) and Theorem (5.26) is proved.

Remark 5.28 *On examination of the proof of Theorem* (5.26), *it will be found that all of the bounds are uniform over Borel subsets S of $\mathbb{R}$. It follows that equation* (5.85) *remains valid if we replace the single Borel set S by an arbitrary family $\{S^N\}$ of Borel sets, parametrized by N. If the family $\{S^N\}$ is chosen in such a way that $\frac{1}{\pi}\int_A \theta_{\sigma_r}(m_+^N(\lambda), S)d\lambda$ converges to a limit as $N \to \infty$, then the measures $\tilde{\nu}_{-s}^N(A)$ will converge to the same limit.*

Chapter 6

Conclusions and further research

In this thesis we generalized the idea of value distribution associated with a Herglotz function, and showed that the resulting Herglotz measure ν_S, defined by (3.9), is related to the measure corresponding to a composed Herglotz function. We thus studied compositions of Herglotz functions, and gave two nice results: one regarding the coefficient $b_{F\circ G}$ of the term linear in z in the representation of a composed Herglotz function, and one regarding the boundary values of a composed Herglotz function. We then derived some important properties of Herglotz measures, obtained some results regarding the generalized value distribution of Herglotz functions, and gave an application to the Schrödinger equation.

An interesting question which arises from Theorem (4.3) is with regard to the constant term $a_{F\circ G}$ in the representation of a composed Herglotz function $F \circ G$: How does this constant depend on the constants a_F, a_G appearing in the representation of the Herglotz functions F and G respectively undergoing composition?

It would also be interesting to see how does the measure ν_S look like for a given Herglotz function F and Herglotz measure $d\sigma$. We have given a result regarding the density function of ν_s in the case when it is absolutely continuous, but this would be best illustrated with some specific examples, by using lemma (3.8).

Finally, developing the theory in the case when the measure $d\sigma$ is singular and obtaining corresponding results is an exciting prospect.

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