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Frobenius manifolds: caustic submanifolds and discriminant almost duality

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Contents

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Abstract

The concept of a Frobenius manifold was invented by Boris Dubrovin as a geometric interpretation of solutions of the WDVV equations with additional constraints. The theory of Frobenius manifolds contains a rich mathematical structure transcending many disparate fields of study. In this work, consideration will be restricted to so called semisimple Frobenius manifolds and their submanifolds.

Chapter 1 introduces the concept of a Frobenius manifold and gives constructions of the closely linked Coxeter group and Hurwitz space based classes. The concept of almost duality is also introduced; this is the notion that from any Frobenius manifold, one may construct a second solution to the WDVV equations adhering to most of the axioms of a Frobenius manifold.

Chapter 2 introduces submanifold geometry and natural submanifolds, on which the induced multiplication coincides with that on the ambient manifold. Such submanifolds are classified in terms of caustics and discriminants. Caustic submanifolds of an arbitrary genus zero Hurwitz space are then considered in chapter 3, extending the idea contained within the main example of [25].

Chapter 4 constructs dual WDVV solutions for A_n Coxeter type and genus zero Hurwitz Frobenius manifolds, including their discriminants. The result of section 4.2 appeared in [21]. It also draws a link, via a twisted Legendre transformation, between certain almost dual solutions. This idea was published in [22].

Finally, chapter 5 deals with the Hurwitz space $H_{1,n}$, which may be thought of in terms of a Jacobi orbit space. In particular, almost dual solutions of the WDVV equations are constructed on the discriminants, giving a generalised version of the result published in [21].

Chapter 1

Frobenius manifolds

1.1 The WDVV equations

The Witten-Dijkgraaf-H. Verlinde-E. Verlinde, or WDVV equations, first appeared in the papers [28] and [8] on topological field theory in the early 1990s. They have been studied from a variety of perspectives including integrable systems, singularity theory, Seiberg-Witten theory and topological quantum field theory.

Definition 1.1 *The* WDVV equations of associativity *are the system of partial differential equations:*

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\alpha}, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n \tag{1.1}$$

In the various applications of these equations, additional constraints (such as quasihomogeneity) are often imposed. The formulation below follows [10]. The equations require that a function $F = F(t^1, t^2, \ldots, t^n)$ be found such that the third derivatives

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{1.2}$$

have the following properties:

W1 Normalization: The matrix with components

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t)$$

must be constant and non-degenerate. This matrix will be used for lowering indices, while the inverse matrix

$$(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}$$

will be used to raise indices.

W2 Associativity: On an n-dimensional space with basis $e_i, \ldots, e_n$, an associative algebra $\mathcal{A}_t$ is defined at a point t by

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma,$$

where $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t)$. These algebras will also be commutative; this is guaranteed by (1.2) above.

W3 Quasihomogeneity: $F(t)$ must satisfy the equation

$$F(c^{d_1}t^1, c^{d_2}t^2, \ldots, c^{d_n}t^n) = c^{d_F} F(t^1, t^2, \ldots, t^n),$$

for some $d_1, \ldots d_n, d_F$ and for all $c \neq 0$.

Note that the conditions W1 and W2 above imply the associativity equations (1.1), though even condition W1 may be weakened, for example by choosing η to take a different form (which may be a function of, or even independent of, F).

Definition 1.2 *Let E be a vector field of the form*

$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha.$$

Then E is the Euler vector field *for F.*

The Euler field implicitly represents the quasihomogeneity condition W3 through the equation

$$\mathcal{L}_E F(t) = E^\alpha \partial_\alpha F(t) = d_F \cdot F(t),$$

where $\mathcal{L}_E$ represents the Lie derivative along E. In the case of $d_\alpha = 0$ for one or more of the coordinate indices α, it is possible to generalise E so that it takes the form:

$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha + \sum_{\alpha | d_\alpha = 0} r_\alpha \partial_\alpha .$$

It should be noted that the Lie derivative along E of the identity vector $e = \partial_1$ is

$$\mathcal{L}_E e = -d_1 e.$$

Attention is now turned to a generalisation of E and of condition W3. If a non-homogenous quadratic function of $t^1, \ldots, t^n$ is added to F, the third derivatives $c_{\alpha\beta\gamma}$ will be unchanged. Hence the associativity equations which define the algebras $\mathcal{A}_t$ are not changed by the addition of this function. In other words, if F is such that

$$\mathcal{L}_E F(t) = d_F F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C,$$

then the $c_{\alpha\beta\gamma}$ would still be quasihomogenous. Provided that $d_F \neq 0$, $d_F - d_\alpha \neq 0$ and $d_F - d_\alpha - d_\beta \neq 0$ for all α, β, then the extra terms in $\mathcal{L}_E F(t)$ above can be killed by adding a quadratic form to F.

Example 1.3 *In two dimensions, the associativity equations are vacuous. However, condition W1 restricts the form solutions of WDVV may take to*

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + f(t_2).$$

Applying W3 and noting the permissible forms of E then restricts the form of $f(t_2)$ *so that all two dimensional solutions to WDVV are equivalent to one of the* *following, as can be seen in [10]:*

$$\begin{aligned}
F(t_1,t_2) &= \frac{1}{2}t_1^2t_2 + t_2^k,\\
F(t_1,t_2) &= \frac{1}{2}t_1^2t_2 + t_2^2\log t_2,\\
F(t_1,t_2) &= \frac{1}{2}t_1^2t_2 + \log t_2,\\
F(t_1,t_2) &= \frac{1}{2}t_1^2t_2 + e^{\frac{2}{r}t_2},\\
F(t_1,t_2) &= \frac{1}{2}t_1^2t_2.
\end{aligned}$$

Note that in the above formulae, the indices of the t^α have been written subscript rather than superscript purely for typographical convenience. This convention will be used in all solutions of WDVV presented in this work.

Example 1.4 *Following [10], let us consider a three dimensional solution to WDVV of the form:*

$$F(t_1,t_2,t_3) = \frac{1}{2}t_1^2t_3 + \frac{1}{2}t_1t_2^2 + f(t_2,t_3).$$

By differentiating F, one can calculate the $c_{\alpha\beta\gamma}$, the non-zero values of which are listed below (recall that the ordering of the indices does not affect the value of $c_{\alpha\beta\gamma}$).

$$\begin{array}{lll}
c_{113}=1 & c_{122}=1 & c_{222}=f_{222}\\
c_{223}=f_{223} & c_{233}=f_{233} & c_{333}=f_{333}
\end{array}$$

In the above, and throughout this example, $f_{ijk}=\frac{\partial^3 f}{\partial t^i\partial t^j\partial t^k}$. In particular, we have

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} = \eta^{\alpha\beta}.$$

One is now able to calculate the $c_{\alpha\beta}^{\gamma}$. The non-zero values are listed below:

$$\begin{array}{lll} c^1_{11} = 1 & c^2_{12} = 1 & c^3_{13} = 1 \\ c^1_{22} = f_{223} & c^2_{22} = f_{222} & c^3_{22} = 1, \\ c^1_{23} = f_{233} & c^2_{23} = f_{223} & c^1_{33} = f_{333}, \\ c^2_{33} = f_{233} & & \end{array}$$

Therefore the multiplication defined by F has the following table:

$\cdot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	$f_{223}e_1 + f_{222}e_2 + e_3$	$f_{233}e_1 + f_{223}e_2$
e_3	e_3	$f_{233}e_1 + f_{223}e_2$	$f_{333}e_1 + f_{233}e_2$

Clearly, e_1 is an identity under this multiplication, so for associativity one needs only to consider the equations

$$e_2^2 \cdot e_3 = e_2 \cdot (e_2 \cdot e_3)$$

and

$$(e_2 \cdot e_3) \cdot e_3 = e_2 \cdot e_3^2,$$

as all of the other associativity equations are trivial.

The first equation expands to

$$\begin{aligned} f_{223}e_3 + f_{222}(f_{233}e_1 + f_{223}e_2) + (f_{333}e_1 + f_{233}e_2) \\ = f_{233}e_2 + f_{223}(f_{233}e_1 + f_{222}e_2 + e_3), \end{aligned}$$

which simplifies to

$$f_{223}^2 = f_{333} + f_{222}f_{233}.$$

It turns out that the second associativity equation simplifies to the same partial differential equation for $f(t_2, t_3)$.

1.2 Definition of a Frobenius manifold

The concept of a Frobenius manifold was first introduced by Boris Dubrovin in [9]. A comprehensive account on the subject was provided by Dubrovin in [10]. To introduce the concept of a Frobenius manifold, one must first define a *Frobenius Algebra.* The following definition and example have both been take from [10].

Definition 1.5 *Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ with a multiplication $\cdot$. $\mathcal{A}$ is said to be a* Frobenius Algebra *if the following four conditions are satisfied:*

FA1 *$\mathcal{A}$ is a commutative algebra.*

FA2 *$\mathcal{A}$ is an associative algebra.*

FA3 *A unit element e exists in $\mathcal{A}$ such that $e \cdot a = a$, for all $a \in \mathcal{A}$.*

FA4 *A non degenerate inner product $<,>: A \times \mathcal{A} \to \mathbb{C}$ exists and satisfies*

$$< ab, c >=< a, bc >,$$

for all $a, b, c \in \mathcal{A}$. This final condition is known as the Frobenius property.

Example 1.6 *Let $\{\mathcal{A}_i\}$, $i = 1, 2, \ldots n$ be a set of n one dimensional semisimple algebras. Then their direct sum*

$$\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{A}_i$$

is a Frobenius algebra. Moreover, a basis $\{e_i\}$, $i = 1, 2, \ldots n$ may be chosen by selecting $e_i \in \mathcal{A}_i$ such that it is a generator for the algebra $\mathcal{A}_i$. In terms of this basis, a suitable multiplication may be defined by

$$e_i e_j = \delta_{ij} e_i.$$

The algebra may then be parameterised by $\mathcal{A} = (t^1, \ldots, t^n)$ where $t^i =< e_i, e_i >$.

Lemma 1.7 *Given a Frobenius algebra $\mathcal{A}$, one may apply rescaling transformations to the multiplication on $\mathcal{A}$, of the form*

$$a \cdot b \to ka \cdot b,$$

and

$$e \to k^{-1} e,$$

for any non-zero k. Under such rescalings, the algebra will still be Frobenius.

Proof: Under such a rescaling, the associativity and commutativity of the algebra will obviously be preserved and an identity element will remain. For the Frobenius property to still hold, we require that

$$< ka \cdot b, c >=< a, kb \cdot c > .$$

But as the inner product is bilinear, we can take the k outside the inner product to leave

$$k < a \cdot b, c >= k < a, b \cdot c >,$$

which is equivalent to the original definition of the Frobenius property.

Having defined a Frobenius algebra, one may now move on to defining a *Frobenius manifold*, again by following [10].

Definition 1.8 *An n-dimensional manifold $\mathcal{M}$ is said to be a* Frobenius Manifold *if the following conditions are satisfied:*

FM1 For every point x on $\mathcal{M}$, a Frobenius algebra may be described on the tangent space $T_x\mathcal{M}$. These algebras must vary smoothly with x.

FM2 The invariant inner product $<,>$ *defines a flat metric on* $\mathcal{M}$. *In the presence of such a metric, a set of coordinates (distinguished to within a linear transformation), known as the* flat coordinates *of* $<,>$ *exist [14] such that the components of* $<,>$ *are constant in these coordinates.*

FM3 A unity vector field e may be defined satisfying

$$\nabla e = 0,$$

for the Levi-Civita connection of the metric.

FM4 The four-tensor

$$(\nabla_z c)(u, v, w)$$

is symmetric in the vectors u, v, w, z *for*

$$c(u, v, w) := < uv, w > .$$

FM5 An Euler vector field E *satisfying*

$$\nabla(\nabla E) = 0$$

may be determined on $\mathcal{M}$ *such that the correspondent one-parameter group of diffeomorphisms acts by conformal transformations of the metric* $<,>$ *and rescalings of the Frobenius algebras* $T_x\mathcal{M}$.

Examples of Frobenius manifolds will be provided later. One may observe obvious similarities between the definition of a Frobenius manifold and the definition of the WDVV equation; for example the requirement that an associative algebra be defined in each case and the presence of an Euler vector field. In fact conditions W1-W3 and Frobenius manifolds are equivalent, as originally shown by Dubrovin and as expressed in the following theorem (taken from [10]).

Theorem 1.9 *Given any solution of the WDVV equations defined in a domain $t \in M$, and subject to $d_1 \neq 0$, the structure of a corresponding Frobenius manifold may be defined. The multiplication of tangent vectors*

$$\partial_\alpha := \frac{\partial}{\partial t^\alpha}$$

is given by

$$\partial_\alpha \cdot \partial_\beta := c^\gamma_{\alpha\beta}\partial_\gamma.$$

The invariant inner product between two tangent vectors on $T_x\mathcal{M}$ vectors is defined as

$$< \partial_\alpha, \partial_\beta >:= \eta_{\alpha\beta}.$$

The identity vector will be given by

$$e := \partial_1.$$

Finally, the Euler vector field will be of the form:

$$\mathcal{L}_E F(t) = d_F F(t) + A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C.$$

Conversely, given a Frobenius manifold $\mathcal{M}$, one may (locally) reconstruct a solution to the WDVV equations by using the above formulae and an Euler vector field of the form

$$E(t) = (q^\alpha_\beta t^\beta + r^\alpha)\partial_\alpha.$$

Proof: It follows immediately from the conditions of the WDVV equations that the multiplication defined on the tangent space will be commutative, associative and will have an identity element $e = \partial_1$. To show that a Frobenius algebra is defined, one requires

$$< \partial_\alpha \cdot \partial_\beta, \partial_\gamma >=< \partial_\alpha, \partial_\beta \cdot \partial_\gamma > .$$

Using the multiplication formula, the left hand side becomes:

$$< c^{\delta}_{\alpha\beta}\partial_\delta, \partial_\gamma >= c^{\gamma}_{\alpha\beta}\eta_{\delta\gamma} = c_{\alpha\beta\gamma}.$$

Similarly, the right hand side becomes:

$$< \partial_\alpha, c^{\delta}_{\beta\gamma}\partial_\delta >= c^{\delta}_{\beta\gamma}\eta_{\alpha\delta} = c_{\beta\gamma\alpha} = c_{\alpha\beta\gamma}.$$

Hence a Frobenius algebra has been determined. FM2 is automatically satisfied, as the components of $\eta_{\alpha\beta}$ are all constants, thus ensuring flatness. FM3 is also automatically satisfied, as $\eta_{\alpha\beta}$ is not only flat but also expressed in terms of its own flat coordinates. As such, covariant derivatives are equal to the corresponding partial derivatives and so the identity field $e = \partial_1$ is covariantly constant. Similarly, condition FM4 is satisfied as

$$\begin{aligned} \nabla_{\partial_\delta} c(\partial_\alpha, \partial_\beta, \partial_\gamma) &= \partial_\delta c_{\alpha\beta\gamma}, \\ &= \frac{\partial^4 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\delta}. \end{aligned}$$

Finally, recalling that $E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha$, it is obvious that FM5 is also satisfied.

To prove the converse statement, one notes that as the metric is flat, a set of flat coordinates $\{t^\alpha\}$ (defined up to a linear transformation) exists such that the components of $\eta_{\alpha\beta}$ are constants. By performing a linear transformation, one may then set $e = \frac{\partial}{\partial t^1}$. In these coordinates, the condition FM4 is equivalent to saying $\partial_\delta c_{\alpha\beta\gamma}$ is symmetric in $\alpha, \beta, \gamma, \delta$. Noting that $c_{\alpha\beta\gamma}$ is also symmetric in all of its indices, FM4 therefore becomes a potentiality condition, i.e. it guarantees the existence of a function $F = F(t)$ such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}.$$

From the definition of a Frobenius manifold, the algebra defined by the $c_{\alpha\beta\gamma}$ will be commutative and associative. Finally, the axiom FM5 ensures that F will be quasihomogenous. Hence conditions W1-W3 have been satisfied.

1.3 The intersection form and flat pencils of metrics

In addition to the metric $\eta_{\alpha\beta}$ defined on a Frobenius manifold in section 1.2, one may introduce a second metric $g_{\alpha\beta}$. It is easier to define this metric as an inner product, $(,)^*$, of two 1-forms on the cotangent bundle $T^*_t\mathcal{M}$:

$$(\omega_1, \omega_2)^* := i_E(\omega_1 \cdot \omega_2).$$

In the above formula, i_E is the operator of contraction of a 1-form along the Euler vector field, whilst the product of two 1-forms is the dual of the product of their dual vectors on $T_t\mathcal{M}$. In the flat coordinates $\{t^i\}$, we may therefore define $g^{\alpha\beta}$ in the obvious way, namely

$$g^{\alpha\beta} := (dt^\alpha, dt^\beta)^*.$$

Definition 1.10 *The metric $g^{\alpha\beta}$, above, is known as the* intersection form *of the Frobenius manifold.*

Where it is defined, the matrix inverse of the intersection form can be used to define a metric $(,)$ on $T_t\mathcal{M}$, that is to say:

$$(\partial_\alpha, \partial_\beta) := g_{\alpha\beta} = (g^{\alpha\beta})^{-1}.$$

This new metric is in fact related to the invariant inner product by the formula

$$(E \cdot u, v) =< u, v > .$$

Note that $g_{\alpha\beta}$ is therefore not defined at those points where E is not invertible.

Definition 1.11 *The locus in $\mathcal{M}$ on which E^{-1} is not defined is known as the* discriminant *of $\mathcal{M}$.*

Lemma 1.12 *The metric* $(,)$ *inherits the flatness of* $<,>$.

Proof: Deferred; this fact will follow automatically from the proof of a stronger statement about flat pencils of metrics in theorem 1.14.

Definition 1.13 *Two metrics* $g_1^{\alpha\beta}$ *and* $g_2^{\alpha\beta}$ *are said to form a* flat pencil *if:*

$$g_\lambda^{\alpha\beta} := g_1^{\alpha\beta} + \lambda g_2^{\alpha\beta}$$

is itself a flat metric for all values of λ *and the Christoffel symbols for the Levi-Civita connection for* $g_\lambda^{\alpha\beta}$ *obey the equations:*

$$\Gamma^{ij}_{\lambda_k} = \Gamma^{ij}_{1k} + \lambda\Gamma^{ij}_{2k},$$

where Γ^{ij}_{1k} *is the Christoffel symbol* Γ^{ij}_k *for the metric* g^{ij} *(likewise* Γ^{ij}_{2k} *for* g_2^{ij} *and* $\Gamma^{ij}_{\lambda k}$ *for* g_λ^{ij}*), defined by:*

$$\Gamma^{ij}_k = -g^{is}\Gamma^j_{ks}.$$

Theorem 1.14 *The metrics* $\eta^{\alpha\beta}$ *and* $g^{\alpha\beta}$ *form a flat pencil.*

Before proving this statement, one recalls some standard facts of differential geometry (see, for example, [14]). For an arbitrary metric g_{ij} in a coordinate system $\{x^i\}$, the Christoffel symbols for the Levi-Civita connection are defined by

$$\Gamma^k_{ij}(x) = \frac{1}{2}g^{ks}(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}).$$

From this definition, it is obvious that they are symmetric in the lower indices, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$. A contravariant Levi-Civita connection for g^{ij} may then be defined by

$$\Gamma^{ij}_k(x) = -g^{is}(x)\Gamma^j_{sk}(x).$$

It is uniquely determined by the equations

$$\begin{aligned} \Gamma_k^{ij} + \Gamma_k^{ji} &= \partial_k g^{ij}, \\ g^{is}\Gamma_s^{jk} &= g^{js}\Gamma_s^{ik}. \end{aligned}$$

One may then calculate the Riemann curvature tensor

$$R_l^{ijk} := g^{is}(\partial_s \Gamma_l^{jk} - \partial_l \Gamma_s^{jk}) + \Gamma_s^{ij}\Gamma_l^{sk} - \Gamma_s^{ik}\Gamma_l^{sj},$$

which will be identically equal to zero for a flat metric.

Proof: One may now prove theorem 1.14. Denote by $\Gamma_{\eta k}^{ij}$ the Christoffel symbols for η^{ij}. Likewise Γ_{gk}^{ij} for g_{ij} and $\Gamma_{\lambda k}^{ij}$ for $(g - \lambda\eta)^{ij}$. We require that

$$\Gamma_{\lambda k}^{ij} = \Gamma_{gk}^{ij} + \lambda \Gamma_{\eta k}^{ij}.$$

If we use the flat coordinates $\{t^\alpha\}$, then $\eta^{\alpha\beta}$ is constant, and so the components of the Levi-Civita connection vanish. Therefore we require

$$\Gamma_{\lambda k}^{ij} = \Gamma_{gk}^{ij},$$

or equivalently

$$\Gamma_{\lambda k}^{ij} + \Gamma_{\lambda k}^{ji} = \partial_k g^{ij}, \tag{1.3}$$

$$g^{is}\Gamma_{\lambda s}^{jk} = g^{js}\Gamma_{\lambda s}^{ik}. \tag{1.4}$$

Expanding the right hand side of

$$\Gamma_{\lambda k}^{ij} + \Gamma_{\lambda k}^{ji} = \partial_k (g - \lambda\eta)^{ij}$$

yields

$$\begin{aligned} \partial_k (g - \lambda\eta)^{ij} &= \partial_k g^{ij} - \lambda \partial_k \eta^{ij}, \\ &= \partial_k g^{ij}, \end{aligned}$$

due to the constancy of η^{ij} in the flat coordinates. Hence equation (1.3) holds true. To prove (1.4), we use the fact that

$$(g - \lambda\eta)^{is}\Gamma_{\lambda s}^{jk} = (g - \lambda\eta)^{js}\Gamma_{\lambda s}^{ik}.$$

Recall from above that

$$(g - \lambda\eta)^{ij} = (t^1\eta - \lambda\eta + \tilde{g}(t^2, \ldots, t^n))^{ij}.$$

By substituting this and comparing the coefficients of the highest power of t^1, it immediately follows that

$$\eta^{is}\Gamma^{jk}_{\lambda s} = \eta^{js}\Gamma^{ik}_{\lambda s},$$

which in turn implies

$$\tilde{g}^{is}\Gamma^{jk}_{\lambda s} = \tilde{g}^{js}\Gamma^{ik}_{\lambda s}.$$

Finally, by recalling that

$$g^{ij} = (t^1\eta + \tilde{g})^{ij},$$

we obtain

$$g^{is}\Gamma^{jk}_{\lambda s} = g^{js}\Gamma^{ik}_{\lambda s},$$

as required.

Corollary 1.15 *For the intersection form g of an arbitrary Frobenius manifold $\mathcal{M}$, there exists a set of distinguished (to within a linear transformation) set of coordinates $\{p^\alpha\}$ such that the matrix $g^{\alpha\beta}(p)$ is constant, the so called* flat coordinates of the intersection form.

Proof: The above lemma implies that g is flat. Therefore the existence of coordinates in which g is constant follows automatically.

Note that the intersection form expressed in its own flat coordinates will be denoted by $G_{\alpha\beta}$.

1.4 Semisimple Frobenius manifolds and canonical coordinates

Definition 1.16 *An arbitrary point t on a Frobenius manifold $\mathcal{M}$ is said to be a* semisimple point *if the Frobenius algebra defined on $T_t\mathcal{M}$ is semisimple, i.e. contains no nilpotents. A Frobenius manifold on which a generic point t is semisimple is known as a* semisimple *(or* massive*)* Frobenius manifold.

Lemma 1.17 *In a neighbourhood of a semisimple point of an n-dimensional Frobenius manifold, there exists a local set of coordinates $\{u^i\}$ such that:*

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}.$$

Proof: As the algebra on $T_u\mathcal{M}$ is semisimple, there exists a set of n vectors which are idempotents of this algebra. Denote these by $\{\partial_i\}$, $i = 1, \ldots, \dim(n)$. By definition, one has

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i.$$

In order for a coordinate system $\{u^i\}$ such that $\partial_i = \frac{\partial}{\partial u^i}$ to exist, one requires that these vectors commute, i.e.

$$[\partial_i, \partial_j] = 0.$$

But this may be shown to be true using the curvature properties of the so called *deformed flat connection*, $\tilde{\nabla}$. The reader is referred to [10] for full details of this.

Definition 1.18 *The coordinates $\{u^i\}$ as defined above are known as* canonical coordinates *for the Frobenius manifold. Note that in general, Roman indices will be used to denote the use of canonical coordinates, whilst Greek indices will denote flat coordinates.*

Lemma 1.19 *In canonical coordinates, the invariant inner product* $<,>$ *takes the form*

$$\eta_{ij} = \eta_{ii}\delta_{ij},$$

the identity field in takes the form

$$e = \sum_i \partial_i,$$

and the Euler field (with a suitable choice of scaling of the coordinates) takes the form

$$E = \sum_i u^i \partial_i.$$

Proof: From the definition of multiplication in canonical coordinates and the Frobenius property of the invariant inner product:

$$\begin{aligned} \eta_{ij} &= <\partial_i, \partial_j>, \\ &= <\partial_i \cdot \partial_i, \partial_j>, \\ &= <\partial_i, \partial_i \cdot \partial_j>, \\ &= <\partial_i, \partial_i \delta_{ij}>, \\ &= \eta_{ii}\delta_{ij}. \end{aligned}$$

The second statement is obvious, as applying e to an arbitrary basis vector ∂_i yields

$$e \cdot \partial_i = \partial_i.$$

and so $e \cdot v = v$ for any $v \in T_u\mathcal{M}$. The final statement is also obvious, as rescalings generated by E act on the idempotents by $\partial_i \to k^{-1}\partial_i$. An appropriate rescaling of u^i provides $u^i \to ku^i$, as required.

As can be seen from the lemmas above, canonical coordinates are an advantageous representation of a Frobenius manifold in that they yield a simple multiplication law, a diagonal metric and identity and Euler fields of a simple form. Furthermore, the intersection form takes a simple form. By considering

$$du^i \cdot du^j = \eta^{ii} du^i \delta_{ij},$$

it immediately follows that

$$g^{ij}(u) = u^i \eta^{ii} \delta_{ij}.$$

However, there are drawbacks to canonical coordinates; whereas in flat coordinates, the invariant inner product is a constant matrix (and automatically defines a flat metric), this is not necessarily the case in canonical coordinates. Therefore it may no longer be possible to express a Frobenius manifold in terms of a prepotential function satisfying the (linear) WDVV equations, as is possible in the flat coordinates $\{t^i\}$. Instead, the structural data of a semisimple Frobenius manifold may be considered in terms of the what Manin describes as the 'Darboux-Egoroff picture' in [19].

Returning to the definition of a Frobenius manifold, one now considers the formulation of a semisimple Frobenius manifold in terms of canonical coordinates on a manifold $\mathcal{M}$. It follows immediately from the definition of multiplication in canonical coordinates that the algebras on the tangent space will be commutative and associative. Showing that the inner product obeys the Frobenius property is also simple. By taking the inner product of three idempotents, we require that:

$$< \partial_i \cdot \partial_j, \partial_k > = < \partial_i, \partial_j \cdot \partial_k >, \forall i, j, k.$$

But both sides of this equation are equal to $\delta_{ijk}\eta_{ii}$, so the metric is Frobenius. Hence in the presence of a covariantly constant identity field (and a flat Euler field), the only obstruction to $\mathcal{M}$ being a Frobenius manifold is flatness of the metric η. In order for a metric to be flat, we require that the $(2,2)$ curvature tensor R^{ij}_{kl} obeys:

$$R^{ij}_{kl} = 0,$$

for all possible combinations of i, j, k and l. We recall from standard Riemannian geometry that for an arbitrary metric g,

$$\begin{aligned} R^{ij}_{kl} &:= g^{is}\left(\partial_k \Gamma^j_{sl} - \partial_l \Gamma^j_{sk} + \Gamma^j_{pk}\Gamma^p_{sl} - \Gamma^j_{pl}\Gamma^p_{sk}\right), \\ \Gamma^k_{ij} &:= \frac{1}{2} g^{kp}\left(\partial_j g_{pi} + \partial_i g_{pj} - \partial_p g_{ij}\right). \end{aligned}$$

For the diagonal metric η_{ij}, the inverse η^{ij} is also diagonal. Hence the above equations simplify to:

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}\eta^{kk}\left(\partial_j \eta_{ki} + \partial_i \eta_{kj} - \partial_k \eta_{ij}\right), \\ R^{ij}_{kl} &= \eta^{ii}\left(\partial_k \Gamma^j_{il} - \partial_l \Gamma^j_{ik} + \Gamma^j_{pk}\Gamma^p_{sl} - \Gamma^j_{pl}\Gamma^p_{sk}\right). \end{aligned}$$

For notational convenience in subsequent calculations, one will rewrite η in terms of squared components, i.e.

$$\eta_{ij} = \delta_{ij}(H_i)^2.$$

It is also convenient for curvature calculations to introduce the *rotation coefficients* of η, defined by

$$\gamma_{ij} := \frac{\partial_j H_i}{H_j}.$$

Note that in general $\gamma_{ij} \neq \gamma_{ji}$. One now proceeds by calculating the Christoffel symbols Γ^k_{ij}. These calculations are split into four parts for varying combinations of distinct and identical indices.

Case I: i,j,k distinct

As the metric is diagonal, all terms vanish so

$$\Gamma^k_{ij} = 0.$$

Case II: $i = j \neq k$

We have

$$\Gamma^k_{ii} = \frac{1}{2}\eta^{kk}(\partial_i \eta_{ki} + \partial_i \eta_{ki} - \partial_k \eta_{ii}).$$

But as η is diagonal, the first two terms in the bracket are zero, so:

$$\Gamma^k_{ii} = -\frac{1}{2}\eta^{kk}\partial_k \eta_{ii},$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{1}{H_k^2}\partial_k(H_i^2),\\
&= -\frac{H_i}{H_k}\frac{\partial_k H_i}{H_k},\\
&= -\frac{H_i}{H_k}\gamma_{ki}.
\end{aligned}
$$

Case III: $i = k \neq j$

Note that as Γ^k_{ij} is symmetric in its lower indices, this is the same as $j = k \neq i$. By similar calculations to those above, we obtain

$$\Gamma^i_{ij} = \frac{H_j}{H_i}\gamma_{ji}.$$

Case IV: $i = j = k$

Again, using similar methods to those above, one obtains the result

$$\Gamma^i_{ii} = \gamma_{ii}.$$

One may then use the Christoffel symbols to calculate the curvature tensor. Again, these calculations split into several cases corresponding to the possible combinations of distinct and identical indices:

Case I: i, j, k, l all distinct:

$$
\begin{aligned}
R^{ij}_{kl} &= \eta^{ii}\left(\partial_k\Gamma^j_{il} - \partial_l\Gamma^j_{ik} + \Gamma^j_{pk}\Gamma^p_{il} - \Gamma^j_{pl}\Gamma^p_{ik}\right),\\
&= 0,
\end{aligned}
$$

as every term in the sum contains a zero value of Γ.

Case II: $i = j$

It is easy to show that in this case,

$$R^{ii}_{kl} = 0,$$

irrespective of the values of k and l (including the cases where they are equal to each other or i).

Case III: $k = l, i \neq j$

Again, it is easy to show that

$$R^{ij}_{kk} = 0.$$

Case IV: $i = k, i \neq j, i \neq l, j \neq l$

We have

$$R^{ij}_{il} = \underbrace{\frac{1}{H_i^2}\left(\partial_i \Gamma^j_{il} - \partial_l \Gamma^j_{ii}\right)}_{A} + \underbrace{\frac{1}{H_i^2}\left(\Gamma^j_{pi}\Gamma^p_{il} - \Gamma^j_{pl}\Gamma^p_{ii}\right)}_{B}.$$

The two parts of this may then be calculated separately.

$$\begin{aligned}
A &= \frac{1}{H_i^2}\left(\partial_l \frac{H_i}{H_j}\gamma_{ji}\right), \\
&= \frac{1}{H_i^2}\left(\partial_l\left(\frac{H_i}{H_j}\right)\gamma_{ji} + \frac{H_i}{H_j}\partial_l\gamma_{ji}\right), \\
&= \frac{1}{H_i^2}\left(\frac{H_j\partial_l H_i - H_i\partial_l H_j}{H_j^2}\gamma_{ji} + \frac{H_i}{H_j}\partial_l\gamma_{ji}\right), \\
&= \frac{H_l}{H_i^2 H_j}\gamma_{li}\gamma_{ji} - \frac{H_l}{H_i H_j^2}\gamma_{lj}\gamma_{ji} + \frac{1}{H_i H_j}\partial_l\gamma_{ji}.
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{H_i^2}\left(\Gamma^j_{pi}\Gamma^p_{il} - \Gamma^j_{pl}\Gamma^p_{ii}\right), \\
&= \frac{1}{H_i^2}\left(\Gamma^j_{ii}\Gamma^i_{il} - \Gamma^j_{jl}\Gamma^j_{ii} - \Gamma^j_{ll}\Gamma^l_{ii}\right), \\
&= \frac{1}{H_i^2}\left(\left(-\frac{H_i}{H_j}\gamma_{ji}\right)\left(\frac{H_l}{H_i}\gamma_{li}\right) - \left(\frac{H_l}{H_j}\gamma_{lj}\right)\left(-\frac{H_i}{H_j}\gamma_{ji}\right) - \left(-\frac{H_l}{H_j}\gamma_{jl}\right)\left(-\frac{H_i}{H_l}\gamma_{li}\right)\right), \\
&= -\frac{H_l}{H_i^2 H_j}\gamma_{ji}\gamma_{li} + \frac{H_l}{H_i H_j^2}\gamma_{ij}\gamma_{jl} - \frac{1}{H_i H_j}\gamma_{jl}\gamma_{li}.
\end{aligned}$$

Adding A and B then yields

$$R^{ij}_{il} = \frac{1}{H_i H_j}\left(\frac{H_l}{H_j}\left(\gamma_{ij}\gamma_{jl} - \gamma_{ji}\gamma_{lj}\right) + \partial_l\gamma_{ji} - \gamma_{jl}\gamma_{li}\right). \tag{1.5}$$

Case V: $i = k \neq j, j = l$

Similar calculations to those in case IV show:

$$R^{ij}_{ij} = \frac{1}{H_i H_j}\left(\partial_i\gamma_{ij} + \partial_j\gamma_{ji} + \sum_{p\neq i,j}\gamma_{pj}\gamma_{pi}\right). \tag{1.6}$$

In order for an arbitrary diagonal metric to be flat, one therefore requires that (1.5) and (1.6) both be equal to zero. In the case of a Frobenius manifold, it transpires that these equations simplify.

Definition 1.20 *A diagonal metric*

$$\eta = \sum_i g_{ii}(du^i)^2$$

is said to be Egoroff *if there exists a metric potential* Φ *such that*

$$g_{ii} = \frac{\partial \Phi}{\partial u^i}.$$

Lemma 1.21 *The rotations coefficients* γ_{ij} *for an Egoroff metric are symmetric in* i *and* j*, i.e.*

$$\gamma_{ij} = \gamma_{ji}.$$

Conversely, any metric whose rotation coefficients are symmetric will be Egoroff.

Proof: For an Egoroff metric, we have

$$\begin{aligned} \gamma_{ij} &= \frac{\partial_j \sqrt{\partial_i \Phi}}{\sqrt{\partial_j \Phi}}, \\ &= \frac{1}{2} \frac{\partial_i \partial_j \Phi}{\sqrt{\partial_i \Phi}\sqrt{\partial_j \Phi}}, \\ &= \frac{1}{2} \frac{\partial_j \partial_i \Phi}{\sqrt{\partial_j \Phi}\sqrt{\partial_i \Phi}}, \\ &= \frac{\partial_i \sqrt{\partial_j \Phi}}{\sqrt{\partial_i \Phi}}, \\ &= \gamma_{ji}. \end{aligned}$$

To prove the converse, one equates γ_{ij} and γ_{ji} to obtain

$$\frac{\partial_j \sqrt{g_{ii}}}{\sqrt{g_{jj}}} = \frac{\partial_i \sqrt{g_{jj}}}{\sqrt{g_{ii}}}.$$

Applying the chain rule to expand $\partial_j \sqrt{g_{ii}}$ and rearranging terms then yields the potentiality condition

$$\partial_j g_{ii} = \partial_i g_{jj},$$

as required.

Lemma 1.22 *The metric on a semisimple Frobenius manifold expressed in terms of canonical coordinates is Egoroff.*

Proof: For a Frobenius manifold, one has $\nabla e = 0$. If $e = \sum e^i \partial_i$, then one may split $\nabla e = 0$ into component equations:

$$\nabla_i e^j = \partial_i e^j + \sum_k \Gamma^j_{ik} e^k,$$

all of which must be equal to zero. Noting that in canonical coordinates, $e^i = 1$ for all i, this simplifies to leave

$$\sum_k \Gamma^j_{ik} = 0.$$

Splitting this sum and using earlier calculations of the Christoffel symbols for a diagonal metric gives the following:

$$\begin{aligned} \sum_k \Gamma^j_{ik} &= \sum_{k \neq i,j} \Gamma^j_{ik} + \Gamma^j_{ii} + \Gamma^j_{jk}, \\ &= \frac{H_j}{H_i}\gamma_{ji} - \frac{H_j}{H_i}\gamma_{ij}. \end{aligned}$$

But as this must be equal to zero, so $\gamma_{ij} = \gamma_{ji}$. Hence the metric is Egoroff by lemma 1.21.

Theorem 1.23 *A diagonal metric on a semisimple Frobenius manifold (expressed in its canonical coordinates) is flat if and only if*

$$\partial_l \gamma_{ij} - \gamma_{il}\gamma_{jl} = 0, \tag{1.7}$$

$$e(\gamma_{ij}) = 0. \tag{1.8}$$

Proof: For the metric to be flat, we require that (1.5) and (1.6) both be equal to zero. The equation (1.7) follows immediately from applying the condition

$\gamma_{ij} = \gamma_{ji}$ to (1.5). Substituting the symmetry of γ_{ij} along with (1.7) into the requirement that (1.6) be equal to zero gives

$$\begin{aligned} 0 &= \partial_i \gamma_{ij} + \partial_j \gamma_{ij} + \sum_{p \neq i,j} \partial_p \gamma_{ij}, \\ &= \sum_p \partial_p \gamma_{ij}, \\ &= e(\gamma_{ij}). \end{aligned}$$

Hence two equations equivalent to the flatness of the metric have been found.

Definition 1.24 *The equations (1.7) and (1.8) above are known as the* Darboux-Egoroff equations.

In addition to the Darboux-Egoroff equations, there is a third equation relating the Euler field of a Frobenius manifold to the rotation coefficients of the metric η.

Lemma 1.25 *For the rotation coefficients γ_{ij} of the diagonal metric η_{ij}, one has:*

$$E(\gamma_{ij}) = -\gamma_{ij}.$$

Proof: In canonical coordinates, the intersection form takes the form

$$g_{ij} = \delta_{ij} u^i \eta_{ii}.$$

If one denotes $u^i \eta_{ii}$ by B_i^2 (so $B_i = (u^i)^{\frac{1}{2}} H_i$), then the rotation coefficients of the intersection form are given by

$$\beta_{ij} = \frac{\partial_j B_i}{B_j} = \left(\frac{u^i}{u^j}\right)^{\frac{1}{2}} \gamma_{ij}. \tag{1.9}$$

Noting that the intersection form is flat, one has $R^{ij}_{kl} = 0$ for all i, j, k, l (where R^{ij}_{kl} now denotes the curvature of g). In particular, one therefore has

$$R^{ij}_{ij} = \frac{1}{B_i B_j} \left(\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{p \neq i,j} \beta_{pi} \beta_{pj} \right) = 0.$$

By substituting in (1.9) and Darboux-Egoroff equations, one obtains

$$\frac{(u^i u^j)^{\frac{1}{2}}}{H_i H_j}\left(\gamma_{ij} + \sum_p u^p \partial_p(\gamma_{ij})\right) = 0.$$

Noting that $E = \sum_p u^p \partial_p$, this implies the required result.

A final property of semisimple Frobenius manifolds is the existence of a Landau-Ginzburg (LG) superpotential. Such a function allows the invariant metric, the intersection form and the trilinear tensor $c(\partial', \partial'', \partial''') =< \partial' \cdot \partial'', \partial''' >$ to be expressed by various residue formulae.

Theorem 1.26 *For a semisimple Frobenius manifold, one is able to construct a function $\lambda = \lambda(z;t)$ such that the following formulae hold:*

$$
\begin{aligned}
< \partial', \partial'' > &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'\lambda \partial''\lambda}{d\lambda} dz, && (1.10)\\
(\partial', \partial'') &= -\sum \operatorname*{res}_{d\log\lambda=0} \frac{\partial'(\log\lambda)\partial''(\log\lambda)}{d(\log\lambda)} dz, && (1.11)\\
c(\partial', \partial'', \partial''') &= -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'\lambda \partial''\lambda \partial'''\lambda}{d\lambda} dz. && (1.12)
\end{aligned}
$$

Proof: One must choose a function $\lambda = \lambda(z;t)$ such that its critical values coincide with the canonical coordinates, i.e.

$$
\begin{aligned}
\lambda(q^i;t) &= u^i,\\
\left.\frac{d\lambda}{dz}\right|_{z=q^i} &= 0.
\end{aligned}
$$

Near any critical point q^i, λ must have an expansion of the form

$$\lambda = u^i - \frac{(z-q_i)^2}{2\eta_{ii}} + O(z-q_i)^3.$$

Hence, near these points, the inverse $z = z(\lambda;t)$ will be of the form

$$z = q_i + 2\sqrt{\eta_{ii}}\sqrt{u_i - \lambda} + O(u_i - \lambda).$$

The existence of such a function will be assumed here; the reader is referred to [10] proof of existence and an explicit construction.

With such a function in place, one may now verify the three formulae above. It is convenient to work in canonical coordinates for these calculations. Firstly, consider

$$-\sum \operatorname*{res}_{d\lambda=0} \frac{\partial_i\lambda\partial_j\lambda}{d\lambda}dz$$

(recalling that $\partial_i = \frac{\partial}{\partial u^i}$ etc). The points at which $d\lambda = 0$ are precisely the $\{q_i\}$. Hence we have

$$< \partial_i, \partial_j >= -\sum_{l=1}^{n} \operatorname*{res}_{z=q_l} \frac{\partial_i\lambda\partial_j\lambda}{d\lambda}dz.$$

Consider the right hand side of this. Near any point $z = q_l$, we have

$$\partial_i\lambda|_{z=q_l} = \delta_{il}.$$

Hence the residues are zero except for when $i = j = l$. At these points, we have

$$\begin{aligned} < \partial_i, \partial_i > &= -\operatorname*{res}_{z=q_i} \frac{1}{\lambda'}, \\ &= \operatorname*{res}_{z=q_i} \frac{1}{\frac{2(z-q_i)}{2\eta_{ii}} + O(z-q_i)^2}, \\ &= \operatorname*{res}_{z=q_i} \frac{1}{z-q_i} \frac{1}{\frac{1}{\eta_{ii}} + O(z-q_i)}, \\ &= \left. \frac{1}{\frac{1}{\eta_{ii}} + O(z-q_i)} \right|_{z=q_i}, \\ &= \eta_{ii}. \end{aligned}$$

Therefore

$$< \partial_i, \partial_j >= \delta_{ij}\eta_{ii},$$

as required. Similar calculations (remembering that $\partial_i \log\lambda = \frac{\partial_i\lambda}{\lambda}$) yield

$$\begin{aligned} (\partial_i, \partial_j) &= \delta_{ij}\frac{\eta_{ii}}{u^i}, \\ (\partial_i, \partial_j, \partial_k) &= \delta_{ij}\delta_{ik}\eta_{ii}. \end{aligned}$$

Definition 1.27 *The function λ, as defined above, is known as the* Landau-Ginzburg (or LG) superpotential *for a Frobenius manifold.*

1.5 Polynomial Frobenius manifolds

Let n be a strictly positive integer. Polynomials of degree $n+1$ may be considered as maps from the Riemann sphere to itself (with an $n+2$ branch point at infinity). If an arbitrary polynomial is of the form

$$p(z) = a_{-1}z^{n+1} + a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

then one may use the freedom $z \to bz + c$ (noting that there will still be an $n+2$ branch point at infinity) to set $a_{-1} = 1$ and $a_0 = 0$. The space of such functions may then be parameterised by the coordinates $a_1, \ldots, a_n$.

Theorem 1.28 *Denote by A_n the affine space of polynomials of the form*

$$p(z) = z^{n+1} + a_1 z^{n-1} + \ldots + a_n.$$

If consideration is restricted to the subset of A_n where

$$p'(z) = \frac{dp}{dz} = (n+1)\prod_{i=1}^{n}(z - \alpha_i)$$

has n distinct (simple) roots, then the structure of a semisimple Frobenius manifold exists with the following structural data:

- *Canonical coordinates $\{u^i\}$ will be defined by*

$$u^i = p(\alpha_i).$$

- *A flat diagonal metric is defined by the residue formula*

$$\eta_{ij} = \operatorname*{res}_{z=\infty} \frac{\frac{\partial p}{\partial u^i}\frac{\partial p}{\partial u^j}}{p'(z)} dz.$$

This metric will be Egoroff, with potential

$$\Phi = \frac{-a_1}{n+1}.$$

- *The identity field will be:*

$$e = \frac{\partial}{\partial a_n}.$$

- *The Euler vector field will be:*

$$\frac{1}{n+1} \sum_i \frac{i+1}{n+1} a_i \frac{\partial}{\partial_{a_i}}.$$

Proof: Firstly, one may use the linear independence of the u^i to calculate $\frac{\partial p}{\partial u^i}$:

$$\begin{aligned} \delta_{ij} &= \frac{\partial u^i}{\partial u^j}, \\ &= \frac{\partial (p(\alpha_i))}{\partial u^j}, \\ &= \frac{\partial p}{\partial u^j}(\alpha_i) + p'(\alpha_i)\frac{\partial \alpha_i}{\partial u^j}, \\ &= \frac{\partial p}{\partial u^j} \end{aligned}$$

Hence $\frac{\partial p}{\partial u^j}$ has $(n-1)$ roots at $z = \alpha_i, i \neq j$ and is equal to 1 at $z = \alpha_i$. Therefore

$$\frac{\partial p}{\partial u^i} = \prod_{i \neq j} \frac{(z - \alpha_j)}{(\alpha_i - \alpha_j)}.$$

Consideration is now turned to the formula for the metric, noting that the residue at infinity of a meromorphic function is equal to the negative of the sum of the residues at all other singularities. As the only places where this can occur are the zeros of $p'(z)$, we therefore have

$$\eta_{ij} = -\sum_k \operatorname*{res}_{z=\alpha_k} \frac{\prod_{l \neq i} \frac{z-\alpha_l}{\alpha_i - \alpha_l} \prod_{m \neq j} \frac{z - \alpha_m}{\alpha_j - \alpha_m}}{(n+1)\prod_q (z - \alpha_q)} dz.$$

In the case that $i \neq j$, then every one of the factors $(z - \alpha_q)$ in the denominator appears at least once in the numerator, hence the residue is of something finite and so is equal to zero. In the case where $i = j$, we have

$$\eta_{ii} = -\operatorname*{res}_{z=\alpha_i} \frac{\prod_{k \neq i} \frac{(z-\alpha_k)^2}{(\alpha_i - \alpha_k)^2}}{(n+1)\prod_l (z - \alpha_l)} dz,$$

$$\begin{aligned} &= -\operatorname*{res}_{z=\alpha_i} \frac{\prod_{k\neq i} \frac{(z-\alpha_k)}{(\alpha_i-\alpha_k)^2}}{(n+1)(z-\alpha_i)} dz, \\ &= -\frac{1}{n+1} \prod_{k\neq i} \frac{(z-\alpha_k)}{(\alpha_i-\alpha_k)^2}\bigg|_{z=\alpha_i}, \\ &= -\frac{1}{n+1} \prod_{k\neq i} \frac{1}{\alpha_i-\alpha_k}. \end{aligned}$$

But noting that

$$p''(z) = (n+1) \sum_j \prod_{k\neq j} (z-\alpha_k),$$

we have

$$\eta_{ii} = -\frac{1}{p''(\alpha_i)}.$$

To show that this metric is Egoroff, firstly recall that

$$\delta_{ij} = \frac{\partial p}{\partial u^j}(\alpha_i).$$

This can be considered as a polynomial in z, so:

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial a_k}{\partial u^j} z^{n-k}\bigg|_{z=\alpha_i}.$$

For a fixed value of j, this is a polynomial of degree $(n-1)$ with zeros at all of the α_i for $i \neq j$. Therefore

$$\sum_{k=1}^{n} \frac{\partial a_k}{\partial u^j} z^{n-k} = c \prod_{i\neq j} (z-\alpha_i).$$

As c is the coefficient of z^{n-1} in the right hand side, equating coefficients yields

$$c = \frac{\partial a_1}{\partial u^j}.$$

But from the fact that

$$\sum_{k=1}^{n} \frac{\partial a_k}{\partial u^j} z^{n-k}\bigg|_{z=\alpha_j} = 1,$$

we can deduce that

$$\begin{aligned} c &= \frac{1}{\prod_{i\neq j}(\alpha_j-\alpha_i)}, \\ &= -(n+1)\eta_{jj}. \end{aligned}$$

Therefore

$$\frac{\partial}{\partial u^j}\frac{-a_1}{n+1}=\eta_{jj},$$

i.e. the metric is Egoroff with potential $-\frac{a_1}{n+1}$.

Similar calculations show the desired results for the identity and Euler fields, as can be seen in [19].

Finally, one must show flatness of the metric η_{ij}. This could be done by using the Darboux-Egoroff equations introduced above. However, one will instead construct flat coordinates $\{t^\alpha\}$ in which the metric has constant coefficients. To do this, a new function is introduced:

$$k(z):=\lambda^{\frac{1}{n+1}}.$$

Near $z=\infty$, this has the Puiseaux inverse series

$$z=k+\frac{t_1}{k}+\ldots+\frac{t_n}{k^n}+O\left(\frac{1}{k^{n+1}}\right).$$

One may then take t_i, $i=1,\ldots,n$ as flat coordinates. To prove that these are flat coordinates, one uses the tangent vectors $\frac{\partial}{\partial t^i}$ in the LG-superpotential formula (1.10) for the metric. The result is the antidiagonal metric

$$\eta_{\alpha\beta}=\delta_{\alpha(n+1-\alpha)}.$$

Details of this calculation are omitted; see [10] for explicit proof.

1.6 Frobenius structure on orbits of Coxeter groups

Definition 1.29 *A* Coxeter group *is a finite group of linear transformations generated by reflections on a vector space* $V \cong \mathbb{R}^n$.

There are many works outlining the full classification of Coxeter groups, e.g. [6]. The action of a Coxeter group can be considered to act on $S(V)$, the group of polynomials of coordinates of V. One may also consider a sub-ring $R = S(V)^W$ of polynomials which are W-invariant. R will be generated by n linearly independent polynomials $y^1, \ldots, y^n$, whose degrees are $d_i, \ldots, d_n$ respectively. The degrees d_i will be fixed by the choice of the group W, and will satisfy the inequality

$$h = d_1 > d_2 > \ldots > d_{n-1} > d_n = 2.$$

They also satisfy the duality condition

$$d_i + d_{n-i+1} = h + 2.$$

Definition 1.30 *The maximal degree of an invariant polynomial of a Coxeter group, denoted above by h, is known as the* Coxeter number*of the group.*

The table below lists all Coxeter groups and the degrees d_i of their invariant polynomials.

Coxeter Group	$d_n, \ldots, d_1$
A_n	2,3,...,n+1
B_n	2,4,6,...,2n
D_{2k}	$2, 4, \ldots, 2k-2, 2k, 2k, 2k+2 \ldots, 4k-2$
D_{2k+1}	$2, 4, \ldots, 2k, 2k+1, 2k+2, 2k+4, \ldots, 4k$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
$I_2(k)$	$2, k$

Example 1.31 *$I_2(k)$ is the symmetry group for the regular k-gon in the plane $\mathbb{R}^2$. This group can be generated by a reflection and a rotation through $\frac{2\pi}{n}$, which on $\mathbb{C} = \mathbb{R}^2$ are defined by the transformations $z \to \overline{z}$ and $z \to e^{\frac{2\pi}{k}} z$ respectively. Under such transformations, it is easy to see that the invariant polynomials are*

$$\begin{aligned} y^1 &= z^k + \overline{z}^k, \\ y^2 &= z\overline{z}. \end{aligned}$$

The action of a Coxeter group W may be extended to a complexified space $M = V \otimes \mathbb{C}/W$. On this space, the invariant polynomials act as coordinates, and are defined up to an invertible transformation

$$y^i \to \tilde{y}^i(y^1, \ldots, y^n),$$

where $\tilde{y}^i$ is a graded homogenous polynomial of degree d_i in the variables $\{y^i\}$. The vector field $\partial_1 := \frac{\partial}{\partial y^1}$ (to within multiplication by a scalar) will be invariant under such transformations, due to the inequality $d_1 > d_2$. Also, the invariant quadratic y^n may be chosen to be

$$y^n = \frac{1}{2h}((x^1)^2 + \ldots + (x^n)^2),$$

where $\{x^i\}$ are orthonormal coordinates with respect to the W-invariant Euclidean metric $(,)$ on V.

Definition 1.32 *A* regular orbit *is an open subset of $\mathcal{M}$, and is the image of precisely h distinct elements $V \otimes \mathbb{C}$ under the quotient map $Q : V \otimes \mathbb{C} \to V \otimes \mathbb{C}/W = \mathcal{M}$.*

Definition 1.33 *The complement of the space of regular orbits is known as the* discriminant locus *of W, or as the* space of irregular orbits.

The Euclidean metric $(,)$ induces a contravariant metric on the space of orbits, defined by

$$g^{ij}(y) = (dy^i, dy^j)^* := \sum_a \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^a}.$$

The components of the Levi-Civita connection are given by

$$\Gamma^{ij}_k(y)dy^k = \frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial x^a \partial x^b}dx^b.$$

Lemma 1.34 *Both the metric g^{ij}, above, and its Christoffel symbols are polynomials in x, and their degrees are given by*

$$\begin{aligned} \deg g^{ij}(y) &= d_i + d_j - 2, \\ \deg \Gamma^{ij}_k(y) &= d_i + d_j - d_k - 2. \end{aligned}$$

Proof: As y^i is a homogenous polynomial in x^a and $\deg y^i = d_i$, it immediately follows that differentiating once with respect to x^a yields a polynomial of degree

$d_i - 1$. Therefore g^{ij}, which is the sum over a of the product of two polynomials of degrees $d_i - 1$ and $d_j - 1$ must be a polynomial of degree

$$(d_i - 1) + (d_j - 1) = d_i + d_j - 2.$$

Similar reasoning shows that

$$\deg \Gamma^{ij}_k(y) dy^k = d_i + d_j - 3.$$

Noting that $\deg y^k = d_k$, one obtains

$$\deg \Gamma^{ij}_k(y) = d_i + d_j - d_k - 2.$$

Corollary 1.35 *The functions $g^{ij}(y)$ and $\Gamma^{ij}_k(y)$ depend at most linearly on y^1.*

Proof: As

$$d_i \leq h,$$

with h being the degree of y^1, so

$$\deg g^{ij} < 2h.$$

Therefore y^1 can only appear in a linear way in $g^{ij}(y)$. Likewise for $\Gamma^{ij}_k(y)$.

Definition 1.36 *The function*

$$D(y) := \det(g^{ij}(y))$$

is known as the discriminant function *of W.*

This term is used because the discriminant locus of W coincides with the region where $D(y) = 0$, i.e. where x^i fail to be local coordinates for V.

Definition 1.37 *A new contravariant metric, known as the* Saito metric, *may be defined by the formula:*

$$\eta^{ij}(y) := \partial_1 g^{ij}(y).$$

Note that the definition above is strictly only of a $(0,2)$ tensor. In order to show that it is a metric, one must also show that it is non-degenerate (see corollary 1.39 below) and symmetric (which follows immediately from the symmetry of g^{ij}).

Lemma 1.38 *The Saito metric will be of upper triangular form*

$$\eta^{ij} = 0, \quad i + j > n + 1,$$

with nonzero antidiagonal elements

$$c_i := \eta^{i(n+1-i)}.$$

Proof: From the definition of η, one has

$$\deg \eta^{ij} = d_i + d_j - h - 2.$$

From the duality condition

$$d_i + d_{(n+1-i)} = h + 2,$$

and the fact that the d_i decrease with i, one has

$$\begin{aligned} \deg(\eta^{ij}) &> 0, \quad i + j < n + 1, \\ &= 0, \quad i + j = n + 1, \\ &< 0, \quad i + j > n + 1. \end{aligned}$$

As a polynomial of negative degree does not exist, one concludes that $\eta^{ij}(y)$ must be zero for $i + j > n + 1$. Hence the triangular form is proved. To show that

$c_i \neq 0$, consider $g^{i(n+1-i)}$. But $\deg g^{i(n+1-i)} = h$, so one may express $g^{i(n+1-i)}$ as a series

$$\sum_{k=1}^{n} a_k y^i,$$

with a_1 being a nonzero constant. Hence

$$\begin{aligned} \eta^{i(n+1-i)} &= = \partial_1 \sum_{k=1}^{n} a_k y^i, \\ &= C_1. \end{aligned}$$

Corollary 1.39 *The function*

$$c := \det(\eta^{ij})$$

is a nonzero constant.

Proof: As η^{ij} is triangular, it follows immediately from the definition of a determinant that

$$c = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} c_i.$$

But from the above lemma, the c_i are all non zero constants, so c is itself a nonzero constant.

Lemma 1.40 *The metric η^{ij} is flat. Moreover, g^{ij} and η^{ij} form a flat pencil.*

Proof: Proof is omitted. The reader is referred to [23], for a full proof.

For the contravariant metric $<,>^*$ defined by η^{ij}, the Christoffel symbols will be

$$\gamma_k^{ij}(y) := \partial_1 \Gamma_k^{ij}(y).$$

Lemma 1.41 *The degree of γ_k^{ij} is given by:*

$$\deg \gamma_k^{ij} = d_i + d_j - d_k - h - 2.$$

Proof: We have

$$\begin{aligned} \deg \gamma_{ij}^k &= \deg \Gamma_k^{ij} - \deg y^1, \\ &= (d_i + d_j - d_k - 2) - h. \end{aligned}$$

Theorem 1.42 *There exists a set of homogenous polynomials $t^1(x), \ldots, t^n(x)$, of degrees $d_1, \ldots, d_n$, such that $\eta^{ij}(t)$ is a constant matrix. They may be chosen in such a way that η takes the antidiagonal form*

$$\eta^{ij} = \delta^{i(n+1-i)}.$$

Proof: From the flatness of η^{ij}, the existence of a set of flat coordinates in which η is constant is assured. These flat coordinates are the solutions to the system:

$$\eta^{is}\partial_s\partial_j t + \gamma_j^{is}\partial_s t = 0.$$

Substituting $\xi_l = \partial_l t$, this becomes

$$\eta^{is}\partial_s\xi_j + \gamma_j^{is}\xi_s = 0.$$

Noting that $(\eta_{ij}(y)) := (\eta^{ij}(y))^{-1}$ exists (and is in fact a polynomial in $\{y^i\}$), the above system may be rewritten as

$$\partial_k\xi_l + \eta_{il}\gamma_k^{is}\xi_s = 0.$$

This overdetermined holonomic system has an n-dimensional space of solutions. If one defines

$$\xi_l^\alpha := \partial_l t^\alpha,$$

then by setting

$$\xi_l^\alpha(0) = \delta_l^\alpha,$$
$$t^\alpha(0) = 0,$$

we have functions which are analytic for small y and the space of solutions is invariant under

$$y^i \to c^{d_i} y^i.$$

Hence $t^\alpha(y)$ are quasihomogenous in y with degrees $d_1, \ldots, d_n$. Therefore they are polynomials in $\{y^i\}$ as the degrees are all positive integers. Therefore the $t^\alpha = t^\alpha(y(x))$ are polynomials of a polynomial function of $\{x^i\}$, so $\{t^\alpha\}$ are polynomial functions of $\{x^i\}$.

Definition 1.43 *The flat coordinates $\{t^\alpha\}$ from the above theorem are known as the* Saito flat coordinates.

One is now able to state the following theorem, first proved by Dubrovin in [12].

Theorem 1.44 *Let $t^1, \ldots, t^n$ be Saito flat coordinates on the space of orbits of a finite Coxeter group and $\eta^{\alpha\beta}$ the Saito metric. Then there exists a quasihomogenous polynomial $F(t)$ such that*

$$(dt^\alpha, dt^\beta)^* = \frac{d_\alpha + d_\beta - 2}{h} \eta^{\alpha\lambda} \eta^{\beta\mu} \partial_\lambda \partial_\mu F(t). \tag{1.13}$$

The function F(t) will be the prepotential for a Frobenius manifold and is uniquely determined (to within an equivalence) by the choice of the Coxeter group W.

Proof: The proof below is an outline of the proof given in [10]. For a rigorous proof, the reader is referred to [10] or [12].

Begin by noting that as η^{ij} and g^{ij} form a flat pencil, with η^{ij} being constant in the coordinates of theorem 1.42, one has

$$\Delta^{\alpha\beta}_{\gamma} = \Gamma^{\alpha\beta}_{\gamma},$$

where $\Delta^{\alpha\beta}_{\gamma}$ is the *difference tensor* introduced in [10], appendix D. Using the equation D.1a from [10], one may express the Christoffel symbols of $g^{\alpha\beta}$ by

$$\Gamma^{\alpha\beta}_{\gamma} = \eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t),$$

for some vector field f^{β}. This is then used in conjunction with the homogeneity of the invariant polynomials to obtain (via an application of the Euler identity) the equation

$$(d_{\gamma} - 1)g^{\beta\gamma} = (d_{\gamma} + d_{\beta} - 2)\eta^{\beta\epsilon}\partial_{\epsilon}f^{\gamma}. \tag{1.14}$$

Using this equation along with the symmetry of $g^{\alpha\beta}$ and $\eta^{\alpha\beta}$, one obtains a new symmetry:

$$\frac{\eta^{\beta\epsilon}\partial_{\epsilon}f^{\gamma}}{d_{\gamma} - 1} = \frac{\eta^{\gamma\epsilon}\partial_{\epsilon}f^{\beta}}{d_{\beta} - 1}.$$

Defining a new field

$$F^{\gamma} := \frac{hf^{\gamma}}{d_{\gamma} - 1},$$

one obtains the symmetry

$$\eta^{\beta\epsilon}\partial_{\epsilon}F^{\gamma} = \eta^{\gamma\epsilon}\partial_{\epsilon}F^{\beta}.$$

But this is integrable; there exists a function $F(t)$ such that

$$F^{\alpha} = \eta^{\alpha\epsilon}\partial_{\epsilon}F(t).$$

It follows clearly (from the homogeneity of the invariant polynomials) that F itself will be quasihomogenous (with degree $2h + 2$). Also, (1.14) implies that F will in fact satisfy (1.13). Having established the existence of F, one must now prove that it is a solution to the WDVV associativity equations. Using the dual structure constants

$$c^{\alpha\beta}_{\gamma} := \eta^{\alpha\delta}\eta^{\beta\epsilon}\partial_{\delta}\partial_{\epsilon}\partial_{\gamma}F,$$

it is possible to show that

$$\Gamma_{\gamma}^{\alpha\beta} = \frac{d_{\beta}-1}{h} c_{\gamma}^{\alpha\beta}.$$

Recalling that in this case, $\Delta_{\gamma}^{\alpha\beta} = \Gamma_{\gamma}^{\alpha\beta}$, one may substitute this into the equation D.2 from [10], appendix D to obtain associativity.

Corollary 1.45 *The Frobenius structure on the orbit space of a finite Coxeter group will be semisimple.*

Proof: The complete proof of this is beyond the scope of this text (see [10] and [12] for full details), but the main ingredient is a polynomial in an auxiliary variable u defined by

$$P(u; y_1, \ldots, y_n) := D(y_1 - u, y_2, \ldots, y_n).$$

If one denotes the discriminant of this with respect to u by $D_0(y_1, \ldots, y_n)$ then one may show that $T_y\mathcal{M}$ has no nilpotents outside of the zeros of D_0 [12, 10]. But D_0 does not vanish identically on the space of orbits, so the algebra on $T_y\mathcal{M}$ will have no nilpotents at a generic point. Hence the desired result.

Example 1.46 *Recalling the $I_2(k)$ example from above, we have invariant polynomials*

$$\begin{aligned} y^1 &= z^k + \bar{z}^k, \\ y^2 &= \frac{1}{2k} z\bar{z}. \end{aligned}$$

In these coordinates, one may calculate g^{ij} to be

$$g^{ij} = \begin{pmatrix} (2k)^{k+1}(y^2)^{k-1} & y^1 \\ y^1 & \frac{2}{k}y^2 \end{pmatrix}.$$

Therefore the Saito metric is given by

$$\eta^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As this is already in constant antidiagonal form, y^1 and y^2 coincide with the Saito flat coordinates. Hence the set of differential equations defining $F(t)$ are

$$\begin{aligned} (2k)^{k+1}(y^2)^{k-1} &= \frac{2k-2}{k}\partial_2\partial_2 F(t), \\ y^1 &= \partial_1\partial_2 F(t), \\ y^2 &= \partial_1\partial_1 F(t). \end{aligned}$$

It is easy to see that

$$F = \frac{1}{2}(y^1)^2(y^2) + \frac{(2k)^{k+1}}{2(k^2-1)}(y^2)^{k+1}$$

is the solution to these equations. Note that this is equivalent to the first of the two dimensional solutions to WDVV listed in section 1.1.

1.7 Hurwitz Frobenius manifolds

Let $H_{g;n_0,\ldots,n_m}$ be the Hurwitz space of equivalence classes $[\lambda : \mathcal{L} \to \mathbb{P}^1]$ of N-fold branched coverings $\lambda : \mathcal{L} \to \mathbb{P}^1$, where $\mathcal{L}$ is a compact Riemann surface of genus g and the holomorphic map λ of degree N is subject to the following conditions:

- it has n (where n is the dimension of the space given by the Riemann Hurwitz formula below) simple ramification points $P_1, \ldots, P_n \in \mathcal{L}$ with distinct *finite* images $l_1, \ldots, l_n \in \mathbb{C} \subset \mathbb{P}^1$;
- the preimage $\lambda^{-1}(\infty)$ consists of $m+1$ points: $\lambda^{-1}(\infty) = \{\infty_0, \ldots, \infty_m\}$, and the ramification index of the map λ at the point ∞_j is $n_j + 1$ $(0 \leq n_j)$.

The ramification index (above) at a point is the number of sheets of the covering which are glued together at that point. A point ∞_j is a ramification point if and only if $n_j > 0$. A ramification point is simple if the corresponding ramification index equals 2.

Such a space will have a dimension n given by the Riemann-Hurwitz formula

$$n = 2g + \sum_{i=0}^{m} n_i + 2m.$$

Example 1.47 *The Hurwitz space $H_{0;1,0}$ consists of all functions of the form*

$$\lambda(z) = z^2 + a + \frac{b}{z - c}.$$

The Hurwitz space $H_{0;n_0}$ consists of all functions of the form

$$\lambda(z) = z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_n.$$

More generally, the Hurwitz space $H_{0;n_0,n_1,\ldots,n_m}$ will be the space of functions of the form

$$\lambda(z) = z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m} \sum_{s=0}^{n_r} \frac{c_{r,s}}{(z - \beta_r)^{s+1}}.$$

Finally, as a higher genus example, the Hurwitz space $H_{g;1,0}$ will consist of hyperelliptic curves of the form

$$\mu^2 = \prod_{i=1}^{2g+1} (\lambda - u^i).$$

Note that in the above example, one has assumed that there is no z^{n_0} term in the genus-zero cases. This will be the case for all Hurwitz spaces, so that one may use the critical values of λ as coordinates on the space, i.e.

$$u^i = \lambda(p_i), \ \frac{d\lambda}{dz}|_{z=p_i} = 0.$$

One will assume that the critical points are distinct, i.e. $p_i \neq p_j$ for $i \neq j$ Similarly, in higher genus cases, one will take critical values of the projection $(\lambda, \mu) \to \lambda$ as coordinates.

Theorem 1.48 *The structure of a semisimple Frobenius manifold exists on an arbitrary Hurwitz space $H_{g;n_0,n_1,\ldots,n_m}$. The coordinates $\{u^i\}$, defined above as the critical values of λ, will be canonical coordinates on the manifold. The function λ will be a Landau-Ginzburg superpotential for the Frobenius manifold.*

Proof: A full proof is given in chapter 5 of [10], the main components of this proof are included below. Dubrovin's proof begins by introducing an *admissible inner product.* Such an inner product, denoted $<,>_{\phi^2}$, will be compatible with the multiplication $\cdot$ in the usual way. Its action on arbitrary tangent vectors ∂' and ∂'' is defined to be

$$< \partial', \partial'' >= \Omega_{\phi^2}(\partial', \partial'').$$

The one-form Ω_{ϕ^2} above is defined in terms of what Dubrovin calls a *primary deifferential*, ϕ, by:

$$\Omega_{\phi^2} = -\sum_{i=1}^{n} \operatorname*{res}_{p_i} \frac{\phi^2}{d\lambda}.$$

Using the covering

$$\hat{H} = \hat{H}_{g;n_0,\dots,n_m} = \{(C;\lambda;k_0,\dots,k_m,a_1,\dots,a_g,b_1,\dots,b_g)\},$$

where k_i is a branched root of λ near ∞_i (i.e. $k_i^{n_i+1} \approx \lambda(z)$ near ∞_i) and $\{a_i, b_i\}$ is a marked symplectic basis.

Dubrovin then gives a list of five admissible primary differentials. They are:

1. Normalised abelian differentials of the second kind with poles only at branch points $\infty_0, \dots, \infty_m$. The orders of these poles will be less than the correspondant orders of the differential $d\lambda$.
2. $$\phi = \sum_{i=1}^{m} \alpha_i \phi_{w^i}.$$

   Here the coefficients δ_i are independent of the point on $\mathcal{M}$. The differentials ϕ_{v^i} are second kind abelian differentials on C, their principal part takes the form

   $$\phi_{v^i} = -d\lambda + \text{regular terms}$$

   near ∞_i, subject also to

   $$\oint_{a_j} \phi_{v^i} = 0.$$

3.

$$\phi = \sum_{i=1}^{m} \alpha_i \phi_{w^i}.$$

As above, the α_i are independent of the point on $\mathcal{M}$. However, the ϕ_{w^i} will now be abelian differentials of the third kind, with simple poles at ∞_0 and ∞_i, whose residues will be -1 and 1 respectively.

4.

$$\phi = \sum_{i=1}^{g} \beta_i \phi_{r^i},$$

with β_i independent of point on $\mathcal{M}$. The components ϕ_{r^i} are normalised multi-valued differentials, with increments along cycles b_j defined by

$$\phi_{r^i}(P + b_j) - \phi_{r^i}(P) = -\delta_{ij} d\lambda,$$

with no singularities other than those prescribed by the line above.

5.

$$\phi = \sum_{i=1}^{g} \gamma_i \phi_{s^i},$$

with γ_i independent of the point on $\mathcal{M}$ and ϕ_{s^i} holomorphic differentials normalized by

$$\oint_{a_j} \phi_{r^i} = \delta_{ij}.$$

By defining

$$z(P) = v.p. \int_{\infty_0}^{P} \phi,$$

one may set $\phi = dz$, which allows λ to be used as a superpotential for the manifold. Dubrovin then goes on to show that the flat coordinates for this Frobenius manifold also consist of five parts:

$$t^A = (t^{i;\alpha}, i = 0, \ldots, m,\ \alpha = 1, \ldots, n_i; p^i; q^i, ; r^j; s^j, j = 1, \ldots, g),$$

where

$$t^{i;\alpha} = \operatorname*{res}_{\infty_i} k_i^{-\alpha} p d\lambda, \tag{1.15}$$

$$p^i = v.p. \int_{\infty_0}^{\infty_i} dp, \tag{1.16}$$

$$q^i = -res_{\infty_i} \lambda dp, \tag{1.17}$$

$$r^i = \oint_{b_i} dp, \tag{1.18}$$

$$s^i = -\frac{1}{2\pi i} \oint_{a_i} \lambda dp. \tag{1.19}$$

The invariant inner product in these coordinates takes the form:

$$\begin{aligned}
\eta_{t^{i;\alpha} t^{i;\beta}} &= \frac{1}{n_i+1} \delta_{ij} \delta_{\alpha+\beta, n_i+1}, \\
\eta_{v^i w^j} &= \frac{1}{n_i+1} \delta_{ij}, \\
\eta_{r^i s^j} &= \frac{1}{2\pi i} \delta_{ij}, \\
\eta_{\alpha\beta} &= 0 \quad \text{otherwise}.
\end{aligned}$$

Example 1.49 *Consider again the Hurwitz space $H_{0;1,0}$. This consists of the space of functions of the form $\lambda = z^2 + a + \frac{b}{z-c}$. For convenience (and without loss of generality) one may instead assume that they are of the form*

$$\frac{1}{2}\lambda = z^2 + a + \frac{b}{z-c}.$$

Using a, b and c as coordinates on this space, one may, using the LG formula (1.10), easily show that

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly, using (1.12), one may show that

$$\begin{aligned}
&c_{aaa} = 1 \quad c_{abc} = 1 \\
&c_{bbb} = \tfrac{1}{b} \quad c_{bcc} = 1 \\
&c_{ccc} = b,
\end{aligned}$$

and is otherwise zero. As the metric is constant, the coordinates are already flat, so these may be integrated up to give (to within addition of quadratic terms) a prepotential

$$F = \frac{1}{6}a^3 + abc + \frac{1}{2}b^2 \log b + \frac{1}{6}bc^3.$$

Some properties of genus zero Hurwitz spaces will be studied in greater detail in chapters 3 and 4. For higher genus examples, one needs to introduce the notion of elliptic functions. A simple example, taken from [10] is given below.

Example 1.50 *The Hurwitz space $H_{1;1}$ is the space of elliptic curves of the form*

$$\mu^2 = 4\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{1.20}$$

Using a Weierstrass normalization and the primary differential

$$dp = \frac{dz}{2\omega},$$

a superpotential

$$\lambda(p) := \wp(2\omega p; \omega, \omega') + c$$

may be constructed. The constant c is such that the squared term in the equation (1.20) expressed in terms of $(\lambda - c)$ vanishes. Dubrovin constructs flat coordinates

$$\begin{aligned} t^1 &= \frac{1}{\pi i}\left(-c + \frac{\eta}{\omega}\right), \\ t^2 &= \frac{1}{\omega}, \\ t^3 &= \frac{\omega'}{\omega}. \end{aligned}$$

These were constructed using modified versions of the formulae (1.15-1.19) such that the metric takes the form

$$ds^2 = dt^1dt^3 + (dt^2)^2 + dt^3dt^1.$$

Further examples of Frobenius manifolds on genus one Hurwitz spaces will be considered in chapter 5.

1.8 Almost duality

Let $\mathcal{M}$ be a Frobenius manifold and denote by Σ the discriminant of $\mathcal{M}$ (recall that the discriminant is the locus where the Euler field E is not invertible). One may define a new multiplication $\star$ on $\mathcal{M}^\star = \mathcal{M}\backslash\Sigma$ by

$$u \star v := E^{-1} \cdot u \cdot v.$$

Lemma 1.51 *This new multiplication, coupled with the intersection form* $(,)$, *defines a Frobenius algebra.*

Proof: Associativity and commutativity follow immediately from the definition of $\star$. Likewise, it is easy to see that E will be a unity element with respect to the new multiplication. Finally, one must show that

$$(u \star v, w) = (u, v \star w).$$

But from the definition of $\star$:

$$\begin{aligned} (u \star v, w) &= (\frac{u \cdot v}{E}, w), \\ &= < \frac{u \cdot v}{E^2}, w >, \\ &= < \frac{u}{E}, \frac{v \cdot w}{E} >, \\ &= (u, v \star w). \end{aligned}$$

Theorem 1.52 *The multiplication* $\star$, *along with* $(,)$ *and a unity* E *(which will also act as the Euler field) satisfy all of the axioms of a Frobenius manifold except for covariant constancy of the unity element.*

Proof: The existence of a Frobenius algebra has been shown above. Furthermore, as the metric is just the intersection form of the original Frobenius manifold, flatness follows automatically. To show that FM4 is satisfied, one will instead consider the (equivalent) condition of

$$\nabla^\gamma c_\rho^{\alpha\beta}$$

being symmetric in α, β, γ, where ∇ is the Levi Civita connection of the intersection form. One has

$$\nabla^\gamma c_\rho^{\alpha\beta} = g^{\gamma\epsilon}\partial_\epsilon c_\rho^{\alpha\beta} - \Gamma_\epsilon^{\gamma\alpha} c_\rho^{\gamma\beta} - \Gamma_\epsilon^{\gamma\alpha} c_\rho^{\alpha\epsilon} + \Gamma_\rho^{\gamma\epsilon} c_\epsilon^{\alpha\beta}$$

Noting that as $\partial_\epsilon c_{\rho\alpha\beta}$ is symmetric in all four indices, one also has:

$$g^{\gamma\epsilon}\partial_\epsilon c_\rho^{\alpha\beta} = g^{\gamma\epsilon}\partial_\rho c_\epsilon^{\alpha\beta}.$$

Coupling this with

$$\partial_\rho(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma}) = g^{\epsilon\gamma}\partial_\rho c_\epsilon^{\alpha\beta} + c_\epsilon^{\alpha\beta}(\Gamma_\rho^{\gamma\epsilon} + \Gamma_\rho^{\epsilon\gamma}),$$

one has

$$\nabla^\gamma c_\rho^{\alpha\beta} = \partial_\rho(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma}) - \Gamma_\epsilon^{\gamma\alpha} c_\rho^{\epsilon\beta} - \Gamma_\epsilon^{\gamma\beta} c_\rho^{\alpha\epsilon} - c_\epsilon^{\alpha\beta}\Gamma_\rho^{\epsilon\gamma}.$$

Using the result from [13] that

$$\Gamma_\gamma^{\alpha\beta} = c_\gamma^{\alpha\epsilon}(\frac{d-1}{2} + \nabla_\eta E)_\epsilon^\beta,$$

where ∇_η is the Levi Civita connection of $<,>$ and associativity, this then becomes

$$\nabla^\gamma c_\rho^{\alpha\beta} = \partial_\rho(c_\epsilon^{\alpha\beta} g^{\epsilon\gamma})\left[c_\epsilon^{\gamma\beta} c_\rho^{\epsilon\lambda}(\frac{d-1}{2}+\nabla_\eta E)_\lambda^\alpha + c_\epsilon^{\alpha\gamma} c_\rho^{\epsilon\lambda}(\frac{d-1}{2}+\nabla_\eta E)_\lambda^\beta + c_\epsilon^{\alpha\beta} c_\rho^{\epsilon\lambda}(\frac{d-1}{2}+\nabla_\eta E)_\lambda\gamma\right].$$

The symmetry of the second term (i.e. the one is quare brackets) is obvious. To show that the first term is symmetric in α, β, γ, note that

$$c_\epsilon^{\alpha\beta} g^{\epsilon\gamma} = i_E(c_\epsilon^{\alpha\beta} dt^\epsilon \cdot dt^\gamma) = i_E(dt^\alpha \cdot dt^\beta \cdot dt^\gamma),$$

which is symmetric in α, β, γ as required.

Corollary 1.53 *A function $F^\star$ exists such that*

$$\frac{\partial^3 F^\star}{\partial p^i \partial p^j \partial p^k} = G_{ia} G_{jb} \frac{\partial t^\gamma}{\partial p^k} \frac{\partial p^a}{\partial t^\alpha} \frac{\partial p^b}{\partial t^\beta} c_\gamma^{\alpha\beta}(t).$$

This function will satisfy the WDVV associativity equations

$$c^\star_{ija} G^{ab} c^\star_{bkl} = c^\star_{lja} G^{ab} c^\star_{bki}.$$

Proof: The existence of a function $F^\star$ whose third derivatives are equal to $c^\star(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}, \frac{\partial}{\partial p^k})$ follows from the above theorem (existence of a prepotential for a Frobenius manifold not being reliant upon covariant constancy of the unity).

Noting that the multiplication $\star$ will define the same multiplication on the cotangent space as $\cdot$ does, one has

$$c_\gamma^{\alpha\beta}(t) = c_\gamma^{\star\alpha\beta}(t).$$

Performing a change of variable from t to p on this $(2,1)$ tensor and lowering the upper indices using

$$G_{ij} = (G^{ij})^{-1},$$

where G^{ij} is the intersection form expressed in terms of its own flat coordinates $\{p^i\}$ provides the desired result.

Example 1.54 *Consider the Frobenius manifold with the prepotential and Euler field given by:*

$$\begin{aligned} F(t_1, t_2) &= \frac{1}{2} t_1^2 t_2 + e^{t_2}, \\ E &= t_1 \partial_1 + 2\partial_2. \end{aligned}$$

The intersection form is

$$g_{\alpha\beta} = \begin{pmatrix} 2e^{t_2} & t_1 \\ t_1 & 2 \end{pmatrix}.$$

One may easily show that the flat coordinates for g are given by

$$\begin{aligned} t_1 &= -\left(e^{z_1}+e^{z_2}\right), \\ t_2 &= z_1+z_2. \end{aligned}$$

In these coordinates, the intersection form takes the form

$$G_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One may then apply the almost duality formula (1.53) to obtain

$$\begin{aligned} c^\star_{111} = -c^\star_{112} = \frac{e^{z_1}}{e^{z_2}-e^{z_1}}, \\ c^\star_{222} = -c^\star_{122} = \frac{e^{z_2}}{e^{z_1}-e^{z_2}}. \end{aligned}$$

These may then be integrated to obtain $F^\star$. In order to do this, one requires the polylogarithm function Li_n, defined by

$$Li_n(z) := \sum_{r=1}^{\infty} \frac{z^r}{r^n}. \tag{1.21}$$

This series is convergent for $|z|<1$ and may be defined by analytic continuation elsewhere. It satisfies

$$\begin{aligned} Li_0(z) &= \frac{z}{1-z}, \\ \frac{d}{dz}Li_n(z) &= \frac{1}{z}Li_{n-1}(z). \end{aligned}$$

Integrating the $c^\star_{ijk}$ three times yields

$$F^\star = \frac{1}{4}z_1z_2\left(z_1^2+z_2^2\right)) - \frac{1}{12}\left(z_1^3+z_2^3\right) + \frac{1}{2}\left(Li_3e^{z_1-z_2}+Li_3e^{z_2-z_1}\right). \tag{1.22}$$

For more detailed examples of such calculations, see section 4.3.

Consideration is now turned to semisimple Frobenius manifolds. On such manifolds, the dual multiplication $\star$ in the canonical coordinates of $\cdot$ is given by:

$$\begin{aligned} \partial_i \star \partial_j &= -\sum_l \frac{1}{u^l}\partial_l(\partial_i \cdot \partial_j), \\ &= \sum_l \frac{1}{u^l}\partial_l \delta_{ij}\partial_i, \\ &= \frac{1}{u^i}\delta_{ij}\partial_i. \end{aligned}$$

Lemma 1.55 *For a semisimple Frobenius manifold, the almost dual multiplication $\star$ will be semisimple. Canonical coordinates for this multiplication are given by*

$$\tau^i = \log u^i.$$

Proof: Using the notation[1]

$$\partial_{\tau^i} = \frac{\partial}{\partial \tau^i},\ \partial_i = \frac{\partial}{\partial u^i},$$

one has

$$\partial_{\tau^i} = \frac{\partial u^i}{\partial \tau^i}\partial_i = u^i \partial_i.$$

Substituting this into formula for the almost dual multiplication yields

$$\begin{aligned} \partial_{\tau^i} \star \partial_{\tau^j} &= \frac{1}{u^i}\delta_{ij}(u^i)^2\partial_i, \\ &= \delta_{ij} u^i \partial_i, \\ &= \delta_{ij}\partial_{\tau^i}. \end{aligned}$$

Hence the multiplication $\star$ is semisimple and $\{\tau^i\}$ are the canonical coordinates for it.

Corollary 1.56 *Let $\mathcal{M}$ be a semisimple Frobenius manifold with an LG superpotential λ. Then the trilinear tensor $c^\star(\partial', \partial'', \partial''')$ may be expressed by the residue formula*

$$c^\star(\partial', \partial'', \partial''') = -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda)\partial''(\log\lambda)\partial'''(\log\lambda)}{d(\log\lambda)} dz. \tag{1.23}$$

Proof: Canonical coordinates will be used to prove this theorem (as only semisimple Frobenius manifolds are being considered, the existence of such coordinates is guaranteed at a generic point). Consider the left hand side of the

[1] Note that throughout this proof, there will be no summation over repeated indices.

above formula:

$$\begin{aligned} c^\star(\partial_i,\partial_j,\partial_k) &= (\partial_i \star \partial_j, \partial_k), \\ &= \frac{1}{u^i}\delta_{ij}(\partial_i,\partial_k), \\ &= \frac{1}{(u^i)^2}\delta_{ij}\delta_{ik}\eta_{ii}. \end{aligned}$$

Next, consider the right hand side:

$$\begin{aligned} -\sum \operatorname*{res}_{d\lambda=0} \frac{\partial'(\log\lambda)\partial''(\log\lambda)\partial'''(\log\lambda)}{d(\log\lambda)}dz &= -\sum_l \operatorname*{res}_{z=q^l} \frac{\partial_i\lambda}{\lambda}\frac{\partial_j\lambda}{\lambda}\frac{\partial_k\lambda}{\lambda}\frac{\lambda}{\lambda'}dz, \\ &= -\sum_l \operatorname*{res}_{z=q^l} \frac{\partial_i\lambda\partial_j\lambda\partial_k\lambda}{\lambda^2\lambda'}dz. \end{aligned}$$

But near $z = q^l$, we have $\partial_i\lambda = \delta_{il}$. Therefore the residues are zero, except for when $i = j = k$, in which case the residue will be non zero at the point $z = q^i$ (but zero everywhere else). Note that at $z = q_i$, λ will be nonzero (in fact it will be equal to u^i). Hence $\frac{1}{\lambda^2}$ may be brought outside of the residue to give

$$c^\star_{ijk} = -\frac{1}{(u^i)^2}\operatorname*{res}_{z=z_i}\frac{\delta_{ij}\delta_{ik}}{\lambda'}dz.$$

But if $i = j = k$, recall from above that this is the same residue as occurred in the calculation of η_{ii}. Therefore

$$c^\star_{ijk} = \delta_{ij}\delta_{jk}\frac{1}{(u^i)^2}\delta_{ij}\delta_{jk}\eta_{ii},$$

as required.

Example 1.57 *Consider the Frobenius manifold on the Hurwitz space $H_{0;0,0}$ with superpotential*

$$\lambda(z) = z + \frac{a}{z-b}.$$

Using the residue formulae (1.10) and (1.12), one may show that a and b are flat coordinates and that the prepotential is

$$F(a,b) = \frac{1}{2}b^2a + \frac{1}{2}a^2\log\left(a - \frac{3}{2}\right).$$

One may also show using the formula (1.11) that the flat coordinates of the intersection form are the zeros of λ*, i.e. if one writes*

$$\lambda(z) = \frac{(z-z_1)(z-z_2)}{z-(z_1+z_2)},$$

then z_1 *and* z_2 *are the flat coordinates of the intersection form. One may therefore use the tangent vectors* $\frac{\partial}{\partial^i_z}$ *in the formula (1.23) to obtain (by simple residue calculations)*

$$\begin{aligned} c^\star_{111} &= \frac{1}{z_1} - \frac{1}{z_1-z_2}, \\ c^\star_{112} &= \frac{1}{z_1-z_2}, \\ c^\star_{122} &= \frac{1}{z_2-z_1}, \\ c^\star_{222} &= \frac{1}{z_2} - \frac{1}{z_2-z_1}. \end{aligned}$$

These may be integrated up to give the almost dual prepotential

$$F^\star = \frac{1}{4}\left(z_1^2 \log(z_1)^2 + z_2^2 \log(z_2)^2 - (z_1-z_2)\log(z_1-z_2)^2\right). \qquad (1.24)$$

Full details of similar calculations can be seen in sections 4.1 and 4.2.

1.9 Legendre transformations

Definition 1.58 *A* symmetry *of the WDVV equations is a transformation of the form*

$$\begin{aligned} t &\rightarrow \hat{t}, \\ \eta_{\alpha\beta} &\rightarrow \hat{\eta}_{\alpha\beta}, \\ F &\rightarrow \hat{F}, \end{aligned}$$

such that $\hat{F}$ *is a solution to the WDVV equations.*

One type of symmetry of the WDVV equations is a Legendre transformation. These were defined in [10] in the following way:

Definition 1.59 *The* Legendre transformation S_k *defines new coordinates*

$$\hat{t}^\alpha = \partial_\alpha \partial_k F(t).$$

The new prepotential $\hat{F}$ *is defined implicitly by the differential equation*

$$\frac{\partial^2 \hat{F}}{\partial \hat{t}^\alpha \partial \hat{t}^\beta} = \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta}, \tag{1.25}$$

whilst the metric remains invariant;

$$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.$$

Note that tangent vectors in the two coordinate systems are linked by the relationship

$$\partial_\alpha = \partial_k \cdot \hat{\partial}_\alpha.$$

Putting $\alpha = k$, one obtains

$$e = \hat{\partial}_k.$$

Finally, consider a new metric defined by

$$< a, b >_k = < \partial_k \cdot \partial_k, a \cdot b > .$$

It follows immediately that this metric will be Frobenius. Moreover, by putting $a = \hat{\partial}_\alpha$ and $b = \hat{\partial}_\beta$, one obtains:

$$\begin{aligned} < \hat{\partial}_\alpha, \hat{\partial}_\beta >_k &= < \partial_k \cdot \partial_k, \hat{\partial}_\alpha \cdot \hat{\partial}_\beta >, \\ &= < \partial_k \cdot \hat{\partial}_\alpha, \partial_k \cdot \hat{\partial}_\beta >, \\ &= < \partial_\alpha, \partial_\beta > . \end{aligned}$$

Hence the new metric $<,>_k$ is $\hat{\eta}_{\alpha\beta}$. Also, as t^α are flat coordinates for $\eta_{\alpha\beta}$, so $\hat{t}^\alpha$ must be flat coordinates for $\hat{\eta}_{\alpha\beta}$.

Example 1.60 *This example appeared in [10], and applies the Legendre transformation S_2 to the prepotential*

$$F = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}.$$

The new variables are defined by

$$\begin{aligned} t^1 &= \hat{t}^2, \\ t^2 &= \log \hat{t}^1, \end{aligned}$$

From (1.25), one obtains

$$\begin{aligned} \hat{F}_{\hat{1}\hat{1}} &= t^2 &= \log \hat{t}^1, \\ \hat{F}_{\hat{1}\hat{2}} &= t^1 &= \hat{t}^2, \\ \hat{F}_{\hat{2}\hat{2}} &= e^{t^2} &= \hat{t}^1. \end{aligned}$$

Integrating yields the prepotential

$$\hat{F} = \frac{1}{2}(\hat{t}^2)^2 \hat{t}^1 + \frac{1}{2}(\hat{t}^1)^2 \left(\log \hat{t}^1 - \frac{3}{2}\right).$$

In summary, the concept of a Frobenius manifold has been introduced and two particular classes of semisimple Frobenius manifolds - those on the orbit space of a Coxeter group and those on a Hurwitz space - have been constructed. These two categories are closely linked, as illustrated by the fact that the polynomial Frobenius manifolds introduced in section 1.5 lies in both classes. The rest of this thesis will be laid out as follows. Chapter 2 will introduce the idea of submanifold geometry for Frobenius manifolds, in particular the idea of natural submanifolds which may be classified in terms of caustics and discriminants. Chapter 3 will then deal with caustic submanifolds of genus zero Hurwitz spaces. Chapter 4 again studies genus zero Hurwitz spaces, this time considering the application of the almost duality of section 1.8 to discriminants. A modified version of the Legendre transformations introduced in section 1.9 will be constructed and used to link certain almost dual solutions of the WDVV equations. Finally, chapter 5 will continue to study almost duality, but this time for a special class of genus one Hurwitz spaces.

Chapter 2

Submanifolds

2.1 Induced structures

The relationship between the geometry of a submanifold and its ambient manifold is one of the oldest problems in the field of differential geometry. Therefore it is an obvious problem, given a Frobenius manifold, to study the geometry of its submanifolds, and whether the rich mathematical structure of a Frobenius manifold carries over to the submanifold. In order to do this, a brief recap of induced structures on a submanifold will be given.

Let $\mathcal{M}$ be a manifold (of dimension m) such that a commutative multiplication

$$\circ : T_p\mathcal{M} \times T_p\mathcal{M} \rightarrow T_p\mathcal{M}$$

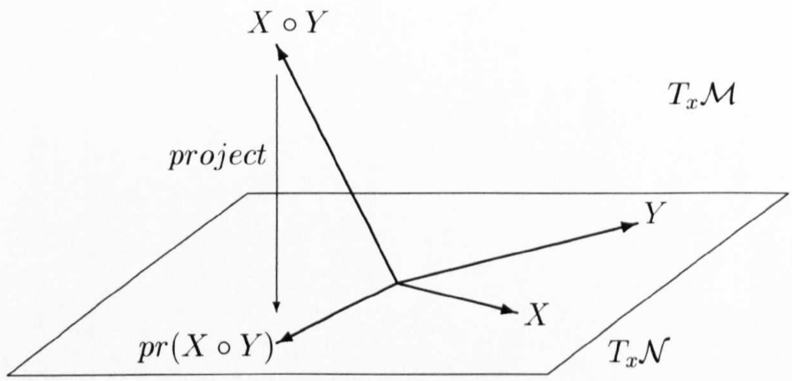
is defined on the tangent space at every point p. Also, assume that a metric $\eta = <, >$ exists on $\mathcal{M}$. Then for any n-dimensional submanifold $\mathcal{N} \subset \mathcal{M}$, one may define an induced multiplication

$$\star : T_p\mathcal{N} \times T_p\mathcal{N} \rightarrow T_p\mathcal{N}$$

by

$$a \star b = pr(a \circ b),$$

where a and b are arbitrary vectors on $T_p\mathcal{N}$ and $pr(\,)$ is the projection using η. This idea is illustrated in the diagram below.



Definition 2.1 *$\mathcal{N}$ is said to be a* natural submanifold *of $\mathcal{M}$ if*

$$a \star b = a \circ b, \quad \forall\, a, b \in T_x N \subset T_x M$$

i.e. no projection is necessary.

The definition above can be extended to manifolds not endowed with a metric by expressing it as the equivalent condition

$$T_p\mathcal{N} \circ T_p\mathcal{N} \subset T_p\mathcal{N}.$$

In addition to the induced multiplication, one may define an induced metric $\eta_{\mathcal{N}}$ on $\mathcal{N}$. Let $\{t^i\}$ be local coordinates on $\mathcal{M}$. Then $\mathcal{N}$ may be parameterised by

$$t^i = t^i(\tau^\alpha),\ \alpha = 1, \ldots, n.$$

Hence $\frac{\partial}{\partial \tau^\alpha}$ forms a basis for $T_p\mathcal{N}$, defined by the formula

$$\frac{\partial}{\partial \tau^\alpha} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial}{\partial t^i}.$$

With these coordinate systems, the components of $\eta_{\mathcal{N}}$, denoted by $\eta_{\alpha\beta}$, are given by the formula

$$\eta_{\alpha\beta} := \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\eta_{ij}. \tag{2.1}$$

One may reconstruct a basis for $T_p\mathcal{M}$ from the basis of $T_p\mathcal{N}$ by adding an orthogonal compliment, i.e.

$$\frac{\partial}{\partial t^i} = A_i^\alpha \frac{\partial}{\partial \tau^\alpha} + n_i^a \frac{\partial}{\partial \nu^a}, \quad a = 1, \ldots, m - n, \quad \frac{\partial}{\partial \nu^a} \in (T_p\mathcal{N})^\perp.$$

Hence, using η and $\eta_{\mathcal{N}}$,

$$A_i^\alpha = \eta^{\alpha\beta}\eta_{ij}\frac{\partial t^j}{\partial \tau^\beta}.$$

Theorem 2.2 *If $\circ$ and η satisfy the Frobenius condition*

$$< a \circ b, c > = < a, b \circ c >,$$

for $a, b \in T_p\mathcal{M}$, then the induced multiplication and metric ($\star$ and $\eta_{\mathcal{N}}$) will satisfy the Frobenius condition on an arbitrary submanifold $\mathcal{N} \subset \mathcal{M}$.

Proof: Let c_{ij}^k denote structure constants defining the multiplication on $T_p\mathcal{M}$ by

$$\partial_i \circ \partial_j := c_{ij}^k \partial_k.$$

Induced structure constants $c_{\alpha\beta}^\gamma$ such that $\partial_\alpha \star \partial_\beta = c_{\alpha\beta}^\gamma \partial_\gamma$ may be defined in the following way:

$$\begin{aligned} \partial_{\tau_\alpha} \circ \partial_{\tau_\beta} &= \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta} c_{ij}^k|_{\mathcal{N}} \partial_k, \\ &= \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta} c_{ij}^k|_{\mathcal{N}} (A_k^\gamma \partial_{\tau_\gamma} + n_k^\gamma \partial_{\nu_\gamma}). \end{aligned}$$

Projecting this onto $\mathcal{N}$ yields

$$\partial_{\tau_\alpha} \star \partial_{\tau_\beta} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta} c_{ij}^k|_{\mathcal{N}} A_k^\gamma \partial_{\tau_\gamma},$$

but

$$A_k^\gamma = \eta^{\gamma\delta}\eta_{kp}\frac{\partial t^q}{\partial \tau^\delta}.$$

Therefore

$$c_{\alpha\beta}^{\gamma} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\frac{\partial t^r}{\partial \tau^\delta}\eta_{kr}\eta^{\gamma\delta}c_{ik}^{k}|_{\mathcal{N}}.$$

Now note that the Frobenius condition is equivalent to symmetry (in all three indices) of the tensor

$$c_{ijk} = \eta_{kl}c_{ij}^{l}.$$

On $\mathcal{N}$, one has

$$c_{\alpha\beta\gamma} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\frac{\partial t^r}{\partial \tau^\delta}\eta_{\gamma\epsilon}\eta_{kr}\eta^{\gamma\delta}c_{ik}^{k}|_{\mathcal{N}}.$$

A suitable reordering of the terms and summations therefore yields

$$c_{\alpha\beta\gamma} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\frac{\partial t^k}{\partial \tau^\gamma}c_{ijk}|_{\mathcal{N}},$$

which is symmetric in all three indices as required.

2.2 Frobenius submanifolds

Definition 2.3 *Let $\mathcal{M}$ be a Frobenius manifold and $\mathcal{N}$ an arbitrary submanifold of $\mathcal{M}$. $\mathcal{N}$ is said to be a* Frobenius submanifold *if it is a Frobenius manifold with respect to the induced structures defined above.*

Recalling the large number of conditions which must be satisfied in order for a manifold to be a Frobenius manifold, it becomes immediately apparent that generally a submanifold of a Frobenius manifold will not be a Frobenius submanifold. For example, the metric on a Frobenius manifold must be flat, but this will not automatically be the case for the induced metric on the submanifold. Similarly, the induced multiplication on the submanifold may not inherit the associativity of the multiplication on the ambient manifold.

If one considers the simplest non-trivial case, namely a two dimensional submanifold of a three dimensional Frobenius manifold, then a suitable condition for a submanifold to be a Frobenius submanifold may be expressed easily by the following lemma.

Lemma 2.4 *Let $\mathcal{M}$ be a three dimensional Frobenius manifold and $\mathcal{N}$ a two dimensional submanifold. If the identity field e is tangential to $\mathcal{N}$ at all points $t \in \mathcal{N}$, then $\mathcal{N}$ is a Frobenius submanifold.*

Proof: The tangentiality of e implies the submanifold may be parameterised

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1(\tau_2) \\ a_2(\tau_2) \\ a_3(\tau_2) \end{pmatrix}.$$

In order for the submanifold to be in its own flat coordinates, we require

$$\eta_{\mathcal{N}} = \eta|_{\mathcal{N}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recalling the equation (2.1) and using $\eta_{\mathcal{N}12} = 1$, one has:

$$\begin{aligned} 1 &= \frac{\partial t^1}{\partial \tau^1}\frac{\partial t^3}{\partial \tau^2} + \frac{\partial t^2}{\partial \tau^1}\frac{\partial t^2}{\partial \tau^2} + \frac{\partial t^3}{\partial \tau^1}\frac{\partial t^1}{\partial \tau^2}, \\ &= \frac{\partial t^3}{\partial \tau^2}. \end{aligned}$$

Hence, taking the constant of integration to be zero,

$$t_3 = \tau_2.$$

Similarly, using $\eta_{\mathcal{N}22} = 0$, along with the above result, one obtains:

$$2\frac{\partial t^1}{\partial \tau^2} + \frac{\partial t^2}{\partial \tau^2}\frac{\partial t^2}{\partial \tau^2} = 0,$$

or

$$a_1 = -\frac{1}{2}\int \left(\frac{\partial t^2}{\partial \tau^2}\right)^2 d\tau_2.$$

One may now consider the algebra on the submanifold. As $\partial_{\tau_1} = e$, it automatically follows that

$$\partial_{\tau_1} \star \partial_{\tau_1} = \partial_{\tau_1}, \quad \partial_{\tau_1} \star \partial_{\tau_2} = \partial_{\tau_2}.$$

Finally, one may show that

$$\begin{aligned}\partial_{\tau_2} \star \partial_{\tau_2} &= \left(-\frac{3}{4} a_2'^4 + a_2'^3 c_{222}|_{\mathcal{N}} + 3 a_2'^2 c_{223}|_{\mathcal{N}} + 3 a_2' c_{233}|_{\mathcal{N}} + c_{333}|_{\mathcal{N}} \right) \partial_{\tau_1} \\ &\quad + (-a_2'^3 + a_2'^2 c_{222}|_{\mathcal{N}} + 2 a_2' c_{223}|_{\mathcal{N}} + c_{233}|_{\mathcal{N}}) \partial_\nu ,\end{aligned}$$

where

$$\partial_\nu = \partial_{t_2} - a_2' \partial_{\tau_1},$$

which is orthogonal to $\mathcal{N}$. Therefore

$$\partial_{\tau_2} \star \partial_{\tau_2} = \left(-\frac{3}{4} a_2'^4 + a_2'^3 c_{222}|_{\mathcal{N}} + 3 a_2'^2 c_{223}|_{\mathcal{N}} + 3 a_2' c_{233}|_{\mathcal{N}} + c_{333}|_{\mathcal{N}} \right) \partial_{\tau_1}.$$

This multiplication is trivially associative (as it is two dimensional) and hence the induced structure is a Frobenius manifold. The prepotential is given by

$$F_{\mathcal{N}}(\tau_1, \tau_2) = \frac{1}{2} \tau_1^2 \tau_2 + \int \int \int \left(-\frac{3}{4} a_2'^4 + a_2'^3 c_{222}|_{\mathcal{N}} + 3 a_2'^2 c_{223}|_{\mathcal{N}} + 3 a_2' c_{233}|_{\mathcal{N}} + c_{333}|_{\mathcal{N}} \right) d\tau_2^3.$$

Corollary 2.5 *The Frobenius manifold from the above theorem is natural if*

$$(-a_2'^3 + a_2'^2 c_{222}|_{\mathcal{N}} + 2 a_2' c_{223}|_{\mathcal{N}} + c_{233}|_{\mathcal{N}}) = 0.$$

Proof: If a_2 satisfies this equation, then it immediately follows that $a \circ b = a \star b$ for all $a, b \in \mathcal{N}$. Hence the submanifold is natural.

Example 2.6 *Consider the Frobenius manifold corresponding to the Coxeter group A_3. In flat coordinates, it has prepotential*

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \frac{1}{4} t_2^2 t_3^2 + \frac{1}{60} t_3^5.$$

The two dimensional submanifold to which $e = \partial_{t_1}$ is always tangential may be parameterised (in its own flat coordinates) by:

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \int (\frac{db}{d\tau_2})^2 d\tau_2 \\ b(\tau_2) \\ \tau_2 \end{pmatrix}.$$

Consideration of the quasihomogeneity conditions then implies that $t_2 = b(\tau_2) = k\tau_2^{\frac{3}{2}}$ *and* $t_1 = \tau_1 - \frac{9}{16}k_2^2\tau_2^2$.

Theorem 2.7 *Let* $\mathcal{M}$ *be an* m*-dimensional Frobenius manifold and* $\mathcal{N}$ *a two dimensional submanifold. If the identity field* e *is tangential to* $\mathcal{N}$ *at all points* $t \in \mathcal{N}$, *then* $\mathcal{N}$ *is a Frobenius submanifold.*

Proof: The proof of this statement is a direct generalisation of the three dimensional case. Begin by parameterising the submanifold by

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} a_1(\tau_2) \\ a_2(\tau_2) \\ \vdots \\ a_m(\tau_2) \end{pmatrix}.$$

The flat coordinates are given by setting

$$\begin{aligned} a_1 &= -\frac{1}{2}\int \sum_{i=2}^{m} a_i' a_{m+1-i}' d\tau_2, \\ a_m &= \tau_2. \end{aligned}$$

A prime denotes differentiation w.r.t. τ_2 in the above formula. The rest of the proof then follows in the same way as in three dimensions.

Example 2.8 *Let* $\mathcal{M}$ *be the four dimensional Frobenius manifold corresponding to the Coxeter group* F_4. *Then its prepotential is*

$$F = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + \frac{1}{6}t_2^3 t_4 + \frac{1}{12}t_3^4 t_4 + \frac{1}{6}t_2 t_3^3 t_4^3 + \frac{1}{60}t_2^2 t_4^5 + \frac{1}{252}t_3^2 t_4^7 + \frac{1}{185328}t_4^{13}.$$

One may parameterise the two dimensional submanifold (in its own flat coordinates by)

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\int a_2' a_3' + a_3' a_2' d\tau_2 \\ a_2(\tau_2) \\ a_3(\tau_2) \\ \tau_2 \end{pmatrix}.$$

The quasihomogeneity condition then implies

$$\begin{aligned} a_2 &= k_2\tau_2^4, \\ a_3 &= k_3\tau_2^3. \end{aligned}$$

This in turn allows one to calculate

$$a_1 = -2k_2k_3\tau_2^6.$$

For submanifolds of dimension greater than two, formulating necessary conditions for a Frobenius submanifold becomes considerably harder, as the associativity is no longer trivially guaranteed. In the case of a natural submanifold however, an elegant expression of necessary conditions for a Frobenius submanifold was provided in [24], and is stipulated in the theorem below.

Theorem 2.9 *Let $\mathcal{N}$ be a flat natural submanifold of a Frobenius manifold $\mathcal{M}$. If the identity field e and Euler field E are both tangential to $\mathcal{N}$ at all $t \in \mathcal{N}$, then $\mathcal{N}$ is a Frobenius submanifold.*

Proof: A sketch of the proof will be given here; see [24] for full details.

To prove that such a submanifold is a Frobenius submanifold, the WDVV formulation of a Frobenius manifold will be used. Firstly, one considers the existence of an induced prepotential $F_{\mathcal{N}}$ satisfying

$$c_{\alpha\beta\gamma} = \frac{\partial F^3}{\partial\tau^\alpha\partial\tau^\beta\partial\tau^\gamma}.$$

The integrability condition for this is

$$\frac{\partial c_{\alpha\beta\gamma}}{\partial\tau^\delta} - \frac{\partial c_{\alpha\beta\delta}}{\partial\tau^\gamma} = 0.$$

Strachan shows how, due to the submanifold being natural, the obstruction to integrability vanishes.

Secondly, the existence of a covariantly constant identity field is proven. This is done by parameterising the submanifold

$$t^1 = \tau^1 + f_1(\tau^2, \ldots, \tau^n), \tag{2.2}$$

$$t^i = f_i(\tau^2, \ldots, \tau^n),\ 2 \leq < n, \tag{2.3}$$

$$t^n = \tau_n. \tag{2.4}$$

Hence

$$e = \frac{\partial}{\partial t^1} = \frac{\partial}{\partial \tau^1}.$$

Also, using the relevant formulae, one has, in these coordinates,

$$c_{1\alpha\beta} = \eta_{\alpha\beta},$$

as required by the WDVV equations.

Finally, Strachan shows that the Euler field on $\mathcal{N}$ is linear in τ^α and satisfies the quasihomogeneity condition

$$\mathcal{L}_E c_{\alpha\beta\gamma} = d_F c_{\alpha\beta\gamma},$$

hence meaning that $F_{\mathcal{N}}$ is not only quasihomogenous but is of the same degree as F.

2.3 Semisimple natural submanifolds

As stated earlier, a submanifold of a Frobenius manifold will not, in general, be a Frobenius submanifold. It is therefore natural to consider which (if any) of the properties of the ambient Frobenius manifold a given submanifold will inherit. The following definitions will be useful in addressing this questions.

Definition 2.10 *A manifold $\mathcal{M}$ endowed with a commutative and associative multiplication*

$$\circ : T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M}$$

defined on the tangent space at every point p is an F-manifold *if*

$$\mathcal{L}_{X \circ Y}(\circ) = X \circ \mathcal{L}_Y(\circ) + Y \circ \mathcal{L}_X(\circ), \ \forall X, Y \in T_p\mathcal{M}.$$

An F_E-manifold *is an F-manifold on which a quasihomogenous Euler field is defined, such that*

$$\mathcal{L}_E(\circ) = d \cdot \circ.$$

An F_η-manifold is an F-manifold on which a metric $\eta = <,>$ satisfying the Frobenius condition is defined.

An $\mathcal{F}$-manifold is a manifold which adheres to the axioms of both an F_E-manifold and an F_η-manifold, as well as the additional condition

$$\mathcal{L}_E <,>= D <,>,$$

for some constant D.

Consideration is now restricted to semisimple F-manifolds. On such a manifold, the tangent space decomposes into one dimensional algebras, and so a set of canonical coordinates $\{u^i\}$ exist such that

$$\partial_i \circ \partial_j = \delta_{ij}\partial_i.$$

This is entirely analogous to semisimple Frobenius manifolds (a Frobenius manifold being a special case of an F-manifold). In the case of a semisimple F_E-manifold, the Euler field takes the familiar form

$$E = \sum_i u^i \partial_i.$$

Similarly, on a semisimple F_η-manifold, the metric will (in canonical coordinates) take the diagonal form

$$\eta_{ij} = \delta_{ij}\eta_{ii}.$$

Definition 2.11 *A submanifold defined by the condition $u^i = 0$, for one or more values of i is a* discriminant hypersurface, *and will be denoted $\mathcal{D}$.*

A submanifold defined by the condition $u^i - u^j = 0$ for some pair u^i and u^j, where $i \neq j$, is known as a caustic, *and will be denoted $\mathcal{K}$.*

A suitable ordering of the coordinates allows such manifolds to parameterised

$$(u^1, \dots, u^m) = (\underbrace{\tau^1, \dots, \tau^1}_{k_1}; \dots; \underbrace{\tau^n, \dots, \tau^n}_{k_n}; \underbrace{0, \dots, 0}_{m-\sum k_i}).$$

The submanifolds may then be denoted

$$(k_1, \dots, k_n, 0, \dots 0).$$

Definition 2.12 *A submanifold of the form $(k_1, \dots, k_n)$ is known as a* pure caustic. *A submanifold of the form $(1, \dots, 1, 0 \dots, 0)$ is known as a* pure discriminant.

Lemma 2.13 *Submanifolds of a semisimple F-manifold of the form $\mathcal{K} \cap \mathcal{D}$ are natural F-manifolds.*

Proof: The proof of this statement follows automatically from the definitions.

Lemma 2.14 *Let $\mathcal{M}$ be a semisimple F_E-manifold. Then any submanifold $\mathcal{N} = \mathcal{K} \cap \mathcal{D}$ will be a natural F_E-manifold. Moreover, the Euler field will be tangential to $\mathcal{N}$, that is $E_{\mathcal{N}} = E|_{\mathcal{N}}$.*

Proof: The Euler field on $\mathcal{M}$ is

$$E = \sum_i u^i \frac{\partial}{\partial u^i}.$$

Noting that on $\mathcal{N}$, $u^i = 0$ for $i \in \mathcal{D}$, this becomes

$$E|_{\mathcal{N}} = \sum_{i \notin \mathcal{D}} u^i \frac{\partial}{\partial u^i}.$$

But $u^i = \tau^{\alpha(i)}$, where $\alpha \in 1, \ldots n$, and

$$\frac{\partial}{\partial u^i} = \frac{\partial \tau^\alpha}{\partial u^i} \frac{\partial}{\partial \tau^\alpha} = \frac{\partial}{\partial \tau^{\alpha(i)}}.$$

Noting that there are k_α of the u^i equal to τ^α, the Euler field is therefore

$$E|_{\mathcal{N}} = \sum_{\alpha} k_\alpha \tau^\alpha \frac{\partial}{\partial \tau^\alpha}.$$

But this is tangential to $\mathcal{N}$, so and so defines an Euler field $E_{\mathcal{N}} = E|_{\mathcal{N}}$. As the multiplication and Euler field on $\mathcal{N}$ are the same as they are on $\mathcal{M}$, so the quasihomogeneity property must also follow.

Lemma 2.15 *Any submanifold of a semisimple $\mathcal{F}$-manifold which is of the form $\mathcal{K} \cap \mathcal{D}$ will be a natural $\mathcal{F}$-manifold.*

Proof: It follows from the above lemma that such a submanifold will be a natural F_E manifold, and the existence of an induced metric ensures it will also be an F_η manifold. Therefore all that is left to prove is that

$$\mathcal{L}_{E|_N} <,>_{\mathcal{N}} = D <,>_{\mathcal{N}}.$$

However, noting that $E_{\mathcal{N}} = E|_{\mathcal{N}}$ and that the induced metric is

$$\eta_{\alpha\beta} = \delta_{\alpha\beta}\eta_{\alpha\alpha},$$

with

$$\eta_{\alpha\alpha} = \sum_{i:\tau^\alpha = u^i} \eta_{ii},$$

the quasihomogeneity property of the ambient manifold is automatically inherited.

Lemma 2.16 *The only natural submanifolds of a semisimple $\mathcal{F}$-manifold are those of the form $\mathcal{K}\cap\mathcal{D}$, i.e. the intersections of caustic and discriminant hypersurfaces.*

Proof: Firstly, note that using the inclusion map $\imath : \mathcal{N} \to \mathcal{M}$, one may push forward vectors from $T_t\mathcal{N}$ to $T_t\mathcal{M}$ by the formula

$$\imath_\star \frac{\partial}{\partial\tau^\alpha} = \frac{\partial u^i}{\partial\tau^\alpha}\frac{\partial}{\partial u^i}.$$

Also, tangent vectors on $T_t\mathcal{M}$ may be orthogonally decomposed:

$$\frac{\partial}{\partial u^i} = A_i^\alpha \frac{\partial}{\partial\tau^\alpha} + N_i^b \frac{\partial}{\partial\nu^b}.$$

Hence one has (noting that the multiplication is canonical)

$$\begin{aligned}\frac{\partial}{\partial\tau^\alpha}\circ\frac{\partial}{\partial\tau^\beta} &= \sum_{i=1}^m \frac{\partial u^i}{\partial\tau^\alpha}\frac{\partial u^i}{\partial\tau^\beta}\frac{\partial}{\partial u^i}\\ &= \sum_{i=1}^m \frac{\partial u^i}{\partial\tau^\alpha}\frac{\partial u^i}{\partial\tau^\beta}\left(A_i^\alpha\frac{\partial}{\partial\tau^\alpha} + N_i^b\frac{\partial}{\partial\nu^b}\right).\end{aligned}$$

As the submanifold is natural, the orthogonal component must vanish to give the equations

$$\Xi^b_{\alpha\beta} = \sum_{i=1}^m \frac{\partial u^i}{\partial\tau^\alpha}\frac{\partial u^i}{\partial\tau^\beta}N_i^b = 0.$$

It is convenient at this point to use a Monge parametrisation[1] of $\mathcal{N}$:

$$\begin{aligned}u^i &= \tau^i,\ i=1,\ldots,n,\\ u^{n+b} &= h^b(\underline{\tau}),\ b=1,\ldots,m-n,\end{aligned}$$

With this parametrisation of $\mathcal{N}$, one has:

$$\begin{aligned}\Xi^b_{\alpha\beta} &= \sum_{i=1}^n \frac{\partial u^i}{\partial\tau^\alpha}\frac{\partial u^i}{\partial\tau^\beta}N_i^b + \sum_{j=n+1}^m \frac{\partial u^j}{\partial\tau^\alpha}\frac{\partial u^j}{\partial\tau^\beta}N_i^b,\\ &= \sum_{i=1}^n \delta^i_\alpha\delta^i_\beta N_i^b + \sum_{j=1}^{m-n}\frac{\partial h^j}{\partial\tau^\alpha}\frac{\partial h^j}{\partial\tau^\beta}N^b_{j+n}\\ &= \delta_{\alpha\beta}N^b_\alpha + \sum_{j=1}^{m-n}\frac{\partial h^j}{\partial\tau^\alpha}\frac{\partial h^j}{\partial\tau^\beta}N^b_{j+n}\end{aligned}$$

[1]Note that if $u^i = 0$ for some $i < n$, this parametrisation breaks down. However, this can be overcome by a suitable reordering of the u^i.

Now observe that $\mathcal{N}$ may be described as the intersection of level sets:

$$\mathcal{N} = \bigcap_{b=1}^{m-n} \{\phi^b = 0\},$$

where $\phi^b = h^b - u^{n+b}$. Noting that $\nabla\phi^b$ is orthogonal to the hypersurface described by $\phi^b = 0$, one may choose the $\frac{\partial}{\partial\nu^b}$ so that

$$N_i^b = \frac{\partial\phi^b}{\partial u^i}.$$

This splits into two cases:

$$\begin{aligned} N_i^b &= \frac{\partial h^b}{\partial u^i}, \quad i = 1, \ldots, n, \\ &= -\delta_i^{b+n}, \quad i = n+1, \ldots, m. \end{aligned}$$

Using these values for N_i^b, the $\Xi^b_{\alpha\beta}$ become:

$$\Xi^b_{\alpha\beta} = \delta_{\alpha\beta}\frac{\partial h^b}{\partial u^\alpha} + \sum_{j=n+1}^{m} \frac{\partial h^j}{\partial \tau^\alpha}\frac{\partial h^j}{\partial \tau^\beta}(-\delta^{b+n}_{j+n}).$$

But this must be equal to zero. Also, note that on $\mathcal{N}$, $u^i = u^i(\underline{\tau})$, and so in particular (by applying the chain rule) one has:

$$\frac{\partial}{\partial u^i} = \frac{\partial}{\partial \tau^i}, \quad i \leq n.$$

Combining this with the above and equating to zero gives

$$\delta_{\alpha\beta}\frac{\partial h^b}{\partial \tau^\alpha} - \frac{\partial h^b}{\partial \tau^\alpha}\frac{\partial h^b}{\partial \tau^\beta} = 0.$$

Hence

$$\begin{aligned} \frac{\partial h^b}{\partial \tau^\alpha} &= 0, \quad \alpha \neq \pi(b), \\ &= 1, \quad \alpha = \pi(b), \end{aligned}$$

for a single value $\pi(b)$ (the existence of which is possible but not guaranteed). Hence either $h^b = a^b$ or $h^b = u^{\pi(b)} + a^b$, where a^b is an arbitrary constant. Now, observing that

$$Eh^b = \sum_{i=1}^{m} u^i \frac{\partial}{\partial u^i} h^b = h^b,$$

h^b must be a homogenous function of degree 1, so $a^b = 0$. Hence either $h^b = 0$ or $h^b = u^{\pi(b)}$. Therefore the conditions $h^b - u^{n+b} = 0$ become the discriminant condition (if h^b=0) or the caustic condition (if $h^b = u^{n+b}$).

Theorem 2.17 *Let $\mathcal{M}$ be a semisimple $\mathcal{F}$-manifold and $\mathcal{N}$ be a natural submanifold. The identity field e will be tangential to $\mathcal{N}$ if and only if $\mathcal{N}$ is a pure caustic.*

Proof: The semisimplicity condition implies that the identity field takes the form

$$e = \sum \frac{\partial}{\partial u^i}$$

on $\mathcal{M}$. On a submanifold $\mathcal{N}$, there will be a unity field

$$e_{\mathcal{N}} = \sum_{\alpha=1}^{n} \frac{\partial}{\partial \tau^\alpha}.$$

On a natural submanifold, the orthogonal decomposition of e may be expressed (in the coordinates and notation of the previous theorem) as

$$\sum_{i=1}^{m} \frac{\partial}{\partial u^i} = \sum_{\alpha=1} \frac{\partial}{\partial \tau^\alpha} - \sum_{b=1}^{n-m} \left(1 - \sum_{j=1}^{n} \frac{\partial h^b}{\partial \tau^i}\right) \frac{\partial}{\partial \nu^b}.$$

Hence, for e to be tangential to $\mathcal{N}$, one must have

$$\sum_{j=1}^{n} \frac{\partial h^b}{\partial \tau^i} = 1, \quad b = 1, \ldots, (m-n).$$

But as $h^b = 0$ or $h^b = u^{\pi(b)}$, this condition is only satisfied in the second case, which corresponds to a caustic. It must be true for all $b = 1, \ldots, (m-n)$, in order for e to be tangential to $\mathcal{N}$. Therefore e being tangential implies $\mathcal{N}$ is a pure caustic.

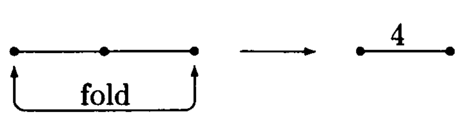
Corollary 2.18 *Any flat pure caustic of a semisimple Frobenius manifold is a natural Frobenius submanifold.*

Proof: Noting that a Frobenius manifold is a special case of an $\mathcal{F}$ manifold, from the theorems above, one has that a pure caustic is natural submanifold and that e and E are tangential to it. Hence the criterium of theorem 2.9 are adhered to. Therefore the flat caustic is a natural Frobenius submanifold.

This result gives a theoretical way to find Frobenius submanifolds, though in reality finding flat caustics is computationally difficult.

2.4 Coxeter subgroups and Frobenius submanifolds

One large class of Frobenius manifolds are those arising from the orbit space of a Coxeter group, as introduced in section 1.6. Where the Coxeter group contains a Coxeter subgroup (which itself corresponds to another Frobenius manifold) one may look for Frobenius submanifolds. In particular, the examples 2.6 and 2.8 respectively correspond to the facts that $I_2(4) \subset A_3$ and $I_2(12) \subset F_4$. These can also be thought of in terms of the foldings of the Dynkin diagrams of the Coxeter groups. For example, the folding of A_3 to give $I_2(4)$ is represented by the diagram below:



A full list of Coxeter groups and their corresponding subgroups is given below.

Coxeter group	Subgroup
A_{2n-1}	B_n
D_{n+1}	B_n
D_4	H_3
E_6	F_4
E_8	H_4
W	$I_2(h)$

In the last line above, W is an arbitrary Coxeter group and h is the Coxeter number of W.

2.5 The Frobenius structure on the A_n caustics

The following example, which originally appeared in [25], is a submanifold of a Frobenius manifold which inherits many of the properties of a Frobenius manifold. However, its curvature prohibits it from being a Frobenius submanifold.

Recall that a Frobenius manifold corresponding to the Coxeter group A_n may be expressed in terms of a superpotential

$$p(z) = z^{n+1} + a_1 z^{n-1} + \ldots + a_n.$$

Differentiating this with respect to z gives

$$p'(z) = (n+1)\prod_{i=1}^{n}(z-\alpha_i).$$

Canonical coordinates were defined on this manifold as the critical values of p, i.e. by the formula

$$u^i = p(\alpha_i).$$

This relied on the assumption the n roots of p' were distinct. If one allows such roots to be equal, however, then $u^i = u^j$ for some $i \neq j$, hence this condition is equivalent to that of a caustic submanifold. Therefore one may define a caustic by the condition

$$p'(z) = (n+1)\prod_{i=1}^{m}(z-\alpha_i)^{k_i},$$

where $\sum k_i = n$ and at least one of the k_i is greater than or equal to 2. Strachan shows in [25] that canonical coordinates $\{\tau^i\}$ may still be defined on such a submanifold. It is also shown that a metric exists, given by the familiar residue formula[2]

$$\eta_{ij} = -\operatorname*{res}_{z=\alpha_i} \frac{\frac{\partial p}{\partial \tau^i}\frac{\partial p}{\partial \tau^j}}{p'(z)}dz,$$

and that this metric is diagonal and has an Egoroff potential

$$\Phi = -\frac{a_1}{n+1}.$$

The submanifold also carries an covariantly constant identity field

$$e = \frac{\partial}{\partial a_n},$$

and an Euler field E. Proof of these statements is deferred to chapter 3, where the same theorem will be proved for the more general case of an arbitrary genus zero Hurwitz space (recall that the A_n type Frobenius manifold corresponds to the simplest class of genus zero Hurwitz space).

2.6 Caustics and discriminants of A_n

Recall from above that A_n type Frobenius manifolds may be expressed in terms of a superpotential λ and that the canonical coordinates are defined as the critical values of λ. A Discriminant submanifold may be described by the condition $u^i = 0$ for some i. This is equivalent to saying λ and λ' have a common root at $z = \alpha_i$. This condition may be defined in terms of the resultant function defined below.

[2]Note that in [25], this formula differs by a factor of (-1).

Definition 2.19 *The resultant of two polynomials* $f(z) = f_m \prod_{i=1}^{m}(z - \alpha_i)$ *and* $g(z) = g_m \prod_j (z - \beta_j)$*, denoted R(f,g), is defined by the formula*

$$R(f,g) = f_m^m g_n^n \prod_{i=0}^{m} \prod_{j=0}^{n} (\alpha_i - \beta_i),$$

where α_i *is a root of* $f(z)$ *and* β_i *is a root of* $g(z)$.

From the definition above, it is obvious that if the resultant of two polynomials is zero, then they must have a common root. A remarkable fact about the resultant function is that it can be defined in a second way. If one considers $f(z)$ as a series, i.e.

$$f(z) = f_m z^m + f_{m-1} z^{m-1} + \ldots + f_0,$$

and likewise

$$g(z) = g_n z^n + g_{n-1} z^{n-1} + \ldots + g_0,$$

then the resultant of f and g is equal to the determinant of the Sylvester matrix defined below.

Definition 2.20 *The* Sylvester matrix *of two polynomials* f *and* g*, of the form above, is an* $(m+n+2) \times (m+n+2)$ *matrix. It is constructed by placing the coefficients* f_m *through to* f_0 *in the first* m *entries of the first row. One moves down a row and right a column and repeats until the entries reach the right hand side. The next row has the coefficients* g_n *to* g_0 *as its first* $n+1$ *entries and the process of moving down a row and right a column is repeated. All other entries are equal to zero. This is easier to understand if visualised as the following matrix:*

This provides a way to define the discriminant locus of an A_n Frobenius manifold in terms of $\{a_i\}$ by the equation

$$R(\lambda, \lambda') = 0.$$

Noting a further property of resultants, namely that for an arbitrary polynomial $f(z)$,

$$R(f,f') = \prod_i (\alpha_i - \alpha_j)^2,$$

the above equation is therefore satisfied if an only if λ has a repeated root.

Example 2.21 *The A_3 superpotential is*

$$\lambda = z^4 + a_1 z^2 + a_2 z + a_3.$$

Differentiating with respect to λ gives

$$\lambda' = 4z^3 + 2a_1 z + a_2.$$

Therefore the discriminant condition is

$$\det \begin{pmatrix} 1 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 & a_2 & a_3 \\ 4 & 0 & 2a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2a_1 & a_2 \end{pmatrix} = 0.$$

Calculating the determinant then leaves

$$-4a_1^3 a_2^2 + 16a_1^4 a_3 + 144a_1 a_2^2 a_3 - 128a_1^2 a_3^2 - 27a_2^4 + 256a_3^3 = 0.$$

This equation gives the swallowtail surface.

One now turns consideration to the caustics of A_n. The condition for a caustic was $u^i - u^j = 0$, for some $i \neq j$. This is automatically satisfied if λ' has a repeated root, i.e. $\alpha_i = \alpha_j$ for some $i \neq j$. In terms of the resultant function, this is equivalent to requiring that

$$R(\lambda', \lambda'') = 0.$$

Example 2.22 *Again, the A_3 case is considered. Differentiating λ' gives*

$$\lambda'' = 12z^2 + 2a_1.$$

One may therefore calculate

$$R(\lambda', \lambda'') = 512a_1^3 + 1728a_2^2.$$

Setting this equal to zero, and noting that a_3 may take an arbitrary value, this surface is the cylinder over a semi cubic parabola. If one writes $\lambda' = (z-\alpha)(z-\alpha)(z+2\alpha)$, canonical coordinates for A_3 are given by

$$\begin{aligned} u^1 &= \lambda(\alpha) &= \tau^1, \\ u^2 &= \lambda(\alpha) &= \tau^1, \\ u^3 &= \lambda(-2\alpha) &= \tau^2. \end{aligned}$$

Hence, tangent vectors are given by

$$\begin{aligned} \partial_{\tau^1} &= \partial_{u^1} + \partial_{u^2}, \\ \partial_{\tau^2} &= \partial_{u^3}. \end{aligned}$$

The multiplication of these vectors (on the ambient manifold) is given by:

$$\begin{aligned} \partial_{\tau^1} \cdot \partial_{\tau^1} &= \partial_{t^1} + \partial_{t^2} &= \partial_{\tau^1}, \\ \partial_{\tau^1} \cdot \partial_{\tau^2} &= 0, \\ \partial_{\tau^2} \cdot \partial_{\tau^2} &= \partial_{t^3} &= \partial_{\tau^2}. \end{aligned}$$

Hence no projection is necessary in order to define the induced multiplication $\star$, so the caustic is a natural submanifold (this fact was already guaranteed by lemma 2.13)

Using theorem 2.17, one also observes that e should be tangential to the caustic. This can easily be verified:

$$e = \partial_{u^1} + \partial_{u^2} + \partial_{u^3} = \partial_{\tau^1} + \partial_{\tau^2}.$$

Therefore the A_3 caustic is a two dimensional submanifold with a tangential identity field, and so is a (natural) Frobenius submanifold by theorem 2.7.

A further case satisfying $u^i - u^j = 0$ for an A_n type Frobenius manifold, without requiring $\alpha_i = \alpha_j$, will now be considered. The locus on which this occurs is known as the *Maxwell strata* and is defined by the condition $\lambda(\alpha_i) = \lambda(\alpha_j)$ for some $\alpha_i \neq \alpha_j$.

Example 2.23 *One again considers the A_3 case. One has*

$$\lambda' = 4(z-\alpha)(z-\beta)(z+\alpha+\beta),$$

with $\alpha \neq \beta$. Comparison of coefficients then gives

$$\begin{aligned} a_1 &= -2(\alpha^2+\alpha\beta+\beta^2), \\ a_2 &= 4\alpha\beta(\alpha+\beta). \end{aligned}$$

Noting that the equation for the Maxwell strata is $\lambda(\alpha) = \lambda(\beta)$, one obtains

$$\alpha^4 + a_1\alpha^2 + a_2\alpha + a_3 = \beta^4 + a_1\beta^2 + a_2\beta + a_3.$$

Substituting the above equations for a_1 and a_2, and noting that the choice of a_3 is arbitrary, one then obtains:

$$(\alpha^4-\beta^4) - 2(\alpha^2-\beta^2)(\alpha^2+\alpha\beta+\beta^2) + 4(\alpha-\beta)\alpha\beta(\alpha+\beta) = 0.$$

This factorises to give

$$(\alpha-\beta)^3(\alpha+\beta) = 0.$$

Hence the only solution (subject to the earlier assumption that $\alpha \neq \beta$) is

$$\alpha = -\beta.$$

Substituting this into the equations for a_1 *and* a_2, *one obtains*

$$\begin{aligned} a_1 &= -\alpha^2, \\ a_2 &= 0. \end{aligned}$$

Therefore the Maxwell strata of A_3 *is a half plane.*

2.7 Genus-zero Hurwitz caustics and discriminants

Recall that for an arbitrary genus-zero Hurwitz space $H_{0;n_0,\ldots,n_m}$, the superpotential was of the form

$$\lambda(z) = z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m}\sum_{s=0}^{n_r} \frac{c_{r,s}}{(z-\beta_r)^{s+1}}.$$

This may be written as the quotient of two polynomials:

$$\lambda(z) = \frac{f(z)}{\prod_{i=1}^{m}(z-\beta_i)^{n_i+1}}.$$

For convenience, denote the denominator of this by $g(z)$. Similarly,

$$\lambda'(z) = (n_0+1)z^{n_0} + \ldots + a_{n_0-1} - \sum_{r=1}^{m}\sum_{s=0}^{n_r} \frac{(s+1)c_{r,s}}{(z-\beta_r)^{s+2}}$$

may be rewritten as

$$\lambda'(z) = \frac{h(z)}{g(z)\prod_{i=1}^{m}(z-\beta_i)}.$$

Hence the discriminant of the Frobenius manifold occurs when

$$R(f(z), h(z)) = 0.$$

However, if one instead obtained λ' by differentiating $\frac{f}{g}$, it would appear in the form

$$\lambda'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

Hence if f and f' have a common root, this will also be a root of λ'. Therefore the discriminant can be found by solving

$$R(f(z), f'(z)) = 0,$$

i.e. by requiring that $f(z)$ has a repeated root.

Similarly, if $h(z)$ has a repeated root, two critical values of λ will coincide, hence this is a condition for finding caustics.

Example 2.24 *Consider the space $H_{0;1,0}$ of functions of the form*

$$\lambda(z) = z^2 + a + \frac{b}{z-c}.$$

This may be rewritten as

$$\lambda(z) = \frac{z^3 - cz^2 + az - (b+ac)}{z-c}.$$

Requiring that the numerator of this has a repeated root gives the condition

$$(ac+b)(9ac+27b+4c^2) = 0.$$

Differentiating λ gives

$$\begin{aligned} \lambda'(z) &= 2z - \frac{b}{z-c^2}, \\ &= \frac{2z^3 - 4z^2c + 2c^2z - b}{(z-c)^2}. \end{aligned}$$

Hence the caustic condition may be calculated using the equation

$$R(2z^3 - 4cz^2 + 2c^2z - b, 6z^2 - 8cz + 2c^2) = 0.$$

Therefore the caustic is defined by the equation

$$27b = 8c^3,$$

with the value of a being arbitrary.

The material contained within this chapter motivates a closer study of caustics and discriminants of Frobenius manifolds. Whilst the theorems have been provided in a very much abstract context, the examples contained within the last two sections show that concrete examples of caustics and discriminants can be defined for certain classes of Frobenius manifolds. Consideration will therefore be given to the Frobenius structures on caustics of a genus zero Hurwitz space in chapter 3. Discriminants of such a Frobenius manifold will then be considered in chapter 4. In particular, the notion of 'almost duality', which conventionally relies on E^{-1} being well defined (recall that E^{-1} is not well defined on a discriminant) will be induced on such discriminants. Finally, an example on a higher genus Hurwitz space will be considered in chapter 5.

Chapter 3

Caustics of genus zero Hurwitz spaces

The result from [25] that caustic submanifolds of a Frobenius manifold corresponding to the Coxeter group A_n carry a set of canonical coordinates, a diagonal Egoroff metric and a covariantly constant identity field has two obvious generalisations. The first is to an arbitrary caustic of any other Coxeter group. However, recalling that the A_n type Frobenius manifold also corresponds to the Hurwitz space $H_{0;n}$, one may attempt to generalise this idea to other Hurwitz spaces. Here, an arbitrary genus zero Hurwitz space will be considered.

Recall that a Frobenius manifold on the Hurwitz space $H_{0;n_0,n_1,\ldots,n_m}$ will have a superpotential of the form

$$\lambda(z) = z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m} \sum_{s=0}^{n_r} \frac{c_{r,s}}{(z-\beta_r)^{s+1}}.$$

The dimension of this space is

$$dim = n_0 + \sum_{i=1}^{m} (n_i + 1) + m.$$

The first n_0 parameters are $\{a_i\}$. There are $\sum_{i=1}^{m}(n_i+1)$ parameters $\{c_{r,s}\}$, whilst the remaining m parameters are $\{\beta_i\}$. Differentiating λ with respect to z gives

$$\frac{d\lambda}{dz} = (n_0+1)z^{n_0} + (n_0-1)z^{n_0-2} + \ldots + a_{n_0-1} - \sum_{r=1}^{m}\sum_{s=0}^{nri} \frac{(s+1)c_{r,s}}{(z-\beta_r)^{s+2}}.$$

Lemma 3.1 $\frac{d\lambda}{dz}$ *may be written as the quotient of two polynomials:*

$$\frac{d\lambda}{dz} = (n_0+1)\frac{\prod_{i=1}^{dim}(z-\alpha_i)}{\prod_{j=1}^{m}(z-\beta_j)^{n_j+2}} = \frac{f(z)}{g(z)}.$$

This quotient will be in its simplest form, i.e. there is no cancellation of factors between the numerator and denominator (which is equivalent to the condition $\alpha_i \neq \beta_j$, *for all possible* i *and* j*).*

Proof: To prove that $\frac{d\lambda}{dz}$ may be written in the quotient form above is trivial; one needs only to multiply every term by $\frac{\prod_{i=1}^{m}(z-\beta_i)^{n_i+2}}{\prod_{j=1}^{m}(z-\beta_j)^{n_j+2}}$. The numerator will therefore be a polynomial of degree $n_0+\sum_{i=1}^{m}(n_i+2) = \dim$. Note also that if one considers

$$f = \lambda' \prod_{i=1}^{m}(z-\beta_i)^{n_i+2},$$

then $(z-\beta_i)$ multiplies every term except $\frac{(n_i+1)c_{i,n_i}}{\prod_{j\neq i}(z-\beta_j)^{n_j+2}}$. Hence

$$f|_{z=\beta_i} = \frac{(n_i+1)c_{i,n_i}}{\prod_{j\neq i}(\beta_i-\beta_j)^{n_j+2}}.$$

But as $c_{i,n_i} \neq 0$ (or the pole at of λ at $z=\beta_i$ would be of a lower degree and so the function λ would belong to a different Hurwitz space), this must be non-zero, and so $(z-\beta_i)$ is not a factor in the numerator of $\frac{d\lambda}{dz}$.

As the critical points of λ are precisely the α_i, the critical values of λ, which are (canonical) coordinates on the manifold are given by $u^i = \lambda(\alpha_i)$. If $\alpha_i = \alpha_j$ for some $i \neq j$, one has $u^i = u^j$ and that point is on a caustic. On a general caustic, one has

$$\frac{d\lambda}{dz} = (n_0+1)\frac{\prod_{i=1}^{q}(z-\alpha_i)^{k_i}}{\prod_{j=1}^{m}(z-\beta_j)^{n_j+2}}, \quad k_i \in \mathbb{N}, \quad \sum k_i = dim.$$

Hence q coordinates can be defined by

$$u^i = \lambda(\alpha_i),\ i = 1, \ldots, q.$$

Lemma 3.2 *The multiplication in these coordinates will be canonical, i.e.*

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}.$$

Proof: If one denotes the original canonical coordinates on the ambient manifold by $\{x^i\}$, then u^i will be equal to k_i of these, i.e.

$$u^i = x^{i_1} = \ldots = x^{i_{k_i}}.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial u^i} &= \sum_k \frac{\partial x^{i_k}}{\partial u^i} \frac{\partial}{\partial x^{i_k}}, \\ &= \sum_k \frac{\partial}{\partial x^{i_k}}. \end{aligned}$$

Multiplying two vectors of this form will then yield

$$\begin{aligned} \frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} &= \sum_k \frac{\partial}{\partial x^{i_k}} \cdot \sum_l \frac{\partial}{\partial x^{j_l}}, \\ &= \sum_k \sum_l \frac{\partial}{\partial x^{i_k}} \cdot \frac{\partial}{\partial x^{j_l}}. \end{aligned}$$

Now, noting that $i_k \neq j_l$ for all k and l (provided $i \neq j$), this must be zero for. For $i = j$,

$$\begin{aligned} \frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^i} &= \sum_k \sum_l \delta_{kl} \frac{\partial}{\partial x^{i_k}}, \\ &= \sum_k \frac{\partial}{\partial x^{i_k}}, \\ &= \frac{\partial}{\partial u^i}. \end{aligned}$$

Lemma is proved.

As the coordinates $\{u^i\}$ are independent, we have

$$\frac{\partial u^j}{\partial u^i} = \delta_{ij}.$$

Now introduce a new function

$$f(z) = \frac{f_1(z)}{f_2(z)} = \frac{\sum_r v_r(\alpha) z^r}{\sum_s w_s(\beta) z^s},$$

where v_r and w_s are functions of $\alpha = (\alpha_1, \ldots, \alpha_q)$. Differentiating $f(\alpha_j)$ with respect to u^i, one has

$$\begin{aligned}
\frac{\partial f(\alpha_j)}{\partial u^i} &= \frac{\partial}{\partial u^i}\left(\frac{\sum_r v_r(\alpha)(\alpha_j)^r}{\sum_s w_s(\beta)(\alpha_j)^s}\right), \\
&= \frac{\frac{\partial}{\partial u^i}\sum_r v_r(\alpha)(\alpha_j)^r}{\sum_s w_s(\beta)(\alpha_j)^s} - \frac{\left(\sum_r v_r(\alpha)(\alpha_j)^r\right)\frac{\partial}{\partial u^i}\sum_s w_s(\beta)(\alpha_j)^s}{\left(\sum_s w_s(\beta)(\alpha_j)^s\right)^2}, \\
&= \frac{\sum_r \frac{\partial v_r(\alpha)\alpha_j^r}{\partial u^i} + \sum_r v_r(\alpha) r (\alpha_j)^{r-1}\frac{\partial \alpha_j}{\partial u^i}}{\sum_s w_s(\beta)(\alpha_j)^s} \\
&\quad - \frac{\left(\sum_r v_r(\alpha)(\alpha_j)^r\right)\left(\sum_s \frac{\partial (w_s(\beta)\alpha_j^s)}{\partial u^i} + \sum_s w_s(\beta) s (\alpha_j)^{s-1}\frac{\partial \alpha_j}{\partial u^i}\right)}{\left(\sum_s w_s(\beta)(\alpha_j)^s\right)^2}, \\
&= \frac{\frac{\partial f_1}{\partial u^i}\big|_{z=\alpha_j} + \frac{\partial f_1}{\partial z}\big|_{z=\alpha_j}\frac{\partial \alpha_j}{\partial u^i}}{f_2|_{z=\alpha_j}} - \frac{f_1|_{z=\alpha_j}\left(\frac{\partial f_2}{\partial u^i}\big|_{z=\alpha_j} + \frac{\partial f_2}{\partial z}\big|_{z=\alpha_j}\frac{\partial \alpha_j}{\partial u^i}\right)}{(f_2|_{z=\alpha_j})^2}, \\
&= \frac{\frac{\partial f_1}{\partial u^i}\big|_{z=\alpha_j}}{f_2|_{z=\alpha_j}} - \frac{f_1|_{z=\alpha_j}\frac{\partial f_2}{\partial u^i}\big|_{z=\alpha_j}}{(f_2|_{z=\alpha_j})^2} + \frac{\partial \alpha_j}{\partial u^i}\left(\frac{\frac{df_1}{dz}\big|_{z=\alpha_j}}{f_2|_{z=\alpha_j}} - \frac{f_1|_{z=\alpha_j}\frac{df_2}{dz}\big|_{z=\alpha_j}}{(f_2|_{z=\alpha_j})^2}\right), \\
&= \frac{\partial f}{\partial u^i}\bigg|_{z=\alpha_j} + \frac{\partial \alpha_j}{\partial u^i}\frac{df}{dz}\bigg|_{z=\alpha_j}.
\end{aligned}$$

If one now sets $f = \lambda$, then

$$\delta_{ij} = \frac{\partial \lambda}{\partial u^i}\bigg|_{z=\alpha_j} + \frac{\partial \alpha_j}{\partial u^i}\frac{d\lambda}{dz}\bigg|_{z=\alpha_i}.$$

But α_j is a root of $\frac{d\lambda}{dz}$, so

$$\delta_{ij} = \frac{\partial \lambda}{\partial u^i}\bigg|_{z=\alpha_j}.$$

This gives q equations for $\frac{\partial \lambda}{\partial u^i}$. Next consider the case of $f = \frac{d^k\lambda}{dz^k}$.

$$\frac{\partial}{\partial u^i}\frac{d^k\lambda}{dz^k} = \frac{d^k}{dz^k}\frac{\partial \lambda}{\partial u^i}.$$

Restricting this to $z = \alpha_j$, one has

$$\begin{aligned}\frac{\partial}{\partial u^i}\frac{d^k\lambda}{dz^k}\Big|_{z=\alpha_j} &= \frac{d^k}{dz^k}\frac{\partial\lambda}{\partial u^i}\Big|_{z=\alpha_j},\\ &= \frac{d^k}{dz^k}\delta_{ij},\\ &= 0, \quad 1 \le k \le k_j - 1.\end{aligned}$$

Hence one has a further $\sum(k_j - 1)$ equations for $\frac{\partial\lambda}{\partial u^i}$. Coupling the equations above with the other q equations, one may determine $\frac{\partial\lambda}{\partial u^i}$. Note in particular that $\frac{\partial\lambda}{\partial u^i}$ will have a zero of degree k_j at α_j, for $j \neq i$.

Lemma 3.3 *$\frac{\partial\lambda}{\partial u^i}$ must be of the form*

$$\frac{\partial\lambda}{\partial u^i} = f_i(z)\prod_{s\neq i}(z-\alpha_s)^{k_s}.$$

The function $f_i(z)$ will be of the form

$$p^{(i)}(z)\prod_{r=1}^{m}(z-\beta_r)^{-(n_r+2)},$$

where $p^{(i)}(z)$ is a polynomial of degree $k_i - 1$.

Proof: Differentiating λ with respect to $u^i = u^i(a, c, \beta)$, one has

$$\begin{aligned}\frac{\partial\lambda}{\partial u^i} &= \frac{\partial}{\partial u^i}\left(z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m}\sum_{s=0}^{n_r}\frac{c_{r,s}}{(z-\beta_r)^{s+1}}\right),\\ &= \frac{\partial a_1}{\partial u^i}z^{n_0-1} + \ldots + \frac{\partial a_{n_0}}{\partial u^i} + \sum_{r=1}^{m}\sum_{s=0}^{n_r}\frac{\partial}{\partial u^i}\frac{c_{r,s}}{(z-\beta_r)^{s+1}}.\end{aligned}$$

But

$$\frac{\partial}{\partial u^i}\frac{c_{r,s}}{(z-\beta_r)^{s+1}} = \frac{\partial c_{r,s}}{\partial u^i}\frac{1}{(z-\beta_r)^{s+1}} - c_{r,s}\frac{\partial\beta_r}{\partial u^i}\frac{1}{(z-\beta_r)^{s+2}}.$$

Putting everything over a common denominator of $\prod_{l=1}^{m}(z-\beta_r)^{n_l+2}$, one obtains

$$\frac{\partial\lambda}{\partial u^i} = \frac{P^{(i)}(z)}{\prod_{l=1}^{m}(z-\beta_l)^{n_l+2}},$$

where $P^{(i)}(z)$ is a polynomial of degree $n_0 - 1 + \sum_{l=1}^{m}(n_l + 2) = dim - 1$. One may now, recalling that the $\frac{\partial\lambda}{\partial u^i}$ have zeros of respective degrees k_s at α_s, $s \neq i$ factor out $\prod_{s\neq i}(z-\alpha_s)^{k_s}$. Hence

$$\frac{\partial\lambda}{\partial u^i} = \frac{p^{(i)}(z)\prod_{s\neq i}(z-\alpha_s)^{k_s}}{\prod_{l=1}^{m}(z-\beta_l)^{n_l+2}}.$$

The degree of $p^{(i)}(z)$ must be

$$deg(p^{(i)}(z)) = (dim - 1) - \sum_{s\neq i} k_s = k_i - 1,$$

as required.

Lemma 3.4 *In the canonical coordinates $\{u^i\}$, a diagonal metric is given by the formula:*

$$\eta_{ij} = -\sum \operatorname*{res}_{d\lambda=0} \frac{\frac{\partial\lambda}{\partial u^i}\frac{\partial\lambda}{\partial u^j}}{\frac{d\lambda}{dz}} dz.$$

Proof: This metric may be calculated explicitly, with consideration being given to the two possible cases of i and j either being distinct or equal.

Case 1: $i \neq j$

$$\eta_{ij} = -\sum_{l=1}^{q} \operatorname*{res}_{z=\alpha_l} \frac{p^{(i)}(z)\prod_{r\neq i}(z-\alpha_r)^{k_i} p^j(z)\prod_{s\neq j}(z-\alpha_s)^{k_s}}{(n_0+1)\prod_{t=1}^{q}(z-\alpha_t)^{k_t}\prod_{u=1}^{m}(z-\beta_u)^{-(n_u+2)}} dz.$$

Noting that $(z-\alpha_l)^{k_l}$ is a factor in the numerator at least once for $l = 1, \ldots, q$, the residues are of a finite function at these points. Hence the residues are all zero and so $\eta_{ij} = 0$ for $i \neq j$.

Case 2: $i = j$

$$\eta_{ii} = -\sum_{l=1}^{q} \operatorname*{res}_{z=\alpha_l} \frac{(p^{(i)})^2\prod_{r\neq i}(z-\alpha_r)^{2k_r}}{(n_0+1)\prod_{t=1}^{q}(z-\alpha_t)^{k_t}\prod_{u=1}^{m}(z-\beta_t)^{-(n_i+2)}} dz.$$

This is finite at all of the α_l where $l \neq i$. Hence the only residue is that at $z = \alpha_i$. But recall from above that $\frac{\partial\lambda}{\partial u^i}|_{z=\alpha_i} = 1$. Hence

$$\eta_{ii} = -\operatorname*{res}_{z=\alpha_i}\left(\frac{1}{\frac{d\lambda}{dz}}\right) dz.$$

Therefore a diagonal metric exists in the coordinates $\{u^i\}$.

Lemma 3.5 *The metric η Egoroff, with potential $\Phi = -\frac{a_1}{n_0+1}$.*

Proof: Firstly, note that

$$\frac{1}{\frac{d\lambda}{dz}} = \frac{\prod_{r=1}^{m}(z-\beta_r)^{n_r+2}\prod_{s\neq i}(z-\alpha_s)^{-k_s}}{(n_0+1)(z-\alpha_i)^{k_i}}.$$

Define a new function

$$h^{(i)} = \prod_{r=1}^{m}(z-\beta_r)^{-(n_r+2)}\prod_{s\neq i}(z-\alpha_s)^{k_s}.$$

As this function is analytic at $z=\alpha_i$, it may be expressed as a Taylor series

$$h^{(i)} = \sum_{s=0}^{\infty}\frac{h_s^{(i)}(z-\alpha_i)^s}{s!}.$$

Also, as $h^{(i)} \neq 0$ in a neighbourhood of α_i, its inverse $h^{(-i)}$ may also be expressed as a Taylor series in $(z-\alpha_i)$:

$$h^{(-i)} = \sum_{s=0}^{\infty}\frac{h_s^{(-i)}(z-\alpha_i)^s}{s!}.$$

Using the trivial fact that $h^{(i)}h^{(-i)} = 1$, one has

$$\begin{aligned} 1 &= \sum_{s=0}^{\infty}\frac{h_s^{(i)}(z-\alpha_i)^s}{s!}\sum_{t=0}^{\infty}\frac{h_t^{(-i)}(z-\alpha_i)^t}{t!}, \\ &= \sum_{r=0}^{\infty}\left(\sum_{s+t=r}\frac{r!h_s^{(i)}h_t^{(-i)}}{s!t!}\right)\frac{(z-\alpha_i)^r}{r!}. \end{aligned}$$

Comparing coefficients of ascending powers of $(z-\alpha_i)$ then yields the system of equations

$$\begin{aligned} 1 &= h_0^{(i)}h_0^{(-i)}, \\ 0 &= h_0^{(i)}h_1^{(-i)} + h_1^{(i)}h_0^{(-i)}, \\ &\vdots \\ 0 &= \sum_{s=0}^{r}\binom{r}{s}h_s^{(i)}h_{r-s}^{(-i)}. \end{aligned}$$

This system determines the $h_s^{(-i)}$ if the $h_s^{(i)}$ are known. One may now substitute the function $h^{(-i)}$ into the formula for η_{ii} to obtain

$$\begin{aligned}
\eta_{ii} &= -\operatorname*{res}_{z=\alpha_i}\left(\frac{1}{(n_0+1)(z-\alpha_i)^{k_i}}h^{(-i)}\right)dz,\\
&= -\frac{1}{n_0+1}\operatorname*{res}_{z=\alpha_i}\left(\frac{1}{(z-\alpha_i)^{k_i}}\sum_s h_s^{(-i)}\frac{(z-\alpha_i)^s}{s!}\right)dz,\\
&= -\frac{1}{n_0+1}\frac{h_{k_i-1}^{(-i)}}{(k_i-1)!}.
\end{aligned}$$

We now recall that

$$\begin{aligned}
\frac{\partial\lambda}{\partial u^i} &= p^{(i)}(z)\prod_{r=1}^{m}(z-\beta_r)^{-(n_r+2)}\prod_{s\neq i}(z-\alpha_s)^{k_s},\\
&= p^{(i)}(z)h^{(i)}(z).
\end{aligned}$$

As $p^{(i)}(z)$ is simply a polynomial of degree (k_i-1), it has a finite Taylor series

$$p^{(i)}(z)=\sum_{r=0}^{k_i-1}p_r^{(i)}\frac{(z-\alpha_i)^r}{r!}.$$

Recalling that

$$\left.\frac{\partial\lambda}{\partial u^i}\right|_{z=\alpha_i}=1,$$

comparison of coefficients yields

$$p_0^{(i)}h_0^{(i)}=1.$$

Similarly,

$$\left.\frac{d^k}{dz^k}\frac{\partial\lambda}{\partial u^i}\right|_{z=\alpha_i}=0,\quad 1\leq k\leq k_i-1$$

becomes

$$\begin{aligned}
0 &= \frac{d^k}{dz^k}\left(p^{(i)}h^{(i)}\right)\Big|_{z=\alpha_i},\\
&= \sum_{r=0}^{k_i-1}\binom{k_i-1}{r}\left(\sum_{s=0}^{r}\binom{r}{s}\frac{d^sp^{(i)}}{dz^s}\frac{d^{r-s}h^{(i)}}{dz^{r-s}}\right)z^{k_i-1-r}\Big|_{z=\alpha_i}.
\end{aligned}$$

This is clearly satisfied by

$$\sum_{s=0}^{r}\binom{r}{s}p_s^{(i)}h_{r-s}^{(i)}=0,\ r\leq k_i-1.$$

Therefore for known $h_s^{(i)}$, the $p_r^{(i)}$ are defined by the system

$$\begin{aligned} 1 &= h_0^{(i)} p_0^{(i)}, \\ 0 &= h_0^{(i)} p_1^{(i)} + h_1^{(i)} p_0^{(i)}, \\ \vdots & \quad \vdots \\ 0 &= \sum_{s=0}^{k_i-1} \binom{k_i-1}{s} h_s^{(i)} p_{k_i-1-s}^{(i)}. \end{aligned}$$

But these are identical to the first k_i equations in the system for $h_s^{(-i)}$. Therefore

$$p_s^{(i)} = h_s^{(-i)}, \quad s \leq k_i - 1.$$

Hence

$$\eta_{ii} = -\frac{1}{n_0+1} \frac{p_{k_i-1}^{(i)}}{(k_i-1)!}.$$

As (z^{k_i-1}) is the highest power of z in $p^{(i)}$, so its coefficient will be the same irrespective of whether the function $p^{(i)}$ is expressed in terms of z or $(z-\alpha_i)$. So

$$\frac{p_{k_i-1}^{(i)}}{(k_i-1)!} = \text{coefficient of}\, z^{k_i-1}\, \text{in}\, p^{(i)}(z).$$

Recalling that

$$\frac{\partial \lambda}{\partial u^i} = p^{(i)}(z) \prod_{r=1}^{m} (z-\beta)^{-(n_i+2)} \prod_{s \neq i} (z-\alpha_s)^{k_s},$$

we see that $p_{k_i-1}^{(i)}$ must also be the coefficient of the highest power of z in $\frac{\partial \lambda}{\partial u^i}$. Therefore

$$p_{k_i-1}^{(i)} = \frac{\partial a_1}{\partial u^i}.$$

But this is true for all i, and so

$$\eta_{ii} = -\frac{1}{n_0+1} \frac{\partial a_1}{\partial u^i} = \frac{\partial}{\partial u^i} \frac{-a_1}{n_0+1}.$$

Therefore the metric is Egoroff with potential

$$\Phi = -\frac{a_1}{n_0+1}.$$

Lemma 3.6 *The identity field may be expressed by*

$$e = \frac{\partial}{\partial a_{n_0}}.$$

Proof: By the chain rule,

$$\frac{\partial}{\partial a_{n_0}} = \frac{\partial u^i}{\partial a_{n_0}} \frac{\partial}{\partial u^i}.$$

But as $u^i = \lambda(\alpha_i)$,

$$\frac{\partial}{\partial a_{n_0}} = \sum_i \frac{\partial}{\partial u^i}.$$

Recalling that $\{u^i\}$ are canonical coordinates, this is the identity field as required.

Lemma 3.7 *The Euler field on the caustic will be the same as the Euler field on the ambient manifold.*

Proof: As $\{u^i\}$ are canonical coordinates, we must have

$$E = \sum_i u^i \frac{\partial}{\partial u^i}.$$

But recalling the definition of $\frac{\partial}{\partial u^i}$ from lemma 2 along with $u^i = x^{i_1} = \ldots = x^{i_{k_i}}$, one may write:

$$\begin{aligned} E &= \sum_i \sum_j u^i \frac{\partial}{\partial x^{i_j}}, \\ &= \sum_i \sum_j x^{i_j} \frac{\partial}{\partial x^{i_j}}. \end{aligned}$$

But this is equal to the Euler field on $H_{0;n_0,\ldots,n_m}$, as required.

Combining the results of this chapter, one obtains the following lemma:

Lemma 3.8 *On caustics of a genus zero Hurwitz space, the following aspects of the structure of a Frobenius manifold remain:*

- *a set of canonical coordinates* $\{u^i\}$,
- *a diagonal Egoroff metric defined by the familiar residue formula and with a potential* $\Phi = -\frac{a_1}{n_0+1}$,
- *an identity field equal to* $\frac{\partial}{\partial a_{n_0}}$.

Proof: The proof of this theorem follows immediately from the lemmas in this chapter, as it is a collation of their results.

Whilst it has been shown that much of the structure of a Frobenius manifold exists on the caustics of $H_{0;n_0,\ldots,n_m}$, an arbitrary caustic will not (in general) be a Frobenius manifold. This is due to curvature; whilst the existence of a diagonal metric has been shown, this metric (just as in the A_n case) will in general be curved.

Example 3.9 *Consider the Hurwitz space* $H_{0;1,0}$. *Recall that this has as superpotential of the form*

$$\lambda(z) = z^2 + a + \frac{b}{z-c}.$$

Also recall from chapter 2 that the caustic condition is $b - \frac{8}{27}c^3$. *Substituting this into* λ', *one obtains*

$$\begin{aligned} \lambda'(z) &= \frac{2z(z-c)^2 - \frac{8}{27}c^3}{(z-c)^2}, \\ &= \frac{\left(z-\frac{c}{3}\right)^2\left(z-\frac{4c}{3}\right)}{(z-c)^2}. \end{aligned}$$

These values of z *may be substituted into* λ *to obtain the canonical coordinates* $\{u^i\}$:

$$\begin{aligned} u^1 &= \frac{c^2}{9} + a + \frac{3b}{2c}, \\ u^2 &= \frac{16c^2}{9} + a + \frac{3b}{c}. \end{aligned}$$

Using the residue formula 1.10, one may calculate the metric:

$$\eta = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{18} \end{pmatrix}.$$

Note that as the coefficients in this metric are constants, this metric is in fact flat. But as curvature was the only obstruction to a caustic submanifold being a Frobenius submanifold, so this caustic is a Frobenius submanifold, as predicted by theorem 2.7. Note, however, that one may not immediately integrate up to a prepotential; whilst the canonical coordinates are a flat set of coordinates for the two dimensional submanifold, they are not the distinguished set of flat coordinates of the WDVV equations.

Having shown which aspects of the structure of a Frobenius manifold are retained by a caustic submanifold of a genus zero Hurwitz space, a natural generalisation would be to consider the same problem for higher genus cases. One would expect similar results (some of which are guaranteed by the theorems in Chapter 2), namely that in addition to an associative algebra being defined on the tangent space, canonical coordinates, a diagonal Egoroff metric and a tangential identity field would still exist. However, as even the simplest example of a higher genus Hurwitz space (namely $H_{1;n}$) involves an elliptic superpotential, calculations in such a case would be problematic, and would become worse for more complicated genus one or even higher genus Hurwitz spaces.

Another interesting point raised by this chapter is that whilst most of the structure of a Frobenius manifold remains intact on a caustic, curvature prevents structure constants being integrated up to a prepotential. This raises an obvious question of how one could find flat caustics. Whilst it is possible to derive conditions for a caustic to be flat, finding specific examples would be computationally very difficult. Note, however, that some examples of flat caustics have been found, e.g. the planes studied by Zuber in [29] by utilising various symmetries on the orbit spaces of Coxeter groups.

Chapter 4

Almost duality for genus zero Hurwitz discriminants

4.1 A_n discriminants

Recall from chapter 1 that the locus in a Frobenius manifold over which the Euler field is not invertible (and so the intersection form is undefined) is known as the discriminant. As the almost duality formulae contained in [13] and outlined in 1.7 are defined in terms of E^{-1} and g_{ij}, the ideas of almost duality may not be applied directly to a discriminant of a generic Frobenius manifold. However, if one restricts consideration to semisimple Frobenius manifolds, one has an alternative tool to describe structural data on an almost dual manifold; namely the residue formula (1.23). This formula does not rely explicitly on E^{-1} or g being well defined.

Also, recall from chapter 2 that for the familiar A_n type Frobenius manifolds, the condition for a discriminant is that the superpotential λ has a repeated root. For an arbitrary discriminant, λ will be of the form

$$\lambda(z) = \prod_{i=0}^{m} (z - z_i)^{k_i},$$

with $\sum k_i = n + 1$ and $\sum z_i k_i = 0$. The second of these two conditions implies that

$$z_0 = z_0(z_1, \ldots, z_m, k_0, \ldots, k_m) = -\sum_{1=1}^{m} \frac{k_i z_i}{k_0}.$$

Noting that $\{z_i\}, i = 1, \ldots, m$ are coordinates for the discriminant submanifold, one may use tangent vectors $\frac{\partial}{\partial z^i}$ in the formula for $c^\star(\partial', \partial'', \partial''')$. For notional convenience, one will denote $c^\star(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})$ by $c^\star_{ijk}$. Hence

$$c^\star_{ijk} = -\sum \operatorname*{res}_{d\log\lambda=0} \left(\frac{\partial \log \lambda}{\partial z^i} \frac{\partial \log \lambda}{\partial z^j} \frac{\partial \log \lambda}{\partial z^k} \right) \frac{\lambda}{\lambda'} dz.$$

Lemma 4.1 *For i, j, k distinct,*

$$c^\star_{ijk} = -\frac{k_i k_j k_k}{k_0^2} \sum_{r=1}^{m} \frac{k_r}{z_0 - z_r} - \frac{k_i k_j k_k}{k_0} \left(\frac{1}{z_0 - z_i} + \frac{1}{z_0 - z_j} + \frac{1}{z_0 - z_k} \right).$$

Proof: One begins by explicitly calculating $\frac{\partial \log \lambda}{\partial z^i}$, taking care to remember the z_i dependence of z_0:

$$\begin{aligned}
\frac{\partial \log \lambda}{\partial z_i} &= \frac{\partial z_0}{\partial z_i} \frac{\partial \log \lambda}{\partial z_0} + \frac{\partial \log \lambda}{\partial z_i}, \\
&= \frac{-k_i}{k_0} \frac{\partial}{\partial z_0} \sum_{j=0}^{m} k_j \log(z - z_j) + \frac{\partial}{\partial z_i} \sum_{j=0}^{m} k_j \log(z - z_j), \\
&= k_i \left(\frac{1}{z - z_0} - \frac{1}{z - z_i} \right).
\end{aligned}$$

Substituting this into the residue formula yields:

$$\begin{aligned}
c^\star_{ijk} &= -k_i k_j k_k \sum \operatorname*{res}_{d\log\lambda=0} \\
&\left(\frac{1}{z - z_0} - \frac{1}{z - z_i} \right) \left(\frac{1}{z - z_0} - \frac{1}{z - z_j} \right) \left(\frac{1}{z - z_0} - \frac{1}{z - z_k} \right) \frac{\lambda(z)}{\lambda'(z)} dz.
\end{aligned}$$

One is now faced with a problem; residues are taken at points where $\frac{d \log \lambda}{dz} = 0$ but the roots of this are not known explicitly. However, by a deformation of contours argument (which relies on the fact that the sum of residues of a meromorphic differential over all points on the Riemann sphere is zero), it can easily be shown that

$$\begin{aligned} c^{\star}_{ijk} &= k_i k_j k_k \left(\operatorname*{res}_{z=z_i} + \operatorname*{res}_{z=z_j} + \operatorname*{res}_{z=z_k} + \operatorname*{res}_{z=z_0} + \operatorname*{res}_{z=\infty} \right) \\ & \quad \left(\frac{1}{z-z_0} - \frac{1}{z-z_i} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_j} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_k} \right) \frac{\lambda(z)}{\lambda'(z)} dz. \end{aligned}$$

As taking residues obeys the distributive law for addition, this can be split up into five separate residue calculations. Firstly, consider the residue at $z = \infty$. Noting that as z becomes very large,

$$\left(\frac{1}{z-z_0} - \frac{1}{z-z_i} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_j} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_k} \right) \frac{\lambda(z)}{\lambda'(z)} \approx \frac{k_i k_j k_k}{z^2},$$

the residue at $z = \infty$ is zero.

Next, consider the residue at $z = z_i$. Observing that $(z - z_i)^{k_i}$ is a factor of λ, so $(z - z_i)^{k_i - 1}$ is a factor of λ', and hence

$$\frac{\lambda(z)}{\lambda'(z)} = (z - z_i)\tilde{\lambda}_i,$$

with $\tilde{\lambda}_i$ being a rational function which is finite and non-zero at $z = z_i$. Hence

$$\begin{aligned} & \operatorname*{res}_{z=z_i} k_i k_j k_k \left(\frac{1}{z-z_0} - \frac{1}{z-z_i} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_j} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_k} \right) \frac{\lambda(z)}{\lambda'(z)} dz, \\ = & \operatorname*{res}_{z=z_i} k_i k_j k_k \left(\frac{1}{z-z_0} - \frac{1}{z-z_i} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_j} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_k} \right) (z - z_i)\tilde{\lambda}_i dz, \\ = & \operatorname*{res}_{z=z_i} k_i k_j k_k \left(\frac{z-z_i}{z-z_0} - 1 \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_j} \right) \left(\frac{1}{z-z_0} - \frac{1}{z-z_k} \right) \tilde{\lambda}_i dz, \\ = & \ 0, \end{aligned}$$

as the function whose residue is being taken at $z = z_i$ is finite there.

Repeating this argument, one can also show that the residues at $z = z_j$ and $z = z_k$ are equal to zero.

Hence

$$
\begin{aligned}
c^\star_{ijk} &= \operatorname*{res}_{z=z_0} k_i k_j k_k \left(\frac{1}{z-z_0}-\frac{1}{z-z_i}\right)\left(\frac{1}{z-z_0}-\frac{1}{z-z_j}\right)\left(\frac{1}{z-z_0}-\frac{1}{z-z_k}\right)\frac{\lambda}{\lambda'}dz,\\
&= k_i k_j k_k \operatorname*{res}_{z=z_0}\Bigg\{\overbrace{\frac{1}{(z-z_0)^3}}^{Q}-\overbrace{\frac{1}{(z-z_0)^2}\left(\frac{1}{z-z_i}+\frac{1}{z-z_j}+\frac{1}{z-z_k}\right)}^{R}\\
&\quad+\underbrace{\frac{1}{z-z_0}\left(\frac{1}{(z-z_i)(z-z_j)}+\frac{1}{(z-z_i)(z-z_k)}+\frac{1}{(z-z_j)(z-z_k)}\right)}_{S}\\
&\quad-\underbrace{\frac{1}{(z-z_i)(z-z_j)(z-z_k)}}_{T}\Bigg\}\frac{\lambda(z)}{\lambda'(z)}dz,\\
&= k_i k_j k_k\left(\operatorname*{res}_{z=z_0} Q\frac{\lambda}{\lambda'}dz-\operatorname*{res}_{z=z_0} R\frac{\lambda}{\lambda'}dz+\operatorname*{res}_{z=z_0} S\frac{\lambda}{\lambda'}dz-\operatorname*{res}_{z=z_0} T\frac{\lambda}{\lambda'}dz\right).
\end{aligned}
$$

One firstly considers Q:

$$
\begin{aligned}
\operatorname*{res}_{z=z_0}\frac{1}{(z-z_0)^3}\frac{\lambda}{\lambda'}dz &= \text{coefficient of}\,\frac{1}{z-z_0}\,\text{in Laurant expansion of}\,\frac{1}{(z-z_0)^3}\frac{\lambda}{\lambda'},\\
&= \text{coefficient of}\,(z-z_0)^2\,\text{in Taylor expansion of}\,\frac{\lambda}{\lambda'},\\
&= \frac{1}{2!}\frac{d^2}{dz^2}\frac{\lambda}{\lambda'}\bigg|_{z=z_0}.
\end{aligned}
$$

If one writes λ as

$$\lambda=(z-z_0)^{k_0}\lambda_0,$$

with

$$\lambda_0=\prod_{i=1}^{m}(z-z_i)^{k_i},$$

then one has

$$\lambda'=(z-z_0)^{k_0}\lambda_0'+k_0(z-z_0)^{k_0-1}\lambda_0.$$

Differentiating $\frac{\lambda}{\lambda'}$ twice therefore yields:

$$
\begin{aligned}
\frac{d^2}{dz^2}\frac{\lambda}{\lambda'}\bigg|_{z=z_0} &= \frac{d^2}{dz^2}\frac{(z-z_0)^{k_0}\lambda_0}{(z-z_0)^{k_0}\lambda_0'+k_0(z-z_0)^{k_0-1}\lambda_0}\bigg|_{z=z_0},\\
&= \frac{d^2}{dz^2}\frac{\overbrace{(z-z_0)\lambda_0}^{p}}{\underbrace{(z-z_0)\lambda_0'+k_0\lambda_0}_{q}}\bigg|_{z=z_0},
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d}{dz}\left(\frac{p'}{q} - \frac{pq'}{q^2}\right)\Big|_{z=z_0}, \\
&= \left(\frac{p''}{q} - \frac{p'q'}{q^2}\right) - \left(\frac{p'q' + pq''}{q^2} - 2\frac{2pq'q'}{q^3}\right)\Big|_{z=z_0}, \\
&= \frac{qp'' - 2p'q' - pq'' + 4p(q')^2}{q^2}\Big|_{z=z_0}, \\
&= \frac{qp'' - 2p'q'}{q^2}\Big|_{z=z_0},
\end{aligned}
$$

The last line above is obtained by noting that $p|_{z=z_0} = 0$. Every term which appears in the last line above may then be considered individually:

$$
\begin{aligned}
q|_{z=z_0} &= k_0\lambda_0|_{z=z_0}. \\
q'|_{z=z_0} &= (\lambda_0' + (z - z_0)\lambda_0'' + k_0\lambda_0')|_{z=z_0}, \\
&= (1 + k_0)\lambda_0'|_{z=z_0}. \\
p'|_{z=z_0} &= ((z - z_0)\lambda_0' + \lambda_0)|_{z=z_0}, \\
&= \lambda_0|_{z=z_0}. \\
p''|_{z=z_0} &= ((z - z_0)\lambda_0'' + 2\lambda_0')|_{z=z_0}, \\
&= 2\lambda_0'|_{z=z_0}.
\end{aligned}
$$

Substituting these into the equation above gives:

$$
\begin{aligned}
\frac{qp'' - 2p'q'}{q^2}\Big|_{z=z_0} &= \frac{2k_0\lambda_0\lambda_0' - 2(1 + k_0)\lambda_0\lambda_0'}{k_0^2\lambda_0^2}\Big|_{z=z_0}, \\
&= -\frac{2\lambda_0'}{k_0^2\lambda_0}\Big|_{z=z_0}.
\end{aligned}
$$

Recalling the definition of λ_0 from above and differentiating with respect to z gives

$$
\lambda_0' = \sum_{r=1}^{m} k_r(z - z_r)^{k_r - 1} \prod_{s \neq r,0} (z - z_s)^{k_s}.
$$

Using this, along with the definition of λ_0, we see that:

$$
-\frac{2\lambda_0'}{k_0^2\lambda_0}\Big|_{z=z_0} = -\frac{2\sum_{r=1}^{m} k_r(z - z_r)^{k_r - 1}\prod_{s\neq r,0}(z - z_s)^{k_s}}{k_0^2\prod_{r=1}^{m}(z - z_r)^{k_r}}\Big|_{z=z_0},
$$

$$= -\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{2k_r}{z-z_r}\Big|_{z=z_0}.$$
$$= -\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{2k_r}{z_0-z_r}.$$

Therefore

$$\operatorname*{res}_{z=z_0} Q\frac{\lambda}{\lambda'}dz = -\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0-z_r}.$$

Next, consider R. Recalling the definition from above

$$\begin{aligned}\operatorname*{res}_{z=z_0} R\frac{\lambda}{\lambda'}dz &= \operatorname*{res}_{z=z_0}\frac{1}{(z-z_0)^2}\left(\frac{1}{z-z_i}+\frac{1}{z-z_j}+\frac{1}{z-z_k}\right)\frac{\lambda}{\lambda'}dz,\\ &= \left(\frac{1}{z-z_0}\left(\frac{1}{z-z_i}+\frac{1}{z-z_j}+\frac{1}{z-z_k}\right)\frac{\lambda}{\lambda'}\right)\Big|_{z=z_0},\\ &= \frac{\lambda}{(z-z_0)\lambda'}\Big|_{z=z_0}\left(\frac{1}{z_0-z_i}+\frac{1}{z_0-z_j}+\frac{1}{z_0-z_k}\right).\end{aligned}$$

Whilst $\frac{\lambda}{(z-z_0)\lambda'}$ may appear singular at $z=z_0$, recall that $(z-z_0)^{k_0}$ is a factor in λ, and so this singularity is cancelled out. Therefore it is possible to evaluate this (by defining the value at $z=z_0$ to be the limit as z tends to z_0):

$$\begin{aligned}\frac{\lambda}{(z-z_0)\lambda'}\Big|_{z=z_0} &= \frac{(z-z_0)^{k_0-1}\prod_{i\neq 0}(z-z_i)^{k_i}}{\sum_{j=0}^{m}k_j(z-z_j)^{k_j-1}\prod_{k\neq j}(z-z_k)^{k_k}}\Big|_{z=z_0},\\ &= \frac{\prod_{i\neq 0}(z-z_i)^{k_i}}{k_0\prod_{j\neq 0}(z-z_j)+(z-z_0)\sum_{k\neq 0}(z-z_k)^{k_k-1}\prod_{l\neq 0,k}(z-z_k)^{k}}\Big|_{z=z_0},\\ &= \frac{1}{k_0}.\end{aligned}$$

Therefore

$$\operatorname*{res}_{z=z_0} R\frac{\lambda}{\lambda'} = \frac{1}{k_0}\left(\frac{1}{z_0-z_i}+\frac{1}{z_0-z_j}+\frac{1}{z_0-z_k}\right).$$

Next, consider the residue

$$\operatorname*{res}_{z=z_0} S\frac{\lambda}{\lambda'}dz = \operatorname*{res}_{z=z_0}\frac{1}{z-z_0}\left(\frac{1}{z-z_i}+\frac{1}{z-z_j}+\frac{1}{z-z_k}\right)\frac{\lambda}{\lambda'}dz,$$

But from above, $\frac{\lambda}{\lambda'(z-z_0)}$ is equal to $\frac{1}{k_0}$ at $z=z_0$, whilst $\frac{1}{z-z_i}$ etc are obviously finite there. Therefore this is a residue of something finite and so must be equal to zero. Finally, it follows trivially that

$$\operatorname*{res}_{z=z_0} T\frac{\lambda}{\lambda'}dz = 0,$$

as T is finite and $\frac{\lambda}{\lambda'} = 0$ at $z = z_0$. Adding these four results, one obtains

$$c^\star_{ijk} = -\frac{k_ik_jk_k}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0-z_r} - \frac{k_ik_jk_k}{k_0}\left(\frac{1}{z_0-z_i}+\frac{1}{z_0-z_j}+\frac{1}{z_0-z_k}\right),$$

as required.

Lemma 4.2 *For precisely two identical indices, $c^\star_{iij}$ takes the form*

$$c^\star_{iij} = -\frac{k_i^2k_j}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0-z_r} - \frac{k_i^2k_j}{k_0}\left(\frac{2}{z_0-z_i}+\frac{1}{z_0-z_j}\right) + k_ik_j\left(\frac{1}{z_i-z_0}-\frac{1}{z_i-z_j}\right).$$

Proof: Using the same reasoning as above, one may immediately write

$$\begin{aligned}
c^\star_{iij} &= \left(\operatorname*{res}_{z=z_i} + \operatorname*{res}_{z=z_0}\right) k_i^2k_j\left(\frac{1}{z-z_0}-\frac{1}{z-z_i}\right)^2\left(\frac{1}{z-z_0}-\frac{1}{z-z_j}\right)\frac{\lambda(z)}{\lambda'(z)}dz,\\
&= \left(\operatorname*{res}_{z=z_i} + \operatorname*{res}_{z=z_0}\right)\Bigg(\overbrace{\frac{1}{(z-z_0)^3} - \frac{1}{(z-z_0)^2}\left(\frac{2}{z-z_i}+\frac{1}{z-z_j}\right)}^{U}\\
&\quad \underbrace{+\frac{1}{z-z_0}\left(\frac{1}{(z-z_i)^2}+\frac{2}{(z-z_i)(z-z_j)}\right) - \frac{1}{(z-z_i)^2(z-z_j)}}_{V}\Bigg)\frac{\lambda}{\lambda'}dz.\\
&= k_i^2k_j\left(\operatorname*{res}_{z=z_i} U\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_i} V\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_0} U\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_0} V\frac{\lambda}{\lambda'}\right)dz.
\end{aligned}$$

Calculating these residues individually, one immediately realises from the results in the previous theorem that:

$$\begin{aligned}
\operatorname*{res}_{z=z_i} U\frac{\lambda}{\lambda'}dz &= 0,\\
\operatorname*{res}_{z=z_0} V\frac{\lambda}{\lambda'}dz &= 0.
\end{aligned}$$

Next, calculate

$$\operatorname*{res}_{z=z_0} U\frac{\lambda}{\lambda'}dz = \operatorname*{res}_{z=z_0}\left(\frac{1}{(z-z_0)^3} - \frac{1}{(z-z_0)^2}\left(\frac{2}{z-z_i}+\frac{1}{z-z_j}\right)\right)\frac{\lambda}{\lambda'}dz,$$

$$= \underbrace{\operatorname*{res}_{z=z_0} \frac{1}{(z-z_0)^3}\frac{\lambda}{\lambda'}dz}_{U_1} - \underbrace{\operatorname*{res}_{z=z_0} \frac{1}{(z-z_0)^2}\left(\frac{2}{z-z_i}+\frac{1}{z-z_j}\right)\frac{\lambda}{\lambda'}dz}_{U_2}.$$

But U_1 is identical to $\operatorname*{res}_{z=z_0} Q\frac{\lambda}{\lambda'}dz$, and so

$$U_1 = -\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0-z_r}.$$

Also, note that

$$U_2 = \operatorname*{res}_{z=z_0} \frac{1}{(z-z_0)^2}\left(\frac{2}{z-z_i}+\frac{1}{z-z_j}\right)\frac{\lambda}{\lambda'}dz,$$

with $z_k = z_i$. This was calculated in lemma 4.1, and so one may immediately write

$$U_2 = \frac{1}{k_0}\left(\frac{2}{z_0-z_i}+\frac{1}{z_0-z_j}\right).$$

Finally, consider

$$\begin{aligned}
&\operatorname*{res}_{z=z_i} V\frac{\lambda}{\lambda'}dz\\
&= \operatorname*{res}_{z=z_i}\left(\frac{1}{(z-z_i)^2}\left(\frac{1}{z-z_0}-\frac{1}{z-z_j}\right)+\frac{2}{(z-z_0)(z-z_i)(z-z_j)}\right)\frac{\lambda}{\lambda'}dz,\\
&= \operatorname*{res}_{z=z_i}\left(\frac{1}{(z-z_i)^2}\left(\frac{1}{z-z_0}-\frac{1}{z-z_j}\right)\frac{\lambda}{\lambda'}\right)dz,\\
&= \left(\frac{1}{z-z_i}\left(\frac{1}{z-z_0}-\frac{1}{z-z_j}\right)\frac{\lambda}{\lambda'}\right)\Big|_{z=z_i},\\
&= \frac{1}{k_i}\left(\frac{1}{z_i-z_0}-\frac{1}{z_i-z_j}\right).
\end{aligned}$$

The last line above was obtained by recalling from earlier that

$$\frac{\lambda}{(z-z_q)\lambda'}\Big|_{z=z_q} = \frac{1}{k_q}.$$

Combining the above results, one obtains

$$c^{\star}_{iij} = k_i^2 k_j\left(\frac{1}{k_i}\left(\frac{1}{z_i-z_0}-\frac{1}{z_i-z_j}\right)-\frac{1}{k_0}\left(\frac{2}{z_0-z_i}+\frac{1}{z_0-z_j}\right)-\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0-z_r}\right).$$

But this is equal to

$$c^\star_{iij} = -\frac{k_i^2 k_j}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0 - z_r} - \frac{k_i^2 k_j}{k_0}\left(\frac{2}{z_0 - z_i} + \frac{1}{z_0 - z_j}\right) + k_i k_j\left(\frac{1}{z_i - z_0} - \frac{1}{z_i - z_j}\right),$$

as required.

Lemma 4.3 *For three identical indices,*

$$c^\star_{iii} = -\frac{k_i^3}{k_0^3}\sum_{r=1}^{m}\frac{k_r}{z_0 - z_s} - \frac{3k_i^3}{k_0}\frac{1}{z_0 - z_i} + 3k_i^2\frac{1}{z_i - z_0} + k_i\sum_{s\neq i}\frac{k_s}{z_i - z_s}.$$

Proof: By applying the same reasoning as in lemma 4.1, it is possible to show that

$$\begin{aligned}
c^\star_{iii} &= \left(\operatorname*{res}_{z=z_i} + \operatorname*{res}_{z=z_0}\right) k_i^3\left(\frac{1}{z - z_0} - \frac{1}{z - z_i}\right)^3 \frac{\lambda}{\lambda'}dz,\\
&= k_i^3\left(\operatorname*{res}_{z=z_i} + \operatorname*{res}_{z=z_0}\right)\Bigg(\overbrace{\frac{1}{(z - z_0)^3} - \frac{3}{(z - z_0)^2(z - z_i)}}^{V}\\
&\qquad + \underbrace{\frac{3}{(z - z_0)(z - z_i)^2} - \frac{1}{(z - z_i)^3}}_{W}\Bigg)\frac{\lambda}{\lambda'}dz,\\
&= k_i^3\left(\operatorname*{res}_{z=z_0} V\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_0} W\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_i} V\frac{\lambda}{\lambda'}dz + \operatorname*{res}_{z=z_i} W\frac{\lambda}{\lambda'}dz\right).
\end{aligned}$$

The four components of the above equation may now be considered individually:

$$\begin{aligned}
\operatorname*{res}_{z=z_0} V\frac{\lambda}{\lambda'}dz &= \operatorname*{res}_{z=z_0}\frac{1}{(z - z_0)^3}\frac{\lambda}{\lambda'}dz - \operatorname*{res}_{z=z_0}\frac{3}{(z - z_0)^2(z - z_i)}\frac{\lambda}{\lambda'}dz,\\
&= -\frac{1}{k_0^2}\sum_{r=1}^{m}\frac{k_r}{z_0 - z_r} - \frac{3}{k_0}\frac{1}{z_0 - z_i},
\end{aligned}$$

by earlier results. Similarly,

$$\begin{aligned}
\operatorname*{res}_{z=z_0} W\frac{\lambda}{\lambda'}dz &= \operatorname*{res}_{z=z_0}\frac{3}{(z - z_0)(z - z_i)^2}\frac{\lambda}{\lambda'}dz - \operatorname*{res}_{z=z_0}\frac{1}{(z - z_i)^3}\frac{\lambda}{\lambda'}dz,\\
&= 0
\end{aligned}$$

may also be obtained by using earlier results. Likewise,

$$\operatorname*{res}_{z=z_i} V\frac{\lambda}{\lambda'}dz = 0$$

follows immediately from the proof of lemma 4.1. Finally, one must calculate

$$\operatorname*{res}_{z=z_i} V\frac{\lambda}{\lambda'}dz = \operatorname*{res}_{z=z_i} \frac{3}{(z-z_0)(z-z_i)^2}\frac{\lambda}{\lambda'}dz - \operatorname*{res}_{z=z_i} \frac{1}{(z-z_i)^3}\frac{\lambda}{\lambda'}dz.$$

But these are analogous to earlier calculations, and so

$$\operatorname*{res}_{z=z_i} W\frac{\lambda}{\lambda'}dz = \frac{3}{k_i}\frac{1}{z_i-z_0} + \frac{1}{k_i^2}\sum_{r\neq 0,i}\frac{k_i}{z_i-z_r}.$$

Therefore combining the various individual resides leaves us with

$$c^\star_{iii} = k_i^3\left(\frac{3}{k_i}\frac{1}{z_i-z_0} + \frac{1}{k_i^2}\sum_{r\neq i}\frac{k_r}{z_i-z_r} - \frac{1}{k_0^2}\sum_{s=1}^{m}\frac{k_s}{z_0-z_s} - \frac{3}{k_0}\frac{1}{z_0-z_i}\right).$$

A simple rearrangement of terms can then show that this is equal to the required result.

Theorem 4.4 *The $c^\star_{ijk}$ calculated above may be integrated up to a prepotential*

$$F^\star = \frac{1}{8}\sum_{r=0}^{m}\sum_{s\neq r} k_r k_s (z_r - z_s)^2 \log(z_r - z_s)^2.$$

Proof: To prove this theorem, one simply needs to differentiate $F^\star$ three times (with all possible permutations of distinct and identical indices) to obtain the required values of $c^\star_{ijk}$. Firstly, note that using the laws of logarithms and rearranging the summation, $F^\star$ may be written in the following way:

$$F^\star = \underbrace{\frac{1}{2}\sum_{r=1}^{m} k_0 k_r (z_0 - z_r)^2 \log(z_0 - z_r)}_{A} + \underbrace{\frac{1}{4}\sum_{t=1}^{m}\sum_{s\neq t,0} k_r k_s (z_r - z_s)^2 \log(z_r - z_s)}_{B}.$$

Recalling that $z_0 = -\frac{1}{k_0}\sum_{p=1}^{m} k_p z_p$ we have

$$\frac{dF^\star}{dz_i} = \frac{\partial z_0}{\partial z_i}\frac{\partial A}{\partial z_0} + \frac{\partial A}{\partial z_i} + \frac{\partial B}{\partial z_i}$$

The three individual components of this may be considered separately:

$$\begin{aligned}
\frac{\partial z_0}{\partial z_i}\frac{\partial A}{\partial z_0} &= -\frac{k_i}{k_0}\frac{1}{2}\sum_{r=1}^{m} k_0 k_r \left(2(z_0-z_r)\log(z_0-z_r)+(z_0-z_r)\right),\\
&= -\frac{k_i}{k_0}\sum_{r=1}^{m} k_0 k_r \left((z_0-z_r)\log(z_0-z_r)+\frac{z_0-z_r}{2}\right).\\
\frac{\partial A}{\partial z_i} &= -\frac{k_0 k_i}{2}\left(2(z_0-z_i)\log(z_0-z_i)+(z_0-z_i)\right),\\
&= -k_0 k_i\left((z_0-z_i)\log(z_0-z_i)+\frac{z_0-z_i}{2}\right).\\
\frac{\partial B}{\partial z_i} &= \frac{\partial}{\partial z_i}\left(\frac{1}{2}\sum_{r\neq i,0} k_r k_i (z_r-z_i)^2\log(z_r-z_i) + \text{terms independent of } z_i\right),\\
&= -\frac{1}{2}\sum_{r\neq i,0} k_r k_i \left(2(z_r-z_i)\log(z_r-z_i)+(z_r-z_i)\right),\\
&= -\sum_{r\neq i,0} k_r k_i\left((z_r-z_i)\log(z_r-z_i)+\frac{z_r-z_i}{2}\right).
\end{aligned}$$

Combining these three results gives

$$\begin{aligned}
\frac{dF^\star}{dz_i} = &\overbrace{-\frac{k_i}{k_0}\sum_{r=1}^{m} k_0 k_r\left((z_0-z_r)\log(z_0-z_r)+\frac{z_0-z_r}{2}\right)}^{D}\\
&\left.-k_0 k_i\left((z_0-z_i)\log(z_0-z_i)+\frac{z_0-z_i}{2}\right)\right\}E\\
&\underbrace{-\sum_{r\neq i,0} k_r k_i\left((z_r-z_i)\log(z_r-z_i)+\frac{z_r-z_i}{2}\right)}_{G}.
\end{aligned}$$

Differentiating again with respect to z_i gives

$$\begin{aligned}
\frac{d^2F^\star}{dz_i^2} &= \frac{\partial z_0}{\partial z_i}\frac{\partial D}{\partial z_0}+\frac{\partial D}{\partial z_i}+\frac{\partial z_0}{\partial z_i}\frac{\partial E}{\partial z_0}+\frac{\partial E}{\partial z_i}+\frac{\partial G}{\partial z_i},\\
&= \left(\frac{k_i}{k_0}\right)^2\sum_{r=1}^{m} k_0 k_r\left(\log(z_0-z_r)+\frac{3}{2}\right)+\left(2k_i^2+k_0k_i\right)\left(\log(z_0-z_i)+\frac{3}{2}\right)\\
&\quad+\sum_{r\neq i,0} k_r k_i\left(\log(z_r-z_i)+\frac{3}{2}\right).
\end{aligned}$$

Differentiating a further time with respect to z_i then gives

$$\frac{\partial^3 F^\star}{\partial z_i^3} = c^\star_{iii},$$

as required. Similarly, by differentiating $\frac{\partial^2 F^\star}{\partial z_i^2}$ with respect to z_j, one obtains

$$\frac{\partial^3 F^\star}{\partial z_j \partial z_i^2} = c^\star_{iij}.$$

Recall $\frac{\partial F^\star}{\partial z_i}$ from above and differentiate with respect to z_j. One obtains:

$$\begin{aligned}\frac{d^2 F^\star}{dz^j dz^i} &= \frac{k_i k_j}{k_0^2}\sum_{r=1}^{m} k_0 k_r \left(\log(z_0 - z_r) + \frac{3}{2}\right) + k_i k_j \left(\log(z_0 - z_j) + \frac{3}{2}\right) \\ &\quad + k_i k_j \left(\log(z_0 - z_i) + \frac{3}{2}\right) - k_i k_j \left(\log(z_i - z_j) + \frac{3}{2}\right).\end{aligned}$$

Differentiating this with respect to z_k then yields the final required result, i.e.

$$\frac{\partial^3 F^\star}{\partial z_k \partial z_j \partial z_i} = c^\star_{ijk}.$$

It should be noted that by setting all of the $k_i = 1$, one is dealing with the original A_n Frobenius manifold. In this special case, the result of theorem 4 agrees with the prepotential derived by Dubrovin in [13]. Also, this function agrees with the results derived geometrically in [16]. It should be noted that as discriminant submanifolds of Frobenius manifolds are not themselves Frobenius manifolds [25], there is no prepotential to which $F^\star$ is itself dual.

4.2 Genus zero Hurwitz space discriminant

Given that A_n type Frobenius manifolds correspond to those on $H_{0;n}$, and obvious generalisation of the ideas in the previous section is to a wider class of Hurwitz spaces. Here, one considers an arbitrary genus zero Hurwitz space. Recall that the discriminant of such a space corresponded to the superpotential expressed in the form

$$\lambda(z) = \frac{f(z)}{g(z)},$$

where f and g are polynomials in z written in their simplest form, having a repeated root (i.e. $f(z)$ having a repeated root). The exact form of f and g is now considered. Begin by taking a superpotential on $H_{0;n_0,\ldots,n_m}$ of the form

$$\lambda(z) = z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m}\sum_{s=0}^{n_r} \frac{c_{r,s}}{(z-\beta_r)^{s+1}}.$$

One may express this as a polynomial by putting every term over a common denominator os $\prod_{r=1}^{m}(z-\beta_i)^{n_i+1}$:

$$\frac{\prod_{r=1}^{m}(z-\beta_r)^{n_r+1}\left(z^{n_0+1} + a_1 z^{n_0-1} + \ldots + a_{n_0} + \sum_{r=1}^{m}\sum_{s=0}^{n_r}\frac{c_{r,s}}{(z-\beta_r)^{s+1}}\right)}{\prod_{r=1}^{m}(z-\beta_r)^{n_r+1}}. \quad (4.1)$$

Noting that the numerator evaluated at $z = \beta_i$ is equal to $c_{i,n_i} \neq 0$, so $(z-\beta_i)$ cannot be a factor in the numerator. Therefore this quotient is in its simplest form. The numerator is a polynomial whose degree, denoted by $(n+1)$, is:

$$n + 1 = n_0 + 1 + \sum_{i=1}^{m}(n_i + 1),$$

i.e.

$$n = \dim - m.$$

Hence the numerator can be expressed as a product

$$\prod_{i=0}^{n}(z-\alpha_i). \quad (4.2)$$

Consideration now the coefficient of z^n in this polynomial. From expanding the numerator in (4.1), one obtains that it must be equal to $-\sum_{i=0}^{m}(n_i+1)\beta_i$. Similarly, from expanding (4.2), this coefficient must be equal to $-\sum_{i=0}^{n}\alpha_i$. Hence one derives the condition

$$\sum_{i=0}^{n}\alpha_i = \sum_{j=1}^{m}(n_j+1)\beta_j,$$

or equivalently

$$\alpha_0 = -\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} (n_j + 1)\beta_i.$$

The Hurwitz space may be parameterised by the $(n+m)$ coordinates $\{\alpha_i\}$, $i \neq 0$ and $\{\beta_i\}$. An arbitrary discriminant corresponds to coinciding roots in the numerator of λ, and so will be of the form

$$\lambda = \frac{\prod_{i=0}^{l}(z - z_i)^{k_i}}{\prod_{j=1}^{m}(z - \beta_j)^{n_j+1}},$$

with $\sum_i k_i = n + 1$. Relabelling β_i as z_{i+l}, and $(n_i + 1)$ as $-k_{i+l}$, one then has

$$\lambda = \prod_{i=0}^{l+m} (z - z_i)^{k_i}.$$

One also has

$$z_0 = z_0(z_1, \ldots, z_N, k_0, \ldots, k_N) = -\sum_{i=0}^{N} \frac{k_i z_i}{k_0}.$$

This is analogous to the superpotential for the A_n discriminants, except that k_i may now take negative values (in fact all k_i will be negative for $i > n$).

Lemma 4.5 *For λ as defined above, one has*

$$\frac{\lambda}{\lambda'} = (z - z_i)\tilde{\lambda}_i,$$

where $\tilde{\lambda}_i$ is a rational function which is finite and non-zero at $z = z_i$.

Proof: Differentiating $\lambda = \prod_{j=0}^{N}(z - z_j)^{k_j}$ with respect to z, one obtains

$$\begin{aligned} \lambda'(z) &= \sum_{j=0}^{N} k_j (z - z_j)^{k_j - 1} \prod_{k \neq j} (z - z_k)^{k_i}, \\ &= (z - z_i)^{k_i - 1} \left(k_i \prod_{j \neq i} (z - z_j)^{k_j} + (z - z_i) \sum_{k \neq i} (z - z_k)^{k_k - 1} \prod_{l \neq i,k} (z - z_k)^{k_k} \right). \end{aligned}$$

Dividing λ by this leaves

$$\frac{\lambda}{\lambda'} = (z - z_i) \frac{\prod_{j \neq i}(z - z_j)^{k_j}}{k_i \prod_{j \neq i}(z - z_j)^{k_j} + (z - z_i) \sum_{k \neq i}(z - z_k)^{k_k - 1} \prod_{l \neq i,k}(z - z_k)^{k_k}}.$$

But $(z - z_i)$ is not a factor in the numerator or denominator of the fraction part of the above equation, so the lemma is proved, with $\tilde{\lambda}_i$ taking the form of this fraction.

Theorem 4.6 *The results of section 4.1 may be generalised to the Hurwitz space* $H_{0;n_0,\ldots,n_m}$. *That is, for a superpotential*

$$\lambda = \prod_{i=0}^{N} (z - z_i)^{k_i}, \; k_i \in \mathbb{Z},$$

an almost dual prepotential exists and is of the form:

$$F^{\star} = \frac{1}{8} \sum_{r=0}^{N} \sum_{s \neq r} k_r k_s (z_r - z_s)^2 \log (z_r - z_s)^2. \tag{4.3}$$

Proof: The proofs of lemmas 4.1-4.3 and theorem 4.4 do not rely explicitly on k_i being positive integers. There is a possibility that λ and λ' having poles other than at $z = \infty$ affects the calculations. However, as their appearance the residue formula is in the from $\frac{\lambda}{\lambda'}$, which according to lemma 5 behaves the same near the z_i when one allows for negative k_i as it does for positive k_i, this does not affect the calculations. Therefore $c^{\star}_{ijk}$ take on the same form as they do in lemmas 4.1-4.3, and so integrating up three times yields the desired result for $F^{\star}$.

It should be noted that the result derived here agrees with that which appeared in [16]. However, the result here allows for negative values off the parameters k_i and offers a geometric interpretation: the negative k_i determine which Hurwitz space one is dealing with, whilst the positive k_i determine which discriminant the solution comes from.

4.3 Extended affine Weyl groups

Let

$$\lambda = e^{-im\phi} \prod_{j=1}^{\dim} (e^{i\phi} - e^{i\phi_j}). \qquad (4.4)$$

From the theory contained within [15], this is the superpotential for a Frobenius manifold corresponding to the extended affine Weyl group $\tilde{W}^m(A_k)$. One will denote such manifolds by $\mathcal{M}(k,m)$. If one replaces $e^{i\phi}$ with z, this appears very similar to the superpotential for $H_{0;k-1,m-1}$, the only difference being the absence of the condition that the roots must sum to zero. In fact the link between such Frobenius manifolds may be formalised:

Lemma 4.7 *Frobenius manifolds on $H_{0;k-1,m-1}$ are linked to those on $\mathcal{M}(k,m)$ by a Legendre transformation.*

Proof: A full proof will not be given. However, noting that the two superpotentials correspond to the same Hurwitz data with a different choice of primary differential, it immediately follows that they are linked by a Legendre transformation s_k [10].

Lemma 4.8 *The discriminants of $\mathcal{M}(k,m)$ correspond to a factor $(e^{i\phi} - e^{i\phi_k})$ in λ being repeated, i.e. an arbitrary discriminant is of the form*

$$\lambda = e^{-im\phi} \prod_{j=1}^{n} (e^{i\phi} - e^{i\phi_j})^{k_j},$$

with at least one of the k_j being greater than or equal to 2.

Proof: Recall from chapter 2 that the discriminant condition is equivalent to λ and λ' having a common root. Differentiating λ with respect to ϕ, one obtains

$$\frac{d\lambda}{d\phi} = -ime^{-im\phi} \prod_{j} (e^{i\phi} - e^{i\phi_j})^{k_j} + ie^{-im\phi} \sum_{p} k_p (e^{i\phi} - e^{i\phi_p})^{k_p - 1} e^{i\phi} \prod_{q \neq p} (e^{i\phi} - e^{i\phi_q})^{k_q},$$

$$= e^{-im\phi}\prod_j (e^{i\phi}-e^{i\phi_j})^{k_j-1}\left(-im\prod_w (e^{i\phi}-e^{i\phi_w}) + ie^{i\phi}\sum_p k_p \prod_{q\neq p}(e^{i\phi}-e^{i\phi_q})\right).$$

Hence for any j such that $k_k \geq 2$, ϕ_j is a common root of both λ and λ'.

Noting that $\{\phi_j\}$ may be taken as coordinates for the discriminant submanifold, it is possible to calculate structure constants for the almost dual manifold using the formula 1.23. It is easy to show that

$$\frac{\partial \log\lambda}{\partial\phi^r} = -\frac{ik_r e^{i\phi_r}}{e^{i\phi}-e^{i\phi_r}},$$

and so the formula 1.23 becomes[1]

$$c^\star_{rst} = -\sum \operatorname*{res}_{d\log\lambda=0} \frac{(-i)^3 k_r k_s k_t e^{i\phi_r} e^{i\phi_s} e^{i\phi_t}}{(e^{i\phi}-e^{i\phi_r})(e^{i\phi}-e^{i\phi_s})(e^{i\phi}-e^{i\phi_t})}\frac{\lambda}{\lambda'}d\phi.$$

From this formula, direct calculation is then possible.

Lemma 4.9 *For distinct indices r, s and t,*

$$c^\star_{rst} = -\frac{ik_r k_s k_t}{m}.$$

Proof: One begins by using a substitution $e^{i\phi} = v$, noting that

$$\begin{aligned} \lambda'(\phi) &= iv\lambda'(v), \\ d\phi &= -\frac{i}{v}dv. \end{aligned}$$

For notational convenience, one will also use $e^{i\phi_r} = v_r$ etc. Hence

$$c^\star_{rst} = \sum \operatorname*{res}_{d_\phi \log\lambda} \frac{iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)v^2}\frac{\lambda(v)}{\lambda'(v)}dv.$$

However, one is now faced with a problem; the zeros of $\frac{d\log\lambda}{d\phi}$ are not known explicitly. However, one can say with certainty that they do not occur at roots of λ (i.e. points where $\phi = \phi_j$ or equivalently $v = v_j$). Therefore the same

[1] As in previous sections, one has written $c^\star_{rst}$ instead of $c^\star(\partial_{\phi^r}, \partial_{\phi^s}, \partial_{\phi^t})$ for notational convenience.

deformation of contours argument that was used in section 4.1 may be applied to obtain

$$c^\star_{rst} = \left(\operatorname*{res}_{v=\infty} + \operatorname*{res}_{v=0} + \operatorname*{res}_{v=v_r} + \operatorname*{res}_{v=v_s} + \operatorname*{res}_{v=v_t}\right) \frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)v^2} \frac{\lambda}{\lambda'} dv.$$

These residues may now be calculated individually. Noting that as v becomes very large,

$$\frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)} \frac{\lambda}{\lambda'} \approx \frac{-iv_r v_s v_t k_r k_s k_t v^{k-m}}{v^5 v^{k-m}}.$$

But this tends to zero as z tends to infinity, so the residue at infinity is zero.

Next, consider the residue at $v = 0$:

$$\begin{aligned}\operatorname*{res}_{v=0} = \frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)v^2} \frac{\lambda}{\lambda'} &= \lim_{v\to 0} \frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)v} \frac{\lambda}{\lambda'}, \\ &= \lim_{v\to 0} \frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_r)(v-v_s)(v-v_t)} \lim_{v\to 0} \frac{\lambda}{v\lambda'}.\end{aligned}$$

Evaluating the first limit in the final line above is possible simply by replacing v with zero. For the second part, noting that $\lambda = v^{-m}\prod(v-v_j)^{k_j}$,

$$\begin{aligned}&\lim_{v\to 0}\frac{\lambda}{v\lambda'} \\ &= \lim_{v\to 0} \frac{v^{-m}\prod_j (v-v_j)^{k_j}}{v\left(v^{-m}\sum_p k_p (v-v_p)^{k_p-1}\prod_{q\neq p}(v-v_q)^{k_q} - mv^{-(m+1)}\prod_w (v-v_w)^{k_w}\right)}, \\ &= \lim_{v\to 0} \frac{\prod_j (v-v_j)^{k_j}}{\left(v\sum_p k_p (v-v_p)^{k_p-1}\prod_{q\neq p}(v-v_q)^{k_q}\right) - m\prod_w (v-v_w)^{k_w}}, \\ &= -\frac{1}{m}.\end{aligned}$$

Therefore the residue at $v = 0$ is

$$\left(\frac{-iv_r v_s v_t k_r k_s k_t}{(-v_r)(-v_s)(-v_t)}\right)\left(-\frac{1}{m}\right) = -\frac{ik_r k_s k_t}{m}.$$

Thirdly, consider the residue at $v = v_r$. Observe that λ is a rational function of v, with $(v - v_r)$ being a factor. Therefore, as in section 4.1, $\frac{\lambda}{\lambda'}$ can be expressed in the form

$$\frac{\lambda}{\lambda'} = (v - v_r)\lambda_r,$$

where λ_r is a rational function which is finite and non zero at $v = v_r$. Therefore

$$\operatorname*{res}_{v=v_r} \frac{-iv_r v_s v_t k_r k_s k_t \lambda}{(v-v_r)(v-v_s)(v-v_t)v^2\lambda'} = \operatorname*{res}_{v=v_r} \frac{-iv_r v_s v_t k_r k_s k_t}{(v-v_s)(v-v_t)v^2}\lambda_r.$$

But this must be zero, as it is the residue of something which is finite at $v = v_r$. Likewise, the residues at $v = v_s$ and $v = v_t$ are zero. Combining these results gives

$$c^\star_{rst} = -\frac{ik_r k_s k_t}{m},$$

as required.

Lemma 4.10 *For two identical indices,*

$$c^\star_{rrs} = -\frac{ik_r^2 k_s}{m} - \frac{ik_r k_s e^{i\phi_s}}{e^{i\phi_r} - e^{i\phi_s}}.$$

Proof: Using the same reasoning as in the previous lemma, one immediately obtains

$$c^\star_{rrs} = -\frac{ik_r^2 k_s}{m} + \left(\operatorname*{res}_{v=v_r} + \operatorname*{res}_{v=v_s}\right) \frac{-iv_r^2 v_s k_r^2 k_s}{(v-v_r)^2(v-v_s)v^2}\frac{\lambda}{\lambda'},$$

where again $v = e^{i\phi}$ etc. Also, it immediately follows that as above, the residue at $v = v_s$ is zero, so only the residue at $v = v_r$ needs to be calculated. But

$$\begin{aligned}\operatorname*{res}_{v=v_r} \frac{-iv_r^2 v_s k_r^2 k_s}{(v-v_r)^2(v-v_s)v^2}\frac{\lambda}{\lambda'} &= \lim_{v\to v_r} \frac{-iv_r^2 v_s k_r^2 k_s}{(v-v_r)(v-v_s)v^2}\frac{\lambda}{\lambda'}, \\ &= \lim_{v\to v_r} \frac{-iv_r^2 v_s k_r^2 k_s}{(v-v_s)v^2} \lim_{v\to v_r} \frac{\lambda}{(v-v_r)\lambda'}.\end{aligned}$$

The first limit above is calculated simply by substituting in $v = v_r$. Calculating the second one is more involved:

$$\begin{aligned}&\lim_{v\to v_r} \frac{\lambda}{(v-v_r)\lambda'} \\ &= \lim_{v\to v_r} \frac{1}{v-v_r} \frac{v^{-m}\prod_j (v-v_j)^{k_j}}{v^{-m}\sum_p k_p (v-v_p)^{k_p-1}\prod_{q\neq p}(v-v_q)^{k_q} - mv^{-(m+1)}\prod_w (v-v_w)^{k_w}}, \\ &= \lim_{v\to v_r} \frac{1}{v-v_r} \frac{\prod_j (v-v_j)}{\sum_p k_p \prod_{q\neq p}(v-v_q) - mv^{-1}\prod_w (v-v_w)},\end{aligned}$$

$$= \lim_{v\to v_r} \frac{\prod_{j\neq r}(v-v_j)}{k_r\prod_{m\neq r}(v-v_m)+\sum_{p\neq r}k_p\prod_{q\neq p}(v-v_q)-mv^{-1}\prod_w(v-v_w)},$$
$$= \frac{1}{k_r}.$$

Therefore

$$\begin{aligned}\operatorname*{res}_{v=v_r}\frac{-iv_r^2v_sk_r^2k_s}{(v-v_r)^2(v-v_s)v^2}\frac{\lambda}{\lambda'} &= \frac{-iv_r^2v_sk_r^2k_s}{(v_r-v_s)v_r^2}\frac{1}{k_r},\\ &= \frac{-iv_sk_rk_s}{(v_r-v_s)}.\end{aligned}$$

Combining this with the residue at $v=0$, and replacing v_r with $e^{i\phi_r}$ etc, one obtains

$$c^\star_{rrs} = \frac{-ik_r^2k_s}{m} - \frac{ik_rk_se^{i\phi_s}}{e^{i\phi_r}-e^{i\phi_s}},$$

as required.

Lemma 4.11 *For three identical indices,*

$$c^\star_{rrr} = ik_r\sum_{q\neq r}\frac{k_qe^{i\phi_r}}{e^{i\phi_r}-e^{i\phi_q}} - \frac{ik_r^3}{m} + 2ik_r^2 - imk_r.$$

Proof: Using the same reasoning as above, it is immediately possible to show that

$$\begin{aligned}c^\star_{rrr} &= \frac{-ik_r^3}{m} + \operatorname*{res}_{v=v_r}\frac{-iv_r^3k_r^3}{(v-v_r)^3v^2}\frac{\lambda}{\lambda'}dv,\\ &= \frac{-ik_r^3}{m} - iv_r^3k_r^3\operatorname*{res}_{v=v_r}\frac{1}{(v-v_r)^3}\frac{\lambda}{v^2\lambda'}dv.\end{aligned}$$

But the residue above is equal to the $(v-v_r)^2$ term in the Taylor expansion of $\frac{\lambda}{v^2\lambda'}$, that is:

$$\operatorname*{res}_{v=v_r}\frac{1}{(v-v_r)^3}\frac{\lambda}{v^2\lambda'}dv = \frac{1}{2!}\frac{d^2}{dv^2}\frac{\lambda}{v^2\lambda'}\bigg|_{v=v_r}.$$

In order to calculate this, observe that $\lambda = (v-v_r)^{k_r}\lambda_r$, with the function λ_r being finite and non zero at $v=v_r$. Differentiating this with respect to v gives

$$\lambda' = (v-v_r)k_r\lambda_r' + k_r(v-v_r)^{k_r-1}\lambda_r,$$

so

$$\frac{\lambda}{v^2\lambda'} = \frac{\overbrace{(v-v_r)\lambda_r}^{p}}{\underbrace{v^2\left((v-v_r)\lambda_r' + k_r\lambda_r\right)}_{q}}$$

Using the same techniques as were used in the proof of lemma 4.1, this reduces to

$$\frac{d^2}{dz^2}\frac{\lambda}{v^2\lambda'}\bigg|_{v=v_r} = \frac{qp''-2p'q'}{q^2}\bigg|_{v=v_r}.$$

Individual components may then be considered:

$$\begin{aligned} p'|_{v=v_r} &= \lambda_r|_{v=v_r}, \\ p''|_{v=v_r} &= 2\lambda_r'|_{v=v_r}, \\ q|_{v=v_r} &= v^2k_r\lambda_r|_{v=v_r}, \\ q'|_{v=v_r} &= v^2(\lambda_r' + k_r\lambda_r') + 2vk_r\lambda_r. \end{aligned}$$

Substituting these in gives:

$$\begin{aligned} \frac{d^2}{dv^2}\frac{\lambda}{v^2\lambda'}\bigg|_{v=v_r} &= -\frac{2v^2\lambda_r\lambda_r'}{v^4k_r^2\lambda_r^2}\bigg|_{v=v_r} - \frac{4vk_r\lambda_r^2}{v^4k_r^2\lambda_r^2}\bigg|_{v=v_r}, \\ &= -\frac{2}{v_r^2k_r^2}\frac{\lambda_r'}{\lambda_r}\bigg|_{v=v_r} - \frac{4}{v_r^3k_r} \end{aligned}$$

But as we know $\lambda_r = v^{-m}\prod_{p\neq r}(v-v_p)^{k_p}$, differentiation yields

$$\lambda' = v^{-m}\sum_{p\neq r}k_p(v-v_p)^{k_p}\prod_{q\neq p,r}(v-v_q)^{k_q} - mv^{-m+1}\prod_{w\neq r}(v-v_w)^{k_w}.$$

Using these explicit expressions, one may then evaluate

$$\frac{\lambda_r'}{\lambda_r}\bigg|_{v=v_r} = \sum_{q\neq r}\frac{k_q}{v_r-v_q} - \frac{m}{v_r}.$$

Combining all of the results above (being careful to remember that the residue at $v=v_r$ was preceded by a factor of $-iv_r^3k_r^3$ etc), $c^\star_{rrr}$ has been calculated explicitly:

$$c^\star_{rrr} = \frac{ik_r^3}{m} + iv_r\sum_{q\neq r}\frac{k_rk_q}{v_r-v_q} - ik_rm + 2ik_r^2.$$

Finally, replacing the substituted $\{v_j\}$ with $\{e^{i\phi_j}\}$ provides the desired result.

As in the A_n and genus zero Hurwitz cases, the coordinates which have been used are a set of flat coordinates for the almost dual manifold, and so it is possible to integrate the $c^\star_{rst}$ three times to obtain a prepotential. This prepotential is expressed in terms of the polylogarithm function Li_n, as introduced in example 1.54

Theorem 4.12 *The almost dual prepotential is:*

$$\begin{aligned} F^\star &= \frac{1}{2}\sum_{p=1}^{n}\sum_{q\neq p} k_p k_q Li_3 e^{i(\phi_p-\phi_q)} && (4.5)\\ &+\sum_{a=1}^{n} A_a\phi_a^3+\sum_{b=1}^{n}\sum_{u\neq b} B_{bu}\phi_b^2\phi_u+\sum_{c=1}^{n-2}\sum_{v>c}\sum_{w>v} C_{cvw}\phi_c\phi_v\phi_w, && (4.6)\end{aligned}$$

where

$$\begin{aligned} A_a &= \frac{1}{3}ik_a^3-\frac{ik_a m}{6}-\frac{ik_a^3}{6m}-\frac{i}{12}\sum_{q\neq a}k_a k_q,\\ B_{bu} &= \frac{ik_b k_u}{4}-\frac{ik_b^2 k_u}{2m},\\ C_{cvw} &= -\frac{ik_c k_v k_w}{m}.\end{aligned}$$

Proof: To prove this theorem, one simply needs to differentiate $F^\star$ three times. Begin by considering the Li_3 part of $F^\star$. Differentiating this once gives

$$\frac{\partial}{\partial\phi_r}\frac{1}{2}\sum\sum k_p k_q Li_3 e^{i(\phi_p-\phi_q)}=\frac{1}{2}\sum_{q\neq r}k_r k_q(iLi_2 e^{i(\phi_r-\phi_q)}-iLi_2 e^{i(\phi_q-\phi_r)}).$$

Differentiating a second time then gives

$$\frac{\partial}{\partial\phi_r^2}\frac{1}{2}\sum_{p}\sum_{q\neq p}k_p k_q Li_3 e^{i(\phi_p-\phi_q)}=\frac{1}{2}\sum_{q\neq r}k_r k_q(\log(1-e^{i(\phi_r-\phi_q)})+\log(1-e^{i(\phi_q-\phi_r)}).$$

But by noting that

$$1-e^{i(\phi_r-\phi_q)}=\frac{e^{i\phi_q}-e^{i\phi_r}}{e^{i\phi_q}}\quad\text{and}\quad 1-e^{i(\phi_q-\phi_r)}=\frac{e^{i\phi_r}-e^{i\phi_q}}{e^{i\phi_r}},$$

one may use the laws of logarithms to show that

$$\frac{\partial^2}{\partial\phi_r^2}\frac{1}{2}\sum\sum k_p k_q Li_3 e^{i(\phi_p-\phi_q)}$$
$$=\frac{1}{2}\sum_{q\neq r} k_r k_q(\log(e^{i\phi_q}-e^{i\phi_r})+\log(e^{i\phi_r}-e^{i\phi_q})-\log e^{i\phi_q}-\log e^{i\phi_r}).$$

Differentiating this a further time with respect to ϕ_r then gives

$$\begin{aligned}\frac{\partial^3}{\partial\phi_r^3}\frac{1}{2}\sum\sum k_p k_q Li_3 e^{i(\phi_p-\phi_q)} &= \frac{1}{2}\sum k_r k_s\left(\frac{-ie^{i\phi_r}}{e^{i\phi_q}-e^{i\phi_r}}+\frac{ie^{i\phi_r}}{e^{i\phi_r}-e^{i\phi_q}}-i\right),\\ &= \sum_{q\neq r} k_r k_s\left(\frac{ie^{i\phi_r}}{e^{i\phi_r}-e^{i\phi_q}}-\frac{i}{2}\right).\end{aligned}$$

The only other non zero third derivative when differentiating three times with respect to ϕ_r is that of $A\phi_r^3$:

$$\begin{aligned}\frac{\partial^3}{\partial\phi_r^3}A\phi_r^3 &= 6A,\\ &= 2ik_r^3-ik_r m-\frac{ik_r^3}{m}+\frac{i}{2}\sum_{q\neq r}k_r k_q.\end{aligned}$$

Adding the two results above gives $c^\star_{rrr}$ as required. Next consider $c^\star_{rrs}$. The only terms in $F^\star$ who which can be non zero after differentiating with respect to ϕ_r twice and ϕ_s once are the Li_3 terms and $B\phi_r^2\phi_s$. It is simple to show that

$$\frac{\partial^3}{\partial\phi_s\partial\phi_r^2}B\phi_r^2\phi_s=2B.$$

For the third derivative of the Li_3 terms, one differentiates $\frac{\partial^2}{\partial\phi_r^2}(Li_3\,\text{type terms})$ from above once with respect to ϕ_s to obtain

$$\frac{\partial^3 F^\star}{\partial\phi_s\partial\phi_r^2}=\frac{1}{2}k_r k_s\left(\frac{e^{i\phi_q}}{e^{i\phi_q}-e^{i\phi_r}}-\frac{e^{i\phi_q}}{e^{i\phi_r}-e^{i\phi_q}}-i\right).$$

Combining these two results provides the correct form for $c^\star_{rrs}$. Finally, one needs to show that

$$\frac{\partial^3 F^\star}{\partial\phi_t\partial\phi_s\partial\phi_r}=c^\star_{rst}.$$

But by observing that the only term containing the three variables ϕ_r, ϕ_s and ϕ_t is $C\phi_r\phi_s\phi_t$, all other terms become zero under differentiation. Therefore

$$\begin{aligned}\frac{\partial^3}{\partial\phi_t\partial\phi_s\partial\phi_r}(C\phi_r\phi_s\phi_t) &= C,\\ &= -\frac{ik_rk_sk_s}{m}.\end{aligned}$$

But this is equal to $c^\star_{rst}$ as required.

Corollary 4.13 *Outside of the discriminants, i.e. when the superpotential is of the form (4.4), the almost dual prepotential will be:*

$$F^\star = \frac{1}{2}\sum_{i\neq j} Li_3 e^{i(\phi_i-\phi_j)} + A\sum_i \phi_i^3 + \sum_{i\neq j}\phi_i^2\phi_j - C\sum_{i\neq j\neq k}\phi_i\phi_j\phi_k, \tag{4.7}$$

with

$$\begin{aligned}A &= -\frac{i}{12m}\left((m-2)(m-1)-mk\right),\\ B &= -\frac{i}{4m}(2-m),\\ C &= -\frac{i}{m}\end{aligned}$$

Proof: This follows immediately from theorem 4.12; it is obtained simply by setting all of the k_i equal to one. in (4.6

This prepotential is very similar to the solutions to the generalised WDVV equations studied in [20], which also consist of a trilogarithm of an exponential term and cubic components. In fact if one considers the 2 dimensional case (with k and m both equal to 1), the solutions are equivalent, as shown in the example below.

Example 4.14 *Set k=1, m=1. By corollary 4.13, the almost dual prepotential is*

$$F^\star = \frac{1}{2}\left(Li_3e^{i(\phi_1-\phi_2)} + Li_3e^{i(\phi_2-\phi_1)}\right) + \frac{i}{12}(\phi_1^3+\phi_2^3) - \frac{i}{4}(\phi_1^2\phi_2+\phi_1\phi_2^2)$$

Under a linear coordinate change

$$\begin{aligned}\phi_1 &\rightarrow -i\omega_1 + \omega_2,\\ \phi_2 &\rightarrow i\omega_1 + \omega_2.\end{aligned}$$

Under this transformation, the prepotential becomes

$$F^{\star} = \frac{1}{2}\left(Li_3 e^{-2\omega_1} + Li_3 e^{2\omega_1}\right) - 2i\left(\frac{1}{6}\omega_2^3 + \frac{1}{2}\omega_2\omega_1^2\right).$$

But this is precisely the 2-dimensional ansatz obtained in [20] with the root system ± 1, $k_\alpha = 4$ *and* $c = 4$.

Whilst this simple example coincides with [20], the presence of two parameters (k and m) ensure that this generates a more generalised set of results. Furthermore, as these solutions are closely linked to sums over root systems, it is likely that a direct link may be drawn with the $\vee$-systems derived by Veselov in [26].

4.4 Twisted Legendre transformations: a link between almost dual solutions

The aim of this section is to construct a link between some of the almost dual solutions constructed in sections 4.2 and 4.3. One begins by recalling from section 1.9 that certain solutions of the WDVV equations are linked via Legendre transformations. To each of two such solutions, F and $\hat{F}$, one may construct almost dual solutions $F^{\star}$ and $\hat{F}^{\star}$. Schematically:

$$\begin{array}{ccc} F & \xrightarrow{\hat{S}_\kappa} & \hat{F} \\ \downarrow & & \downarrow \\ F^{\star} & & \hat{F}^{\star} \end{array}$$

Lemma 4.15 *The exists a transformation $\hat{s}_\kappa$ mapping $F^\star$ to $\hat{F}^\star$.*

Proof: The almost duality map from F to $F^\star$ is invertible [13]. Composing this inverse with S_κ and the almost duality transformation from $\hat{F}$ to $\hat{F}^\star$ will provide the desired transformation.

However, whilst this abstract proof ensures the existence of such a function, it is not clear whether such a transformation may be expressed explicitly in a simple form. However, it turns out that it can, as shown in the theorem below (also see [22]). Before stating this theorem, however, one notes the form which the multiplication and metric on the tangent space of $\hat{F}^\star$ must take:

$$\begin{array}{ccc} F:\{<a,b>,\circ,E\} & \overset{S_\kappa}{\longrightarrow} & \hat{F}:\{<a,b>_\kappa:=<\partial_\kappa\circ\partial_\kappa,a\circ b>,\circ,E\} \\ \downarrow & & \downarrow \\ F^\star:\left\{\begin{array}{ccc}(a,b) & := & <E^{-1}\circ a,\, b> \\ a\star b & := & E^{-1}\circ a\circ b\end{array}\right\} & & \hat{F}^\star:\left\{\begin{array}{ccc}(a,b)_\kappa & := & <E^{-1}\circ a,\, b>_\kappa \\ a\star b & := & E^{-1}\circ a\circ b\end{array}\right\} \end{array}$$

Theorem 4.16 *The vector field*

$$\hat{\partial}_\kappa = E\cdot\partial_\kappa$$

generates a twisted Legendre transformation $\hat{s}_\kappa$ *from $F^\star$ to $\hat{F}^\star$ so that the metric $(,)_k$ is defined by*

$$(a,b)_k = (\hat{\partial}_\kappa\star\hat{\partial}_\kappa, a\star b).$$

Proof: Using the definition of the intersection form and $\star$, one has:

$$\begin{aligned}(a,b)_k &= <E^{-1}\cdot a, b>_k, \\ &= <\partial_\kappa\cdot\partial_\kappa, E^{-1}a\cdot b>, \\ &= <\partial_\kappa\cdot\partial_\kappa, a\star b>, \\ &= (E\cdot\partial_\kappa\cdot\partial_\kappa, a\star b), \\ &= ((E\cdot\partial_\kappa)\star(E\cdot\partial_\kappa), a\star b).\end{aligned}$$

But as $\hat{\partial}_\kappa = E \cdot \partial_\kappa$, this is the required result.

Corollary 4.17 *Almost dual prepotentials of the form (4.6) and (4.3) are linked by a twisted Legendre transformation.*

Proof: According to lemma 4.7, the original Frobenius manifolds are linked by a Legendre transformation. Hence applying theorem 4.16 immediately yields the desired result.

Example 4.18 *The prepotential*

$$F = \frac{1}{2}(t^1)^2 t^2 + e^{t^2},$$

with Euler field $E = t^1\partial_1 + 2\partial_2$ generates an almost dual prepotential

$$F^\star = \frac{1}{4} z_1 z_2 \left(z_1^2 + z_2^2\right)) - \frac{1}{12}\left(z_1^3 + z_2^3\right) + \frac{1}{2}\left(Li_3 e^{z_1 - z_2} + Li_3 e^{z_2 - z_1}\right),$$

as shown in example 1.54. Alternatively, one may apply the Legendre transformation S_2 to F (example 1.60) to obtain

$$\hat{F} = \frac{1}{2}(\hat{t}^2)^2 \hat{t}^1 + \frac{1}{2}(\hat{t}^1)^2\left(\log \hat{t}^1 - \frac{3}{2}\right).$$

But it was shown in example 1.57 that the almost dual prepotential to this (1.24) is

$$\hat{F} = \frac{1}{4}\left(\hat{t}_1^2 \log(\hat{t}_1)^2 + \hat{t}_2^2 \log(\hat{t}_2)^2 - (\hat{t}_1 - \hat{t}_2)^2 \log(\hat{t}_1 - \hat{t}_2)^2\right).$$

By theorem 4.16, $F^\star$ and $\hat{F}^\star$ above are linked by a twisted Legendre transformation generated by the vector

$$\hat{\partial}_2 = (t^1\partial_1 + 2\partial_2)\cdot\partial_2.$$

Note that $\hat{\partial}_2 \neq \frac{\partial}{\partial \hat{t}^2}$.

To summarise this chapter, an almost dual prepotential has been explicitly constructed for an arbitrary genus zero Hurwitz space (and any of its discriminants).

A natural subject for further study would be to generalise this idea to a higher genus Hurwitz space. The simplest such example, namely $H_{1;n}$ will be considered in the next chapter.

Chapter 5

Almost duality for a genus one Hurwitz space

In this chapter, almost dual solutions to the WDVV equations will be calculated for the Hurwitz space $H_{1;n}$. The approach will mirror that used in chapter 4; one will use a superpotential λ and the formula (1.23) to calculate dual structure constants $c^{\star}_{\alpha\beta\gamma}$ corresponding to third derivatives of the prepotential $F^{\star}$.

One begins by considering the form that such a superpotential must take. The Hurwitz space $H_{1;n}$ consists of holomorphic maps from the (complex) torus to the Riemann sphere with $(n+2)$ simple ramification points and n sheets glued at ∞. The functions in such a space will be elliptic. But an elliptic function is determined by:

- the locations of its zeros and poles,
- a modular parameter τ and
- a general scaling factor u, which may be expressed in the period form $e^{2i\pi u}$.

Now note that without a loss of generality, one may choose the branch point of the Hurwitz space to be zero, and so the superpotential will be of the form

$$\lambda(v) = e^{2\pi iu} \prod_{i=0}^{n} \frac{\theta_1(v - z_i|\tau)}{\theta_1(v|\tau)}, \quad \sum_{i=0}^{n} z_i = 0 \tag{5.1}$$

Note that by the Riemann-Hurwitz formula, this space is of dimension $n + 2$.

This case may alternatively be thought of as the orbit space of a Jacobi group. Such a space, denoted $\Omega/J(\mathfrak{g})$, where $\mathfrak{g}$ is a complex finite dimensional Lie algebra with Weyl group W, carries the structure of a Frobenius manifold [4, 5]. If one uses a Lie algebra with Weyl group A_n, then (in an abused notation) the Jacobi orbit space $\Omega/J(A_n)$ is a Frobenius manifold with a superpotential of the form (5.1) [4, 5]. As such, the contents of this chapter may be thought of as a generalisation of the Coxeter group construction to Jacobi orbit spaces as well as a generalisation from a genus zero to a genus one Hurwitz space.

A digression is now made to consider the function θ_1 in greater detail. A comprehensive introduction to this (along with the other Jacobi theta functions) may be found in [27]. The notations used in [27] will be used here.

Definition 5.1 *One defines θ_1 by the infinite sum*

$$\theta_1(v, q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)v. \tag{5.2}$$

The variable q above is known as the nome. *Note that θ_1 may instead by expressed in terms of the* half-period ratio τ. *In such cases, it will be denoted $\theta_1(v|\tau)$. The nome and half-period ratio are connected by the equation*

$$q = e^{i\pi\tau}.$$

The function θ_1 is doubly quasi-periodic, obeying the equations:

$$\begin{aligned} \theta_1(v + 2\pi|\tau) &= \theta_1(v|\tau), \\ \theta_1(v + 2\pi\tau|\tau) &= e^{-2\pi i\tau} e^{-2iv} \theta_1(v|\tau). \end{aligned}$$

It follows from (5.2) that $\theta_1(v|\tau)$ is an odd function and so may be expressed as a power series of the form

$$\theta_1(v|\tau) = \theta_1'(0|\tau)v + \frac{1}{6}\theta_1'''(0|\tau)v^3 + O(v^5). \tag{5.3}$$

As θ_1 is an odd function, it also follows that $\theta_1(0|\tau) = 0$. As such, the z_i are the roots of λ. Hence, analogous to the genus zero case, setting $z_i = z_j$ for some $i \neq j$ creates a repeated root in the superpotential and so is the condition for a discriminant. Therefore for an arbitrary discriminant, the superpotential will be of the form

$$\lambda(v) = e^{2\pi i u} \prod_{i=0}^{m} \frac{\theta_1(v - z_i|\tau)^{k_i}}{\theta_1(v|\tau)^{k_i}}, \tag{5.4}$$

subject to the constraints

$$\sum k_i = n + 1, \quad \sum k_i z_i = 0.$$

Returning to the properties of θ_1, note that it is possible to express θ_1 as an infinite product rather than a series:

$$\theta_1(v|\tau) \;=\; 2Gq^{\frac{1}{4}} \sin v \prod_{n=1}^{\infty}(1 - q^{2n}e^{2izv}) \prod_{n=1}^{\infty}(1 - q^{2n}e^{-2iv}), \tag{5.5}$$

where

$$G = \prod_{n=1}^{\infty}(1 - q^{2n}) \tag{5.6}$$

In addition to being able to differentiate θ_1 with respect to v, one may also differentiate with respect to the half period τ. This gives [27] the heat equation:

$$\frac{\partial\theta_1(v|\tau)}{\partial\tau} = -\frac{i}{4}\pi\theta_1''. \tag{5.7}$$

Similarly, one may consider the function λ and its derivative with respect to τ. In order to do this, one utilises the *elliptic connection* which sends modular functions of order k to modular functions of order $k + 2$ (see, for example, [4][1]).

[1]Note that the factor preceding the F' is $\frac{1}{2i\pi}$ in [4] rather than the $\frac{\pi}{2i}$ used here due to a different normalisation of the θ_1 function being used there.

Definition 5.2 *The* elliptic connection *is defined by its action on modular function F of weight k by:*

$$\begin{aligned}(D^{(k)}F)(v|\tau) &= \nabla_\tau F(v|\tau) - \frac{\theta_1'(v|\tau)}{\theta_1(v|\tau)}\frac{\pi}{2i}F'(v|\tau), \\ &= \eta^{2k}\partial_\tau\left(\eta^{-2k}F(v|\tau)\right) - \frac{\theta_1'(v|\tau)}{\theta_1(v|\tau)}\frac{\pi}{2i}F'(v|\tau),.\end{aligned}$$

Here η denotes Dedekind's η function. *A definition of this function can be found in [3]. However, as it will only appear to the power zero below, its precise definition is not necessary here.*

Noting that λ is a modular function of weight 0, the elliptic connection takes a particularly convenient form[2] when applied to λ

$$D\lambda = \frac{\partial\lambda}{\partial\tau} - \frac{1}{2i\pi}\frac{\Theta}{\lambda'}.$$

Using the facts above, one may attempt to calculate the intersection form in terms of coordinates $\{u, \tau, z_i\,(i = 1, \ldots, m)\}$. This will be done using the discriminant superpotential 5.4. However, as the formula 1.11 contains terms of the form $\partial' \log\lambda$, it is necessary to calculate these derivatives, as shown the lemma below.

Lemma 5.3 *The derivatives of* $\log\lambda$ *with respect to the coordinates u, τ and z_i are*

$$\frac{\partial\log\lambda}{\partial u} = 2\pi i, \tag{5.8}$$

$$\frac{\partial\log\lambda}{\partial\tau} = -\frac{i\pi}{4}\sum_{i=0}^{m}k_i\left(\frac{\theta_1''(v-z_i|\tau)}{\theta_1(v-z_i|\tau)}\right) - (n+1)\frac{\theta_1''(v|\tau)}{\theta_1(v|\tau)}, \tag{5.9}$$

$$\frac{\partial\log\lambda}{\partial z^i} = k_i\left(\frac{\theta_1'(v-z_0|\tau)}{\theta_1(v-z_0|\tau)} - \frac{\theta_1'(v-z_i|\tau)}{\theta_1(v-z_i|\tau)}\right). \tag{5.10}$$

Proof: Noting that

$$\log\lambda = 2\pi iu + \sum_{i=0}^{m}k_i\log\theta_1(v-z_i|\tau) - (n+1)\log\theta_1(v|\tau),$$

[2]For convenience the function $\frac{\theta_1'(0|\tau)}{\theta_1(0|\tau)}$ has been denoted by Θ.

the equation (5.8) follows immediately. Recalling (5.7), one may also immediately write (5.9). Finally, in order to obtain (5.10), note that $z_0 = -\sum_{i=1}^{n}\frac{k_i z_i}{k_0}$ and so (in a slightly abused notation):

$$\frac{d}{dz^i} = \frac{\partial}{\partial z^i} + \frac{\partial z_0}{\partial z^i}\frac{\partial}{\partial z^0}.$$

The desired result follows from differentiating $\log\lambda$ with respect to z_i in this way.

For notational convenience, a new function

$$\Theta_i := \frac{\theta_1'(v - z_i|\tau)}{\theta_1(v|\tau)}$$

is defined so that

$$\frac{\partial\log\lambda}{\partial z^i} = k_i(\Theta_0 - \Theta_i).$$

It is also useful at this point to consider the function Θ_i in slightly more detail. Using the power series of the odd function θ_1 around $v = z_i$, one may also express θ_1' as a power series in even powers of $v - z_i$, and so

$$\begin{aligned}\Theta_i &= \frac{\theta_1'(0|\tau) + \frac{1}{2!}\theta_1'''(0|\tau)(v-z_i)^2 + O(v-z_i)^4}{\theta_1'(0|\tau)(v-z_i) + \frac{1}{3!}\theta_1'''(0|\tau)(v-z_i)^3 + O(v-z_i)^5},\\ &= \frac{1}{v-z_i} + \frac{1}{3}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}(v-z_i) + O(v-z_i)^3.\end{aligned}$$

One may now move onto calculating the components of g in the coordinates $\{u, \tau, z^i\}$, as shown in the lemmas below.

Lemma 5.4 *For components relating to z_i, $i = 1, \ldots, m$, the components of g are:*

$$g_{ij} = \frac{k_i k_j}{k_0} + \delta_{ij}k_i.$$

Proof: Consider firstly $i \neq j$.

$$\begin{aligned}g_{ij} &= -\sum \operatorname*{res}_{d\log\lambda=0}\left(\frac{\partial\log\lambda}{\partial z^i}\frac{\partial\log\lambda}{\partial z^j}\frac{\lambda}{\lambda'}\right)dv,\\ &= -\sum \operatorname*{res}_{d\log\lambda=0} k_i k_j\left(\Theta_0 - \Theta_i\right)\left(\Theta_0 - \Theta_j\right)\frac{\lambda}{\lambda'}dv.\end{aligned}$$

By similar reasoning to that used in chapter 4, one may apply a deformation of contours argument (noting that Θ_i is singular at $v = z_i$ and that λ is singular at $v = 0$) to obtain

$$g_{ij} = \left(\operatorname*{res}_{v=v_i} + \operatorname*{res}_{v=v_j} + \operatorname*{res}_{v=v_0} + \operatorname*{res}_{v=0} \right) k_i k_j \left(\Theta_0 - \Theta_i\right)\left(\Theta_0 - \Theta_j\right) \frac{\lambda}{\lambda'} dv.$$

But whilst λ is singular at $v = 0$, the quotient $\frac{\lambda}{\lambda'} \sim 0$ there, and so the residue at $v = 0$ is zero. Expanding the remaining terms and using the fact that the residues at points with no singular terms will automatically be zero, the equation above becomes

$$\begin{aligned} g_{ij} &= \operatorname*{res}_{v=z_i} k_i k_j \Theta_i (\Theta_j - \Theta_0) \frac{\lambda}{\lambda'} dv + \operatorname*{res}_{v=z_j} k_i k_j \Theta_j (\Theta_i - \Theta_0) \frac{\lambda}{\lambda'} dv \\ &\quad + \operatorname*{res}_{v=z_0} k_i k_j \Theta_0^2 \frac{\lambda}{\lambda'} dv - \operatorname*{res}_{v=z_0} k_i k_j \Theta_0 (\Theta_i + \Theta_j) \frac{\lambda}{\lambda'} dv. \end{aligned}$$

These terms may be considered individually. Beginning with the residue at z_i note that Θ_j and Θ_0 are finite at $v = z_i$, so the only singularity arises from the $\theta_1(v - z_i|\tau)$ in the denominator of Θ_i. However, as $\theta(v - z_i|\tau)$ is a factor in λ, this singularity is cancelled out. Therefore the residue at z_i is of something which is finite there and so must be equal to zero. Likewise, the second and fourth residues in the expression above are equal to zero. In considering the third term, one utilises the power series expansion of Θ_0^2. Using (5.3), it is easy to show that

$$\Theta_i^2 = \frac{1}{(v - z_0)^2} + O(1).$$

Therefore

$$\begin{aligned} \operatorname*{res}_{v=z_0} k_i k_j \Theta_0^2 \frac{\lambda}{\lambda'} dv &= \text{coefficient of } (v - z_0) \text{ in expansion of } k_i k_j \frac{\lambda}{\lambda'}, \\ &= k_i k_j \left(\frac{\lambda}{\lambda'} \right)' \Big|_{v=z_0}. \end{aligned}$$

In order to calculate this, write

$$\begin{aligned} \lambda &= \theta_1^{k_0}(v - z_0|\tau)\lambda_0, \\ \lambda' &= k_0 \theta_1^{k_0 - 1}(v - z_0|\tau)\theta_1'(v - z_0|\tau)\lambda_0 + \theta_1^{k_0}(v - z_0|\tau)\lambda_0'. \end{aligned}$$

By writing the quotient and dividing the top and bottom by $\theta_1^{k_0-1}(v-z_i|\tau)$, one has

$$\frac{\lambda}{\lambda'} = \frac{\theta_1(v-z_0|\tau)\lambda_0}{k_0\theta_1'(v-z_0|\tau)\lambda_0 + \theta_1(v-z_0|\tau)\lambda_0'} = k_ik_j\frac{p}{q}.$$

But differentiating this gives[3]

$$\left(k_ik_j\frac{p}{q}\right)' = \left(\frac{\theta_1'\lambda_0 + \theta_1\lambda_0'}{k_0\theta_1'\lambda_0 + \theta_1\lambda_0'} - \frac{(k_0(\theta_1''\lambda_0 + \theta_1'\lambda_0') + \theta_1'\lambda_0' + \theta_1\lambda_0'')\theta_1\lambda_0}{(k_0\theta_1'\lambda_0 + \theta_1\lambda_0')^2}\right).$$

Evaluating this at $v = z_0$, noting that $\theta_1(0|\tau) = \theta_1''(0|\tau) = 0$, one obtains

$$\left.\left(k_ik_j\frac{p}{q}\right)'\right|_{v=z_0} = \frac{1}{k_0}.$$

Therefore

$$g_{ij} = \frac{k_ik_j}{k_0}.$$

Similar reasoning to that used above implies that in the case of $i = j$,

$$\begin{aligned} g_{ii} &= \left(\operatorname*{res}_{v=z_0} + \operatorname*{res}_{v=z_i}\right) k_i^2\left(\Theta_0 - \Theta_i\right)^2 \frac{\lambda}{\lambda'}dv, \\ &= \operatorname*{res}_{v=z_0} k_i^2\Theta_i^2\frac{\lambda}{\lambda'}dv + \operatorname*{res}_{v=z_0} k_i^2\Theta_0^2\frac{\lambda}{\lambda'}dv, \\ &= \frac{k_i^2}{k_0} + k_i. \end{aligned}$$

Lemma is proved.

Lemma 5.5 *For terms in g relating to the coordinate u, one has*

$$g_{uu} = 0, \tag{5.11}$$

$$g_{ui} = 0, \tag{5.12}$$

$$g_{u\tau} = -\pi^2. \tag{5.13}$$

Proof: The equation (5.11) may be obtained simply:

$$\begin{aligned} g_{uu} &= -\sum \operatorname*{res}_{d\log\lambda=0}(2\pi i)^2\frac{\lambda}{\lambda'}dv, \\ &= \operatorname*{res}_{v=0}(2\pi i)^2\frac{\lambda}{\lambda'}dv, \\ &= 0. \end{aligned}$$

[3]NB $\theta_1(v-z_0|\tau)$ has been denoted θ_1 for typographical convenience here.

Similarly, by using the arguments contained within the proof of lemma 5.4, one may immediately write

$$\begin{aligned} g_{ui} &= -\sum \operatorname*{res}_{d\log\lambda=0} k_i(\Theta_0-\Theta_i)2\pi i\frac{\lambda}{\lambda'}dv, \\ &= \left(\operatorname*{res}_{v=z_i} + \operatorname*{res}_{v=z_0} + \operatorname*{res}_{v=0}\right) k_i(\Theta_0-\Theta_i)2\pi i\frac{\lambda}{\lambda'}dv, \\ &= 0. \end{aligned}$$

Finally consider

$$\begin{aligned} g_{u\tau} &= -\sum \operatorname*{res}_{d\log\lambda=0} 2\pi i\frac{\frac{\partial\lambda}{\partial\tau}}{\lambda}\frac{\lambda}{\lambda'}dv, \\ &= -2\pi i\sum \operatorname*{res}_{d\log\lambda=0} \frac{D\lambda-\frac{i\pi}{2}\Theta\lambda'}{\lambda'}dv, \\ &= -2\pi i\sum \operatorname*{res}_{d\log\lambda=0}\left(\frac{D\lambda}{\lambda'}-\frac{i\pi}{2}\Theta\right)dv, \\ &= -2\pi i\sum \operatorname*{res}_{d\log\lambda=0}\frac{D\lambda}{\lambda'}dv, \end{aligned}$$

the last line following from the fact that Θ is analytic everywhere except $v=0$ (and so is analytic where $d\log\lambda=0$). The fact that $D\lambda$ is elliptic allows the usual deformation of contours argument to be applied to give:

$$\begin{aligned} g_{u\tau} &= 2\pi i\operatorname*{res}_{v=0}\frac{D\lambda}{\lambda'}dv, \\ &= 2\pi i\operatorname*{res}_{v=0}\left(\frac{\partial_\tau\lambda}{\lambda'}+\frac{i\pi}{2}\Theta\right)dv. \end{aligned}$$

This may then be split into two individual parts:

$$\begin{aligned} \operatorname*{res}_{v=0}\frac{\partial_\tau\lambda}{\lambda} &= \operatorname*{res}_{v=0}\frac{\partial\log\lambda}{\partial\tau}\frac{\lambda}{\lambda'}, \\ &= 0, \end{aligned}$$

as all parts are analytic at $v=0$. Using the expansion of Θ, it is also possible to calculate the second part of the residue:

$$\begin{aligned} \operatorname*{res}_{v=0}\frac{i\pi}{2}\Theta dv &= \frac{i\pi}{2}\operatorname*{res}_{v=0}\left(\frac{1}{v}+O(v)\right), \\ &= \frac{i\pi}{2}. \end{aligned}$$

Remembering the preceding factor of $2i\pi$, one therefore has

$$g_{uT} = -\pi^2,$$

as required.

Lemma 5.6 *The remaining components of g are zero; that is to say*

$$g_{Ti} = 0, \tag{5.14}$$

$$g_{TT} = 0. \tag{5.15}$$

Proof: Using the usual residue formula,

$$\begin{aligned} g_{Ti} &= -\sum \underset{d\log\lambda=0}{\text{res}} (\Theta_0 - \Theta_i)\left(-\frac{i\pi}{4}\overbrace{\left(\sum_{i=0}^{n}\frac{\theta_1''(v-z_i|\tau)}{\theta_1(v-z_i|\tau)} - (n+1)\frac{\theta_1''(v|\tau)}{\theta_1(v|\tau)}\right)}^{A}\right)\frac{\lambda}{\lambda'}dv, \\ &= \frac{1\pi}{4}\left(\underset{v=z_i}{\text{res}}\,\Theta_i A\frac{\lambda}{\lambda'}dv - \underset{v=z_0}{\text{res}}\,\Theta_0 A\frac{\lambda}{\lambda'}dv\right). \end{aligned}$$

Noting that A is analytic near $v = z_0$ and $v = z_i$, one may therefore use the argument that the singularities of Θ_0 and Θ_i cancel with the zeros of λ. This both of the residues are zero, and so g_{Ti} must also be zero.

Moving on to g_{TT}, applying the formula (1.11) yields:

$$\begin{aligned} g_{TT} &= -\sum \underset{d\log\lambda=0}{\text{res}} \frac{(\partial_\tau\lambda)^2}{\lambda\lambda'}, \\ &= -\sum \underset{d\log\lambda=0}{\text{res}} \frac{(D\lambda - \frac{i\pi}{2}\Theta\lambda')^2}{\lambda\lambda'}, \\ &= -\sum \underset{d\log\lambda=0}{\text{res}} \left(\underbrace{\frac{(D\lambda)^2}{\lambda\lambda'}}_{P} - \underbrace{i\pi\frac{D\lambda\Theta}{\lambda}}_{Q} - \underbrace{\frac{\pi^2}{4}\frac{\Theta^2\lambda'}{\lambda}}_{R}\right). \end{aligned}$$

But as λ' does not appear in the denominators of Q and R, they are analytic at points where $d\log\lambda = 0$, and so the residues of Q and R are zero. By then applying the usual deformation of contours argument, the remaining residues may be moved to give

$$\begin{aligned}
g_{\tau\tau} &= \left(\sum \operatorname*{res}_{\lambda=0} + \operatorname*{res}_{v=0}\right) \frac{(D\lambda)^2}{\lambda\lambda'}, \\
&= \left(\sum \operatorname*{res}_{\lambda=0} + \operatorname*{res}_{v=0}\right) \frac{(\partial_\tau \lambda + \frac{i\pi}{2}\Theta\lambda')^2}{\lambda\lambda'}, \\
&= \overbrace{\sum \operatorname*{res}_{\lambda=0} (\partial_\tau \log\lambda)^2 \frac{\lambda}{\lambda'}}^{U} + \overbrace{\sum \operatorname*{res}_{\lambda=0} i\pi\Theta(\partial_\tau \log\lambda)}^{V} - \overbrace{\sum \operatorname*{res}_{\lambda=0} \frac{\pi^2}{4}\Theta^2 \frac{\lambda'}{\lambda}}^{W}, \\
&+ \underbrace{\operatorname*{res}_{v=0} (\partial_\tau \log\lambda)^2 \frac{\lambda}{\lambda'}}_{X} + \underbrace{\operatorname*{res}_{v=0} i\pi\Theta(\partial_\tau \log\lambda)}_{Y} - \underbrace{\operatorname*{res}_{v=0} \frac{\pi^2}{4}\Theta^2 \frac{\lambda'}{\lambda}}_{Z}.
\end{aligned}$$

These residues may then be considered individually. Using similar techniques to those employed earlier in this chapter, it is possible to show that

$$\begin{aligned}
U &= 0, \\
V &= 0, \\
W &= -\frac{\pi^2}{4}\sum_{i=0}^{n} \left(\frac{\theta_1'(z_i|\tau)}{\theta_1(z_i|\tau)}\right)^2, \\
X &= 0, \\
Y &= \frac{\pi^2}{4}\sum_{i=0}^{n} \frac{\theta_1''(z_i|\tau)}{\theta_1(z_i|\tau)} - (n+1)\frac{\pi^2}{4}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}, \\
Z &= \frac{\pi^2}{4}\sum_{i=0}^{n} \left(\left(\frac{\theta_1'(z_i|\tau)}{\theta_1(z_i|\tau)}\right)^2 - \frac{\theta_1''(z_i|\tau)}{\theta_1(z_i|\tau)}\right) \\
&\quad +(n+1)\frac{\pi^2}{4}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}.
\end{aligned}$$

Summing these then gives the desired result of $g_{\tau\tau} = 0$.

Lemma 5.7 *In the coordinates* $\{u, \tau, z_i\,(i = 1, \dots, m)\}$*, the intersection form is*

$$g = \sum_{i=1}^{n}\sum_{i\neq j}^{n} \frac{k_i k_j}{k_0} dz_i dz_j + \sum_{k=1}^{n} \left(\frac{k_i^2}{k_0} + k_i\right) dz_i^2 - 2\pi^2 du d\tau.$$

Proof: This theorem follows immediately by combining the results of the lemmas above.

Corollary 5.8 *The coordinates $\{u, \tau, z_i\}$ are flat coordinates of the intersection form.*

Proof: From the lemma above, the components of the intersection form are all constant in these coordinates. Therefore it follows automatically that they are flat coordinates of g.

As $\{u, \tau, z_i\}$ are flat coordinates for the intersection form, it follows that the tensor $c^\star(\partial_\alpha, \partial_\beta, \partial_\gamma)$ coincides with the dual structure constants $c^\star_{\alpha\beta\gamma}$, where ∂_α, ∂_β and ∂_γ are basis vectors in this coordinate system. Therefore it is possible to use such vectors in the formula (1.23) to calculate the $c^\star_{\alpha\beta\gamma}$. These calculations will again be performed in such a way that they are applicable on an arbitrary discriminant.

Lemma 5.9 *For distinct z_i, z_j and z_k,*

$$\begin{aligned} c^\star_{ijk} &= \frac{k_i k_j k_k}{k_0^2}\left((n+1)\Theta(z_0) - \sum_{r\neq 0} k_r \Theta(z_0 - z_r)\right) \\ &\quad - \frac{k_i k_j k_k}{k_0}\left(\Theta(z_0 - z_i) + \Theta(z_0 - z_j) + \Theta(z_0 - z_k)\right), \end{aligned}$$

where

$$\Theta(z) = \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}.$$

Proof: From the formula 1.23, we have

$$c^\star_{ijk} = -\sum \operatorname*{res}_{d\log\lambda=0} k_i k_j k_k (\Theta_0 - \Theta_i)(\Theta_0 - \Theta_j)(\Theta_0 - \Theta_k)\frac{\lambda}{\lambda'} dv.$$

Applying a deformation of contours argument and expanding the brackets, this becomes

$$\begin{aligned} c^\star_{ijk} &= k_i k_j k_k \left(\operatorname*{res}_{v=v_0} + \operatorname*{res}_{v=v_i} + \operatorname*{res}_{v=v_j} + \operatorname*{res}_{v=v_k} + \operatorname*{res}_{v=0}\right) \\ &\quad \left(\Theta_0^3 - \Theta_0^2(\Theta_i + \Theta_j + \Theta_k) + \Theta_0(\Theta_i\Theta_j + \Theta_j\Theta_k + \Theta_k\Theta_i) - \Theta_i\Theta_j\Theta_k\right)\frac{\lambda}{\lambda'} dv. \end{aligned}$$

Note that as $\frac{\lambda}{\lambda'}$ is finite (in fact zero) at $v = 0$ and all other terms are finite there, the residues at $v = 0$ are all themselves zero. Note also that where $v = v_i$ and Θ_i appears only to the power one, the singularity caused by $\theta_1(v - z_i|\tau)$ in the denominator of Θ_i is cancelled by a corresponding zero in $\frac{\lambda}{\lambda'}$, and so such residues also vanish. This leaves

$$c^\star_{ijk} = k_i k_j k_k \left(\operatorname*{res}_{v=z_0} \Theta_0^3 \frac{\lambda}{\lambda'} dv - \operatorname*{res}_{v=z_0} \Theta_0^2 (\Theta_i + \Theta_j + \Theta_k) \frac{\lambda}{\lambda'} dv \right).$$

Consider the first residue by using the expansion of Θ_0:

$$\operatorname*{res}_{v=z_0} \Theta_0^3 \frac{\lambda}{\lambda'} dv = \operatorname*{res}_{v=z_0} \left(\frac{1}{(v-z_0)^3} + \frac{1}{v-z_0} + O(1) \right) \frac{\lambda}{\lambda'}.$$

As $\frac{\lambda}{\lambda'}$ is zero at $v = z_0$, the simple pole is cancelled out and so

$$\begin{aligned} \operatorname*{res}_{v=z_0} \Theta_0^3 \frac{\lambda}{\lambda'} dv &= \operatorname*{res}_{v=z_0} \frac{1}{(v-z_0)^3} \frac{\lambda}{\lambda'} dv, \\ &= \text{coefficient of } (v-z_0)^2 \text{ in the expansion of } \frac{\lambda}{\lambda'}, \\ &= \frac{1}{2} \left(\frac{d^2}{dv^2} \frac{\lambda}{\lambda'} \right) \Big|_{v=z_0}. \end{aligned}$$

By writing

$$\frac{\lambda}{\lambda'} = \frac{p}{q}$$

in the same way as in the calculation of g_{ij}, and using similar reasoning to that used in chapter 4, one may immediately show that differentiating this twice and evaluating at $v = z_0$ gives:

$$\left(\frac{d^2}{dv^2} \frac{\lambda}{\lambda'} \right) \Big|_{v=z_0} = \frac{p''q - 2p'q'}{q^2} \Big|_{v=z_0}.$$

Considering the components of this individually (noting that as θ_1 is an odd function, $\theta_1(0|\tau) = \theta_1''(0|\tau) = 0$), we obtain:

$$\begin{aligned} p'|_{v=z_0} &= \theta_1'(v - z_0|\tau)\lambda_0|_{v=z_0}, \\ p''|_{v=z_0} &= 2\theta_1'(v - z_0|\tau)\lambda_0'|_{v=z_0}, \\ q|_{v=z_0} &= k_0\theta_1'(v - z_0|\tau)\lambda_0|_{v=z_0}, \\ q'|_{v=z_0} &= (1 + k_0)\theta_1'(v - z_0|\tau)\lambda_0'|_{v=z_0}. \end{aligned}$$

Therefore

$$\left.\frac{p''q-2p'q'}{q^2}\right|_{v=z_0} = -\left.\frac{2\lambda_0'}{k_0^2\lambda_0}\right|_{v-z_0}, \\ = -\frac{2}{k_0^2}\left.\frac{d\log\lambda_0}{dv}\right|_{v=v_0}.$$

But from its definition,

$$\lambda_0 = \frac{\prod_{r\neq 0}\theta_1^{k_r}(v-z_r|\tau)}{\theta_1^{n+1}(v|\tau)}.$$

Therefore

$$\frac{d\log\lambda_0}{dv} = \sum_{r\neq 0} k_r\Theta_r - (n+1)\Theta,$$

and so[4]

$$\operatorname*{res}_{v=v_0}\Theta_0^3\frac{\lambda}{\lambda'}dv = \frac{1}{k_0^2}\left((n+1)\Theta(z_0) - \sum_{r\neq 0} k_r\Theta(z_0-z_r)\right). \tag{5.16}$$

Next consider, again by using the expansion of Θ_0, the residue

$$\operatorname*{res}_{v=z_0}\Theta_0^2(\Theta_i+\Theta_j+\Theta_k)\frac{\lambda}{\lambda'}dv \\ = \operatorname*{res}_{v=z_0}\left(\frac{1}{(v-z_0)^2}+O(1)\right)(\Theta_i+\Theta_j+\Theta_k)\frac{\lambda}{\lambda'}dv, \\ = \text{coefficient of}\,(v-z_0)\,\text{in expansion of}\ (\Theta_i+\Theta_j+\Theta_k)\,\frac{\lambda}{\lambda'}.$$

Noting that Θ_i, Θ_j and Θ_k are analytic at $v=z_0$ one may, therefore, show that

$$\operatorname*{res}_{v=z_0}(\Theta_i+\Theta_j+\Theta_k)\Theta_0^2\frac{\lambda}{\lambda'} = \frac{\Theta(z_0-z_i)+\Theta(z_0-z_j)+\Theta(z_0-z_k)}{k_0},$$

by using the same technique as in the calculation of g_{ij}.

Combining this with 5.16 (being careful to remember the appropriate factors which multiply them), one therefore obtains the desired result. Lemma is proved.

Lemma 5.10 *For precisely two repeated indices, i.e. for distinct z_i and z_j, the dual structure constant takes the form:*

[4]Here the function $\Theta(z)$ will denotes $\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}$.

$$
\begin{aligned}
c^\star_{iij} &= \frac{k_i^2 k_j}{k_0^2}\left((n+1)\Theta(z_0) - \sum_{r\neq 0} k_r \Theta(z_0 - z_r)\right) \\
&\quad - \frac{k_i^2 k_j}{k_0}\left(2\Theta(z_0 - z_i) - \Theta(z_0 - z_j)\right) + k_i k_j \left(\Theta(z_i - z_0) - \Theta(z_i - z_j)\right).
\end{aligned}
$$

Proof: By the same reasoning as used in lemma 5.9, it follows that

$$
c^\star_{ijk} = k_i^2 k_j \left(\operatorname*{res}_{v=0} + \operatorname*{res}_{v=z_0} + \operatorname*{res}_{v=z_i} + \operatorname*{res}_{v=z_j}\right)(\Theta_0 - \Theta_i)^2(\Theta_0 - \Theta_j)\frac{\lambda}{\lambda'}dv,
$$

$$
= k_i^2 k_j \left(\operatorname*{res}_{v=z_i} \Theta_i^2(\Theta_0 - \Theta_j)\frac{\lambda}{\lambda'}dv - \operatorname*{res}_{v=z_0} \Theta_0^2(2\Theta_i + \Theta_j)\frac{\lambda}{\lambda'}dv + \operatorname*{res}_{v=z_0} \Theta_0^3 \frac{\lambda}{\lambda'}dv\right).
$$

But using the same techniques as those used in lemma 5.9, it can be shown that

$$
\begin{aligned}
\operatorname*{res}_{v=z_i} \Theta_i^2(\Theta_0 - \Theta_j)\frac{\lambda}{\lambda'}dv + &= \frac{1}{k_i}\left(\Theta(z_i - z_0) - \Theta(z_i - z_j)\right), \\
\operatorname*{res}_{v=z_0} \Theta_0^2(2\Theta_i + \Theta_j)\frac{\lambda}{\lambda'}dv &= \frac{1}{k_0}\left(2\Theta(z_0 - z_i) + \Theta(z_0 - z_j)\right), \\
\operatorname*{res}_{v=z_0} \Theta_0^3 \frac{\lambda}{\lambda'}dv &= \frac{1}{k_0^2}\left((n+1)\Theta(z_0) - \sum_{r\neq 0} k_r \Theta(z_0 - z_r)\right).
\end{aligned}
$$

Substituting these in above will then yield the desired equation.

Lemma 5.11 *For three identical indices, the dual structure constants are*

$$
\begin{aligned}
c^\star_{iii} &= \frac{k_i^3}{k_0^2}\left((n+1)\Theta(z_0) - \sum_{r\neq 0} k_r \Theta(z_0 - z_r)\right) - \frac{3k_i^3}{k_0}\Theta(z_0 - z_i) \\
&\quad + 3k_i^2\Theta(z_i - z_0) + k_i\left(\sum_{s\neq i} k_s \Theta(z_i - z_s) - (n+1)\Theta(z_i)\right).
\end{aligned}
$$

Proof: Again using the same reasoning and techniques as in lemma 5.9, one may immediately write

$$
\begin{aligned}
c^\star_{iii} &= k_i^3\left(\operatorname*{res}_{v=v_i} + \operatorname*{res}_{v=v_0}\right)(\Theta_0 - \Theta_i)^3 \frac{\lambda}{\lambda'}dv, \\
&= k_i^3\left(\operatorname*{res}_{v=v_0} \Theta_0^3 \frac{\lambda}{\lambda'}dv - \operatorname*{res}_{v=z_0} 3\Theta_0^2\Theta_i \frac{\lambda}{\lambda'}dv\right) \\
&\quad + k_i^3\left(\operatorname*{res}_{v=z_i} \Theta_i^2\Theta_0 \frac{\lambda}{\lambda'}dv - \operatorname*{res}_{v=z_i} \Theta_i^3 \frac{\lambda}{\lambda'}dv\right).
\end{aligned}
$$

The calculations for the residues above are entirely analogous to those in the previous two lemmas, and so one may immediately derive the desired result.

Having constructed the structure constants relating solely to the z_i coordinates, it is possible to introduce a function $f(\mathbf{z}|\tau)$ such that

$$\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial z^i \partial z^j \partial z^k} = c^\star_{ijk}(\mathbf{z}|\tau). \tag{5.17}$$

In order to do this, we begin by noting the following fact. The function $\log\theta_1(z|\tau)$ may be expressed as the following:

$$\log\theta_1(z|\tau) = \log(iGq^{\frac{1}{4}}) - iz - \left(\sum_{n=0}^{\infty} Li_1(q^{2n}e^{2iz}) + \sum_{n=1}^{\infty} Li_1(q^{2n}e^{-2iz})\right), \tag{5.18}$$

where, as above, G takes the form (5.6) and $q = e^{i\pi\tau}$. To show this, one uses the infinite product representation (5.5):

$$\begin{aligned}
\theta_1(z|\tau) &= 2Gq^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty}(1-q^{2n}e^{2iz}) \prod_{n=1}^{\infty}(1-q^{2n}e^{-2iz}), \\
&= 2Gq^{\frac{1}{4}} \frac{e^{iz}-e^{-iz}}{2i} \frac{1}{1-e^{2iz}} \prod_{n=0}^{\infty}(1-q^{2n}e^{2iz}) \prod_{n=1}^{\infty}(1-q^{2n}e^{-2iz}), \\
&= iGq^{\frac{1}{4}} e^{-iz} \prod_{n=0}^{\infty}(1-q^{2n}e^{2iz}) \prod_{n=1}^{\infty}(1-q^{2n}e^{-2iz}).
\end{aligned}$$

Hence, by taking the logarithm of this,

$$\log\theta_1(z|\tau) = \log(iGg^{\frac{1}{4}}) - iz + \sum_{n=0}^{\infty}(1-q^{2n}e^{2iz}) + \sum_{n=1}^{\infty}\log(1-q^{2n}e^{-2iz}).$$

But recalling that

$$Li_1(z) = -\log(1-z),$$

the desired result is obtained. The expression of $\log\theta_1$ above will offer a convenient representation of the function Λ_1, defined (for an arbitrary integer suffix) by the infinite series below.

Definition 5.12 *A new function Λ_N is defined by the series*

$$\Lambda_N(z,q) = -\frac{1}{2}\frac{(2iz)^N}{N!} - \left(\sum_{n=0}^{\infty} Li_N(q^{2n}e^{2iz}) + (-1)^{N+1}\sum_{n=1}^{\infty} Li_N(q^{2n}e^{-2iz})\right).$$

This function is, to within a polynomial, the elliptic polylogarithm function (see [18]) and so shares its convergence properties.

The function Λ_1 is related to $\theta_1(z|\tau)$ by the following equation:

$$\Lambda_1(z,q) = \log\theta_1(z|\tau) - \log(iGq^{\frac{1}{4}}). \tag{5.19}$$

This equation follows immediately from the definition of Λ_1 and (5.18). Consideration is now moved to differentiating Λ_N with respect to z. This gives rise to a simple differential equation. The derivative of $\Lambda_N(z,q)$ with respect to z is linked to Λ_{N-1} by the differential equation

$$\frac{d}{dz}\Lambda_N(z,q) = 2i\Lambda_{N-1}. \tag{5.20}$$

To prove this, note that from the definition of Λ_N, it immediately follows that

$$\Lambda_{N-1}(z,q) = -\frac{1}{2}\frac{(2iz)^{N-1}}{(N-1)!} - \left(\sum_{n=0}^{\infty} Li_{N-1}(q^{2n}e^{2iz}) + (-1)^N \sum_{i=1}^{\infty} Li_{N-1}(q^{2n}e^{-2iz})\right).$$

Differentiating $\Lambda_N(z,q)$ with respect to z yields

$$\begin{aligned}
&\frac{d}{dz}\Lambda_N(z,q)\\
&= -\frac{1}{2}\frac{2iN(2i)^{N-1}}{N!}\\
&- \left(\sum_{n=0}^{\infty} 2iq^{2n}e^{2iz}\frac{Li_{N-1}(q^{2n}e^{2iz})}{q^{2n}e^{2iz}} + (-1)^{N+1}\sum_{n=1}^{\infty} -2iq^{2n}e^{-2iz}\frac{Li_{N-1}(q^{2n}e^{-2iz})}{q^{2n}e^{-2iz}}\right),\\
&= 2i\left(-\frac{1}{2}\frac{(2iz)^{N-1}}{(N-1)!}\right) - 2i\left(\sum_{n=0}^{\infty} Li_{N-1}(q^{2n}e^{2iz}) + (-1)^N\sum_{n=1}^{\infty} Li_{N-1}(q^{2n}e^{-2iz})\right),\\
&= 2i\Lambda_{N-1}(z,q),
\end{aligned}$$

as required. Note also that the function Λ_0 satisfies

$$\Lambda_0(z,q) = \frac{1}{2i}\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}. \tag{5.21}$$

This follows immediately from differentiating (5.19) with respect to z using the lemma above. One may then use the function Λ_3, along with the lemmas above, to introduce a function $f(\mathbf{z}|\tau)$ satisfying the equation (5.17).

Lemma 5.13 *The function*

$$f(\mathbf{z}|\tau) := \frac{n+1}{4}\sum_{p} k_p\Lambda_3(z_p) - \frac{1}{8}\sum_{r\neq s} k_r k_s \Lambda_3(z_r - z_s)\Big|_{\sum k_i z_i = 0}$$

where indices for the sums run from 0 *to* m, *satisfies (5.17).*

Proof: To prove this, one needs simply to differentiate f three times with respect to z_i, z_j and z_k and show that the result (for appropriate combinations of distinct and identical indices) coincides with $c^{\star}_{ijk}$. Firstly, differentiate f with respect to z_i:

$$\begin{aligned}\frac{\partial f(\mathbf{z}|\tau)}{\partial z^i} &= \frac{2i(n+1)}{4}\left(k_i\Lambda_2(z_i) - \frac{k_i}{k_0}k_0\Lambda_2(z_0)\right) + \frac{i}{4}\sum_{p\neq i} k_i k_p(\Lambda_2(z_p - z_i) - \Lambda_2(z_i - z_p)) \\ &+ \frac{i}{4}\frac{k_i}{k_0}\sum_{p\neq 0} k_0 k_p(\Lambda_2(z_0 - z_p) - \Lambda_2(z_p - z_0).\end{aligned}$$

Differentiate again with respect to z_j, $i \neq j$ to obtain

$$\begin{aligned}\frac{\partial^2 f(\mathbf{z}|\tau)}{\partial z^j \partial z^i} &= -(n+1)\frac{k_i k_j}{k_0}\Lambda_1(z_0) - \frac{1}{2}k_i k_j(\Lambda_1(z_j - z_i) + \Lambda_1(z_i - z_j) \\ &+ \frac{1}{2}\frac{k_j}{k_0}k_0 k_i(\Lambda_1(z_0 - z_i) + \Lambda_1(z_i - z_0)) \\ &+ \frac{1}{2}\frac{k_i}{k_0}k_j k_0(\Lambda_1(z_0 - z_j) + \Lambda_1(z_j - z_0)) + \\ &\frac{1}{2}\frac{k_i k_j}{k_0^2}\sum_{p\neq 0} k_p k_0(\Lambda_1(z_0 - z_p) + \Lambda_1(z_p - z_0)).\end{aligned} \tag{5.22}$$

Differentiating a further time, with respect to z_k, $k \neq i, j$ gives

$$\begin{aligned}\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial z^k \partial z^j \partial z^i} &= 2i(n+1)\frac{k_i k_j k_k}{k_0^2}\Lambda_0(z_0) - i\frac{k_i k_j k_k}{k_0}(\Lambda_0(z_0 - z_i) - \Lambda_0(z_i - z_0)) \\ &- i\frac{k_i k_j k_k}{k_0}(\Lambda_0(z_0 - z_j) - \Lambda_0(z_j - z_0) - i\frac{k_i k_j k_k}{k_0}(\Lambda_0(z_0 - z_k) - \Lambda_0(z_k - z_0)) \\ &- i\frac{k_i k_j k_k}{k_0^2}\sum_{p\neq 0} k_p(\Lambda_0(z_0 - z_p) - \Lambda_0(z_p - z_0).\end{aligned}$$

Substituting in the result from equation (5.21) that

$$\Lambda_0(z, q) = \frac{1}{2i}\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)},$$

and noting that

$$\frac{\theta_1'(-z|\tau)}{\theta_1(-z|\tau)} = -\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)},$$

one obtains the desired result of

$$\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial z^k \partial z^j \partial z^k} = c^\star_{ijk}.$$

Similar reasoning and techniques may be applied to show that this holds for $i = j \neq k$ and for $i = j = k$.

Recall that the third derivatives of $F^\star$ must satisfy

$$\frac{\partial F^\star}{\partial\alpha\partial\beta\partial\gamma} = c^\star_{\alpha\beta\gamma}$$

for all α, β, γ including u and τ. The function $f(\mathbf{z}|\tau)$ does not, therefore, necessarily coincide with $F^\star$. One now moves on to considering $c^\star_{u\alpha\beta}$, where α and β may be any of the coordinates.

Lemma 5.14 *The non-zero dual structure constants* $c^\star_{u\alpha\beta}$ *are:*

$$\begin{aligned} c^\star_{uij} &= 2\pi i\left(\frac{k_i k_j}{k_0} + k_i\delta_{ij}\right), \\ c^\star_{uu\tau} &= -2i\pi^3, \end{aligned}$$

with

$$c^\star_{uui} = c^\star_{ui\tau} = c^\star_{u\tau\tau} = c^\star_{uuu} = 0.$$

Proof: Applying the formula (1.23), one may easily show that for α, β being any of the coordinates z_i, u, τ, the following formula holds.

$$\begin{aligned} c^\star_{u\alpha\beta} &= -\sum \operatorname*{res}_{d\log\lambda=0} \pi i\frac{\partial\lambda}{\partial\alpha}\frac{\partial\lambda}{\partial\beta}\frac{\lambda}{\lambda'}dv, \\ &= 2\pi i g_{\alpha\beta}. \end{aligned}$$

The lemma follows automatically from this.

Lemma 5.15 *The function*

$$g^\star(u, z_i, \tau) = 2\pi i \left(u \sum_{i=i}^{m} \sum_{j>i} \frac{k_i k_j}{k_0} z_i z_j + u \sum_{i=1}^{m} \left(k_i + \frac{k_i^2}{k_0} \right) \frac{z_i^2}{2} - \frac{1}{2} \pi^2 \tau u^2 \right) \quad (5.23)$$

satisfies the differential equations

$$\frac{\partial^3 g^\star}{\partial u \partial \alpha \partial \beta} = c^\star_{u\alpha\beta}.$$

Proof: Proof is obvious; differentiating $g^\star$ immediately yields the desired results.

Note that the entire u-dependence of $F^\star$ must therefore be included in $g^\star$ (ignoring any quadratic terms which vanish under triple differentiation). Hence if one writes

$$F^\star = g^\star + F_1^\star,$$

the function $F_1^\star$ must be independent of u. Also, observe that

$$\frac{\partial^3 g^\star}{\partial \alpha \partial \beta \partial \gamma} = 0, \quad \alpha, \beta, \gamma \neq u.$$

Therefore

$$\frac{\partial^3 F^\star}{\partial \alpha \partial \beta \partial \gamma} = \frac{\partial^3 F_1^\star}{\partial \alpha \partial \beta \partial \gamma}, \quad \alpha, \beta, \gamma \neq u.$$

But this must be equal to $c^\star_{\alpha\beta\gamma}$. Now write

$$F_1^\star = f + f_1.$$

From theorem 5.13 and the line above, one must have

$$\frac{\partial^3}{\partial z_i \partial z_j \partial z_k} (F_1^\star - f) \;=\; 0. \quad (5.24)$$

Integrating this gives rise to the following lemma.

Lemma 5.16 *The function $f_1(\mathbf{z}|\tau)$ above is of the following form:*

$$f_1(\mathbf{z}|\tau) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} z_i z_j(\tau) + \sum_{i=1}^{m} B_i(\tau) z_i + C(\tau).$$

Proof: The proof of this follows immediately from integrating the equations (5.24) to obtain:

$$F_1^\star(\mathbf{z}|\tau) - f(\mathbf{z}|\tau) = \frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} A_{ij}z_iz_j(\tau) + \sum_{i=0}^{m} B_i(\tau)z_i + C(\tau).$$

But as $F_1^\star - f = f_1$, the lemma is proved.

Note that the functions A_{ij}, B_i and C are functions only of τ, i.e. they are independent of the z_i. Differentiating $F_1^\star$ with respect to z_i, z_j and τ, one obtains

$$c^\star_{\tau ij}(\mathbf{z}|\tau) = \frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau\partial z_i\partial z_j} + \frac{dA_{ij}}{d\tau}.$$

But as A_{ij} is independent of z, one may evaluate $f(\mathbf{z}|\tau)$ at $\mathbf{z} = \mathbf{0}$ to obtain

$$A'_{ij}(\tau) = c^\star_{\tau ij}(\mathbf{0}|\tau) - \left.\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau\partial z^i\partial z^j}\right|_{\mathbf{z}=\mathbf{0}}.$$

Similar reasoning leads to

$$\begin{aligned} B''_i(\tau) &= c^\star_{\tau\tau i}(\mathbf{0}|\tau) - \left.\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau^2\partial z^i}\right|_{\mathbf{z}=\mathbf{0}}, \\ C'''(\tau) &= c^\star_{\tau\tau\tau}(\mathbf{0}|\tau) - \left.\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau^3}\right|_{\mathbf{z}=\mathbf{0}}. \end{aligned}$$

It transpires that the third derivatives in the above equations all take simple forms.

Lemma 5.17 *The derivatives*

$$\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau\partial z^i\partial z^j} \quad \text{and} \quad \frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau^2\partial z^i}$$

are both zero when $\mathbf{z} = \mathbf{0}$.

Proof: From the proof of lemma 5.13, we already know that $\frac{\partial^2 f(\mathbf{z}|\tau)}{\partial z^i\partial z^j}$ is of the form (5.22) Using the fact that

$$\left.\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau\partial z^i\partial z^j}\right|_{\mathbf{z}=\mathbf{0}} = \frac{\partial}{\partial\tau}\left(\left.\frac{\partial^2 f(\mathbf{z}|\tau)}{\partial z^i\partial z^j}\right|_{\mathbf{z}=\mathbf{0}}\right),$$

one may evaluate the second derivative of f and then differentiate the result with respect to τ. But

$$\begin{aligned}\left.\frac{\partial^2 f(\mathbf{z}|\tau)}{\partial z^i \partial z^j}\right|_{\mathbf{z}=0} &= \lim_{\mathbf{z}\to\mathbf{0}}\Bigg(-(n+1)\frac{k_i k_j}{k_0}\Lambda_1(z_0) - \frac{1}{2}k_i k_j(\Lambda_1(z_j - z_i) + \Lambda_1(z_i - z_j) \\ &+\frac{1}{2}\frac{k_j}{k_0}k_0 k_i(\Lambda_1(z_0 - z_i) + \Lambda_1(z_i - z_0)) \\ &+\frac{1}{2}\frac{k_i}{k_0}k_j k_0(\Lambda_1(z_0 - z_j) + \Lambda_1(z_j - z_0)) + \\ &\frac{1}{2}\frac{k_i k_j}{k_0^2}\sum_{p\neq 0} k_p k_0(\Lambda_1(z_0 - z_p) + \Lambda_1(z_p - z_0))\Bigg)\end{aligned}$$

But noting that the all of the $\Lambda_1(z_i - z_j)$ terms are functions of linear functions of the z_i, they will all (assuming that $\mathbf{z} \to \mathbf{0}$ in a suitable way) behave like $\lim_{z\to 0}\lambda_1(z)$ as $\mathbf{z} \to \mathbf{0}$. Therefore they may be grouped in the limit, and so one obtains

$$\begin{aligned}\left.\frac{\partial^2 f(\mathbf{z}|\tau)}{\partial z^i \partial z^j}\right|_{\mathbf{z}=0} &= \lim_{z\to 0}\left(\left((n+1)\frac{k_i k_j}{k_0} + k_i - k_i - \frac{k_i k_j k_0}{k_0} - \frac{k_i k_j}{k_0}\sum_{p\neq 0} k_p\right)\Lambda_1(z)\right), \\ &= \frac{k_i k_j}{k_0}((n+1) - k_0 - (n+1-k_0))\lim_{z\to 0}\Lambda_1(z), \\ &= 0\lim_{z\to 0}\Lambda_1(z) \\ &= 0.\end{aligned}$$

Differentiating this with respect to τ therefore trivially yields

$$\left.\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial\tau\partial z^i \partial z^j}\right|_{\mathbf{z}=0} = 0.$$

By recalling $\frac{\partial f}{\partial z^i}$ and applying similar reasoning to that used above, it is possible to immediately write:

$$\begin{aligned}\left.\frac{\partial f(\mathbf{z}|\tau)}{\partial z^i}\right|_{z=0} &= \lim_{z\to 0}\Bigg(\frac{2i(n+1)}{4}\left(k_i\Lambda_2(z) - \frac{k_i}{k_0}k_0\Lambda_2(z)\right) + \frac{i}{4}\sum_{p\neq i} k_i k_p(\Lambda_2(z) - \Lambda_2(z)) \\ &+\frac{i}{4}\frac{k_i}{k_0}\sum_{p\neq 0} k_0 k_p(\Lambda_2(z) - \Lambda_2(z))\Bigg), \\ &= 0.\end{aligned}$$

Differentiating this twice with respect to τ trivially gives zero, as required.

It follows automatically from this lemma that

$$\begin{aligned} A'_{ij}(\tau) &= c^\star_{\tau ij}(\mathbf{0}|\tau) \quad \text{and} \\ B''_i(\tau) &= c^\star_{\tau\tau i}(\mathbf{0}|\tau) \end{aligned}$$

In order to proceed, one needs to evaluate $c^\star_{\tau ij}(\mathbf{0}|\tau)$, $c^\star_{\tau\tau i}(\mathbf{0}|\tau)$ and $c^\star_{\tau\tau\tau}(\mathbf{0}|\tau)$, though it is not necessary to calculate these functions everywhere.

Lemma 5.18 *For $i \neq j$:*

$$c^\star_{\tau ij}(\mathbf{0}|\tau) = 0.$$

Similarly, in the case where i and j coincide, one has

$$c^\star_{\tau ii}(\mathbf{0}|\tau) = 0.$$

Proof: By using the formula (1.23), one obtains

$$\begin{aligned} c^\star_{\tau ij}(\mathbf{z}|\tau) &= -\sum \operatorname*{res}_{d\log\lambda=0} k_i k_j (\Theta_0 - \Theta_i)(\Theta_0 - \Theta_j) \\ & \left(\underbrace{\frac{-i\pi}{4} \sum_{r=0}^{m} k_r \frac{\theta_1''(v - z_r|\tau)}{\theta_1(v - z_r|\tau)} - (n+1)\frac{\theta_1''(v|\tau)}{\theta(v|\tau)}}_{A} \right) \frac{\lambda}{\lambda'} dv \end{aligned}$$

But A is finite everywhere (with the possible exception of infinity), so by the usual deformation of contours argument, this becomes

$$\begin{aligned} c^\star_{\tau ij}(\mathbf{z}|\tau) &= \left(\operatorname*{res}_{v=z_0} + \operatorname*{res}_{v=z_i} + \operatorname*{res}_{v=z_j} \right) k_i k_j (\Theta_0 - \Theta_i)(\Theta_0 - \Theta_j) A \frac{\lambda}{\lambda'} dv, \\ &= k_i k_j A\big|_{v=z_0}, \\ &= \frac{-i\pi}{4}\frac{k_i k_j}{k_0} \left(\sum_{r=0}^{m} k_r \frac{\theta_1''(z_0 - z_r|\tau)}{\theta_1(z_0 - z_r|\tau)} - (n+1)\frac{\theta_1''(z_0|\tau)}{\theta_1(z_0|\tau)} \right). \end{aligned} \tag{5.25}$$

Note that

$$\lim_{v\to 0} \frac{\theta_1''(v|\tau)}{\theta_1(v|\tau)}$$

is well defined, and so one may take this value to be equal to $\frac{\theta_1''(0|\tau)}{\theta_1(0|\tau)}$. Hence if one evaluates the above at $\mathbf{z}=\mathbf{0}$, there are $\sum k_r = (n+1)$ positive $\frac{\theta_1''(0|\tau)}{\theta_1(0|\tau)}$ terms and $(n+1)$ negative terms. These cancel out to give

$$c^\star_{\tau ij}(\mathbf{0}|\tau)=0.$$

Similar calculations can show that

$$\begin{aligned} c^\star_{\tau ii}(\mathbf{z}|\tau) &= \frac{-i\pi}{4}k_i^2\left(\sum_{r=0}^{m} k_r\frac{\theta_1''(z_0-z_r|\tau)}{\theta_1(z_0-z_r|\tau)}+\frac{\theta_1''(z_i-z_r|\tau)}{\theta_1(z_i-z_r|\tau)}\right)\\ &+\frac{i\pi}{4}k_i^2(n+1)\left(\frac{\theta_1''(z_0|\tau)}{\theta_1(z_0|\tau)}+\frac{\theta_1''(z_i|\tau)}{\theta_1(z_i|\tau)}\right).\end{aligned}$$

Again it is easy to show that evaluating this at $\mathbf{z}=\mathbf{0}$ gives $c^\star_{\tau ii}(\mathbf{0}|\tau)=0$.

Whilst in the above lemma, $c^\star_{\tau ij}$ was calculated everywhere and then evaluated at $\mathbf{z}=\mathbf{0}$, such calculations for $c^\star_{\tau\tau i}$ and $c^\star_{\tau\tau\tau}$ grow in complexity, and so a different approach will be used in proving the following lemma.

Lemma 5.19 *Evaluating $c^\star_{\tau\tau i}$ at $\mathbf{z}=\mathbf{0}$ leads to:*

$$c^\star_{\tau\tau i}(\mathbf{0}|\tau)=\mathbf{0}.$$

Proof: One begins by utilising the WDVV associativity equations. In particular, consider the equation

$$c^\star_{ij\alpha}c^\star_{j\tau\beta}g^{\alpha\beta}=c^\star_{jj\alpha}c^\star_{i\tau\beta}g^{\alpha\beta}.$$

Noting for which α and β the components $g^{\alpha\beta}$ are zero, this becomes

$$(c^\star_{ij\tau}c^\star_{j\tau u}+c^\star_{iju}c^\star_{j\tau\tau})g^{\tau u}+\sum_{p,q}c^\star_{ijp}c^\star_{\tau jq}g^{pq}=(c^\star_{jj\tau}c^\star_{i\tau u}+c^\star_{jju}c^\star_{i\tau\tau})g^{\tau u}+\sum_{p,q}c^\star_{jjp}c^\star_{i\tau q}g^{pq}.$$

But $c^\star_{j\tau u}=c^\star_{i\tau u}=0$ and $c^\star_{uij}=2\pi i g_{ij}$. Furthermore, one may easily show that $g^{\tau u}=\frac{1}{\pi^2}$ Substituting these in and rearranging the terms yields

$$\frac{2i}{\pi k_0}(k_ik_jc^\star_{\tau\tau j}-2k_j^2c^\star_{\tau\tau i})=\underbrace{\sum_{p,q}(c^\star_{jjp}c^\star_{\tau iq}-c^\star_{ijp}c^\star_{\tau jq})g^{pq}}_{P_{ij}}.$$

Similarly, one may show that

$$P_{ji} = \frac{2i}{\pi k_0}(k_i k_j c^\star_{\tau\tau i} - 2k_i^2 c^\star_{\tau\tau j}).$$

Adding $2k_iP_{ij}$ to k_jP_{ji} therefore gives

$$\frac{-6ik_ik_j^2}{\pi k_0}c^\star_{\tau\tau i} = \sum_{p,q}\left(2k_ic^\star_{jjp}c^\star_{\tau iq} - 2k_ic^\star_{ijp}c^\star_{\tau jq} + k_jc^\star_{iip}c^\star_{\tau jq} - k_jc^\star_{ijp}c^\star_{\tau iq}\right)g^{pq}.$$

One may obtain $c^\star_{\tau\tau i}(\mathbf{0}|\tau)$ by evaluating this at $\mathbf{z} = \mathbf{0}$.

Consider now the individual terms, e.g. $c^\star_{ijp}c^\star_{\tau jq}$. We know that $c^\star_{\tau jq}(\mathbf{0}|\tau) = \mathbf{0}$ and it is easy to see that $c^\star_{ijp}(\mathbf{0}|\tau)$ is singular. In order to show the behaviour of their product near $\mathbf{z} = \mathbf{0}$, one considers the terms from which they are formed. Begin with $c^\star_{ijp}$, which is the sum of terms of the form

$$\frac{\theta_1'(\tilde{z}|\tau)}{\theta_1(\tilde{z}|\tau)},$$

where $\tilde{z}$ is something linear in the z_i. The expansion of this near $\tilde{z} = 0$ is of the form

$$\frac{1}{\tilde{z}} + O(\tilde{z}).$$

Hence $c^\star_{ijp}$ is the sum of terms which have simple poles at $\mathbf{z} = \mathbf{0}$, and so $c^\star_{ijp}$ must itself have a simple pole there. Move onto, $c^\star_{\tau jq}$, which is the sum of terms of the form

$$\frac{\theta_1''(\tilde{z}|\tau)}{\theta_1(\tilde{z}|\tau)},$$

which have expansions of the form

$$c + O(\tilde{z}^2).$$

But $c^\star_{\tau jq}(\mathbf{0}|\tau) = \mathbf{0}$ (obtained by adding the same number of 'positive' and 'negative' terms), and so it must have a zero of order 2 at $\mathbf{z} = \mathbf{0}$. Therefore at $\mathbf{z} = \mathbf{0}$,

$$\begin{aligned} c^\star_{ijp}c^\star_{\tau jq} &= (\text{pole of order 1}) \times (\text{zero of order 2}), \\ &= (\text{zero of order 1}). \end{aligned}$$

Hence $c^{\star}_{\tau\tau i}$ is the sum of terms which all individually have zeroes at $\mathbf{z} = \mathbf{0}$ and so must itself have a zero (of order one) there.

Lemma 5.20 *The final structure constant, $c^{\star}_{\tau\tau\tau}(\mathbf{z}|\tau)$ adheres to*

$$c^{\star}_{\tau\tau\tau}(\mathbf{0}|\tau) \;=\; 0.$$

Proof: The modularity properties of $c^{\star}_{\alpha\beta\gamma}$ will be considered. Begin by noting that the function $\theta_1(v|\tau)$ obeys the modularity equation (where A is a constant, the precise value of which is not needed here):

$$A\theta_1(v|\tau) = -ie^{\frac{-iv^2}{\pi\tau}}\theta_1(\frac{v}{\tau}|\frac{-1}{\tau}).$$

Differentiating this with respect to v, one obtains

$$\begin{aligned} A\theta_1'(v|\tau) &= -i\frac{-2iv}{\pi\tau}e^{\frac{-iv^2}{\pi\tau}}\theta_1(\frac{v}{\tau}|\frac{-1}{\tau}) - ie^{\frac{-iv^2}{\pi\tau}}\frac{1}{\tau}\theta_1'(\frac{v}{\tau}|\frac{-1}{\tau}), \\ &= \frac{-2v}{\pi\tau}e^{\frac{-iv^2}{\pi\tau}}\theta_1(\frac{v}{\tau}|\frac{-1}{\tau}) - ie^{\frac{-iv^2}{\pi\tau}}\frac{1}{\tau}\theta_1'(\frac{v}{\tau}|\frac{-1}{\tau}). \end{aligned}$$

Dividing this by $A\theta_1(v|\tau)$ gives

$$\frac{\theta_1'(v|\tau)}{\theta_1(v|\tau)} = \frac{-2iv}{\pi\tau} + \frac{1}{\tau}\frac{\theta_1'(\frac{v}{\tau}|\frac{-1}{\tau})}{\theta_1(\frac{v}{\tau}|\frac{-1}{\tau})},$$

or upon a simple rearrangement of terms:

$$\frac{\theta_1'(\frac{v}{\tau}|\frac{-1}{\tau})}{\theta_1(\frac{v}{\tau}|\frac{-1}{\tau})} = \frac{2iv}{\pi} + \tau\frac{\theta_1'(v|\tau)}{\theta_1(v|\tau)}.$$

Note that for small v, this is therefore a modular function of weight 1. Similar reasoning leads to the analogous relationship

$$\frac{\theta_1''(\frac{v}{\tau}|\frac{-1}{\tau})}{\theta_1(\frac{v}{\tau}|\frac{-1}{\tau})} = \tau^2\frac{\theta_1''(v|\tau)}{\theta_1(v|\tau)} + \frac{6i\tau}{\pi} + O(v^2). \tag{5.26}$$

for small v.

One now moves on to considering the modular properties of $c^\star_{ijk}$. Begin with the case of i, j and k being distinct:

$$\begin{aligned} c^\star_{ijk}(\frac{\mathbf{z}}{\tau}|\frac{-1}{\tau}) &= \frac{k_i k_j k_k}{k_0^2}\frac{2i}{\pi}\left((n+1)(z_0) - \sum_{r\neq 0} k_r(z_0 - z_r)\right) \\ &\quad - \frac{k_i k_j k_k}{k_0}\frac{2i}{\pi}((z_0 - z_i) + (z_0 - z_j) + (z_0 - z_k)) + \tau c^\star_{ijk}(\mathbf{z}|\tau), \\ &= -\frac{k_i k_j k_k}{k_0}\frac{2i}{\pi}((z_0 - z_i) + (z_0 - z_j) + (z_0 - z_k)) + \tau c^\star_{ijk}(\mathbf{z}|\tau). \end{aligned}$$

Noting that $c^\star_{ijk}$ is singular at $\mathbf{z} = \mathbf{0}$, one now considers the limit[5] as $\mathbf{z} \to \mathbf{0}$ of the ratio

$$\frac{c^\star_{ijk}(\frac{\mathbf{z}}{\tau}|\frac{-1}{\tau})}{c^\star_{ijk}(\mathbf{z}|\tau)} = \frac{-\frac{k_i k_j k_k}{k_0}\frac{2i}{\pi}((z_0 - z_i) + (z_0 - z_j) + (z_0 - z_k))}{c^\star_{ijk}(\mathbf{z}|\tau)} + \tau.$$

It is easy to see that as $\mathbf{z} \to \mathbf{0}$, the right hand side tends to τ.

Similar calculations show the same result for $c^\star_{iij}$ and $c^\star_{iii}$. Consideration is now turned to the modularity properties of $c^\star_{\tau ij}$.

Substituting the modularity equation (5.26) into (5.25) gives (for small z)

$$\begin{aligned} c^\star_{\tau ij}(\frac{\mathbf{z}}{\tau}|\frac{1}{\tau}) = \frac{-i\pi}{4}\frac{k_i k_j}{k_0}&\left(\sum_{r=0}^{m} k_r \tau^2 \frac{\theta_1''(z_0 - z_r|\tau)}{\theta_1(z_0 - z_r|\tau)} - (n+1)\tau^2 \frac{\theta_1''(z_0|\tau)}{\theta_1(z_0|\tau)}\right) \\ &- \frac{i\pi}{4}\frac{k_i k_j}{k_0}\frac{6i\tau}{\pi}\left(\sum_{r=0}^{m} k_r - (n+1)\right) + O(v^2). \end{aligned}$$

But as $\sum k_r = (n+1)$, one has

$$c^\star_{\tau ij}(\frac{\mathbf{z}}{\tau}|\frac{-1}{\tau}) = \tau^2 c^\star_{\tau ij}(\mathbf{z}|\tau) + O(z^2),$$

for small z. Hence $c^\star_{\tau ij}$ is modular of weight 2 near $\mathbf{z} = 0$. In order to obtain the modularity properties of $c^\star_{\tau\tau i}(\mathbf{0}|\tau)$ in the absence of an explicit formula for $c^\star_{\tau\tau i}(\mathbf{z}|\tau)$, one utilises the WDVV associativity equations. Recall from the previous lemma that

$$\frac{-6ik_i k_j^2}{\pi k_0} c^\star_{\tau\tau i} = \sum_{p,q}\left(2k_i c^\star_{jjp} c^\star_{\tau iq} - 2k_i c^\star_{ijp} c^\star_{\tau jq} + k_j c^\star_{iip} c^\star_{\tau jq} - k_j c^\star_{ijp} c^\star_{\tau iq}\right) g^{pq}.$$

[5]Note that as one is dealing with a multivariate limit, one must be careful as to how the limit is taken.

One now restricts this to near $\mathbf{z} = \mathbf{0}$ and considers the individual terms[6]. From the equations above, one has

$$\frac{c^\star_{jjp}(\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau})c^\star_{\tau iq}((\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau})}{c^\star_{jjp}(\mathbf{0}|\tau)} = \tau^2 c^\star_{\tau iq}(\mathbf{0}|\tau)\tau.$$

If one multiplies both sides by $c^\star_{jjp}(\mathbf{0}|\tau)$, the right hand side is finite (from the proof of lemma 5.18). Hence

$$c^\star_{jjp}c^\star_{\tau iq}(\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau}) = \tau^3 c^\star_{jjp}c^\star_{\tau iq}(\mathbf{0}|\tau).$$

Therefore $c^\star_{jjp}c^\star_{\tau iq}(\mathbf{0}|\tau)$ is a modular function of weight 3. Moreover, $c^\star_{\tau\tau i}(\mathbf{0}|\tau)$ is a sum of modular functions of weight 3, and so is itself a modular function of weight 3.

Attention is now turned to the modularity properties of $c^\star_{\tau\tau\tau}(\mathbf{0}|\tau)$. By considering the associativity equation

$$c^\star_{\tau\tau\alpha}c^\star_{ii\beta}g^{\alpha\beta} = c^\star_{\tau i\alpha}c^\star_{\tau i\beta}g^{\alpha\beta},$$

one may (by substituting in known $g^{\alpha\beta}$ and $c^\star_{\alpha\beta\gamma}$) obtain the equation:

$$c^\star_{\tau\tau\tau} = \frac{\pi k_0}{ik_i^2}\sum_{p,q}(c^\star_{\tau ip}c^\star_{\tau iq} - c^\star_{\tau\tau p}c^\star_{iiq})g^{pq}.$$

At $\mathbf{z} = \mathbf{0}$, the $c^\star_{\tau ip}$ terms all vanish. By applying similar logic to that used in lemma 5.19, it is possible to show that

$$\begin{aligned} c^\star_{\tau\tau p}c^\star_{iiq}(\mathbf{0}|\tau) &= (\text{zero of order 1}) \times (\text{pole of order 1}), \\ &= \text{something finite.} \end{aligned}$$

Also observe that

$$\frac{c^\star_{\tau\tau p}(\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau})c^\star_{iiq}(\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau})}{c^\star_{iiq}(\mathbf{0}|\tau)} = \tau^3 c^\star_{\tau\tau p}(\mathbf{0}|\tau)\tau,$$

which implies

$$c^\star_{\tau\tau p}c^\star_{iiq}(\frac{\mathbf{0}}{\tau}|\frac{-1}{\tau}) = \tau^4 c^\star_{\tau\tau p}c^\star_{iiq}(\mathbf{0}|\tau).$$

[6] More formally, one should take the limit as $\mathbf{z} \to \mathbf{0}$ in a suitable way.

Therefore $c^\star_{\tau\tau p} c^\star_{iiq}(0|\tau)$ is a modular function of weight 4. Hence $c^\star_{\tau\tau\tau}$ is a sum of modular functions of weight 4 and so is itself a modular function of weight 4.

Having established that $c^\star_{\tau\tau\tau}$ is a modular function of τ (of weight 4), we now consider its behaviour as $q \to 0$ (or equivalently $\tau \to i\infty$). Begin by considering $c^\star_{ijk}(\mathbf{z}|\tau)$, which is the sum of terms of form $\frac{\theta_1'}{\theta_1}$. But from [27], it follows that

$$\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = \cot z + 4\sum_{n=1}^{\infty} \frac{q^{2n}\sin 2nz}{1-q^{2n}}.$$

Therefore for small q,

$$\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = O(1),$$

and so $c^\star_{ijk}(\mathbf{z}|\tau) = O(1)$ also. Moving on to $c^\star_{\tau ij}$, which contains terms of the form $\frac{\theta_1''}{\theta_1}$, note that differentiating

$$\theta_1' = \frac{\theta_1'}{\theta_1}\theta_1,$$

and dividing through by θ_1 gives

$$\frac{\theta_1''(z|\tau)}{\theta_1(z|\tau)} = \left(-\frac{1}{\sin^2 z} + 8\sum_{n=1}^{\infty} \frac{nq^{2n}\cos 2nz}{1-q^{2n}}\right) + \left(\cot z + 4\sum_{n=1}^{\infty} \frac{q^{2n}\sin 2nz}{1-q^{2n}}\right)^2.$$

Hence as $q \to 0$,

$$\frac{\theta_1''(z|\tau)}{\theta_1(z|\tau)} \to -1 + O(q^2).$$

Substituting this into the formula (5.25) for $c^\star_{\tau ij}$ and noting that there are $n + 1 - (n+1) = 0$ constant terms, one therefore has that as a function of q, and for small q,

$$c^\star_{\tau ij}(\mathbf{z}|\tau) = O(q^2).$$

The same argument holds for $c^\star_{\tau ii}$ (which contains precisely twice as many constant terms which cancel out with each other).

In order to consider the small q behaviour of $c^\star_{\tau\tau i}$, recall that from the associativity equations (where k_{pq} is the appropriate constant obtained by substituting in all known constants):

$$c^\star_{\tau\tau i}(\mathbf{z}|\tau) = \sum_{p,q} k_{pq} c^\star_{ijp} c^\star_{\tau jq}.$$

Hence for small q,

$$\begin{aligned} c^{\star}_{\tau\tau i} &= \sum_{p,q} k_{pq} O(1) O(q^2), \\ &= O(q^2). \end{aligned}$$

Similarly,

$$\begin{aligned} c^{\star}_{\tau\tau\tau} &= \sum_{p,q} k_{pq} (c^{\star}_{\tau ip} c^{\star}_{\tau iq} - c^{\star}_{\tau\tau p} c^{\star}_{iiq}), \\ &= O(q^2). \end{aligned}$$

Therefore $c^{\star}_{\tau\tau\tau}(\mathbf{0}|\tau)$ is a modular function of τ which tend to zero as q tends to zero. This means it is a *cusp form* [17]. But the only cusp form of weight $k < 12$ is the zero function [1]. Hence

$$c^{\star}_{\tau\tau\tau}(\mathbf{0}|\tau) = 0,$$

as required.

Corollary 5.21 *The functions A_{ij}, B_i and C are (to within a quadratic term):*

$$\begin{aligned} A_{ij} &= 0, \\ B_i &= 0, \\ C &= -f(\mathbf{z}|\tau)|_{\mathbf{z}=\mathbf{0}}. \end{aligned}$$

Proof: The first two equations above follow immediately from integrating $A'_{ij} = 0$ and $B'_i = 0$, taking the constants of integration to be zero. The final line comes from integrating

$$\begin{aligned} C''' &= -\frac{\partial^3 f(\mathbf{z}|\tau)}{\partial \tau^3}\bigg|_{\mathbf{z}=\mathbf{0}}, \\ &= -\frac{\partial^3}{\partial \tau^3} \left(f(\mathbf{z}|\tau)|_{\mathbf{z}=\mathbf{0}} \right), \end{aligned}$$

again taking the constants of integration to all be zero.

Noting that

$$f_1^\star(\mathbf{z}|\tau)|_{z=0} = \frac{n+1}{4}\sum_i k_i\Lambda_3(0) - \frac{1}{8}\sum_{i\neq j} k_ik_j\Lambda_3(0),$$

the function $F_1^\star$ must therefore be:

$$\begin{aligned} F_1^\star &= f(\mathbf{z}|\tau)|_{z=0} & (5.27)\\ &= \frac{n+1}{4}\sum_i (k_i\Lambda_3(z_i) - \Lambda_3(0)) - \frac{1}{8}\sum_{i\neq j} k_ik_j(\Lambda_3(z_i - z_j) - \Lambda_3(0)) & (5.28)\end{aligned}$$

This may be rewritten in a more convenient form by introducing a new function, known as the *elliptic polylogarithm*, as constructed in [18] and [2].

Definition 5.22 *The* elliptic polylogarithm *function Li_r is defined for odd r by the series*

$$Li_r(q,\zeta) = \sum_{n=0}^{\infty} Li_r(q^2\zeta) + \sum_{n=1}^{\infty}(q^n\zeta^{-1}) - \chi_r(q,\zeta),$$

where $\chi_r(q,\zeta)$ is defined, with B_i being the Bernoulli numbers, as:

$$\chi_r(q,\zeta) = \sum_{n=0}^{r} \frac{B_{n+1}}{(r-n)!(n+1)!}(\log\zeta^{r-n})(\log q)^j.$$

Lemma 5.23 *The third elliptic polylogarithm satisfies*

$$Li_3(q^2, e^{2iz}) = -\Lambda_3(z,q) + \frac{1}{3}(\log q)^2 z^2 + \frac{1}{90}(\log q)^3.$$

Proof: The proof of this follows immediately from the definitions of $Li_r(q,\zeta)$ and $\Lambda_N(z,q)$.

Lemma 5.24 *The function $F_1^\star$ may be expressed in terms of the elliptic polylogarithm. Explicitly,*

$$\begin{aligned} F_1^\star &= \frac{1}{8}\sum_{i\neq j}\left(Li_3(q^2, e^{2i(z_i - z_j)}) - Li_3(q^2,1)\right),\\ &\quad -\frac{n+1}{4}\sum_i \left(Li_3(q^2, e^{2iz_i}) - Li_3(q^2,1)\right).\end{aligned}$$

Proof: It follows immediately from lemma 5.23 the equation above may be rewritten as:

$$\begin{aligned} F_1^\star &= \frac{1}{8}\sum_{i\neq j} k_i k_j \left(-\Lambda_3(z_i - z_j) + \frac{1}{3}(\log q)^2 (z_i - z_j)^2 + \Lambda_3(0)\right) \\ &\quad -\frac{n+1}{4}\sum_i k_i \left(-\Lambda_3(z_i) + \frac{1}{3}(\log q)^2 z_i^2 + \Lambda_3(0)\right). \end{aligned}$$

Counting the number of $(\log q)^2$ terms, one finds that there are of N them, where

$$N = \frac{1}{24}\sum_{i\neq j} k_i k_j (z_i - z_j)^2 - \frac{(n+1)}{12}\sum_i k_i z_i^2.$$

But using $\sum k_i = (n+1)$ and $\sum k_i z_i = 0$, N may be simplified:

$$\begin{aligned} 24N &= \sum_{i\neq j} k_i k_j (z_i^2 - 2z_i z_j + z_j^2) - 2(n+1)\sum_s k_s z_s^2, \\ &= 2\sum_{i\neq j} k_i k_j z_i^2 - 2\sum_{i\neq j} k_i k_j z_i z_j + 2\sum_r k_r \sum_s k_s z_s^2, \\ &= 2\sum_{i\neq j} k_i k_j z_i^2 - 2\sum_{i\neq j} k_i k_j z_i z_j - 2\sum_{r\neq s} k_r k_s z_s^2 - 2\sum_r k_r^2 z_r^2, \\ &= -2\left(\sum_r \sum_s k_r z_r k_s z_s\right), \\ &= 0. \end{aligned}$$

Hence $F_1^\star$ expressed as a sum of elliptic polylogarithms (above) agrees with (5.27).

Adding this to $g^\star$ therefore gives the almost dual prepotential for the discriminants of $H_{1;n}$.

Theorem 5.25 *The almost dual prepotential is*

$$\begin{aligned} F^\star &= \frac{1}{8}\sum_{i\neq j} k_i k_j \left(Li_3(q^2, e^{2i(z_i - z_j)}) - Li_3(q^2, 1)\right) \\ &\quad -\frac{n+1}{4}\sum_i k_i \left(Li_3(q^2, e^{2iz_i}) - Li_3(q^2, 1)\right) \\ &\quad +2\pi i u \left(\sum_{i=i}^{m}\sum_{j>i} k_i k_j z_i z_j + \sum_{i=1}^{m}\left(k_i + \frac{k_i^2}{k_0}\right)\frac{z_i^2}{2} - \frac{1}{2}\pi^2 \tau u\right). \end{aligned}$$

Proof: Proof is obvious; this theorem is a the collation of the results of the lemmas in this chapter.

Note that this theorem generalises the result of [21] to an arbitrary discriminant of $H_{1;n}$. As with the genus zero case, as the discriminant submanifold this was calculated on is not actually a Frobenius manifold, there is again no prepotential to which this function is actually dual. However, if all of the $k_i = 1$, the superpotential (5.4) defines a Frobenius manifold rather than a discriminant submanifold, and in such a case the solution $F^\star$ agrees with the result in [21]. An obvious generalisation to this theorem would be to find an analogous result for an arbitrary genus one Hurwtiz space. Though this will not be considered here, one would expect conjecturally that such a function, like in the genus zero case, would be of the same form but would allow negative values of the parameters k_i. Beyond this, an obvious generalisation would be a Hurwitz space of an arbitrary genus.

Chapter 6

Conclusion

The work contained in this thesis has been based around the idea that a Frobenius manifold with a polynomial superpotential is the simplest example of a construction on a Hurwitz space (namely $H_{0;n}$). This affords a generalisation in one of two directions; to an arbitrary Hurwitz space of the same genus or to a simple Hurwitz space of a higher genus. Chapters 3 and 4 generalised their respective ideas of induced Frobenius structures on caustic submanifolds and almost duality for discriminants in the former of the two directions, whilst chapter five generalised discriminant almost duality in the latter. Schematically:

$$\begin{array}{ccc} \{H_{0;n}\} & \longrightarrow & \{H_{0;n_0,\dots,n_m}\}_{\text{chapters 3 \& 4}} \\ \downarrow & & \downarrow \\ \{H_{1;n}\}_{\text{chapter 5}} & \longrightarrow & \{H_{1;n_0,\dots,n_m}\} \end{array}$$

The first obvious extension to the this work is to generalise the ideas of chapter 3 to $H_{1;n}$. As mentioned earlier, one would expect that this would become computationally difficult due to the superpotential being elliptic. Similarly, the ideas of chapter 5 may be generalised to a higher genus Hurwitz space $H_{g;n}$. Such a construction would be based around a superpotential, which one would expect to be expressed in terms of higher genus functions. As such, some aspects of the construction of an almost dual prepotential would mirror the $H_{1;n}$ case. However, technical difficulties would be expected, for example where the elliptic connection was used in chapter 5, one may need to use some sort of higher genus analogous connection in its place.

Alternatively, analogous to generalising $H_{0;n}$ to $H_{0;n_0,\ldots,n_m}$, one may generalise the ideas of chapter 5 to $H_{1;n_0,\ldots,n_m}$ (which would complete the diagram above). One would expect that if a suitable superpotential were to be constructed in terms of the function $\theta_1(v|\tau)$ that the calculations involved would follow very closely from those contained within chapter 5.

Finally, going back to the motivating example of a Frobenius manifold with polynomial superpotential, recall that this also corresponds to the orbit space $\mathbb{C}^n/A_n$. From this perspective, the obvious generalisation is to extend the ideas of discriminant almost duality to other Coxeter groups. Results for this, based around deformed root systems, can be found in [16]. However, such results could be instead derived by direct calculation using an LG superpotential. For example, in the case of B_n the superpotential would be of the form[1].

$$\lambda = \prod_{i=1}^{n}(z^2 - z_i^2) = \prod_{i=1}^{n}(z - z_i)(z + z_i).$$

Note that this is the same as the superpotential for A_{2n-1} subject to a certain constraint on the $\{z_i\}$. This is expected though; recall that $B_n \subset A_{2n-1}$. Note also that in this particular case, the calculation of structure constants $c^{\star}_{ijk}$ would actually become easier than in section 4.1, as the sum of the roots would automatically be zero, thus removing the condition $z_0 = z_0(z_1, \ldots, z_n)$.

[1]In order for the superpotential to define a discriminant, one would require repeated roots, i.e. $z_i = z_j$ for some $i \neq j$

Analogous to the genus zero extension to other Coxeter groups, one notes that $H_{1;n} \cong \Omega/J(A_N)$, and so the ideas of chapter 5 may be extended to other Jacobi orbit spaces. One would expect that this again would be possible in terms of root systems, particularly if one notes that terms in $F^\star$ for $H_{1;n}$ are functions of $z_i - z_j$ and so appear to be linked to the A_n root system.

Finally, one moves on to considering applications of the results derived here. The ideas of chapter 3 may be applied to bi-Hamiltonian structures; as the caustics are natural submanifolds. It is already guaranteed by [25] that the submanifolds considered will be bi-Hamiltonian, and although no construction was given, one may show that the induced intersection form is (in the canonical submanifold coordinates τ^i:

$$g = \sum_i \tau^i \eta_{ii} (d\tau^i)^2.$$

The ideas of chapter 5 have a perhaps surprising application in 6d Seiberg-Witten theory. Discussion of this is given in [7]. Finally, the ideas of chapters 4 and 5 appear closely linked to the ideas of deformed root systems discussed in [26, 16] and may provide a way of finding further examples of $\vee$-systems. The result (4.3) extends the result in [16] to include negative integer values of the parameters k_i. It also provides a geometric interpretation of this result, with the negative k_i determining which Hurwitz space the solution comes from and the positive k_i determining the precise discriminant in that space.

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