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Applications of Nonstandard Analysis in Differential Game Theory

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by

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Abstract

In this study we look at optimal control theory and differential game theory. In the control section, to illustrate some of the nonstandard methods which we will be using, we give existence and uniqueness proofs for standard and Loeb measurable controls. The standard existence is a well-known result, the proof we give is is due to Keisler; this proof was given by him in previously unpublished lecture notes at the University of Wisconsin ([27]). The uniqueness proof is a simple application of Gronwall's Lemma ([31]).

We then show that there is always an optimal Loeb control even in situations where there is no optimal Lebesgue control. Using this result we are then able to show the well known result that there is always a standard optimal relaxed control.

In the games section, by using nonstandard analysis we show that, under certain circumstances, we have the existence of value for two player, zero-sum differential games played over the unit time interval. We follow the work of Elliott and Kalton and, as they did, we show that if the Isaacs condition holds then the game has value in the sense of Friedman. Over the relaxed controls the Isaacs condition is always satisfied and so there is always value for relaxed controls. Like Elliott and Kalton, we do not need Friedman's hypothesis that the variables appear separated in the dynamics and payoff. By using nonstandard methods we are, unlike Elliott and Kalton, able to show these results without using the Isaacs-Bellman equation, other than to explain what the Isaacs condition is. We also find it unnecessary to impose as many restrictions on the functions as Elliott and Kalton.

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To Granny

Contents

Introduction

Two player differential game theory was developed to study competitive contests whereas optimal control theory investigates one player optimization problems.

In differential game theory we are looking at a dynamical situation described by differential equations and, at the end of a fixed period of time, which we take to be one unit of time, a payoff is computed. (This is equivalent to the cost in control theory.) We are studying zero-sum games, that is, games where one player is trying to minimize the payoff and the other to maximize the payoff.

Differential game theory was first studied in the 1950's by Isaacs, though his work was not published until 1965 ([26]). His main contribution was to derive, heuristically, a differential equation, known as the Isaacs-Bellman equation, which the value of the game should satisfy. The equation however cannot be guaranteed to have solutions.

Fleming, Friedman and then Elliott and Kalton use this equation to attain value. We however, manage to achieve the same results without using the Isaacs-Bellman equation.

A strategy for a player in a game is, roughly speaking, a rule which tells him what to do on the basis of what has happened so far in the game. Because of the continuity of time this is a difficult notion to make precise. Fleming ([13], [14], [15] and [16]) avoided this difficulty by studying a sequence of discrete time games and approximating the differential equations by difference equations.

Each approximating game has an upper and lower value, W_n^+ and W_n^-, depending on which player goes first, the minimizer or the maximizer.

Two problems arise, first, do the the values W_n^+ and W_n^- tend to limits as the time between stages decreases to zero ? This is what Fleming refers to as the convergence problem.

Secondly, are the two limits, if they exist, equal i.e. does $W^+ = W^-$?

If the opposing control variables appear separated in the dynamics and payoff, Fleming ([13]) was able to give positive answers to both of these questions.

Another approach by Fleming ([14], [15]) was to introduce noise into the game and so into the approximating discrete difference games. Although the Isaacs-Bellman equation cannot be guaranteed to have solutions, with this small amount of noise added, the upper and lower value functions satisfy a non-linear parabolic equation and work done by Friedman ([18]) or Oleinik and Kruzhkov ([29]) shows that such equations have unique solutions. Therefore, Fleming was able to show that the upper and lower values of the approximating games approached the solution of the corresponding non-linear parabolic equation and was able to prove the convergence of the values as the noise tended to zero.

Thus again he provided positive answers to both of the questions. However Fleming's functions had to satisfy a constant Lipschitz condition. We go on to use the idea of adding noise into the game in section 6.3 but we do not do it in the context of the Isaacs-Bellman equation.

Friedman ([20], [21], [22]) studied differential games directly not by approximating them by difference equations; he did however find it necessary to approximate the idea of a strategy by upper and lower strategies, varying only at a finite number of division points throughout the interval.

Again, depending on which player goes first, Friedman obtained upper and lower values V_n^+ and V_n^- for the game. These functions are monotonic and so tend to limits V^+ and V^-. (These values are not necessarily the same as the values W^+ and W^- obtained by Fleming.)

In order to show that the game G has value i.e. that $V^+ = V^-$, Friedman also had to assume the opposing variables appear separated in the dynamics and payoff however Friedman only required the functions to satisfy a weaker Lipschitz condition and his payoff was more general than that used by Fleming.

Elliott and Kalton ([10]) give a definition of strategy and then reformulate Friedman's result using approximating games and relaxed controls, without the separation of variables. They then relate the upper and lower values as obtained by Friedman to those obtained by Fleming. To do this they use the Isaacs-Bellman equation and the results of Fleming. Their main result is to show that there is value when the Isaacs condition is satisfied and since relaxed controls always satisfy the Isaacs condition, that there is always value with relaxed controls.

Elliott and Kalton first prove these results with a constant Lipschitz condition but by approximation arguments they are then able to assume a weaker condition and have a much more general payoff than Friedman.

In this study we use nonstandard analysis to provide the same results without using the Isaacs-Bellman equation or assuming as many restrictions on our functions. We begin by looking at differential control theory (Chapter 1) and then extend the study to differential game theory by bringing in another controller. Many of the results obtained in the control section carry over naturally to the game theory.

In the control section we start by showing the existence and uniqueness of solutions with standard Lebesgue controls we then show, by an example, that there is not always a standard ordinary optimal control.

We look at the idea of extending the class of admissible controls to include Loeb controls. To be able to do this we have to adapt the dynamics and cost functions to accommodate nonstandard times (section 1.3). Once this has been done we can show the existence and uniqueness of solutions corresponding to Loeb controls. We show that with Loeb controls the minimum cost is the same as with standard controls. We are able to show that there is always an optimal Loeb control even in situations were there is no ordinary optimal Lebesgue control.

By using Loeb controls we are able to show that the minimum cost with relaxed controls is the same as with ordinary controls or Loeb controls and then we give a nonstandard proof of the well-known result that there is always a standard optimal relaxed control.

Having got the fact that there is always an optimal relaxed control we then look at the well-known result which states that there is always a standard ordinary optimal control if we have convexity.

In section 1.5, we give a proof of this result using the results obtained so far, however our proof relies on an, as yet, unproven conjecture. (See Lemma 1.5.1 .)

Having studied the differential control problem we extend our study by bringing in another controller this then becomes differential game theory. We study two player differential game theory.

We follow the work done by Elliott and Kalton [10], Fleming [14] and Friedman [20] and study two player zero-sum games.

Like Elliott and Kalton, we are able to show that there is always value if the Isaacs condition is satisfied and so there is always value with relaxed controls. We however do not use the Isaacs-Bellman equation at all and find it unnecessary to assume as many restrictions on the functions. In fact by introducing a new game $H^{+}_{m,n}$ (Chapter 5) and relaxed controls, we are able to show that the values W^{+}_{1} (W^{+}_{12}) and similarly W^{-}_{2} (W^{-}_{12}) exist by only assuming one of Fleming's five conditions, (F1), that is we assume that f satisfies a constant Lipschitz condition but it is not hard to see that these results would still hold if we had a weaker condition on f.

In section 6.2 we recall briefly the Isaacs-Bellman equation, this is only included so that we can define the Isaacs condition, we never actually use the Isaacs-Bellman to attain any of our results. In section 6.3 we use Fleming's idea of introducing noise into the game, by doing this we are able to show that there is always value if the Isaacs condition holds and so, since over relaxed controls the Isaacs condition is always satisfied, we can show the existence of value for relaxed controls.

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Preliminaries

We assume knowledge of the basics of nonstandard analysis as expounded in [1], [7], [8] and [25]. For standard measure theory we refer the reader to [28] and [38].

Here, to set our notation, we give a brief recollection of some Theorems and Definitions which we shall be using, we give them in the form in which they will be used.

Given a finite member of $x \in {}^*\mathbb{R}$ we shall use both notations, ${}^\circ x$ and $st(x)$ to mean the **standard part** of x.

We shall work in a nonstandard superstructure constructed over the reals, $\mathbb{R}$. We assume $\aleph_1$**−saturation** and note that this is equivalent to countable comprehension, so that given a function $f : \mathbb{N} \to A$, where A is an internal set, the function may be extended to an internal function $F : {}^*\mathbb{N} \to A$.

We will also use **overflow**; recall that this states that if we have an internal $A \subseteq {}^*\mathbb{N}$ and $n \in A$ for all finite $n \in {}^*\mathbb{N}$ then there is an infinite $N \in A$.

Throughout this work we frequently refer to the concept of $\mathcal{S}$**-continuity** as in ([8]). Recall that given an internal function $F : {}^*[0,1] \to {}^*\mathbb{R}$, F is said to be $\mathcal{S}$-continuous if for all $x, y \in {}^*[0,1]$, $F(x) \approx F(y)$ whenever $x \approx y$. We note that given an $\mathcal{S}$-continuous function $F : {}^*[0,1] \to {}^*\mathbb{R}$ if $F(0)$ is finite then there exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that $f(t) = {}^\circ F(t)$ for all $t \in [0,1]$. ($f({}^\circ\tau) = {}^\circ F(\tau)$ for all $\tau \in {}^*[0,1]$.)

We also observe that if we have a function $F : {}^*[0,1] \to {}^*\mathbb{R}$ defined by $F(\tau) = \int_0^\tau \theta(\sigma)d\sigma$ for some internal function θ then F is $\mathcal{S}$-continuous if θ is bounded; this can clearly be seen by the following. If θ is bounded by κ and $\tau' > \tau$, $\tau' \approx \tau$ then

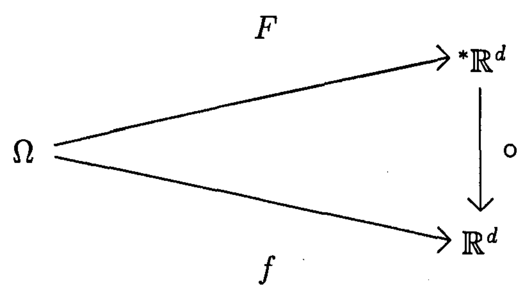
$$|F(\tau') - F(\tau)| = |\int_0^{\tau'} \theta(\sigma)d\sigma - \int_0^{\tau} \theta(\sigma)d\sigma| \leq \int_\tau^{\tau'} |\theta(\sigma)|d\sigma \leq \kappa(\tau' - \tau) \approx 0.$$

We note that given a compact metric space M, functions $g : \mathbb{R}^d \times M \to \mathbb{R}^d$ and

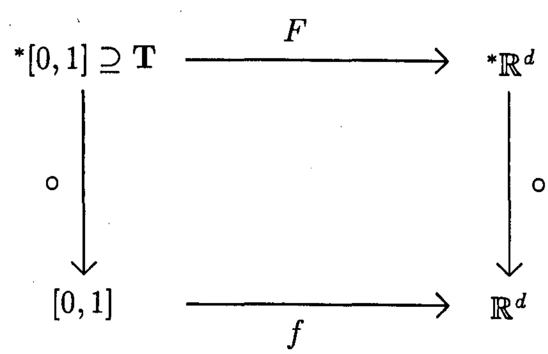
$G: {}^*\mathbb{R}^d \times {}^*M \to {}^*\mathbb{R}^d$ are close (i.e. $G \approx g$) in the sense of the compact open topology iff ${}^\circ G(X, A) = g({}^\circ X, {}^\circ A)$ for all finite $X \in {}^*\mathbb{R}^d$ and all $A \in {}^*M$.

We shall also assume the knowledge of the **Loeb measure construction** which, given a standard measure, constructs a Loeb measure from it – we shall denote the Loeb measure of a standard measure μ by μ_L. We shall also assume familiarity with **Loeb integration** and lifting theorems. Recall two definitions of lifting:

(1) Given a measure space $(\Omega, \mathcal{A}, \mu)$ and functions $f: \Omega \to \mathbb{R}$ and $F: \Omega \to {}^*\mathbb{R}$, F is called a **lifting** of f if F is internal and ${}^\circ F(\omega) = f(\omega)$ for a.a. $\omega \in \Omega$ with respect to the Loeb measure μ_L.



(2) Given a discrete time line $\mathbf{T} = \{0, \Delta t, 2\Delta t, \cdots, 1\}$, where $\Delta t = \frac{1}{N}$ for some infinite $N \in {}^*\mathbb{N}$, and functions $f: [0,1] \to \mathbb{R}$ and $F: \mathbf{T} \to {}^*\mathbb{R}$, F is a **lifting** of f if F is internal and ${}^\circ F(t) = f({}^\circ t)$ for a.a. $t \in \mathbf{T}$ with respect to the measure μ_L on ${}^*[0,1]$ where μ is the internal counting measure on ${}^*[0,1]$.



(This is sometimes called a two-legged lifting.)

In particular we shall use **Anderson's Lifting Theorem** which states that
(1) Given a hyperfinite probability space $(\Omega, \mathcal{A}, p)$ and its corresponding Loeb

space $(\Omega, L(\mathcal{A}), p_L)$ a function $f : \Omega \to \mathbb{R}$ is Loeb measurable if and only if it has a lifting.

(2) Given the standard Lebesgue space $([0,1], \mathcal{B}([0,1]), \lambda)$ and $(\mathbf{T}, L(\mathcal{A}), \mu_L)$, the Loeb space associated with the discrete time line $\mathbf{T} = \{0, \Delta t, 2\Delta t, \cdots, 1\}$ where $\Delta t \approx 0$, and μ is the counting measure, a function $f : [0,1] \to \mathbb{R}$ is Lebesgue measurable iff f has a lifting $F : \mathbf{T} \to {}^*\mathbb{R}$.

Note, in (2) above, $\mathbb{R}$ can be replaced by any separable metric space M. (see [1, page 69]) The space $\mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$ in the compact open topology is a separable metrizable space ([1, page 58]), we can therefore employ this result with $\mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$ in the compact open topology and then we can say that $f : [0,1] \to \mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$ is Lebesgue measurable iff f has a lifting $F : \mathbf{T} \to {}^*\mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$.

Another result which we shall use frequently is **Anderson's Lusin Theorem**, this states that given a Radon space $(X, \mathcal{B}, \mu)$ with completion $(X, \mathcal{C}, \mu)$ and a function $f : X \to Y$, where Y is a topological space, then if Y is Hausdorff with a countable basis of open sets and f is $\mathcal{C}$-measurable then

$$^\circ({}^*f(x)) = f({}^\circ x) \qquad \text{for a.a. } x \in X .$$

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Chapter 1

Introduction to the theory of control

In control theory we have a **process**, that is some action taking place in time. Along with a process we have a set of **controls** which can be used to influence the behaviour of the process in question. We have a structure which governs the **state** of the process, which we call the **dynamics** of the process. When a choice of control has been made the dynamics provide a means by which, given the state, $x(t)$, of the process for times $t \leq t'$ for some arbitrary time t' in our time set, we can determine the evolution of $x(t)$ for $t > t'$.

The next element required in the formulation of a control problem is the **objective**, that is we set some goal to be achieved by our process by applying the controls. An objective is usually specified by some desired target states of the process.

One question which arises naturally is whether the means of influencing the process are strong enough to achieve the desired objective. If such means exist then we have properly formulated a control structure.

Starting from some arbitrary initial state for the process we may consider the set of all possible states which may be attained through the influences available to us. Such a set is called the **reachable set** for the process defined relative to some initial state.

Therefore a more precise meaning of a properly formulated control structure is when an objective state lies in the reachable set relative to the present state.

There are a number of ways in which the objective may be achieved, for example one may systematically choose the 'best' approach with respect to some performance criterion.

If with respect to some performance criterion we seek, in the set of all influencing policies for achieving an objective, the one that is 'best' then the formulation is an **optimal control problem.**

A common formulation of the dynamics is in the form of an **ordinary differential equation**; in this case the control, whose range is contained in some pre-assigned **control region**, is a function belonging to a certain **admissible class.**

The performance criterion for such a system is usually the **integral of some real valued function.** This is the class of optimal control problems we discuss in this Chapter.

Some questions which arise in control theory are whether a given initial point can be 'steered to the target' using a certain control, whether controls required to belong to some special class of functions would also steer this point to the target and whether an optimal control exists. [For more information on this see [12], [17], [23] and [33].]

1.1 Deterministic control problem

In this section we consider a control problem over a fixed time interval $I = [0,1]$. At each time $t \in I$, the controller picks an element $u(t)$ from a fixed (separable) compact metric space M i.e. $u : [0,1] \to M$. Functions of this form play the role of our controls. The set of all measurable controls u is denoted by $\mathcal{U}$. The situation we look at is when the dynamics are given by the following equation

$$x(t) = x(0) + \int_0^t f(s, x(s), u(s))ds. \tag{1.1}$$

Here, $x(t) \in \mathbb{R}^d$ and the function $f : I \times \mathbb{R}^d \times M \to \mathbb{R}^d$ is such that there exists $\kappa < \infty$ satisfying the following conditions

(i) $f(t, \bullet, \bullet)$ is continuous for each $t \in I$

(ii) f is measurable

(iii) $|f(t, x, a)| \leq \kappa(1 + |x|)$

(iv) $|f(t, x, a) - f(t, y, a)| \leq \kappa|x - y|$

whenever $x, y \in \mathbb{R}^d$, $a \in M$, and $t \in I$. Condition (iii) is known as a growth condition on f and condition (iv) is known as a Lipschitz condition on f; in (iv) κ is a Lipschitz constant for f.

Before we think about our objectives for the process, to illustrate some of the nonstandard methods which we will be using, we give a nonstandard proof of the Carathéodory Existence Theorem. This is a well-known result which states that, given a control $u \in \mathcal{U}$ and an initial state for the process, i.e. $x(0)$, the dynamics equation (1.1) has a solution $x(t)$. The proof we give is due to Keisler ([27]).

1.1.1 Existence and uniqueness of solutions

Theorem 1.1.1

For each control $u \in \mathcal{U}$, equation (1.1) has a solution $x(t)$.

Proof: If, for any x, we write $g(t, x(t)) = f(t, x(t), u(t))$ then equation (1.1) becomes

$$x(t) = x(0) + \int_0^t g(s, x(s))ds \tag{1.2}$$

and g is such that there exists a constant κ satisfying the following conditions:

(i′) $g(t, \bullet)$ is continuous for all $t \in I$

(ii′) g is measurable

(iii′) $|g(t, x)| \leq \kappa(1 + |x|)$

(iv′) $|g(t, x) - g(t, y)| \leq \kappa|x - y|$

whenever $x, y \in \mathbb{R}^d$ and $t \in I$.

We want to show that a solution to equation (1.2) exists under conditions (i′)-(iv′).

Choose $N \in {}^*\mathbb{N}$ such that N is infinite and let $\Delta t = \frac{1}{N} \approx 0$. Define a hyperfinite time line by $\mathbf{T} = \{0, \Delta t, 2\Delta t, \cdots, 1\}$.
Let $\mathcal{C} = \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$, the class of all continuous functions $q : \mathbb{R}^d \to \mathbb{R}^d$.
Let

$$\hat{g}(t) = g(t, \bullet) \qquad \text{for each } t \in [0, 1] \tag{1.3}$$

then by (i′)

$$\hat{g} : [0, 1] \to \mathcal{C}$$

i.e. $\hat{g}(t) \in \mathcal{C}$ for each $t \in [0, 1]$. Now, by Anderson's Lifting Theorem (see [1]), we can take a lifting $\hat{G}$ of $\hat{g}$,

$$\hat{G} : \mathbf{T} \to {}^*\mathcal{C}$$

such that for a.a. $t \in \mathbf{T}$

$$\hat{G}(t) \approx \hat{g}({}^\circ t)$$

in the sense of the compact open topology i.e. for a.a. $t \in \mathbf{T}$

$$\hat{G}(t)(Y) \approx \hat{g}({}^\circ t)({}^\circ Y) \tag{1.4}$$

for all finite $Y \in {}^*\mathbb{R}^d$. i.e. for a.a. $t \in \mathbf{T}$ the following diagram commutes

$$\begin{array}{ccc} {}^*\mathbb{R}^d & \xrightarrow{\hat{G}(t)} & {}^*\mathbb{R}^d \\ \Big\downarrow{\scriptstyle \circ} & & \Big\downarrow{\scriptstyle \circ} \\ \mathbb{R}^d & \xrightarrow[\hat{g}({}^\circ t)]{} & \mathbb{R}^d \end{array} .$$

Now define $G : \mathbf{T} \times {}^*\mathbb{R}^d \to {}^*\mathbb{R}^d$ by

$$G(t, Y) = \hat{G}(t)(Y)$$

for all $Y \in {}^*\mathbb{R}^d$. Then, by (1.4) we have, for a.a. $t \in \mathbf{T}$,

$$G(t, Y) \approx \hat{g}({}^\circ t)({}^\circ Y)$$

for all finite $Y \in {}^*\mathbb{R}^d$ i.e. , by (1.3), for a.a. $t \in \mathbf{T}$ and for all finite $Y \in {}^*\mathbb{R}^d$

(1.5) $$G(t, Y) \approx g({}^\circ t, {}^\circ Y).$$

Therefore, G is a two-legged lifting of g (see [1]), i.e. the following diagram commutes for a.a. $t \in \mathbf{T}$, for all finite $Y \in {}^*\mathbb{R}^d$:

$$\begin{array}{ccc} \mathbf{T} \times {}^*\mathbb{R}^d & \xrightarrow{G} & {}^*\mathbb{R}^d \\ \Big\downarrow{\scriptstyle \circ} & & \Big\downarrow{\scriptstyle \circ} \\ [0,1] \times \mathbb{R}^d & \xrightarrow[g]{} & \mathbb{R}^d \end{array} .$$

Now define

$$X(0) = x(0)$$

$$X(t + \Delta t) = X(t) + G(t, X(t))\Delta t$$

for each $t \in \mathbf{T}$. i.e. for each $t \in \mathbf{T}$,

(1.6) $$X(t) = x(0) + \sum_{\{s \in \mathbf{T} : 0 \le s < t\}} G(s, X(s))\Delta t.$$

We claim that $X(t)$ is $\mathcal{S}$-continuous i.e. $X(t) \approx X(t')$ if $t \approx t'$, where $t,\ t' \in \mathbf{T}$, and if we define $x : [0,1] \to \mathbb{R}^d$ by letting $x(s) = {}^\circ X(t)$ for each $s \in [0,1]$ when $s \approx t$, $(t \in \mathbf{T})$ then $x(s)$ is a solution to equation (1.2).

First we consider the case where g is bounded. If $|g| \leq \kappa < \infty$ then we can take G such that $|G| \leq \kappa$, therefore

$$(1.7) \qquad |G(t, X(t))| \leq \kappa \qquad \text{for all } t \in \mathbf{T}.$$

Take $s > t$, $s \approx t$, where $s,\ t \in \mathbf{T}$ then,

$$\begin{aligned} |X(s) - X(t)| &= \Big| \sum_{t \leq u < s} G(u, X(u)) \Delta t \Big| \quad \text{by (1.6)} \\ &\leq \sum_{t \leq u < s} |G(u, X(u))| \Delta t \\ &\leq \kappa(s - t) \qquad \text{by (1.7)} \\ &\approx 0 \qquad (\text{since } s \approx t \text{ and } \kappa < \infty). \end{aligned}$$

Therefore, $X(t)$ is $\mathcal{S}$-continuous when g is bounded.

Now, since $X(t)$ is finite for all $t \in \mathbf{T}$, we can take standard parts and define $x \in \mathcal{C}([0,1], \mathbb{R}^d)$ by

$$(1.8) \qquad x(s) = {}^\circ X(t) \text{ when } s \approx t$$

i.e. $X \approx x$ in the sense of the uniform topology.

Now, remembering that g is still assumed to be bounded here, we define

$$(1.9) \qquad H(t) = G(t, X(t)) \qquad \text{for all } t \in \mathbf{T}$$

and

$$(1.10) \qquad h(t) = g(t, x(t)) \qquad \text{for all } t \in [0,1]$$

then, by (1.5) and (1.8) above, H is a bounded lifting of h and

$$\begin{aligned}
x({}^\circ t) &= {}^\circ X(t) \\
&= {}^\circ(x(0) + \sum_{0\leq s<t} G(s, X(s))\Delta t) && \text{by (1.6)} \\
&= {}^\circ(x(0) + \sum_{0\leq s<t} H(s)\Delta t) && \text{by (1.9)} \\
&= x(0) + {}^\circ(\sum_{0\leq s<t} H(s)\Delta t) \\
&= x(0) + \int_0^{{}^\circ t} h(s)ds && \text{(by Loeb Theory)} \\
&= x(0) + \int_0^{{}^\circ t} g(s, x(s))ds && \text{by (1.10).}
\end{aligned}$$

Thus, $x(t)$ is a solution to equation (1.2).

Therefore, when g is bounded we have shown that a solution to equation (1.2) exists.

Now we consider the case where g is unbounded. Each g (bounded or unbounded) has linear growth by (iii′) on page 11 i.e.

(1.11) $$|g(s, x(s))| \leq \kappa(1 + |x(s)|) \qquad \text{for all } s \in [0, 1].$$

Suppose $x(t)$ is a solution to equation (1.2) then,

$$x(t) = x(0) + \int_0^t g(s, x(s))ds$$

therefore,

$$\begin{aligned}
|x(t)| &= |x(0) + \int_0^t g(s, x(s))ds| \\
&\leq |x(0)| + |\int_0^t g(s, x(s))ds| \\
&\leq |x(0)| + \int_0^t |g(s, x(s))|ds \\
&\leq |x(0)| + \kappa \int_0^t (1 + |x(s)|)ds && \text{by (1.11).}
\end{aligned}$$

Now let $y(t) = |x(t)|$ for all $t \in [0, 1]$; then y is continuous and

$$y(t) \leq y(0) + \kappa \int_0^t (1 + y(s))ds.$$

Using Gronwall's Lemma (see [31] and Lemma B.2.2) we get

(1.12) $$y(t) \leq (y(0) + \kappa t)e^{\kappa t}$$

so we know that for all $t \in I$, $y(t) \le L$ for some constant $L < \infty$, i.e.

$$|x(t)| \le L \qquad \text{for all } t \in I. \tag{1.13}$$

Note: Equation (1.13) holds for a solution corresponding to a bounded or unbounded g.

Now if we truncate our unbounded g to $\bar{g}$ where

$$\bar{g}(t,x) = \begin{cases} g(t,x) & \text{if} \quad |g(t,x)| \le \kappa(1+L) \\ \kappa(1+L) & \text{if} \quad g(t,x) > \kappa(1+L) \\ -\kappa(1+L) & \text{if} \quad g(t,x) < -\kappa(1+L) \end{cases}$$

then solve

$$\bar{x}(t) = x(0) + \int_0^t \bar{g}(s, \bar{x}(s))ds, \tag{1.14}$$

by the above we see that there is a solution, $\bar{x}(t)$, to (1.14) (since now $\bar{g}$ is bounded) and by (1.13), this solution is bounded by L i.e.

$$|\bar{x}(t)| \le L \quad \text{for all } t \in [0,1].$$

So we lose nothing when truncating g to $\bar{g}$. This means

$$\bar{g}(s, \bar{x}(s)) = g(s, \bar{x}(s)) \qquad \text{for all } s \in [0,1]$$

and we have

$$\bar{x}(t) = x(0) + \int_0^t g(s, \bar{x}(s))ds.$$

Therefore, when g is unbounded, $\bar{x}(t)$ is a solution to equation (1.2). Thus we have shown that a solution to equation (1.1) exists for any f satisfying (i)-(iv).

□

Now, for completeness, we go on to give a proof of the well-known result that for each control $u \in \mathcal{U}$, given an initial state $x(0)$, equation (1.1) has a unique solution. This a standard result that can be found in many control theory books; the proof is a simple application of Gronwall's Lemma.

Theorem 1.1.2

Given a control $u \in \mathcal{U}$ and an initial state $x(0)$, equation (1.1) has a unique solution which we denote by $x_u(t)$.

Proof: In Theorem 1.1.1 we showed that a solution to (1.1) exists, we now need to show the uniqueness of a solution to equation (1.1). To do this, we show equation (1.2) has a unique solution.

Suppose there are two solutions, $x(t)$ and $x'(t)$, then

$$\begin{aligned} x(t) &= x(0) + \int_0^t g(s, x(s))ds \\ x'(t) &= x(0) + \int_0^t g(s, x'(s))ds, \end{aligned}$$

where, as before, $g(s, x(s)) = f(s, x(s), u(s))$.

Let $z(t) = |x(t) - x'(t)|$ then we have

$$\begin{aligned} z(t) &= |\int_0^t g(s, x(s))ds - \int_0^t g(s, x'(s))ds| \\ &\leq \int_0^t |g(s, x(s)) - g(s, x'(s))|ds \\ &\leq \kappa \int_0^t |x(s) - x'(s)|ds \qquad \text{by (iv}'\text{)} \\ &= \kappa \int_0^t z(s)ds \end{aligned}$$

so we have

$$0 \leq z(t) \leq \kappa \int_0^t z(s)ds. \tag{1.15}$$

Since $z(t)$ is continuous, we can apply Gronwall's Lemma and show that

$$z(t) \leq 0 \qquad \text{for all } t \in I. \tag{1.16}$$

Therefore, by (1.15) and (1.16), we have

$$z(t) = 0 \qquad \text{for all } t \in I,$$

i.e.

$$x(t) = x'(t) \qquad \text{for all } t \in I.$$

So we have shown that, given an initial state $x(0)$, for each control $u \in \mathcal{U}$, there is a unique solution to equation (1.1).

□

Now we can start to think about objectives for the process.

1.1.2 The cost of controls

Definition 1.1.3

Associated with each control $u \in \mathcal{U}$ for the process, there is a **cost** which we denote by $J(u)$. We assume here that the cost is defined by the following equation

$$J(u) = \int_0^1 h(s, x_u(s), u(s))ds + \bar{h}(x_u(1)) \tag{1.17}$$

where, $h : [0,1] \times \mathbb{R}^d \times M \longrightarrow \mathbb{R}$ and $\bar{h} : \mathbb{R}^d \longrightarrow \mathbb{R}$ are measurable real valued functions satisfying $h \geq 0$ and $\bar{h} \geq 0$. Here, x_u is the unique solution to (1.1) corresponding to the control $u \in \mathcal{U}$.

Our objective is to achieve the lowest cost for our process, so we define J_0 as follows

$$J_0 = \inf_{u \in \mathcal{U}} J(u). \tag{1.18}$$

In general, if we restrict ourselves to ordinary controls of the form $u \in \mathcal{U}$, it is not always possible to achieve this objective.

To illustrate this we now give a well-known simple example where, using standard ordinary controls, the minimum cost is not attainable.

Later, in this Chapter, we will show that, by extending the class of admissible controls to include Loeb measurable controls (explained later), the minimum cost is always attainable.

Example 1.1.4

We consider the following control problem:

The dynamics are given by

$$\frac{dx}{dt} = u(t). \tag{1.19}$$

The class of admissible controls, $\mathcal{U}$, is the set of all measurable controls of the form

$$u : [0,1] \to \{-1,1\}. \tag{1.20}$$

The cost of a control $u \in \mathcal{U}$ is given by

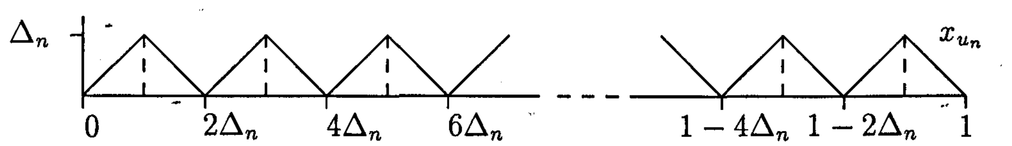
$$J(u) = \int_0^1 |x_u(t)|dt,$$

where $x_u(t)$ is the unique trajectory corresponding to the control $u \in \mathcal{U}$. Note, $J(u) \geq 0$ for all $u \in \mathcal{U}$.

Now, for each integer $n \in \mathbb{N}$, let $\Delta_n = 2^{-n}$ and define the control u_n by

$$u_n(t) = \begin{cases} 1 & \text{if} \quad t \in [0, \Delta_n] \\ (-1)^j & \text{if} \quad t \in]j\Delta_n, (j+1)\Delta_n] \quad \text{for } j = 1, 2, \cdots 2^n - 1 \end{cases}$$

i.e. the control u_n alternates from 1 to -1 on intervals of length Δ_n starting with value 1 on $[0, \Delta_n]$.



For each integer $n \in \mathbb{N}$, the cost of the control u_n is given by

$$\begin{aligned} J(u_n) &= \int_0^1 |x_{u_n}(t)|dt \\ &= \frac{\Delta_n}{2} \\ &= \frac{1}{2^{n+1}}, \end{aligned}$$

where x_{u_n} denotes the unique trajectory corresponding to the control u_n.

Thus, as $n \to \infty$, $J(u_n) \to 0$. Therefore, since $J(u) \geq 0$ for all $u \in \mathcal{U}$, we have

$$J_0 = \inf_{u \in \mathcal{U}} J(u) = 0.$$

We now show that there is no control $u \in \mathcal{U}$ satisfying $J(u) = 0$.

Suppose there is a control $\bar{u} \in \mathcal{U}$ such that $J(\bar{u}) = 0$. Then, $x_{\bar{u}}(t) = 0$ for all $t \in [0,1]$ which means $\frac{dx_{\bar{u}}}{dt} = 0$ for all $t \in [0,1]$, but by (1.19), this would imply that $\bar{u}(t) = 0$ for all $t \in [0,1]$ – however we know this cannot be true since, by (1.20), $\bar{u}(t) \in \{-1, 1\}$ and so we have a contradiction. Therefore we cannot have an optimal control in $\mathcal{U}$.

□

1.2 Relaxed controls

Since, in the standard situation, the minimum cost is not always attainable, the idea of extending the class of admissible controls was introduced. It was found that if the class of admissible controls was extended to include measurable controls of the form

$$\nu : [0,1] \to \Lambda(M), \tag{1.21}$$

where $\Lambda(M)$ is the set of all probability measures on the compact space M, and the definition of the function f was extended to $F : [0,1] \times \mathbb{R}^d \times \Lambda(M) \to \mathbb{R}^d$, by the following definition,

$$F(t,x,\mu) = \int_M f(t,x,a)d\mu(a) \tag{1.22}$$

then an optimal control could always be found. We give a proof of this well-known result later in section 1.4 (Corollary 1.4.5).

Controls of the from (1.21) are called **relaxed controls** and we shall denote the class of all such measurable controls by $\mathcal{R}$. Relaxed controls were first used in control theory in [37].

Note, by identifying each $a \in M$ with the probability measure δ_a concentrated at a, it can be shown that M is a closed subset of $\Lambda(M)$ and we have by (1.22),

$$F(t,x,\delta_a) = \int_M f(t,x,a)d\delta_a(a) = f(t,x,a).$$

Thus we can denote the extension of f by f. For each control $u \in \mathcal{U}$ we have

$$F(t,x,\delta_{u(t)}) = f(t,x,u(t))$$

and so we have $\mathcal{U} \subset \mathcal{R}$, using the above identification.

Since $\Lambda(M)$ is a compact metric space (with the Prohorov metric ([34])) , and the extended function f satisfies (i)-(iv) (as on page 10), all of the above results hold for relaxed controls.

We shall return to relaxed controls later in section 1.4; first we look at Loeb measurable controls.

1.3 Loeb measurable controls

In this section we look at the idea of extending the set of admissible controls to include measurable controls of the form

(1.23) $$v : {}^*[0,1] \to M.$$

We denote this extended collection of all such Loeb measurable controls of the form (1.23) by $\mathcal{V}$ and define the minimum cost, $\hat{J}_0$, as follows

(1.24) $$\hat{J}_0 = \inf_{v \in \mathcal{V}} J(v)$$

where $J(v)$ denotes the cost of the Loeb measurable control v.
(**Note:** We have not yet extended the dynamics or the cost function to deal with Loeb measurable controls, this will be done later.)
We ask if, with this extended collection of admissible controls, we can lower the minimum cost i.e. can we get $\hat{J}_0 < J_0$. We also ask if there is a $v \in \mathcal{V}$ which satisfies $J(v) = \hat{J}_0$ i.e. is $\hat{J}_0$ in our reachable set when the class of admissible controls is $\mathcal{V}$?

1.3.1 The dynamics equation for Loeb measurable controls

Before we can find answers to the above questions, we first have to extend our definitions of dynamics and cost to cope with times $\tau \in {}^*[0,1]$ and Loeb measurable controls of the form (1.23).

We need to adapt the functions f, g and h to accommodate times $\tau \in {}^*[0,1]$.

We need a function $F : {}^*[0,1] \times \mathbb{R}^d \times M \to \mathbb{R}^d$ which naturally extends f so that the dynamics can be defined by the following equation

$$x(\tau) = x(0) + \int_0^\tau F(\sigma, x(\sigma), v(\sigma))d\sigma_L \tag{1.25}$$

where σ_L is the Loeb measure associated with *Lebesgue measure.

There are two natural choices for F

(i) $F_1(\tau, x, a) = {}^\circ({}^*f(\tau, x, a))$

(ii) $F_2(\tau, x, a) = f({}^\circ\tau, x, a)$.

We shall show that these are in fact equivalent in the sense that, for almost all $\tau \in {}^*[0,1]$, the following diagram commutes

$$\begin{array}{ccc}
{}^*[0,1] \times \mathbb{R}^d \times M & \xrightarrow{\ {}^*f\ } & {}^*\mathbb{R}^d \\
\Big\downarrow {\scriptstyle \circ} & & \Big\downarrow {\scriptstyle \circ} \\
[0,1] \times \mathbb{R}^d \times M & \xrightarrow[\ f\ ]{} & \mathbb{R}^d
\end{array}.$$

We actually prove a more general result and note afterwards, in Corollary 1.3.2, that the result we require is an application of this more general result.

Proposition 1.3.1

For almost all $\tau \in {}^*[0,1]$

$$^\circ({}^*f(\tau, X, A)) = f({}^\circ\tau, {}^\circ X, {}^\circ A)$$

for all finite $X \in {}^*\mathbb{R}^d$ and all $A \in {}^*M$.

Proof: Given $f : [0,1] \times \mathbb{R}^d \times M \to \mathbb{R}^d$, let $\mathcal{C}(\mathbb{R}^d \times M, \mathbb{R}^d) = Y$ and define $\hat{f} : [0,1] \to Y$ by

$$\hat{f}(t) = f(t, \bullet, \bullet) \tag{1.26}$$

i.e. $\hat{f}(t) : \mathbb{R}^d \times M \to \mathbb{R}^d$ for each $t \in [0,1]$.

Using Anderson's Lusin Theorem (see [3]) we see that for a.a. $\tau \in {}^*[0,1]$

$$^*\hat{f}(\tau) \approx \hat{f}({}^\circ\tau)$$

in the sense of the compact open topology i.e. for a.a. $\tau \in {}^*[0,1]$

$$^\circ({}^*\hat{f}(\tau)(X, A)) = \hat{f}({}^\circ\tau)({}^\circ X, {}^\circ A)$$

for all finite $X \in {}^*\mathbb{R}^d$ and all $A \in {}^*M$. i.e. by (1.26), for a.a. $\tau \in {}^*[0,1]$,

$$^\circ({}^*f(\tau, X, A)) = f({}^\circ\tau, {}^\circ X, {}^\circ A)$$

for all finite $X \in {}^*\mathbb{R}^d$ and all $A \in {}^*M$.

□

Corollary 1.3.2

For almost all $\tau \in {}^*[0,1]$,

$$^\circ({}^*f(\tau, x, a)) = f({}^\circ\tau, x, a).$$

for all $x \in \mathbb{R}^d$ and all $a \in M$.

Proof: Take $x = X \in \mathbb{R}^d$ and $A = a \in M$ in Proposition 1.3.1. Then, $x = X = {}^\circ X$ and $a = A = {}^\circ A$ and so for a.a. $\tau \in {}^*[0,1]$,

$$^\circ({}^*f(\tau, x, a)) = f({}^\circ\tau, x, a)$$

for all $x \in \mathbb{R}^d$ and all $a \in M$.

□

We will use the notation of the second definition and so we define

$$F(\tau, x, a) = f({}^\circ\tau, x, a).$$

Thus, by (1.25), for a given Loeb measurable control $v \in \mathcal{V}$, the dynamics are given by

$$x(\tau) = x(0) + \int_0^\tau f({}^\circ\sigma, x(\sigma), v(\sigma))d\sigma_L \tag{1.27}$$

where the integral is a Loeb integral.

Definition 1.3.3

We say that $x(\tau)$ is a solution to the dynamics equation (1.27) if
(i) $x(\tau)$ is Loeb measurable.
(ii) $f({}^\circ\sigma, x(\sigma), v(\sigma))$ is Loeb integrable.
(iii) $x(\tau)$ satisfies (1.27).

Remarks 1.3.4

(a) $x(\tau') = x(\tau)$ if $\tau \approx \tau'$.
(b) If we let $\bar{x} : [0,1] \to \mathbb{R}^d$ be defined by $\bar{x}(t) = x(t)$ then $\bar{x}$ is continuous.

Proof of (a): Let $g(\sigma) = f({}^\circ\sigma, x(\sigma), v(\sigma))$, then if $\tau \approx \tau'$ where $\tau \geq \tau'$

$$\begin{aligned} x(\tau) - x(\tau') &= x(0) + \int_0^\tau g(\sigma)d\sigma_L - x(0) - \int_0^{\tau'} g(\sigma)d\sigma_L \\ &= \int_{\tau'}^\tau g(\sigma)d\sigma_L \\ &= 0 \qquad \text{since } g \text{ is Loeb integrable and } [\tau', \tau] \text{ is null.} \end{aligned}$$

Proof of (b): Consider a sequence of times in $^*[0,1]$ decreasing to a time t i.e.

$$t_n \downarrow t \qquad \text{as } n \to \infty \tag{1.28}$$

then for each n we have

$$x(t_n) - x(t) = \int_t^{t_n} g(\sigma) d\sigma_L$$

(where as before $g(\sigma) = f(^\circ\sigma, x(\sigma), v(\sigma))$) and so, since by (1.28)

$$\sigma_L([t, t_n]) \to 0 \qquad \text{as } n \to \infty,$$

we have

$$x(t_n) - x(t) = \int_{[t,t_n]} g(\sigma) d\sigma_L \to 0 \quad \text{as } n \to \infty. \tag{1.29}$$

Now, $\bar{x}(t) = x(t)$ for all $t \in [0,1]$ therefore, by (1.29), $\bar{x}$ is continuous.

□

1.3.2 Existence and uniqueness for Loeb controls

Our aim now is to show the existence and uniqueness of solutions for Loeb measurable controls.

In Theorem 1.1.1 it was shown that for each control $u \in \mathcal{U}$ there exists a standard solution to (1.1) i.e. a solution to

$$x(t) = x(0) + \int_0^t f(s, x(s), u(s)) ds$$

and, by (1.12), for all $t \in [0,1]$

$$|x(t)| \leq (|x(0)| + \kappa t) e^{\kappa t} \tag{1.30}$$

where κ is the growth constant for f as in (iii). Therefore, by the transfer principle, we know that for each control $U \in {}^*\mathcal{U}$, there exists a nonstandard solution, $X(\tau)$, satisfying

$$X(\tau) = x(0) + \int_0^\tau {}^*f(\sigma, X(\sigma), U(\sigma)) d\sigma \tag{1.31}$$

and that, by (1.30), for all $\tau \in {}^*[0,1]$

$$|X(\tau)| \leq (|x(0)| + \kappa\tau)e^{\kappa\tau}. \tag{1.32}$$

So we have the existence of finitely bounded nonstandard solutions to (1.31). We consider these finitely bounded nonstandard solutions which are also $\mathcal{S}$-continuous and show that their standard parts are solutions to equation (1.27).

Proposition 1.3.5

Given a control $v \in \mathcal{V}$, if $U \in {}^*\mathcal{U}$ satisfies

$$v(\tau) = {}^\circ U(\tau) \text{ for a.a. } \tau \in {}^*[0,1], \tag{1.33}$$

then the standard part of the nonstandard solution to (1.31), X_U, corresponding to the control $U \in {}^*\mathcal{U}$ is itself a solution to the dynamics equation (1.27) when the control is v.

Proof: X_U is a solution to (1.31) and so we have

$$\begin{aligned}
{}^\circ X_U(\tau) &= {}^\circ(x(0) + \int_0^\tau {}^*f(\sigma, X_U(\sigma), U(\sigma))d\sigma) \\
&= x(0) + \int_0^\tau {}^\circ({}^*f(\sigma, X_U(\sigma), U(\sigma)))d\sigma_L \quad \text{(by Loeb theory)} \\
&= x(0) + \int_0^\tau f({}^\circ\sigma, {}^\circ X_U(\sigma), {}^\circ U(\sigma))d\sigma_L \\
&\quad \text{(by Anderson's Lusin Theorem (see [3]) and continuity of } f\text{)} \\
&= x(0) + \int_0^\tau f({}^\circ\sigma, {}^\circ X_U(\sigma), v(\sigma))d\sigma_L \\
&\quad \text{(since } v(\tau) = {}^\circ U(\tau) \quad \text{for a.a. } \tau \in {}^*[0,1]).
\end{aligned}$$

□

We now show the existence of a solution when the class of admissible controls is $\mathcal{V}$.

Theorem 1.3.6

For each Loeb measurable control $v \in \mathcal{V}$, equation (1.27) has a solution $x(\tau)$.

University

Proof: If v is a Loeb measurable control of the form

$$v : {}^*[0,1] \rightarrow M$$

then, by Anderson's Lifting Theorem, v has a lifting U i.e. there exists a control $U \in {}^*\mathcal{U}$ satisfying (1.33), therefore, by Proposition 1.3.5, we see that ${}^\circ X_U$ is a solution to the dynamics equation (1.27) where ${}^\circ X_U$ is the standard part of the nonstandard solution to (1.31) corresponding to this control U.

□

Therefore, we have shown the existence of a solution to the dynamics equation when the class of controls is $\mathcal{V}$.

We now go on to show that for any control $v \in \mathcal{V}$ this solution is unique.

Theorem 1.3.7

For each $v \in \mathcal{V}$ there is a unique solution to the dynamics equation (1.27) which we denote by $x_v(\tau)$.

Proof: We know by Theorem 1.3.6 that for each $v \in \mathcal{V}$ there exists a solution to (1.27).

Suppose there are two solutions $x(\tau)$ and $x'(\tau)$ then

$$x(\tau) = x(0) + \int_0^\tau g(\sigma, x(\sigma))d\sigma_L$$
$$x'(\tau) = x(0) + \int_0^\tau g(\sigma, x'(\sigma))d\sigma_L$$

where

$$g(\sigma, x(\sigma)) = f({}^\circ\sigma, x(\sigma), v(\sigma)).$$

Let $z(\tau) = |x(\tau) - x'(\tau)|$ then we have

$$\begin{aligned} z(\tau) &= \left|\int_0^\tau g(\sigma, x(\sigma))d\sigma_L - \int_0^\tau g(\sigma, x'(\sigma))d\sigma_L\right| \\ &\leq \int_0^\tau |g(\sigma, x(\sigma)) - g(\sigma, x'(\sigma))|d\sigma_L \\ &\leq \kappa \int_0^\tau |x(\sigma) - x'(\sigma)|d\sigma_L \qquad (1.34) \\ &= \kappa \int_0^\tau z(\sigma)d\sigma_L. \end{aligned}$$

Note: We cannot use Gronwall's Lemma directly here since the integral is a Loeb integral.

By Remarks 1.3.4 (a) we see that $x(\tau)$ and $x'(\tau)$ are constant on monads. Therefore, $z(\tau) = |x(\tau) - x'(\tau)|$ is also constant on monads.

Now, if we define

$$\bar{z}(t) = |\bar{x}(t) - \bar{x}'(t)| \qquad \text{for all } t \in [0,1]$$

where $\bar{x}$ and $\bar{x}'$ are related to x and x' in the sense of Remarks 1.3.4 (b), then since $\bar{x}$ and $\bar{x}'$ are continuous, $\bar{z}$ is continuous and clearly $\bar{z}(t) = z(t)$ for all $t \in [0,1]$.

We need to show that $\bar{z}(t) = 0$ for all $t \in [0,1]$. Then we would have $\bar{x}(t) = \bar{x}'(t)$ for all $t \in [0,1]$ and so, $x(\tau) = x'(\tau)$ for all $\tau \in {}^*[0,1]$.

Now, by (1.34), for all $t \in [0,1]$,

$$\bar{z}(t) \leq \kappa \int_0^t z(\sigma) d\sigma_L.$$

If we had

$$\bar{z}(t) \leq \kappa \int_0^t \bar{z}(s) ds$$

then we could apply the standard form of Gronwall's Lemma to get

$$\bar{z}(t) = 0 \qquad \text{for all } t \in [0,1].$$

Therefore, we are done if we can show that

(1.35) $$\int_0^t z(\sigma) d\sigma_L \leq \int_0^t \bar{z}(s) ds.$$

We can actually show equality in equation (1.35).

Note, since $z(\sigma) = z({}^\circ\sigma) = \bar{z}({}^\circ\sigma)$ for all $\sigma \in {}^*[0,1]$,

(1.36) $$\int_0^t z(\sigma) d\sigma_L = \int_0^t \bar{z}({}^\circ\sigma) d\sigma_L.$$

Now, $\bar{z}$ is continuous, therefore

(1.37) $$\begin{aligned} \int_0^t \bar{z}({}^\circ\sigma) d\sigma_L &= \int_0^t {}^\circ({}^*\bar{z}(\sigma)) d\sigma_L \\ &= {}^\circ\left(\int_0^t {}^*\bar{z}(\sigma) d\sigma\right) \quad \text{(since } \bar{z} \text{ is bounded)} \\ &= \int_0^t \bar{z}(s) ds. \end{aligned}$$

This, (1.37), together with (1.36) establishes (1.35) and so

$$0 \leq \bar{z}(t) \leq \kappa \int_0^t \bar{z}(s)ds.$$

We now have a standard integral and so, since $\bar{z}$ is continuous, we may use Gronwall's Lemma to obtain

$$\bar{z}(t) = 0 \quad \text{for all } t \in [0,1]$$

i.e.

$$\begin{aligned} \bar{x}(t) &= \bar{x}'(t) \quad \text{for all } t \in [0,1] \\ \Longrightarrow x(\tau) &= x'(\tau) \quad \text{for all } \tau \in {}^*[0,1] \end{aligned}$$

so the solution is unique.

□

Therefore we have shown the existence and uniqueness of solutions to the dynamics equation (1.27) for Loeb measurable controls.

Corollary 1.3.8

Given a Loeb measurable control $v \in \mathcal{V}$, if $U \in {}^*\mathcal{U}$ is a nonstandard control satisfying (1.33) then,

(1.38) $$^\circ X_U(\tau) = x_v(\tau) \quad \text{for all } \tau \in {}^*[0,1]$$

where x_v is the unique Loeb solution corresponding to the control $v \in \mathcal{V}$ and X_U is the nonstandard solution to (1.31) corresponding to the control U.

Proof: Clearly, x_v is a solution to (1.27) and in Proposition 1.3.5 it was shown that, if U is the control in ${}^*\mathcal{U}$ satisfying (1.33) then $^\circ X_U$ is a solution to (1.27), where $^\circ X_U$ is the standard part of the nonstandard solution corresponding to this control U. In the proof of Theorem 1.3.7 it was shown that this equation has a unique solution and so $^\circ X_U = x_v$.

□

1.3.3 The cost of Loeb measurable controls

Now, as for the standard controls, associated with each $v \in \mathcal{V}$, there is a cost $J(v)$. To be able to define this cost, we have to adapt our functions h and $\bar{h}$ to accommodate $\tau \in {}^*[0,1]$. We adapt h and $\bar{h}$ in exactly the same way that we adapted our function f.

Definition 1.3.9

For each control $v \in \mathcal{V}$, the cost $J(v)$ is given by

$$J(v) = \int_0^1 h({}^\circ\sigma, x_v(\sigma), v(\sigma))d\sigma_L + \bar{h}(x_v(1)) \tag{1.39}$$

where x_v is the unique solution to (1.27) corresponding to the control v.

Proposition 1.3.10

Given a control $v \in \mathcal{V}$, if $U \in {}^*\mathcal{U}$ satisfies (1.33) i.e.

$$v(\tau) = {}^\circ U(\tau) \qquad \text{for a.a. } \tau \in {}^*[0,1]$$

then

$$J(v) = {}^\circ({}^*J(U)).$$

Proof: For $U \in {}^*\mathcal{U}$,

$${}^*J(U) = \int_0^1 {}^*h(\sigma, X_U(\sigma), U(\sigma))d\sigma + {}^*\bar{h}(X_U(1))$$

where X_U is the solution to (1.31) corresponding to the control $U \in {}^*\mathcal{U}$. Now, by Corollary 1.3.8, since U satisfies (1.33), $x_v(\tau) = {}^\circ X_U(\tau)$ for all $\tau \in {}^*[0,1]$ (where

X_U is any nonstandard solution corresponding to the control U) so we have

$$\begin{aligned}
{}^\circ({}^*J(U)) &= {}^\circ\left(\int_0^1 {}^*h(\sigma, X_U(\sigma), U(\sigma))d\sigma + {}^*\bar{h}(X_U(1))\right) \\
&= \int_0^1 {}^\circ({}^*h(\sigma, X_U(\sigma), U(\sigma)))d\sigma_L + {}^\circ({}^*\bar{h}(X_U(1))) \\
&\quad \text{(by Loeb Theory)} \\
&= \int_0^1 h({}^\circ\sigma, {}^\circ X_U(\sigma), {}^\circ U(\sigma))d\sigma_L + \bar{h}({}^\circ X_U(1)) \\
&\quad \text{(by Anderson's Lusin Theorem)} \\
&= \int_0^1 h({}^\circ\sigma, x_v(\sigma), v(\sigma))d\sigma_L + \bar{h}(x_v(1)) \\
&\quad \text{(by Corollary 1.3.8 and the fact that } v(\tau) = {}^\circ U(\tau) \text{ for a.a. } \tau) \\
&= J(v).
\end{aligned}$$

□

Now recall our objectives; we asked if $\hat{J}_0 < J_0$ and if there exists a $v \in \mathcal{V}$ such that $J(v) = \hat{J}_0$.

Remember, J_0 and $\hat{J}_0$ are defined as follows

(1.40) $$J_0 = \inf_{u \in \mathcal{U}} J(u)$$

(1.41) $$\hat{J}_0 = \inf_{v \in \mathcal{V}} J(v).$$

We now show that for each control $U \in \mathcal{U}$ there is a corresponding control $V \in \mathcal{V}$ which has the same cost.

Proposition 1.3.11

For each $u \in \mathcal{U}$, there is a $v \in \mathcal{V}$ such that

$$J(u) = J(v)$$

namely the $v \in \mathcal{V}$ given by

$$v(\tau) = {}^\circ({}^*u(\tau)).$$

Proof: If $v(\tau) = {}^\circ({}^*u(\tau))$ then, since ${}^*u \in {}^*\mathcal{U}$, by Proposition 1.3.10 we have

$$J(v) = {}^\circ({}^*J({}^*u)) = J(u).$$

□

Theorem 1.3.12 $\hat{J}_0 = J_0$

Proof: By Proposition 1.3.11 above we know

$$\hat{J}_0 = \inf_{v \in \mathcal{V}} J(v) \leq \inf_{u \in \mathcal{U}} J(u) = J_0$$

i.e.

$$\hat{J}_0 \leq J_0. \tag{1.42}$$

Now we show that $\hat{J}_0 \geq J_0$. Take any $v \in \mathcal{V}$, $v : {}^*[0,1] \to M$, now take a lifting U of v, $U : {}^*[0,1] \to {}^*M$, then, $U \in {}^*\mathcal{U}$.

Now,

$$u \in \mathcal{U} \implies J(u) \geq J_0$$

therefore by transfer,

$$U \in {}^*\mathcal{U} \implies {}^*J(U) \geq J_0.$$

Now, by Proposition 1.3.10, $J(v) \approx {}^*J(U) \geq J_0$ hence by taking standard parts we have

$$J(v) \geq J_0.$$

Therefore we have

$$\inf_{v \in \mathcal{V}} J(v) = \hat{J}_0 \geq J_0 \tag{1.43}$$

i.e. by (1.42) and (1.43) we have

$$\hat{J}_0 = J_0.$$

□

Hence we cannot lower the minimum cost by using this extended class of controls. i.e. we do not have $\hat{J}_0 < J_0$.

We now look at the other question, is there a $v \in \mathcal{V}$ such that $J(v) = \hat{J}_0$? The answer is yes as we now show.

Theorem 1.3.13

There is a $v \in \mathcal{V}$ such that $J(v) = \hat{J}_0$.

Proof: Let

$$u_n : [0,1] \to M$$

be a sequence of standard controls i.e.

$$u : \mathbb{N} \to \mathcal{U} \qquad \text{where } u(n) = u_n \text{ for each } n \in \mathbb{N}$$

such that

$$J(u_n) \downarrow J_0 \qquad \text{as } n \to \infty.$$

By taking liftings of these u_n, $n \in \mathbb{N}$ we have

$$U_n : {}^*[0,1] \to {}^*M,$$

a sequence of nonstandard controls, i.e.

$$U : \mathbb{N} \to {}^*\mathcal{U} \qquad \text{where} \qquad U(n) = U_n \qquad \text{for each } n \in \mathbb{N}.$$

Now, ${}^*\mathcal{U}$ is internal and so, by $\aleph_1$–saturation, we can extend the sequence $U : \mathbb{N} \to {}^*\mathcal{U}$ to an internal sequence

$$U : {}^*\mathbb{N} \to {}^*\mathcal{U}$$

so we have a sequence $(U_n)_{n \in {}^*\mathbb{N}}$ where,

$$U_n : {}^*[0,1] \to {}^*M \qquad \text{for each } n \in {}^*\mathbb{N}.$$

Note, for all finite $n \in {}^*\mathbb{N}$, $u_n = {}^\circ U_n$ a.s. therefore, by Proposition 1.3.10, we have

$$J(u_n) = {}^\circ({}^*J(U_n))$$

for all finite $n \in {}^*\mathbb{N}$. Now, without loss of generality, we can arrange the u_n's so that $J(u_n) < J_0 + \frac{1}{n}$ therefore, we see that

$$J_0 \leq {}^*J(U_n) < J_0 + \frac{1}{n}$$

for all finite $n \in {}^*\mathbb{N}$ so, by overflow, there exists an infinite $N \in {}^*\mathbb{N}$ such that

$$J_0 \leq {}^*J(U_N) < J_0 + \frac{1}{N}$$

so we see that

$${}^*J(U_N) \approx J_0.$$

Now let $v = {}^\circ U_N$ for this infinite N then, by Proposition 1.3.10 and Theorem 1.3.12 we see that

$$J(v) = {}^\circ({}^*J(U_N)) = J_0 = \hat{J}_0,$$

i.e. we have shown that there exists a $v \in \mathcal{V}$ such that

$$J(v) = \hat{J}_0 = J_0.$$

□

Therefore, when we let our dynamics be determined by equation (1.27) and our set of admissible controls be $\mathcal{V}$, if we let our objective be to achieve the lowest cost, then we have a properly formulated optimal control problem since using this class of admissible controls, the lowest cost is in our reachable set. i.e. there is always an optimal Loeb control.

Now, to illustrate this, we go back to Example 1.1.4, where there is no standard optimal control.

Example 1.3.14

The dynamics are given by

$$\frac{dx}{d\tau} = v(\tau).$$

The class of admissible controls, $\mathcal{V}$, is now the set of all measurable controls of the form

$$v : {}^*[0,1] \to \{-1, 1\}.$$

The cost is given by

$$J(v) = \int_0^1 |x_v(\tau)| d\tau_L,$$

where x_v is the unique trajectory corresponding to the control v.

By Example 1.1.4 and Theorem 1.3.12

$$\hat{J}_0 = \inf_{v \in \mathcal{V}} J(v) = 0.$$

We now give details of optimal controls for this example.

For each integer $n \in {}^*\mathbb{N}$ consider the control $U_n \in {}^*\mathcal{U}$ as given in Example 1.1.4 i.e. the control U_n alternates from 1 to -1 on intervals of length Δ_n starting with value 1 on $[0, \Delta_n]$, where $\Delta_n = 2^{-n}$. Then,

$${}^*J(U_n) = \int_0^1 |X_{U_n}(\tau)| d\tau = \frac{1}{2^{n+1}}$$

where X_{U_n} denotes the trajectory corresponding to the control U_n. Now, let

$$v_n = {}^\circ U_n \qquad \text{for each } n \in {}^*\mathbb{N}$$

then $v_n \in \mathcal{V}$ for each n and by Proposition 1.3.10, the cost of v_n is given by

$$J(v_n) = {}^\circ({}^*J(U_n)) = {}^\circ(\int_0^1 |X_{U_n}(\tau)| d\tau) = {}^\circ(\frac{1}{2^{n+1}})$$

so for any infinite $N \in {}^*\mathbb{N}$,

$$J(v_N) = {}^\circ(\frac{1}{2^{N+1}}) = 0$$

i.e. when N is infinite, v_N is an optimal control for this example.

□

Notation 1.3.15

Here we give a summary of the notation used for the different classes of controls. We denote by $\mathcal{U}$ the class of ordinary controls of the form

$$u : [0, 1] \longrightarrow M$$

and

$$J_0 = \inf_{u \in \mathcal{U}} J(u).$$

We denote by $\mathcal{R}$ the class of relaxed controls of the form

$$\nu : [0,1] \to \Lambda(M)$$

and

$$J_0^{\mathcal{R}} = \inf_{\nu \in \mathcal{R}} J(\nu).$$

We denote by $\mathcal{V}$ the class of Loeb controls of the form

$$v : {}^*[0,1] \to M$$

and

$$\hat{J}_0 = \inf_{v \in \mathcal{V}} J(v).$$

We denote by $\mathcal{S}$ the class of relaxed Loeb controls of the form

$$\xi : {}^*[0,1] \to \Lambda(M)$$

and

$$\hat{J}_0^{\mathcal{S}} = \inf_{\xi \in \mathcal{S}} J(\xi).$$

By identifying each $a \in M$ with the Dirac measure δ_a concentrated at a we have $M \subseteq \Lambda(M)$ so $\mathcal{U} \subseteq \mathcal{R}$ similarly, $\mathcal{V} \subseteq \mathcal{S}$.

1.4 The existence of a relaxed optimal control

We now go back to relaxed controls. Using Theorem 1.3.13, we give a proof of the well-known result that there is always an optimal relaxed control. To do this we first show that from a relaxed Loeb control, $\xi \in \mathcal{S}$, we can obtain a standard relaxed control, ν_ξ , which gives the same trajectory and has the same cost as ξ.

The proof of this is based on work done by Cutland ([6]).

A weak *topology (see [35]) is defined on $\mathcal{U}$ by means of a set $\mathcal{K}$ defined as follows:

(1) $\mathcal{K}$ is the set of bounded measurable functions

$$g : [0,1] \times M \to \mathbb{R}$$

with $g(t, \bullet)$ continuous for each $t \in [0,1]$.

(2) For $u \in \mathcal{U}$ and $g \in \mathcal{K}$, the action of u on g is defined by

$$u(g) = \int_0^1 g(t, u(t))dt.$$

(3) The $\mathcal{K}$-topology on $\mathcal{U}$ has as subbase of open neighbourhoods the sets

$$\{u : |u(g)| < \epsilon\}_{g \in \mathcal{K}, \epsilon > 0}.$$

The topology on $\mathcal{U}$ is extended to $\mathcal{R}$ by extending each $g \in \mathcal{K}$ to $[0,1] \times \Lambda(M)$ with the definition

$$g(t, \mu) = \int_M g(t, a)d\mu(a) \tag{1.44}$$

for $\mu \in \Lambda(M)$, and so, for $\nu \in \mathcal{R}$ and $g \in \mathcal{K}$

$$\begin{aligned} \nu(g) &= \int_0^1 g(t, \nu(t))dt \\ &= \int_0^1 (\int_M g(t, a)d\nu(t)(a))dt \qquad \text{by (1.44).} \end{aligned} \tag{1.45}$$

Lemma 1.4.1

Given any relaxed Loeb measurable control $\xi : {}^*[0,1] \to \Lambda(M)$, let Q^ξ be the measure on ${}^*[0,1] \times M$ defined by

$$Q^\xi(C \times D) = \int_C \xi(\tau)(D)d\tau_L \tag{1.46}$$

for Loeb measurable $C \subseteq {}^*[0,1]$ and Borel $D \subseteq M$. From this define a standard measure q^ξ on $[0,1] \times M$ by

$$q^\xi(B \times D) = Q^\xi(st^{-1}(B) \times D) \tag{1.47}$$

for $B \subseteq [0,1]$ and $D \subseteq M$. Then we have the following

(i) For a Borel set $A \subseteq [0,1]$, $q^\xi(A \times M) = \lambda(A)$ where λ denotes Lebesgue measure.

(ii) q^ξ is a probability measure on $[0,1] \times M$.

Proof of (i): $st^{-1}(A)$ is Loeb measurable for a Borel set $A \subseteq [0,1]$. So

$$\begin{aligned} q^{\xi}(A \times M) &= Q^{\xi}(st^{-1}(A) \times M) && \text{by (1.47)} \\ &= \int_{st^{-1}(A)} \xi(\tau)(M) d\tau_L && \text{by (1.46)} \\ &= \int_{st^{-1}(A)} d\tau_L && (\text{since } \xi(\tau)(M) = 1) \\ &= {}^*\lambda_L(st^{-1}(A)) \\ &= \lambda(A). \end{aligned}$$

Therefore, for Borel sets $A \subseteq [0,1]$ we have

(1.48) $$q^{\xi}(A \times M) = \lambda(A).$$

Proof of (ii):

(a) $q^{\xi}([0,1] \times M) = \lambda([0,1]) = 1$ by (1.48)

(b) σ-additivity of q^{ξ} follows from the σ-additivity of Q^{ξ}.

□

Lemma 1.4.2

Given any Loeb measurable control $\xi \in \mathcal{S}$, the measure q^{ξ} as constructed in Lemma 1.4.1 can be disintegrated to give $\nu_{\xi} : [0,1] \to \Lambda(M)$, i.e. $\nu_{\xi} \in \mathcal{R}$, with the property

(1.49) $$q^{\xi}(A \times B) = \int_A \nu_{\xi}(t)(B) dt$$

for Borel $A \subseteq [0,1]$ and $B \subseteq M$.

Proof: See [6] and [34].

□

Lemma 1.4.3

Given a Loeb measurable control $\xi \in \mathcal{S}$, for any $g \in \mathcal{K}$, let $\xi(g)$ be given by

(1.50) $$\xi(g) = \int_0^1 g({}^{\circ}\tau, \xi(\tau)) d\tau_L$$

where

$$g({}^\circ\tau, \xi(\tau)) = \int_M g({}^\circ\tau, a) d\xi(\tau)(a) \tag{1.51}$$

then, for all $g \in \mathcal{K}$, we have

$$\xi(g) = \nu_\xi(g) \tag{1.52}$$

where ν_ξ is constructed from ξ as in Lemma 1.4.2 i.e. for all $g \in \mathcal{K}$,

$$\int_0^1 g({}^\circ\tau, \xi(\tau)) d\tau_L = \int_0^1 g(t, \nu_\xi(t)) dt. \tag{1.53}$$

Proof:

$$\begin{aligned}
\xi(g) &= \int_0^1 g({}^\circ\tau, \xi(\tau)) d\tau_L && \text{by (1.50)} \\
&= \int_0^1 (\int_M g({}^\circ\tau, a) d\xi(\tau)(a)) d\tau_L && \text{by (1.51)} \\
&= \int_{{}^*[0,1]\times M} g({}^\circ\tau, a) dQ^\xi(\tau, a) && \\
&= \int_{[0,1]\times M} g(t, a) dq^\xi(t, a) && \\
&= \int_0^1 (\int_M g(t, a) d\nu_\xi(t)(a)) dt && \\
&= \int_0^1 g(t, \nu_\xi(t)) dt && \text{by (1.44)} \\
&= \nu_\xi(g) && \text{by (1.45).}
\end{aligned}$$

□

Proposition 1.4.4

Given a relaxed Loeb measurable control $\xi \in \mathcal{S}$, there is a corresponding control $\nu_\xi \in \mathcal{R}$ satisfying

$$x_\xi(t) = x_{\nu_\xi}(t) \qquad \text{for all } t \in [0,1]$$

and

$$J(\xi) = J(\nu_\xi).$$

Consequently, given an ordinary Loeb control $v \in \mathcal{V}$ there is a corresponding control $\nu_v \in \mathcal{R}$ such that $J(v) = J(\nu_v)$.

Proof: Given a control $\xi \in \mathcal{S}$, by Theorem 1.3.7 there exists a unique solution $x_\xi(\tau)$ corresponding to this control, and

$$(1.54) \qquad x_\xi(\tau) = x_\xi({}^\circ\tau) \qquad \text{for all } \tau \in {}^*[0,1].$$

From Lemma 1.4.3 it can be seen that given a control $\xi \in \mathcal{S}$ we can construct a control ν_ξ such that

$$(1.55) \qquad \int_0^1 g({}^\circ\tau, \xi(\tau))d\tau_L = \int_0^1 g(t, \nu_\xi(t))dt$$

for all $g \in \mathcal{K}$, where $\mathcal{K}$ is the set of all bounded measurable functions

$$g : [0,1] \times \Lambda(M) \rightarrow \mathbb{R}$$

with $g(t, \bullet)$ continuous for each $t \in [0,1]$.

Now, given $f : [0,1] \times \mathbb{R}^d \times \Lambda(M) \rightarrow \mathbb{R}^d$, for each $t \in [0,1]$, $x \in \mathbb{R}^d$ and $\mu \in \Lambda(M)$,

$$(1.56) \qquad f(t, x, \mu) = (f_1(t, x, \mu), \cdots, f_d(t, x, \mu))$$

where $f_i : [0,1] \times \mathbb{R}^d \times \Lambda(M) \rightarrow \mathbb{R}$ for $i = 1, \cdots, d$. Now, for each fixed $s \in [0,1]$ if, for each $i = 1, \cdots, d$ in turn, we take

$$g(t, \mu) = \begin{cases} f_i(t, x_\xi(t), \mu) & \text{if } t \leq s \\ 0 & \text{otherwise} \end{cases}$$

for $t \in [0,1]$ and $\mu \in M$ where $x_\xi(\tau)$ denotes the trajectory corresponding to the control $\xi \in \mathcal{S}$ then, by (1.55), we have

$$(1.57) \qquad \int_0^s f({}^\circ\tau, x_\xi({}^\circ\tau), \xi(\tau))d\tau_L = \int_0^s f(t, x_\xi(t), \nu_\xi(t))dt.$$

for all $s \in [0,1]$.

Now, $x_\xi(0) = x_{\nu_\xi}(0) = x(0)$. The above means that

$$\begin{aligned} x_\xi(t) &= x(0) + \textstyle\int_0^t f({}^\circ\tau, x_\xi(\tau), \xi(\tau))d\tau_L && \text{(by definition)} \\ &= x(0) + \textstyle\int_0^t f({}^\circ\tau, x_\xi({}^\circ\tau), \xi(\tau))d\tau_L && \text{by (1.54)} \\ &= x(0) + \textstyle\int_0^t f(s, x_\xi(s), \nu_\xi(s))ds && \text{by (1.57)} \end{aligned}$$

for all t, so x_ξ is the unique solution for the control ν_ξ i.e.

$$x_\xi(t) = x_{\nu_\xi}(t) \qquad \text{for all } t \in [0,1]. \tag{1.58}$$

Similarly, by considering

$$g(t,\mu) = h(t, x_\xi(t), \mu) \qquad \text{for } t \in [0,1] \text{ and } \mu \in \Lambda(M)$$

it can be seen that

$$\begin{aligned}
J(\xi) = \quad & \textstyle\int_0^1 h({}^\circ\tau, x_\xi(\tau), \xi(\tau))d\tau_L + \bar{h}(x_\xi(1)) && \text{(by definition)} \\
= \quad & \textstyle\int_0^1 h({}^\circ\tau, x_\xi({}^\circ\tau), \xi(\tau))d\tau_L + \bar{h}(x_\xi(1)) && \text{by (1.54)} \\
= \quad & \textstyle\int_0^1 h(t, x_\xi(t), \nu_\xi(t))dt + \bar{h}(x_\xi(1)) && \text{by (1.55)} \\
= \quad & \textstyle\int_0^1 h(t, x_{\nu_\xi}(t), \nu_\xi(t))dt + \bar{h}(x_{\nu_\xi}(1)) && \text{by (1.58).} \\
= \quad & J(\nu_\xi) && \text{(by definition).}
\end{aligned}$$

The fact that $\mathcal{V} \subseteq \mathcal{S}$ gives us the second part of this Proposition.

□

Corollary 1.4.5

There is always an optimal relaxed control i.e. there is a $\nu \in \mathcal{R}$ such that

$$J(\nu) = J_0.$$

Proof: By Theorem 1.3.13, there exists an optimal Loeb measurable control $\bar{v} \in \mathcal{V}$. Let $\nu \in \mathcal{R}$ be given by $\nu = \nu_{\bar{v}}$ then, by Proposition 1.4.4

$$J(\nu_{\bar{v}}) = J(\bar{v}) = \hat{J}_0 = J_0.$$

□

This Corollary leaves open the possibility that a lower cost can be achieved by using relaxed controls – however it is well known that this cannot happen. We complete the picture by giving a proof of this using the above results and the fact that Loeb controls cannot lower the cost.

Definition 1.4.6

A control $u \in \mathcal{U}$ is a step control if there exist times

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $u(t)$ is constant on each interval $]t_n, t_{n+1}]$.

Proposition 1.4.7

Given a standard relaxed control $\nu \in \mathcal{R}$, there is a corresponding Loeb measurable control $v_\nu \in \mathcal{V}$ satisfying

$$x_\nu(t) = x_{v_\nu}(t) \qquad \text{for all } t \in [0,1]$$

and

$$J(\nu) = J(v_\nu).$$

Proof: By restricting the work done by Cutland in [6] to [0,1], it can be seen that given a $\nu \in \mathcal{R}$, there is a step control $U_\nu \in {}^*\mathcal{U}$ satisfying

$$U_\nu({}^*g) \approx \nu(g) \qquad \text{for all } g \in \mathcal{K}$$

i.e.

$$\text{(1.59)} \qquad \int_0^1 {}^*g(\tau, U_\nu(\tau))d\tau \approx \int_0^1 g(t, \nu(t))dt \qquad \text{for all } g \in \mathcal{K}.$$

Now, let ${}^\circ U_\nu$ be denoted by v_ν then, using Anderson's Lusin Theorem and Loeb Theory, this gives us

$$\text{(1.60)} \qquad \int_0^1 g({}^\circ\tau, v_\nu(\tau))d\tau_L = \int_0^1 g(t, \nu(t))dt \qquad \text{for all } g \in \mathcal{K}.$$

Now, by Section 1.2, M is a closed subset of $\Lambda(M)$ and by (1.56)

$$f(t,x,\mu) = (f_1(t,x,\mu), \cdots, f_d(t,x,\mu))$$

for all $t \in [0,1]$, $x \in \mathbb{R}^d$ and $\mu \in \Lambda(M)$. Therefore, if for each fixed $s \in [0,1]$, we take, for each $i = 1, \cdots, d$ in turn,

$$g(t,\mu) = \begin{cases} f_i(t, x_{v_\nu}(t), \mu) & \text{if } t \le s \\ 0 & \text{otherwise} \end{cases}$$

for $t \in [0,1]$ and $\mu \in \Lambda(M)$ where x_{v_ν} denotes the trajectory corresponding to the control $v_\nu \in \mathcal{V}$ then, by (1.60) we have

$$(1.61) \qquad \int_0^s f({}^\circ\tau, x_{v_\nu}({}^\circ\tau), v_\nu(\tau))d\tau_L = \int_0^s f(t, x_{v_\nu}(t), \nu(t))dt.$$

for all $s \in [0,1]$. We know that v_ν is a Loeb measurable control and so, by Theorem 1.3.7, there exists a unique solution $x_{v_\nu}(\tau)$ corresponding to this control and

$$(1.62) \qquad x_{v_\nu}(\tau) = x_{v_\nu}({}^\circ\tau) \qquad \text{for all } \tau \in {}^*[0,1].$$

We also know that $x_{v_\nu}(0) = x_\nu(0) = x(0)$, therefore we have

$$\begin{aligned} x_{v_\nu}(t) &= x(0) + \int_0^t f({}^\circ\sigma, x_{v_\nu}(\sigma), v_\nu(\sigma))d\sigma_L && \text{(by definition)} \\ &= x(0) + \int_0^t f({}^\circ\sigma, x_{v_\nu}({}^\circ\sigma), v_\nu(\sigma))d\sigma_L && \text{by (1.62)} \\ &= x(0) + \int_0^t f(s, x_{v_\nu}(s), \nu(s))ds && \text{by (1.61)} \end{aligned}$$

for all t, so x_{v_ν} is the unique solution for the control ν i.e.

$$(1.63) \qquad x_{v_\nu}(t) = x_\nu(t) \qquad \text{for all } t \in [0,1].$$

Similarly, by letting $g(t,\mu) = h(t, x_\nu(t), \mu)$ for $t \in [0,1]$ and $\mu \in \Lambda(M)$, we see that

$$\begin{aligned} J(v_\nu) = & \int_0^1 h({}^\circ\tau, x_{v_\nu}(\tau), v_\nu(\tau))d\tau_L + \bar{h}(x_{v_\nu}(1)) && \text{(by definition)} \\ = & \int_0^1 h({}^\circ\tau, x_{v_\nu}({}^\circ\tau), v_\nu(\tau))d\tau_L + \bar{h}(x_{v_\nu}(1)) && \text{by (1.62)} \\ = & \int_0^1 h(t, x_{v_\nu}(t), \nu(t))dt + \bar{h}(x_{v_\nu}(1)) && \text{by (1.60)} \\ = & \int_0^1 h(t, x_\nu(t), \nu(t))dt + \bar{h}(x_\nu(1)) && \text{by (1.63)} \\ = & J(\nu) && \text{(by definition)} \end{aligned}$$

□

Corollary 1.4.8

$$J_0 = \hat{J}_0 = \hat{J}_0^{\mathcal{S}} = J_0^{\mathcal{R}}.$$

Proof: We already have (Theorem 1.3.12) $J_0 = \hat{J}_0$. By Proposition 1.4.4 and the fact that $\mathcal{V} \subset \mathcal{S}$ we have

$$J_0^{\mathcal{R}} \le \hat{J}_0^{\mathcal{S}} \le \hat{J}_0$$

and by Proposition 1.4.7

$$\hat{J}_0 \leq J_0^{\mathcal{R}}.$$

From this we see that

$$J_0^{\mathcal{R}} = \hat{J}_0 = \hat{J}_0^{\mathcal{S}} = J_0.$$

□

1.5 The existence of ordinary optimal controls

In this section we look at, in the context of the preceding work, the well-known result that if the set $\{f(t,x,a) : a \in M\}$ is convex, for each fixed t and x, then we can find an ordinary control $u \in \mathcal{U}$ which is optimal.

Lemma 1.5.1

Given any relaxed control $\nu \in \mathcal{R}$ if the set $\{f(t,x,a) : a \in M\}$ is convex then there exists a control $u_\nu \in \mathcal{U}$ satisfying

$$x_\nu(t) = x_{u_\nu}(t) \qquad \text{for all } t \in [0,1]$$

and

$$J(\nu) = J(u_\nu).$$

Note: The 'proof' we are about to give involves defining a function $u_\nu : [0,1] \to M$ (see (1.65)) we would like this to be a control function; for this to be true we need u_ν to be measurable and as yet we have not been able to show that this is true. Therefore the following 'proof' relies on a conjecture.

'Proof': Fix $\nu \in \mathcal{R}$. For each $t \in [0,1]$. Fix $t \in [0,1]$ then by (1.22) we have

$$f(t, x(t), \nu(t)) = \int_M f(t, x(t), a) d\nu(t)(a). \tag{1.64}$$

Now, let

$$f(t, x(t), a) = g_t(a) \qquad \text{for each } a \in M$$

then by (1.64) we have

$$f(t,x(t),\nu(t)) = \int_M g_t(a)d\nu(t)(a).$$

Now, by Lemma E.1.1 we can find $a_1,\cdots,a_N \in {}^*M$ such that $M = \{{}^\circ a_1,\cdots,{}^\circ a_N\}$ and by Lemma E.1.2 we can find an internal sequence of disjoint *Borel subsets of ${}^*M$, $A_1,\cdots,A_N$, such that

$$\int_M g_t(a)d\nu(t)(a) = {}^\circ(\sum_{i=1}^{N} {}^*g_t(a_i)\,{}^*\nu(t)(A_i)).$$

Therefore, by transfer of the convexity of the set $\{g_t(a) : a \in M\}$, we have

$$\begin{aligned}\int_M g_t(a)d\nu(t)(a) &= {}^\circ({}^*g_t(\bar{a}_t)) \qquad \text{for some } \bar{a}_t \in {}^*M \\ &= g_t({}^\circ\bar{a}_t) \qquad \text{(since } g_t \text{ is continuous).}\end{aligned}$$

So in terms of f, for each fixed $t \in [0,1]$ we have

$$f(t,x(t),\nu(t)) = f(t,x(t),{}^\circ\bar{a}_t) \qquad \text{for some } \bar{a}_t \in {}^*M.$$

Now, if we define $u_\nu : [0,1] \to M$ by

(1.65) $$u_\nu(t) = {}^\circ\bar{a}_t \qquad \text{for each } t \in [0,1]$$

then we have

$$f(t,x(t),\nu(t)) = f(t,x(t),u_\nu(t)) \qquad \text{for each } t \in [0,1].$$

From this it can be seen that

(1.66) $$x_{u_\nu}(t) = x_\nu(t) \qquad \text{for all } t \in [0,1].$$

Similarly, by considering

(1.67) $$h(t,x_\nu(t),\nu(t)) = \int_M h(t,x_\nu(t),a)d\nu(t)(a),$$

and, for each fixed t letting

$$h(t,x_\nu(t),a) = g_t(a) \qquad \text{for each } a \in M$$

we see that

$$\int_M g_t(a)d\nu(t)(a) = g_t({}^\circ\bar{a}_t) \qquad \text{for some } \bar{a}_t \in {}^*M$$

i.e. in terms of h, by (1.65) we see that this is

$$h(t, x_\nu(t), \nu(t)) = h(t, x_\nu(t), u_\nu(t))$$

and so, by (1.66), since $x_{u_\nu}(t) = x_\nu(t)$ for all $t \in [0,1]$, we see that

$$h(t, x_\nu(t), \nu(t)) = h(t, x_{u_\nu}(t), u_\nu(t)) \qquad \text{for each } t \in [0,1] \; .$$

From this it can be seen that

$$J(u_\nu) = J(\nu).$$

□

Proposition 1.5.2

Given a compact metric space M, if the set

$$\{f(t, x, a) : a \in M\}$$

is convex then there exists a standard ordinary optimal control of the form

$$u : [0,1] \rightarrow M$$

i.e. $u \in \mathcal{U}$ and $J(u) = J_0$.

Proof: By Corollary 1.4.5 there exists a control $\bar{\nu} \in \mathcal{R}$ satisfying

$$J(\bar{\nu}) = J_0 \tag{1.68}$$

and, by Lemma 1.5.1 (**Note:** Lemma 1.5.1) relies on a conjecture), if the set $\{f(t, x, a) : a \in M\}$ is convex then there exists a control $u_{\bar{\nu}} \in \mathcal{U}$ satisfying

$$J(u_{\bar{\nu}}) = J(\bar{\nu}) \tag{1.69}$$

i.e. $u_{\bar{\nu}} \in \mathcal{U}$ and by (1.68) and (1.69)

$$J(u_{\bar{\nu}}) = J_0.$$

□

1.6 General cost functions

In this section we consider what happens if we bring in a more general cost function.

Consider the above situation except that now, the cost $J(u)$ is given by the following equation

$$J(u) = \int_0^1 h(s, x_u(s), u(s))ds + \mu(x_u(s)) \tag{1.70}$$

where h is as before but now, $\bar{h}(x(1))$ has been replaced by a more general function μ, where μ is a continuous real valued function on the space of continuous functions $x : [0,1] \rightarrow \mathbb{R}^d$.

We see that Proposition 1.3.10 still holds with this more general cost function, Proposition 1.3.5 states that when $v = {}^\circ U$, $x_v = {}^\circ X_U$ and so since μ is continuous in the uniform topology we have

$$^\circ({}^*\mu(X_U(\tau))) = \mu({}^\circ X_U(\tau)) = \mu(x_v(\tau))$$

for all $\tau \in {}^*[0,1]$.
Therefore **all** of the results above hold for this more general cost function. This more general cost function will be used in the proof of Proposition 2.5.3.

In the next Chapter we consider what happens if we bring in another controller with his own set of controls. This is known as a two player game.

———— oOo ————

Chapter 2

Differential game theory

Game theory can be regarded as control theory with two or more controllers or players. We, as in the control theory section, consider a dynamical situation governed by a differential equation.

The game is played for some fixed time, say one unit of time, then a payoff is computed, this payoff is usually in the form of an integral of a real valued function.

We shall be looking at two-player zero-sum games, that is games where one player is trying to minimize and the other to maximize the payoff.

A strategy for a player is a rule telling the player what to do next on the basis of what he and the other player have done previously in the game. [For more information on this see [10], [20].]

2.1 Definition of the game G

We consider a general game G being played by two players J_1 and J_2 over the fixed time interval $I = [0, 1]$. At each time $t \in I$, J_1 chooses an element $y(t)$ from a fixed (separable) compact metric space $\mathcal{Y}$, and J_2 chooses an element $z(t)$ from a similar space $\mathcal{Z}$, in such a way that the functions $y : t \to y(t)$ and $z : t \to z(t)$ are measurable. The functions $y(t)$ and $z(t)$ are the controls for the players J_1 and

J_2 respectively.

Notation 2.1.1

$\mathcal{M}_1$ is the set of all measurable functions $y : I \to \mathcal{Y}$, modulo the identification of any two functions equal almost everywhere. This is the class of admissible controls for player J_1.
The class of admissible controls for player J_2, $\mathcal{M}_2$, is defined similarly with $\mathcal{Z}$ replacing $\mathcal{Y}$.

The dynamics of G are determined by the following equation

$$x(t) = x(0) + \int_0^t f(s, x(s), y(s), z(s))ds \tag{2.1}$$

where $x(t) \in \mathbb{R}^d$ and

$$f : I \times \mathbb{R}^d \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}^d$$

is a continuous function such that there exists a constant $\kappa < \infty$ satisfying the following Lipschitz condition

$$|f(t, x_1, y, z) - f(t, x_2, y, z)| \leq \kappa |x_1 - x_2| \tag{2.2}$$

whenever $x_1, x_2 \in \mathbb{R}^d, t \in I$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$; κ is known as a Lipschitz constant for the function f.

□

From (2.2), the Lipschitz condition on f, it can be seen that there is a growth condition on f. To show this we need the following Lemma.

Lemma 2.1.2

Given any $t \in [0, 1]$, $x \in \mathbb{R}^d$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, for each fixed constant $L < \infty$, if $|x| \leq L$ then there exists a constant $R_L < \infty$ satisfying

$$|f(t, x, y, z)| \leq R_L$$

Proof: Fix $L < \infty$. Since f is continuous on the compact set

$$C = [0,1] \times \{x \in \mathbb{R}^d : |x| \leq L\} \times \mathcal{Y} \times \mathcal{Z}$$

the set

$$\{|f(t,x,y,z)| : (t,x,y,z) \in C\}$$

has a greatest element which we shall denote by R_L .

□

We can now show that the Lipschitz condition on f implies the following growth condition on f.

Proposition 2.1.3

There exists $\hat{\kappa} < \infty$ such that

$$|f(t,x,y,z)| \leq \hat{\kappa}(1+|x|) \tag{2.3}$$

whenever $x \in \mathbb{R}^d$, $t \in I$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$; here $\hat{\kappa}$ is known as a growth constant for the function f.

Proof: Let $x_1 = x$ and $x_2 = 0$ in the Lipschitz condition, (2.2), then we have

$$|f(t,x,y,z) - f(t,0,y,z)| \leq \kappa|x|$$

hence

$$\begin{aligned} |f(t,x,y,z)| &\leq |f(t,0,y,z)| + \kappa|x| \\ &\leq R_0 + \kappa|x| \qquad \text{(by Lemma 2.1.2 ; } R_0 < \infty \text{ is a constant)} \\ &\leq \hat{\kappa}(1+|x|) \end{aligned}$$

for some constant $\hat{\kappa} < \infty$. Hence we have a growth condition on f.

□

From here onwards we shall simply take our Lipschitz and Growth constants for f to be a common value $\kappa < \infty$. (Clearly this can be done since we can just take $\max\{\kappa, \hat{\kappa}\}$)

2.1.1 Existence and uniqueness of solutions

We now have a Lipschitz and growth condition on f and so by considering the product space $(\mathcal{Y} \times \mathcal{Z})$ it can be seen from Theorems 1.1.1 and 1.1.2 that for any pair of control functions, $y(t)$ and $z(t)$, corresponding to any given initial state $x(0)$, there is a unique solution to equation (2.1). Throughout this thesis we shall take $x(0) = 0$. (This is not necessary, if $x(0)$ is restricted to some bounded region everything still works.)

Notation 2.1.4

The resulting solution $x(t)$ is called the trajectory corresponding to $(y(t)$,$z(t))$ and is, where necessary, denoted by $x^{y,z}(t)$, to identify which controls the solution corresponds to – however in most cases this will not be necessary as it will be clear from the context which controls are being used and so it will simply be denoted by $x(t)$ to simplify later notation.

2.1.2 Conditions on f

Here we show that, since during this work we will only be interested in

$$f(t, x, y(t), z(t))$$

when $x \in \mathbb{R}^d$ is the solution $x^{y,z}(t)$ corresponding to the controls $y \in \mathcal{M}_1$ and $z \in \mathcal{M}_2$ i.e.

$$f(t, x^{y,z}(t), y(t), z(t)),$$

we can essentially take f to be bounded. To do this we need the following Lemma.

Lemma 2.1.5

The solution $x^{y,z}(t)$ is uniformly bounded for all $t \in [0, 1]$ and all controls $y \in \mathcal{M}_1$ and $z \in \mathcal{M}_2$.

Proof: Let

$$A = \{x^{y,z}(t) : t \in [0,1], \quad y \in \mathcal{M}_1, \quad z \in \mathcal{M}_2\}$$

then,

$$^*A = \{X^{Y,Z}(\tau) : \tau \in {}^*[0,1], \quad Y \in {}^*\mathcal{M}_1, \quad Z \in {}^*\mathcal{M}_2\}$$

By (1.13) we know that $X^{Y,Z}(\tau)$ is finite for all $\tau \in {}^*[0,1]$ i.e. for all $a \in {}^*A$, a is finite therefore,

$$^*A \subseteq [0,N] \qquad \text{for all infinite } N \in {}^*\mathbb{N}$$

so by overflow,

$$^*A \subseteq [0,n] \qquad \text{for some finite } n \in {}^*\mathbb{N}$$

therefore by transfer we have

$$A \subseteq [0,n] \qquad \text{for this } n.$$

This means that $x^{y,z}(t)$ is uniformly bounded for all $t \in [0,1]$, $y \in \mathcal{M}_1$ and $z \in \mathcal{M}_2$ i.e. for some constant $L < \infty$,

$$|x^{y,z}(t)| \leq L$$

for all controls $y \in \mathcal{M}_1$, $z \in \mathcal{M}_2$ and all $t \in [0,1]$.

□

We now show that f is essentially bounded.

Lemma 2.1.6

There exists $R_L < \infty$ such that

$$|f(t, x^{y,z}(t), y(t), z(t))| \leq R_L$$

for each $y \in \mathcal{M}_1$, $z \in \mathcal{M}_2$ and $t \in [0,1]$.

Proof: By Lemma 2.1.5, we know that for $y \in \mathcal{M}_1$, $z \in \mathcal{M}_2$ and $t \in [0,1]$, there exists a constant $L < \infty$ such that

$$(t, x^{y,z}(t), y(t), z(t)) \in [0,1] \times \{x \in \mathbb{R}^d : |x| \leq L\} \times \mathcal{Y} \times \mathcal{Z} = C$$

and so, by Lemma 2.1.2, there exists a constant $R_L < \infty$ such that

$$|f(t, x^{y,z}(t), y(t), z(t))| \leq R_L.$$

□

Therefore the conditions we are assuming on the function

$$f : I \times \mathbb{R}^d \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}^d$$

are, without loss of generality, that there exists a constant $\kappa < \infty$ and a constant $R < \infty$ satisfying

(1) f is continuous

(2) f satisfies the following Lipschitz condition

$$|f(t, x_1, y, z) - f(t, x_2, y, z)| \leq \kappa |x_1 - x_2| \tag{2.4}$$

whenever x_1, $x_2 \in \mathbb{R}^d, t \in I$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.

(3) f is bounded i.e.

$$|f(t, x, y, z)| \leq R \tag{2.5}$$

for each $t \in [0, 1]$, $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ and $x \in \mathbb{R}^d$.

2.1.3 The payoff for the game G

Given a pair of controls for the game G, we assume there is a payoff, $p(y, z)$, which is given by

$$p(y, z) = \int_0^1 h(t, x^{y,z}(t), y(t), z(t))dt + \mu(x^{y,z}(t)) \tag{2.6}$$

where h is a continuous function

$$h : I \times \mathbb{R}^d \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$$

and μ is a continuous real-valued function on the Banach space $[\mathcal{C}(I)]^d$ of continuous functions $x : I \to \mathbb{R}^d$.

Note: Here, if we deal with the functions h and μ in the same way as we did the function f, without loss of generality, h and μ can be assumed to be bounded.

The game is zero-sum so that the objective of player J_1 is to maximize the payoff p, while the objective of player J_2 is to minimize p. At each time t, both players are aware of the complete history of the game as played so far.

2.2 Value in the sense of Elliott and Kalton

In this section we give a brief summary of the idea of value used by Elliott and Kalton ([10]).

2.2.1 Strategies for the game G

Any map $\alpha : \mathcal{M}_2 \to \mathcal{M}_1$ is a **pseudo-strategy** for J_1. It gives a means by which J_1 may determine his own choice of control function given J_2's choice of control function.

Each pseudo-strategy α has a value

$$u(\alpha) = \inf_{z \in \mathcal{M}_2} p(\alpha z, z) \tag{2.7}$$

which gives the worst possible outcome of the game for J_1 if he uses the pseudo-strategy α. Similarly, a pseudo-strategy for J_2 is a map $\beta : \mathcal{M}_1 \to \mathcal{M}_2$ and its value is given by

$$v(\beta) = \sup_{y \in \mathcal{M}_1} p(y, \beta y). \tag{2.8}$$

In practice not all pseudo-strategies are 'reasonable', for they imply foreknowledge of the other players choice of control function so the following definition is made.

Definition 2.2.1

The map $\alpha : \mathcal{M}_2 \to \mathcal{M}_1$ is a **strategy** if whenever $0 < T \leq 1$

$$z_1(t) = z_2(t) \qquad \text{a.e. } 0 \leq t \leq T$$

$$\Longrightarrow \alpha z_1(t) = \alpha z_2(t) \qquad \text{a.e } 0 \leq t \leq T.$$

A strategy for player J_2 is similarly defined.

The set of all strategies for J_1 is denoted by Γ, and the set of strategies for J_2 by Υ.

2.2.2 Value for the game G

The **value** of the game G to J_1 is the best he can force by using a strategy i.e.

$$U = \sup_{\alpha \in \Gamma} u(\alpha) \tag{2.9}$$

while the value to J_2 is

$$V = \inf_{\beta \in \Upsilon} v(\beta). \tag{2.10}$$

If $U = V$ then the game G is said to have value, in the sense that neither player can force a better result for themselves than V, and both players can (almost) force V.

It is not always true that $U = V$, as illustrated by the following classical example due to Berkovitz ([4]) – also cited in Elliott and Kalton ([10]).

Example 2.2.2

Let $\mathcal{Y} = \mathcal{Z} = [-1, 1]$ and suppose the dynamics of G are given by

$$\frac{dx}{dt} = (y - z)^2$$

where $x \in \mathbb{R}$. Let the payoff be given by

$$p(y, z) = \int_0^1 x(t)dt.$$

It is easy to see that a best strategy $\hat{\alpha}$ for J_1 is given by

$$\hat{\alpha} z(t) = \begin{cases} 1 & \text{if } z(t) < 0 \\ -1 & \text{if } z(t) \geq 0 \end{cases}$$

while a best strategy $\hat{\beta}$ for J_2 is given by

$$\hat{\beta}y(t) = y(t).$$

Then

$$\begin{aligned} U &= \sup_{\alpha\in\Gamma} \inf_{z\in\mathcal{M}_2} p(\alpha z, z) \\ &= p(\hat{\alpha}z_0, z_0) \qquad\qquad \text{where } z_0 \equiv 0 \\ &= \int_0^1 t dt \\ &= \frac{1}{2} \end{aligned}$$

and similarly,

$$\begin{aligned} V &= \inf_{\beta\in\Upsilon} \sup_{y\in\mathcal{M}_1} p(y, \beta y) \\ &= 0. \end{aligned}$$

Clearly, for this example $U \neq V$.

□

Note: The above example can easily be changed to make it have value. All that is required is to change $\mathcal{Z}$ from [-1,1] to [0,1], then U and V have the common value of $\frac{1}{2}$.

2.3 Value in the sense of Friedman

In this section we describe an alternative definition of value used by Friedman ([20]) and discussed by Elliott and Kalton ([10]). This involves a game which we shall denote by E_n^+; this game relates very closely to those described by Friedman ([20]). This game will then be adapted in section 2.4 to make a new game which we will denote by $\bar{E}_n^+$.

Then for completeness we give our own proofs to results stated but not proved in [10]. We then go on to give alternative proofs to those included in [10] using nonstandard methods.

Later in our work we need to distinguish between different partitions of [0,1]; to do this we use some notation which we shall introduce here.

Notation 2.3.1

Throughout this work we shall use a function which we define by

$$\Delta_n = 2^{-n} \qquad \text{for each integer } n \in \mathbb{N}.$$

We determine a discrete time line $\mathbf{T}_n$ by

$$\mathbf{T}_n = \{0, \Delta_n, 2\Delta_n, 3\Delta_n, \cdots, 1\} \qquad \text{for each integer } n$$

and given an integer n, we denote the divisions of [0,1] by

$$\begin{aligned} {}^n I_1 &= [0, \Delta_n] \\ {}^n I_j &=](j-1)\Delta_n, j\Delta_n] \end{aligned}$$

for $j = 2, 3, \cdots, 2^n$.

2.3.1 The game E_n^+

Here we give a brief description of the game E_n^+ as in [10], we then go on to show that as n increases, the value of the game decreases.

Definition 2.3.2 (The game E_n^+)

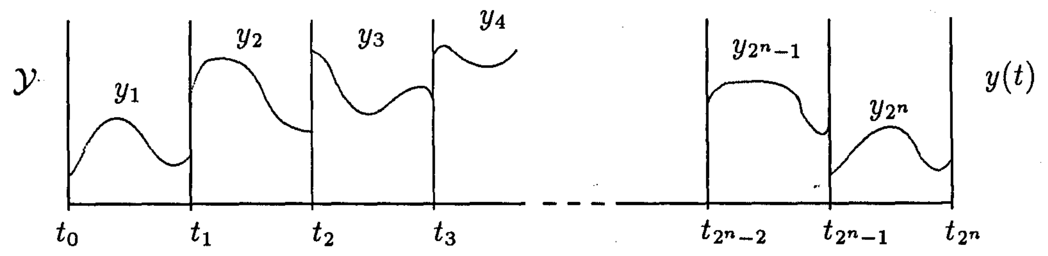
Let n be an integer, then define ${}^n I_j$ as above, i.e.

$$\begin{aligned} {}^n I_1 &= [0, t_1] \\ {}^n I_j &=]t_{j-1}, t_j] \end{aligned}$$

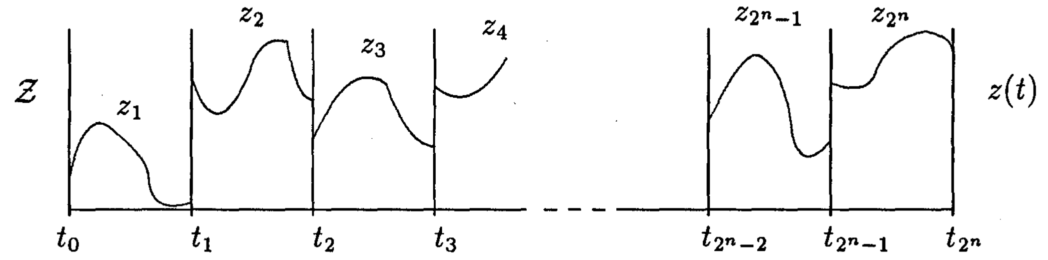
for $j = 2, \cdots, 2^n$ where $t_j = j\Delta_n$.

The game E_n^+ has the same dynamics, initial condition and payoff as G given by equations (2.1) and (2.4)-(2.6) but is played in the following manner: J_2 first selects his control z_1 on ${}^nI_1$ and then J_1 selects his control function y_1 on ${}^nI_1$, and the players play alternately, J_2 selecting his control z_j on ${}^nI_j$ before J_1 selects his control y_j on ${}^nI_j$ at the j^{th} step

i.e. J_1 plays a varying control $y(t)$



and J_2 plays a varying control $z(t)$



Notation 2.3.3

${}^n\mathcal{M}_1^j$ and ${}^n\mathcal{M}_2^j$ denote the spaces of measurable functions

$$y_j : {}^nI_j \to \mathcal{Y} \qquad \text{and} \qquad z_j : {}^nI_j \to \mathcal{Z}$$

respectively, for $j = 1, \cdots, 2^n$ in which as before, two functions which are equal almost everywhere are identified.

Then, ${}^n\mathcal{M}_1$ is the class of controls for J_1 of the form $y = (y_1, \cdots, y_{2^n})$ where for each $j = 1, \cdots, 2^n$, $y_j \in {}^n\mathcal{M}_1^j$

Similarly, ${}^n\mathcal{M}_2$ is the class of controls for J_2 of the form $z = (z_1, \cdots, z_{2^n})$ where for each $j = 1, \cdots, 2^n$, $z_j \in {}^n\mathcal{M}_2^j$.

2.3.2 Strategies for the game E_n^+

Later in our work we find it necessary to distinguish between strategies for different games E_n^+ and E_m^+ with $n \neq m$; because of this we find it necessary to introduce some slightly different notation to that of [10], we simply tag each symbol with an n if it refers to the game E_n^+.

A strategy for J_1 in the game E_n^+ is a collection of maps

$$\Sigma = (\Sigma_1, \cdots, \Sigma_{2^n})$$

where

$$\Sigma_j : {}^n\mathcal{M}_2^1 \times \cdots \times {}^n\mathcal{M}_2^j \rightarrow {}^n\mathcal{M}_1^j$$

for each $j = 1, \cdots, 2^n$. Similarly, a strategy Π for J_2 in E_n^+ is an element z_1 of ${}^n\mathcal{M}_2^1$ together with a collection of maps $(\Pi_2, \cdots, \Pi_{2^n})$ where for each $j = 2, \cdots, 2^n$

$$\Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \rightarrow {}^n\mathcal{M}_2^j.$$

i.e. $\Pi_1 = z_1$ where $z_1 \in {}^n\mathcal{M}_2^1$.

Let Γ^n denote the class of all strategies for player J_1 in the game E_n^+ and similarly, let Υ^n denote the class of all strategies for J_2 in the game E_n^+.

A pair of strategies $\Sigma \in \Gamma^n$ and $\Pi \in \Upsilon^n$ determine rules of procedure for J_1 and J_2 respectively. The game has alternate play and so, if two such rules, Σ and Π, are played against each other in the game E_n^+ then a pair of controls, $y \in {}^n\mathcal{M}_1$ and $z \in {}^n\mathcal{M}_2$ i.e.

$$y = (y_1, y_2, \cdots, y_{2^n}) \quad \text{and} \quad z = (z_1, z_2, \cdots, z_{2^n})$$

are generated and a payoff

$$p_E^n(y, z) = p_E^n(z_1, y_1, \cdots, z_{2^n}, y_{2^n}) = p_E^n(\Sigma, \Pi) \tag{2.11}$$

can be computed, where $\Sigma \in \Gamma^n$ and $\Pi \in \Upsilon^n$ are the strategies used to select the controls y and z respectively.

(We have tagged the payoff symbol with the letter E for the game E_n^+ since later we will need to be able to distinguish between the payoffs for different games.)

2.3.3 Value for the game E_n^+

From the theory of alternate move games we have the following result

$$\inf_{\Pi\in\Upsilon^n}\sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi) = \sup_{\Sigma\in\Gamma^n}\inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi). \tag{2.12}$$

Let this value be denoted by V_n^+. The game E_n^+ has value V_n^+. [For details of this approach see [20].]

Note: Since, by (2.11)

$$p_E^n(\Sigma,\Pi) = p_E^n(y,z)$$

where $y = \Pi z \in {}^n\mathcal{M}_1$ and $z = \Sigma y \in {}^n\mathcal{M}_2$ we have

$$p_E^n(\Sigma,\Pi) = p_E^n(y,\Pi) \quad \text{where } y = \Sigma(\Pi) \in {}^n\mathcal{M}_1 \tag{2.13}$$

and

$$p_E^n(\Sigma,\Pi) = p_E^n(\Sigma,z) \quad \text{where } z = \Pi(\Sigma) \in {}^n\mathcal{M}_2 \tag{2.14}$$

Therefore the value V_n^+ can be written in several ways including

$$\begin{aligned} V_n^+ &= \inf_{\Pi\in\Upsilon^n}\sup_{y\in\mathcal{M}_1} p_E^n(y,\Pi) && \text{by (2.13)} \\ &= \sup_{y\in{}^n\mathcal{M}_1}\inf_{\Pi\in\Upsilon^n} p_E^n(y,\Pi) && \text{by (2.13)} \\ &= \sup_{\Sigma\in\Gamma^n}\inf_{z\in\mathcal{M}_2} p_E^n(\Sigma,z) && \text{by (2.14).} \\ &= \inf_{z\in{}^n\mathcal{M}_2}\sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,z) && \text{by (2.14).} \end{aligned} \tag{2.15}$$

For completeness, we include our own proof of (2.12) in Proposition 2.3.6 below. First we give a Lemma which we then go on to use in the proof of this Proposition.

Lemma 2.3.4

Given a control $y \in {}^n\mathcal{M}_1$ for J_1 and a control $z \in {}^n\mathcal{M}_2$ for J_2 in the game E_n^+ the following two statements hold

(i) $\forall z_1\exists y_1\cdots\forall z_{2^n}\exists y_{2^n} p_E^n(z_1,y_1,\cdots,z_{2^n},y_{2^n})\geq r \Leftrightarrow \exists\Sigma\in\Gamma^n\forall\Pi\in\Upsilon^n p_E^n(\Sigma,\Pi)\geq r$

(ii)$\exists z_1 \forall y_1 \cdots \exists z_{2^n} \forall y_{2^n} p_E^n(z_1, y_1, \cdots, z_{2^n}, y_{2^n}) < r \Leftrightarrow \exists \Pi \in \Upsilon^n \forall \Sigma \in \Gamma^n p_E^n(\Sigma, \Pi) < r$.

Proof: (i) '$\Leftarrow$'

If the right hand side holds then we can find $\Sigma' \in \Gamma^n$ for J_1 which, when played against any $\Pi \in \Upsilon^n$ which J_2 uses, forces the payoff greater or equal to r.

Now, given any $z_1, z_2, \cdots, z_{2^n}$ we can consider the strategy $\Pi_z \in \Upsilon^n$ which tells J_2 to do $z_1, z_2, \cdots, z_{2^n}$ no matter what J_1 does, if we fix $\Sigma = \Sigma'$ and play it against this Π_z we have

$$\forall z_1 \cdots \forall z_{2^n} p_E^n(z_1, {}^n\Sigma'_1(z_1), \cdots, z_{2^n}, {}^n\Sigma'_{2^n}(z_1, \cdots, z_{2^n})) \geq r$$

(since ${}^n\Sigma'$ against any $\Pi \in \Upsilon^n$ makes the payoff greater than or equal to r)

$$\Longrightarrow \forall z_1 \cdots \forall z_{2^n} \exists y_{2^n} p_E^n(z_1, {}^n\Sigma'_1(z_1), \cdots, {}^n\Sigma'_{2^n-1}(z_1, \cdots, z_{2^n-1}), z_{2^n}, y_{2^n}) \geq r$$

(take $y_{2^n} = \Sigma'_{2^n}(z_1, \cdots, z_{2^n})$)

$$\Longrightarrow \forall z_1 \cdots \forall z_{2^n-1} \exists y_{2^n-1} \forall z_{2^n} \exists y_{2^n} p_E^n(z_1, \Sigma'_1(z_1), \cdots, z_{2^n-1}, y_{2^n-1}, z_{2^n}, y_{2^n}) \geq r$$

(take $y_{2^n-1} = \Sigma'_{2^n-1}(z_1, \cdots, z_{2^n-1})$)

We continue in this way replacing $\Sigma'_i(z_1, \cdots, z_i)$ by y_i for all $i = 1, 2, \cdots, 2^n$ until we have

$$\forall z_1 \exists y_1 \forall z_2 \exists y_2 \cdots \forall z_{2^n} \exists y_{2^n} p_E^n(z_1, y_1, z_2, y_2, \cdots, z_{2^n}, y_{2^n}) \geq r.$$

'$\Rightarrow$' Conversely, if the left hand side holds

$$\text{(2.16)} \qquad \forall z_1 \exists y_1 \forall z_2 \exists y_2 \cdots \forall z_{2^n} \exists y_{2^n} p_E^n(z_1, y_1, z_2, y_2, \cdots, z_{2^n}, y_{2^n}) \geq r.$$

Let $\Sigma' \in \Gamma^n$ be the strategy which, at the j^{th} stage, tells J_1 to do a $y_j \in {}^n\mathcal{M}_1^j$ in response to the $(z_1, \cdots, z_j)$ played by J_2, thus making $p_E^n(z_1, y_1, \cdots, z_{2^n}, y_{2^n}) \geq r$. (We know there is one by (2.16))

i.e. let $\Sigma'_j(z_1, \cdots, z_j) = y_j$ for $j = 1, 2, \cdots, 2^n$ as given by (2.16), then we have

$$\text{(2.17)} \qquad \forall z_1 \cdots \forall z_{2^n-1} \forall z_{2^n} p_E^n(z_1, \Sigma'_1(z_1), \cdots, z_{2^n}, \Sigma'_{2^n}(z_1, \cdots, z_{2^n})) \geq r.$$

Now suppose we are given a strategy $\Pi \in \Upsilon^n$ for J_2. If we put $z_1 = \Pi_1$, $\Pi_2(\Sigma'_1(z_1)), \cdots, \Pi_{2^n}(\Sigma'_{2^n-1}(\cdots \Sigma'_1(z_1)) \cdots)$ as $z_1, z_2, \cdots, z_{2^n}$ respectively i.e. for $j = 2, \cdots, 2^n$ put

$$z_j = \Pi_j(\Sigma'_{j-1}(\Pi_{j-1}(\cdots(\Pi_2(\Sigma'_1(z_1)))\cdots)))$$

(recall Π_1 is just a z_1) then by (2.17) we have

$$\forall\Pi_1,\cdots,\forall\Pi_{2^n}p_E^n(\Pi_1,\Sigma_1'(\Pi_1),\Pi_2(\Sigma_1'(\Pi_1)),\cdots,\Sigma_{2^n}'(\Pi_{2^n}(\cdots)))\geq r$$

$$\text{i.e.}\quad \forall\Pi\in\Upsilon^n p_E^n(\Sigma',\Pi)\geq r$$

and so we see that

$$\text{i.e.}\exists\Sigma\in\Gamma^n\forall\Pi\in\Upsilon^n p_E^n(\Sigma,\Pi)\geq r.$$

Proof: (ii) Similar.

□

From Lemma 2.3.4 above, we obtain the following Corollary.

Corollary 2.3.5

Given $r\in\mathbb{R}$, either (a) or (b) below holds

(a) $\exists\Sigma\in\Gamma^n\forall\Pi\in\Upsilon^n p_E^n(\Sigma,\Pi)\geq r$

(b) $\exists\Pi\in\Upsilon^n\forall\Sigma\in\Gamma^n p_E^n(\Sigma,\Pi)< r$

Proof: By Lemma 2.3.4 we see that

(i)$\forall z_1\exists y_1\cdots\forall z_{2^n}\forall y_{2^n}p_E^n(z_1,y_1,\cdots,z_{2^n},y_{2^n})\geq r\Leftrightarrow\exists\Sigma\in\Gamma^n\forall\Pi\in\Upsilon^n p_E^n(\Sigma,\Pi)\geq r$

and

(ii)$\exists z_1\forall y_1\cdots\exists z_{2^n}\forall y_{2^n}p_E^n(z_1,y_1,\cdots,z_{2^n},y_{2^n})<r\Leftrightarrow\exists\Pi\in\Upsilon^n\forall\Sigma\in\Gamma^n p_E^n(\Sigma,\Pi)<r$

therefore, since

$$\neg(\forall z_1\exists y_1\cdots\forall z_{2^n}\exists y_{2^n}p_E^n(z_1,y_1,\cdots,z_{2^n},y_{2^n})\geq r)$$

$$=\exists z_1\forall y_1\cdots\exists z_{2^n}\forall y_{2^n}p_E^n(z_1,y_1,\cdots,z_{2^n},y_{2^n})<r$$

we see that by (i) and (ii), if (a) does not hold then (b) must hold.

□

We now go on to prove (2.12) using Corollary 2.3.5 above.

Proposition 2.3.6

For strategies $\Sigma \in \Gamma^n$ and $\Pi \in \Upsilon^n$ for J_1 and J_2 in the game E_n^+

$$\inf_{\Pi\in\Upsilon^n} \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi) = \sup_{\Sigma\in\Gamma^n} \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi).$$

Proof: Fix $\Pi = \Pi' \in \Upsilon^n$, then for each fixed $\Sigma' \in \Gamma^n$ we have

$$p_E^n(\Sigma',\Pi') \leq \sup_{\Sigma} p_E^n(\Sigma,\Pi')$$

$$\Longrightarrow \inf_{\Pi} p_E^n(\Sigma',\Pi) \leq \inf_{\Pi} \sup_{\Sigma} p_E^n(\Sigma,\Pi) \qquad \text{for each } \Sigma'$$

(2.18)
$$\Longrightarrow \sup_{\Sigma} \inf_{\Pi} p_E^n(\Sigma,\Pi) \leq \inf_{\Pi} \sup_{\Sigma} p_E^n(\Sigma,\Pi).$$

Conversely, suppose that

(2.19)
$$\sup_{\Sigma\in\Gamma^n} \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi) < \inf_{\Pi\in\Upsilon^n} \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi)$$

then there exists $r \in \mathbb{R}$ such that

(2.20)
$$\sup_{\Sigma\in\Gamma^n} \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi) < r < \inf_{\Pi\in\Upsilon^n} \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi).$$

Now , by Corollary 2.3.5 we see that either (a) or (b) below holds

(a) $\exists\Sigma \in \Gamma^n \forall\Pi \in \Upsilon^n p_E^n(\Sigma,\Pi) \geq r$

(b) $\exists\Pi \in \Upsilon^n \forall\Sigma \in \Gamma^n p_E^n(\Sigma,\Pi) < r$

Now, if (a) holds then we have

$$\exists\Sigma \in \Gamma^n : \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi) \geq r$$

so we have

$$\sup_{\Sigma\in\Gamma^n} \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi) \geq r$$

which contradicts (2.20), therefore (b) must hold, but if (b) holds we see that

$$\exists\Pi \in \Upsilon^n : \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi) \leq r$$

so we have

$$\inf_{\Pi\in\Upsilon^n} \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi) \leq r$$

which also contradicts (2.20), therefore (2.19) cannot hold and so by (2.18) we have

$$\inf_{\Pi\in\Upsilon^n} \sup_{\Sigma\in\Gamma^n} p_E^n(\Sigma,\Pi) = \sup_{\Sigma\in\Gamma^n} \inf_{\Pi\in\Upsilon^n} p_E^n(\Sigma,\Pi).$$

□

2.3.4 Friedman's upper and lower values

In [10] it is stated without proof that $V_n^+ \geq V_{n+1}^+$ for all n, i.e. when the length of the intervals ${}^nI_j$ are decreased, the value of the game decreases. For completeness, we include our own proof of this result. (For an alternative proof leading to this result we refer the reader to [20].)

Theorem 2.3.7

$V_n^+ \geq V_{n+1}^+$ for all integers n.

Proof: A play of the game E_n^+ with J_1 using a strategy $\Sigma \in \Gamma^n$ and J_2 using a strategy $\Pi \in \Upsilon^n$ generates a pair of controls $y \in {}^n\mathcal{M}_1$ and $z \in {}^n\mathcal{M}_2$ i.e.

$$y = (y_1, y_2, \cdots, y_{2^n}) \quad \text{and} \quad z = (z_1, z_2, \cdots, z_{2^n})$$

where $y_j : {}^nI_j \to \mathcal{Y}$ and $z_j : {}^nI_j \to \mathcal{Z}$ for each $j = 1, \cdots, 2^n$. Corresponding to this pair of controls and strategies there is a payoff $p_E^n(y,z) = p_E^n(z_1, y_1, \cdots, z_{2^n}, y_{2^n}) = p_E^n(\Sigma, \Pi)$. Note,

$$p_E^n(z_1, y_1, z_2, y_2, \cdots, z_{2^n}, y_{2^n}) = p_E^n((y_1 y_2 \cdots y_{2^n}), (z_1 z_2 \cdots z_{2^n})).$$

By the definition of V_n^+, equation (2.12), given any $\epsilon > 0$,

$$\exists \Pi \in \Upsilon^n \forall \Sigma \in \Gamma^n p_E^n(\Sigma, \Pi) < V_n^+ + \epsilon$$

therefore, by Lemma 2.3.4 (ii) we have

$$\exists z_1 \forall y_1 \exists z_2 \forall y_2 \cdots \exists z_{2^n} y_{2^n} (p_E^n(z_1, y_1, \cdots, z_{2^n}, y_{2^n}) < V_n^+ + \epsilon)$$

(2.21)

$$\Longrightarrow \exists z_1' z_1'' \forall y_1 \exists z_2' z_2'' \forall y_2 \cdots \exists z_{2^n}' z_{2^n}'' \forall y_{2^n} (p_E^n(z_1' z_1'', y_1, \cdots, z_{2^n}' z_{2^n}'', y_{2^n}) < V_n^+ + \epsilon)$$

(since for each $j = 1, \cdots, 2^n$, we can split z_j into z_j' and z_j'' over the two halves of ${}^nI_j$ i.e. z_j' acts on ${}^{n+1}I_{2j-1}$ and z_j'' acts on ${}^{n+1}I_{2j}$)

$$\Longrightarrow \exists z_1' \exists z_1'' \forall y_1' \forall y_1'' \cdots \exists z_{2^n}' \exists z_{2^n}'' \forall y_{2^n}' \forall y_{2^n}'' (p_E^{n+1}(z_1', z_1'', y_1', y_1'', \cdots, y_{2^n}', y_{2^n}'') < V_n^+ + \epsilon)$$

(since, for each $j = 1, \cdots, 2^n$, any pair y_j' on ${}^{n+1}I_{2j-1}$ and y_j'' on ${}^{n+1}I_{2j}$ is simply a y_j on ${}^{n}I_j$ and by (2.21) it holds for all such y_j)

(2.22)

$$\Longrightarrow \exists z_1' \forall y_1' \exists z_1'' \forall y_1'' \cdots \exists z_{2^n}'' \forall y_{2^n}'' p_E^{n+1}(z_1', y_1', z_1'', y_1'', \cdots, z_{2^n}'', y_{2^n}'') < V_n^+ + \epsilon.$$

Therefore, by letting

$$(\bar{y}_1, \bar{y}_2, \bar{y}_3, \cdots) = (y_1', y_1'', y_2', \cdots) \quad \text{and} \quad (\bar{z}_1, \bar{z}_2, \bar{z}_3, \cdots) = (z_1', z_1'', z_2', \cdots)$$

we have $\bar{y}_j \in {}^{n+1}\mathcal{M}_1^j$ and $\bar{z}_j \in {}^{n+1}\mathcal{M}_2^j$, i.e. $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_{2^{n+1}}) \in {}^{n+1}\mathcal{M}_1$ and $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_{2^{n+1}}) \in {}^{n+1}\mathcal{M}_2$. Therefore, $\bar{y}$ and $\bar{z}$ are controls for the game E_{n+1}^+ and by (2.22), we have

$$\exists \bar{z}_1 \forall \bar{y}_1 \exists \bar{z}_2 \forall \bar{y}_2 \cdots \bar{z}_{2^{n+1}} \forall \bar{y}_{2^{n+1}} p_E^{n+1}(\bar{z}_1, \bar{y}_2, \cdots, \bar{z}_{2^{n+1}}, \bar{y}_{2^{n+1}}) < V_n^+ + \epsilon$$

so we have, by Lemma 2.3.4 (ii)

$$\begin{aligned} \exists \Pi \in \Upsilon^{n+1} \forall \Sigma \in \Gamma^{n+1} p_E^{n+1}(\Sigma, \Pi) &< V_n^+ + \epsilon \\ \Rightarrow \exists \Pi \in \Upsilon^{n+1} : \sup_{\Sigma \in \Gamma^{n+1}} p_E^{n+1}(\Sigma, \Pi) &\leq V_n^+ + \epsilon \\ \Rightarrow \inf_{\Pi \in \Upsilon^{n+1}} \sup_{\Sigma \Gamma^{n+1}} p_E^{n+1}(\Sigma, \Pi) &\leq V_n^+ + \epsilon \end{aligned}$$

i.e.

$$V_{n+1}^+ \leq V_n^+.$$

□

We now define Friedman's upper and lower values.

Definition 2.3.8

Since V_n^+ decreases with increase in n, we can take the limit

(2.23)
$$V^+ = \lim_{n \to \infty} V_n^+.$$

V^+ is the **upper value** of G in the sense of Friedman ([20]).

The game E_n^- is defined to be the game played in the same way as E_n^+ except that at each stage J_1 plays first. By similar methods to those above, it can be shown that E_n^- has a value V_n^- with

$$V_n^- \leq V_{n+1}^- \qquad \text{for all integers } n$$

and so we may take the limit. The **lower value** of G in the sense of Friedman, is defined by

$$V^- = \lim_{n\to\infty} V_n^-.$$

We immediately have that $V_n^- \leq V_n^+$ for all n and so $V^- \leq V^+$.

Definition 2.3.9

G is said to have value in the sense of Friedman ([20]) if $V^- = V^+$.

2.4 Friedman's upper and lower values in terms of pseudo-strategies

Elliott and Kalton ([10]) give a re-interpretation of the Friedman values $V^-$and V^+ in terms of pseudo-strategies. Since some of the main results we go on to give proofs of rely upon this work we find it necessary to include a summary of this section of their work for the readers convenience.

2.4.1 Pseudo-strategies and reaction times

Definition 2.4.1

For $-1 \leq s \leq 1$, $\Gamma(s)$ is defined to be the set of pseudo-strategies α for J_1 such that for all $T > 0$

$$z_1(t) = z_2(t) \qquad \text{a.e. } 0 \leq t \leq T$$

$$\Longrightarrow \qquad \alpha z_1(t) = \alpha z_2(t) \qquad\qquad \text{a.e. } 0 \leq t \leq \min(T+s, 1).$$

$\Upsilon(s)$ for J_2 is defined similarly. Thus $\Gamma(s)$ is the set of strategies available to J_1 if he has reaction time s (which may be negative, in which case he is anticipating his opponents play).

Definition 2.4.2

Let

$$U(s) = \sup_{\alpha \in \Gamma(s)} u(\alpha) \tag{2.24}$$

$$V(s) = \inf_{\beta \in \Upsilon(s)} v(\beta)\ , \tag{2.25}$$

then, since U and V are monotone functions (U decreases and V increases with increase in s.) the following definitions are made

$$U^+(s) = \lim_{t \downarrow s} U(t) \tag{2.26}$$

$$U^-(s) = \lim_{t \uparrow s} U(t) \tag{2.27}$$

$$V^+(s) = \lim_{t \downarrow s} V(t) \tag{2.28}$$

$$V^-(s) = \lim_{t \uparrow s} V(t). \tag{2.29}$$

It is from here onwards that nonstandard methods come into use.

2.4.2 The game $\bar{E}_n^+$

Elliott and Kalton ([10]) show that $V^+(0) = V^+$ and $V^-(0) = V^-$, later we shall give a proof of this using nonstandard methods. First we use ideas related to those of Elliott and Kalton ([10]) to introduce a new game which we denote by $\bar{E}_n^+$. We shall then use this game to show the above result.

Definition 2.4.3 (The game $\bar{E}_n^+$)

The game $\bar{E}_n^+$ has the same dynamics, initial condition and payoff as the game E_n^+. The game is played in the following manner: J_2 first selects his control z_1 on ${}^nI_1$ then J_1 selects his control y_1 on ${}^nI_1$, the players continue playing alternately, with J_2 choosing his control z_j on ${}^nI_j$ at the j^{th} stage before J_1 chooses his control y_j on ${}^nI_j$, for $j = 1, \cdots, 2^n$.

The difference between this game and the game E_n^+ is that in this game the choice of controls available to player J_2 is restricted.

2.4.3 Controls for the game $\bar{E}_n^+$

In the game $\bar{E}_n^+$, player J_1 is still using the class of controls ${}^n\mathcal{M}_1$ i.e. measurable functions of the form $y : [0,1] \to \mathcal{Y}$ with $y = (y_1, \cdots, y_{2^n})$ where $y_j \in {}^n\mathcal{M}_1^j$ for each $j = 1, \cdots, 2^n$ and ${}^n\mathcal{M}_1^j$ is the space of all measurable functions of the form $y_j : {}^nI_j \to \mathcal{Y}$.

Player J_2 however, is restricted to those controls $z \in {}^n\mathcal{M}_2$ which satisfy

$$z(t) = c \qquad \text{if } t \in]t_{j-1}, t_{j-1} + (\Delta_n)^2] \quad \text{for } j = 1, \cdots, 2^n$$

where c is a member of $\mathcal{Z}$ which is fixed throughout the game.

We denote this restricted class of controls by ${}^n\bar{\mathcal{M}}_2$ and observe that ${}^n\bar{\mathcal{M}}_2 \subset {}^n\mathcal{M}_2$. Similarly, for each $j = 1, \cdots, 2^n$, we define ${}^n\bar{\mathcal{M}}_2^j$ to be the class of all controls $z_j \in {}^n\mathcal{M}_2^j$ which satisfy

$$z_j(t) = c \qquad \text{if } t \in]t_{j-1}, t_{j-1} + (\Delta_n)^2].$$

Note: Given a pair of controls $y \in {}^n\mathcal{M}_1$ and $z \in {}^n\mathcal{M}_2$ for the game E_n^+, there is a naturally corresponding pair of controls $y \in {}^n\mathcal{M}_1$ and $\bar{z} \in {}^n\bar{\mathcal{M}}_2$ for the game $\bar{E}_n^+$, y is the same control in both games and the control $\bar{z}$ for the game $\bar{E}_n^+$ is given by

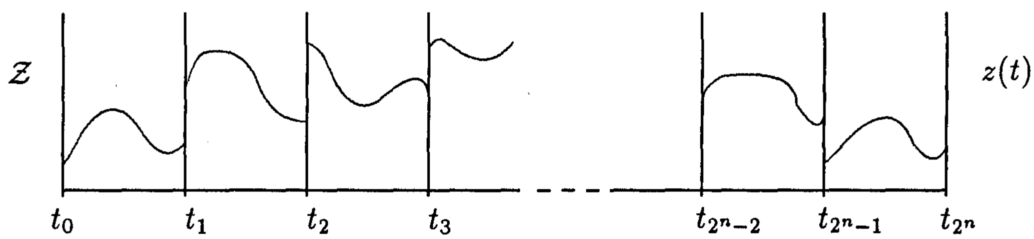
$$\bar{z}(t) = \begin{cases} c & \text{if } t \in]t_{j-1}, t_{j-1} + (\Delta_n)^2] \\ z(t) & \text{if } t \in]t_{j-1} + (\Delta_n)^2, t_j] \end{cases} \tag{2.30}$$

where z is the control in the game E_n^+.

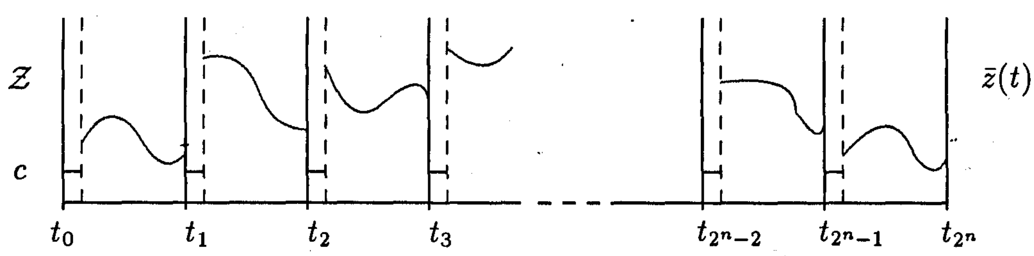
Note: We need the closed interval in the definition for the case $j = 1$ so as to get

$$\bar{z}(t) = \begin{cases} c & \text{if } t \in [0, (\Delta_n)^2] \\ z(t) & \text{if } t \in](\Delta_n)^2, t_1] \end{cases}$$

i.e. when $z(t)$ for the game E_n^+ is given by



$\bar{z}(t)$ for the game $\bar{E}_n^+$ is given by



2.4.4 Strategies for the game $\bar{E}_n^+$

A strategy for J_1 in the game $\bar{E}_n^+$ is a collection of maps

$$\Sigma = (\Sigma_1, \cdots, \Sigma_{2^n})$$

where for each $j = 1, \cdots, 2^n$

$$\Sigma_j : {}^n\bar{\mathcal{M}}_2^1 \times \cdots \times {}^n\bar{\mathcal{M}}_2^j \to {}^n\mathcal{M}_1^j.$$

Similarly, a strategy for J_2 in the game $\bar{E}_n^+$ is a member of ${}^n\bar{\mathcal{M}}_2^1$ together with a collection of maps $(\Pi_2, \cdots, \Pi_{2^n})$ where for each $j = 2, \cdots, 2^n$

$$\Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\bar{\mathcal{M}}_2^j.$$

Let $\bar{\Gamma}^n$ denote the class of all strategies for J_1 in the game $\bar{E}_n^+$ and similarly, let $\bar{\Upsilon}^n$ denote the class of all strategies for J_2 in the game $\bar{E}_n^+$.

2.4.5 Value for the game $\bar{E}_n^+$

Playing a pair of strategies $\Sigma \in \bar{\Gamma}^n$ and $\Pi \in \bar{\Upsilon}^n$ in the game $\bar{E}_n^+$ generates a pair of controls $y \in {}^n\mathcal{M}_1$ and $z \in {}^n\bar{\mathcal{M}}_2$. A payoff can be computed

$$p_E^n(y, z) = p_E^n(\Sigma, \Pi)$$

where $y = \Sigma z$ and $z = \Pi y$.

The game $\bar{E}_n^+$ has value given by

$$\bar{V}_n^+ = \inf_{\Pi \in \bar{\Upsilon}^n} \sup_{\Sigma \in \bar{\Gamma}^n} p_E^n(\Sigma, \Pi) = \sup_{\Sigma \in \bar{\Gamma}^n} \inf_{\Pi \in \bar{\Upsilon}^n} p_E^n(\Sigma, \Pi). \tag{2.31}$$

Note: Given a strategy $\Pi \in \Upsilon^n$ for J_2 in the game E_n^+, there is a naturally corresponding strategy $\bar{\Pi} \in \bar{\Upsilon}^n$ for J_2 in the game $\bar{E}_n^+$. Here, $\bar{\Pi}$ consists of $(\bar{\Pi}_1, \cdots, \bar{\Pi}_{2^n})$ where for each $j = 1, \cdots, 2^n$

$$\bar{\Pi}_j(y_1, \cdots, y_{j-1}) = \begin{cases} c & \text{if } t \in]t_{j-1}, t_{j-1} + (\Delta_n)^2] \\ \Pi_j(y_1, \cdots, y_{j-1}) & \text{if } t \in]t_{j-1} + (\Delta_n)^2, t_j] \end{cases} \tag{2.32}$$

i.e. the strategy $\bar{\Pi} \in \bar{\Upsilon}^n$ corresponding to the strategy $\Pi \in \Upsilon^n$ tells J_2 to respond to a control $y \in {}^n\mathcal{M}_1$ in the game $\bar{E}_n^+$ in exactly the same way as he would by using Π in the game E_n^+ except that for $j = 1, \cdots, 2^n$, he must play the constant c for a short time at the beginning of each interval ${}^nI_j$.

Note: If $\bar{\Pi} \in \bar{\Upsilon}^n$ corresponds to $\Pi \in \Upsilon^n$ in the sense of (2.32), then for each $y \in {}^n\mathcal{M}$

$$\bar{\Pi}y = \overline{\Pi y} \in {}^n\bar{\mathcal{M}}_2. \tag{2.33}$$

Note:

$$\bar{\Upsilon}^n \subset \Upsilon^n \qquad \text{for each integer } n \tag{2.34}$$

$$\text{since if } \Pi \in \Upsilon^n \text{ then} \qquad \Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\mathcal{M}_2^j$$

$$\text{and if } \Pi \in \bar{\Upsilon}^n \text{ then} \qquad \Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\bar{\mathcal{M}}_2^j$$

and we know that ${}^n\bar{\mathcal{M}}_2^j \subset {}^n\mathcal{M}_2^j$ for each $j = 1, \cdots, 2^n$.

Therefore we have the following result

Lemma 2.4.4

$$V_n^+ \le \bar{V}_n^+. \tag{2.35}$$

Proof: By (2.15) V_n^+ can be written as

$$V_n^+ = \inf_{\Pi \in \Upsilon^n} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi y)$$

and $\bar{V}_n^+$ can be written as

$$\bar{V}_n^+ = \inf_{\Pi \in \bar{\Upsilon}^n} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi y)$$

so it follows immediately from (2.34) that

$$V_n^+ \le \bar{V}_n^+.$$

□

Note: We could similarly define a game $\bar{E}_n^-$ based on the game E_n^-.

2.5 The game E_n^+ compared to the game $\bar{E}_n^+$

In this section we compare the two games E_n^+ and $\bar{E}_n^+$. We show that if we use internal controls and let $N \in {}^*\mathbb{N}$ be infinite then the value in E_N^+ is infinitely close to the value in $\bar{E}_N^+$.

First we show that for a fixed pair of controls, the trajectories and payoffs in the two games are infinitely close, in the sense of the uniform topology, when N is infinite.

To do this we introduce some notation.

Notation 2.5.1

Recall, for a fixed integer n, the dynamics for the game E_n^+ and $\bar{E}_n^+$ are given by

$$x^{y,z}(t) = x(0) + \int_0^t f(s, x^{y,z}(s), y(s), z(s))ds.$$

For a fixed pair of controls $y \in {}^n\mathcal{M}_1$ and $z \in {}^n\mathcal{M}_2$, let $\bar{z} \in {}^n\bar{\mathcal{M}}_2$ be as given by (2.30) then, we denote the solution corresponding to y and z in E_n^+ by $x^{y,z}(t)$ and the solution in $\bar{E}_n^+$ corresponding to y and $\bar{z}$ by $x^{y,\bar{z}}(t)$.

Using the above notation in the nonstandard setting we have the following result.

Proposition 2.5.2

For a fixed infinite N, given a fixed pair of controls $Y \in {}^{N*}\mathcal{M}_1$ and $Z \in {}^{N*}\mathcal{M}_2$ i.e.

(2.36) $$Y : {}^*[0,1] \to {}^*\mathcal{Y} \quad \text{and} \quad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

for each initial state, $x(0)$, $X^{Y,Z} \approx X^{Y,\bar{Z}}$ in the sense of the uniform topology, i.e.

$$\sup_{\tau \in {}^*[0,1]} |X^{Y,Z}(\tau) - X^{Y,\bar{Z}}(\tau)| \approx 0.$$

Proof: Let $U : {}^*[0,1] \to {}^*M$ be given by

$$(2.37) \qquad U(\tau) = (Y(\tau), Z(\tau)) \qquad \text{for all } \tau \in {}^*[0,1]$$

where ${}^*M = {}^*(\mathcal{Y} \times \mathcal{Z})$, then ${}^\circ U : {}^*[0,1] \to M$ is a Loeb measurable control in $\mathcal{V}$. Similarly, let $\bar{U} : {}^*[0,1] \to {}^*M$ be given by

$$(2.38) \qquad \bar{U}(\tau) = (Y(\tau), \bar{Z}(\tau)) \qquad \text{for all } \tau \in {}^*[0,1]$$

where $\bar{Z}$ corresponds to Z in the sense of (2.30), then ${}^\circ\bar{U} : {}^*[0,1] \to M$ is a Loeb measurable control in $\mathcal{V}$.

Now, $U(\tau) = \bar{U}(\tau)$ a.a. $\tau \in {}^*[0,1]$ since the controls only differ on the intervals $]t_{j-1}, t_{j-1} + (\Delta_N)^2]$ for $j = 1, \cdots, 2^N$ therefore, if we let $v : {}^*[0,1] \to M$ be given by $v = {}^\circ U$ then

$$(2.39) \qquad v(\tau) = {}^\circ U(\tau) = {}^\circ\bar{U}(\tau) \quad \text{for a.a. } \tau \in {}^*[0,1]$$

From (2.39) we see, by Theorem 1.3.7 and Corollary 1.3.8, that

$$x_v(\tau) = {}^\circ X_U(\tau) = {}^\circ X_{\bar{U}}(\tau) \quad \text{for all } \tau \in {}^*[0,1]$$

i.e.

$${}^\circ X^{Y,Z}(\tau) = {}^\circ X^{Y,\bar{Z}}(\tau) \quad \text{for all } \tau \in {}^*[0,1].$$

□

We now show that the payoffs in the games are infinitely close when N is infinite.

Using the above notation in the nonstandard setting we have the following result.

Proposition 2.5.3

For each fixed infinite $N \in {}^*\mathbb{N}$ and fixed pair of controls $Y \in {}^{N*}\mathcal{M}_1$ and $Z \in {}^{N*}\mathcal{M}_2$

$$(2.40) \qquad P_E^N(Y,Z) \approx P_E^N(Y,\bar{Z}).$$

Proof: Let $U \in {}^*\mathcal{U}$ and $\bar{U} \in {}^*\mathcal{U}$ be given by (2.37) and (2.38) then, by (2.39) and Proposition 1.3.10 with the generalised cost function (see section 1.6) we have

$$J(v) = {}^\circ({}^*J(U)) = {}^\circ({}^*J(\bar{U}))$$

i.e.

$$P_E^N(Y, Z) \approx P_E^N(Y, \bar{Z}).$$

□

Corollary 2.5.4

For a fixed infinite $N \in {}^*\mathbb{N}$, given any strategy $\Pi \in \Upsilon^N$ let $\bar{\Pi} \in \bar{\Upsilon}^N$ be as given by (2.32), then

$$P_E^N(Y, \Pi Y) \approx P_E^N(Y, \bar{\Pi} Y)$$

for all $Y \in {}^{N\,*}\mathcal{M}_1$.

Proof: Fix $Y \in {}^{N\,*}\mathcal{M}_1$. Given $\Pi \in \Upsilon^N$

$$P_E^N(Y, \Pi Y) \approx P_E^N(Y, \overline{\Pi Y}) \qquad \text{by (2.40)}$$

and if $\bar{\Pi} \in \bar{\Upsilon}^N$ is related to $\Pi \in \Upsilon^N$ in the sense of (2.32) then

$$P_E^N(Y, \overline{\Pi Y}) = P_E^N(Y, \bar{\Pi} Y) \qquad \text{by (2.33)}$$

i.e.

$$P_E^N(Y, \Pi Y) \approx P_E^N(Y, \bar{\Pi} Y).$$

□

We have shown that the trajectories and payoffs in the two games are infinitely close when N is infinite, so we can now go on to show that the values are close.

Theorem 2.5.5

For all infinite $N \in {}^*\mathbb{N}$

$$\bar{V}_N^+ \approx V_N^+.$$

Proof: Fix $N \in {}^*\mathbb{N}$ infinite and fix $Y \in {}^{N*}\mathcal{M}_1$. By (2.34), we know that

$$\inf_{\Pi \in \bar{\Upsilon}^N} P_E^N(Y, \Pi Y) \geq \inf_{\Pi \in \Upsilon^N} P_E^N(Y, \Pi Y)$$

now suppose there exists $0 < r \in \mathbb{R}$ such that

$$\inf_{\Pi \in \bar{\Upsilon}^N} P_E^N(Y, \Pi Y) \geq \inf_{\Pi \in \Upsilon^N} P_E^N(Y, \Pi Y) + r \tag{2.41}$$

then, by the definition of infimum

$$\exists (\Pi' \in \Upsilon^N) : P_E^N(Y, \Pi' Y) < \inf_{\Pi \in \Upsilon^N} P_E^N(Y, \Pi Y) + \frac{r}{2}$$

and by Corollary 2.5.4

$$\exists (\bar{\Pi}' \in \bar{\Upsilon}^N) : P_E^N(Y, \bar{\Pi}' Y) \approx P_E^N(Y, \Pi' Y)$$

Now, by (2.41),

$$|P_E^N(Y, \bar{\Pi}' Y) - P_E^N(Y, \Pi' Y)| \geq \frac{r}{2}$$

which is a contradiction, therefore (2.41) cannot hold and so we have

$$\inf_{\Pi \in \bar{\Upsilon}^N} P_E^N(Y, \Pi Y) \approx \inf_{\Pi \in \Upsilon^N} P_E^N(Y, \Pi Y).$$

And so , since the operation sup preserves the infinite closeness (see Lemma D.1.1) we have

$$\sup_{Y \in {}^{N*}\mathcal{M}_1} \inf_{\Pi \in \bar{\Upsilon}^N} P_E^N(Y, \Pi Y) \approx \sup_{Y \in {}^{N*}\mathcal{M}_1} \inf_{\Pi \in \Upsilon^N} P_E^N(Y, \Pi Y).$$

i.e.

$$\bar{V}_N^+ \approx V_N^+.$$

□

Note: By comparing the games $\bar{E}_n^-$ and E_n^- it could be seen that

$$\bar{V}_N^- \approx V_N^-.$$

2.5.1 The connection between $V^+(0)$ and V^+

Here we give a nonstandard proof of $V^+(0) = V^+$. To do this we use the following two propositions which build on ideas of Elliott and Kalton ([10]).

Proposition 2.5.6

For each fixed integer n

$$V_n^+ \leq V(\Delta_n).$$

Proof: We first show that each $\beta \in \Upsilon(\Delta_n)$ gives a strategy $\Pi_\beta \in \Upsilon^n$ such that $p_E^n(y, \Pi_\beta y) = p_E^n(y, \beta y)$ for all $y \in {}^n\mathcal{M}_1$.

Given $\beta \in \Upsilon(\Delta_n)$ we define Π_β by

$$\Pi_\beta y = \beta y \qquad \text{for each } y \in {}^n\mathcal{M}_1.$$

We now have to show that Π_β is in Υ^n.

We have $\beta \in \Upsilon(\Delta_n)$ therefore, by Definition 2.4.1, we have

$$y_1(t) = y_2(t) \qquad \text{a.e. } 0 \leq t \leq T, \text{ where } T > 0$$

$$\Longrightarrow \quad \beta y_1(t) = \beta y_2(t) \qquad \text{a.e. } 0 \leq t \leq \min(T + \Delta_n, 1)$$

i.e.

$$\Pi_\beta y_1(t) = \Pi_\beta y_2(t) \qquad \text{a.e. } 0 \leq t \leq \min(T + \Delta_n, 1).$$

This means that for each $j = 1, \cdots, 2^n$ if J_2 knows what J_1 has done on all of ${}^nI_j$ i.e. up to time t_j, then he knows what $z_{j+1} = \Pi_\beta(y_1, \cdots, y_j)$ to play on the interval ${}^nI_{j+1}$, therefore $\Pi_\beta \in \Upsilon^n$.

Therefore we have shown that for each $\beta \in \Upsilon(\Delta_n)$ there is a corresponding $\Pi_\beta \in \Upsilon^n$ such that

$$p_E^n(y, \beta y) = p_E^n(y, \Pi_\beta y) \qquad \text{for each fixed } y \in {}^n\mathcal{M}_1$$

and so for each $\beta \in \Upsilon(\Delta_n)$ there is a corresponding $\Pi_\beta \in \Upsilon^n$such that

$$\sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \beta y) = \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi_\beta y).$$

Now, there are more Π's in Υ^n than those which correspond to a $\beta \in \Upsilon(\Delta_n)$ so we have

$$\inf_{\Pi \in \Upsilon^n} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi y) \leq \inf_{\beta \in \Upsilon(\Delta_n)} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \beta y)$$

i.e.

$$V_n^+ \leq V(\Delta_n).$$

□

Proposition 2.5.7

For each fixed integer n

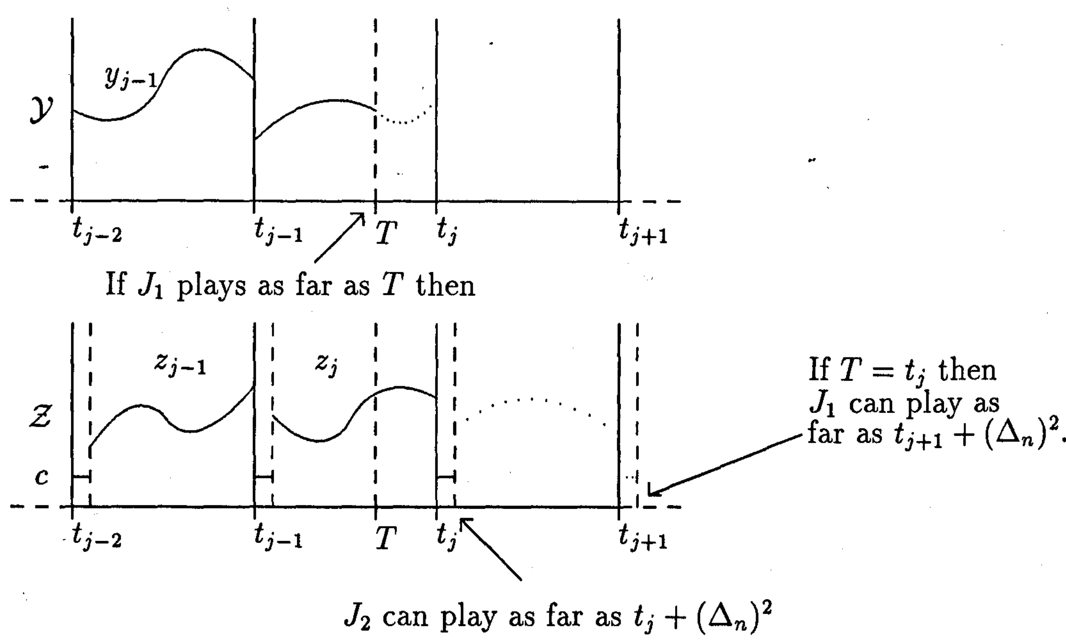
$$V((\Delta_n)^2) \leq \bar{V}_n^+.$$

Proof: First we show that each $\Pi \in \bar{\Upsilon}^n$ gives a strategy $\beta_\Pi \in \Upsilon((\Delta_n)^2)$ such that $p_E^n(y, \beta_\Pi y) = p_E^n(y, \Pi y)$ for all $y \in {}^n\mathcal{M}_1$.

Given a strategy $\Pi \in \bar{\Upsilon}^n$ we define β_Π by

$$\beta_\Pi y = \Pi y \qquad \text{for each fixed } y \in {}^n\mathcal{M}_1$$

then, $\beta_\Pi : {}^n\mathcal{M}_1 \to {}^n\bar{\mathcal{M}}_2$. We now have to show that β_Π is in $\Upsilon((\Delta_n)^2)$. For each $j = 1, \cdots, 2^n$ given $y(t)$ on all of ${}^nI_j$, J_2 knows $z = \Pi y$ on ${}^nI_{j+1} + (\Delta_n)^2$ but, if he only knows what J_1 has done on part of ${}^nI_j$ i.e. up to a time T where $t_{j-1} \leq T \leq t_j$ then J_2 only knows $z = \Pi y$ as far as $t_j + (\Delta_n)^2$ (he can go beyond t_j to $t_j + (\Delta_n)^2$ since he knows that he is going to have to play the constant c on the interval $]t_j, t_j + (\Delta_n)^2]$) he can only go beyond $t_j + (\Delta_n)^2$ if $T = t_j$.



Now, $t_j + (\Delta_n)^2$ could be as small as $T + (\Delta_n)^2$ since T could be equal to $j\Delta_n$, so given y up to time T, J_2 can only be certain of $\beta_\Pi y = \Pi y$ up to time $T + (\Delta_n)^2$ and so β_Π is in $\Upsilon((\Delta_n)^2)$.

Therefore we have shown that for each $\Pi \in \bar{\Upsilon}^n$ there is a corresponding $\beta_\Pi \in \Upsilon((\Delta_n)^2)$ satisfying

$$p_E^n(y, \beta_\Pi y) = p_E^n(y, \Pi y) \qquad \text{for each fixed } y \in {}^n\mathcal{M}_1$$

and so for each $\Pi \in \bar{\Upsilon}^n$ there is a corresponding $\beta_\Pi \in \Upsilon((\Delta_n)^2)$ such that

$$\sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \beta_\Pi y) = \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi y).$$

Now, there are more β's in $\Upsilon((\Delta_n)^2)$ than those which correspond to a strategy $\Pi \in \bar{\Upsilon}^n$, therefore we have

$$\inf_{\beta \in \Upsilon((\Delta_n)^2)} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \beta y) \leq \inf_{\Pi \in \bar{\Upsilon}^n} \sup_{y \in {}^n\mathcal{M}_1} p_E^n(y, \Pi y)$$

i.e.

$$V((\Delta_n)^2) \leq \bar{V}_n^+.$$

□

If, in the nonstandard setting, we put these two Propositions together with Theorem 2.5.5, we have the following result.

Theorem 2.5.8

$V^+(0) = V^+$.

Proof: By (2.28) and (2.23), we have

$$V^+ = \lim_{n \to \infty} V_n^+ \quad \text{and} \quad V^+(0) = \lim_{t \downarrow 0} V(t),$$

therefore, if $N \in {}^*\mathbb{N}$ is infinite we have $\Delta_N \approx 0$ then,

$$V_N^+ \approx V^+ \quad \text{and} \quad V(\Delta_N) \approx V^+(0) \approx V((\Delta_N)^2).$$

By Theorem 2.5.5, Proposition 2.5.6 and Proposition 2.5.7, with N infinite, we have

$$\bar{V}_N^+ \approx V_N^+ \leq V(\Delta_N) \approx V^+(0) \approx V((\Delta_N)^2) \leq \bar{V}_N^+$$

and so we have

$$V^+(0) \approx V_N^+ \approx V^+$$

i.e.

$$V^+(0) = V^+.$$

□

Note: Similarly, by considering the game E_n^-, it can be seen that $V^-(0) = V^-$.

Note: As observed by Elliott and Kalton ([10]), V^+ is the value to player J_1 if he can in some sense anticipate the actions of player J_2 since $V^+ = U^-(0)$. The value V is the value to player J_1 if his reactions are instantaneous. The smallest value V^- is the realistic value to J_1, it is obtained by giving J_1 a reaction time and letting this reaction time tend to zero.

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Chapter 3

Discrete time games

In this chapter we show, using nonstandard methods, the results which appear in [10], that the Friedman values V^+ and V^- of the game G, may be obtained by considering discrete versions H_n^+ and H_n^- of the games E_n^+ and E_n^- respectively. We also give details of the game K_n^+ which appears in [10] – we will use this game in subsequent chapters.

We assume throughout this section that the payoff function for the game G is given by (2.6).

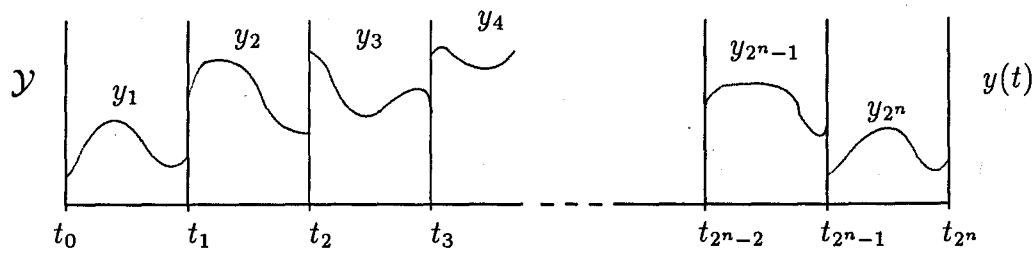
3.1 The game H_n^+

Here we give a brief description of the game H_n^+ as in [10].

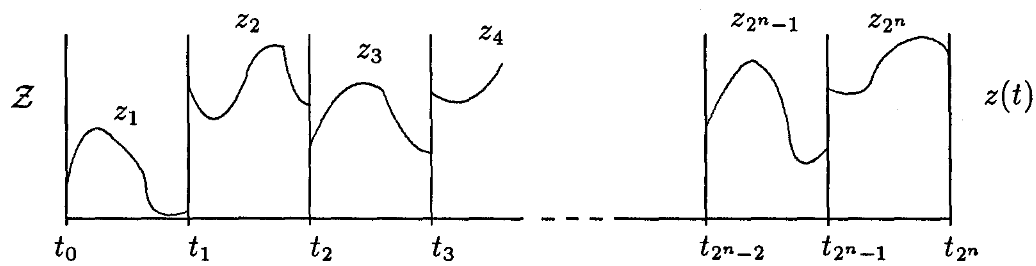
Definition 3.1.1 (The game H_n^+)

The class of controls for each player in the game H_n^+ is exactly the same as in E_n^+. The players play the game H_n^+ in exactly the same way that they would play the game E_n^+ i.e. they play alternately with J_2 selecting his control $z_j \in {}^n\mathcal{M}_2^j$ on ${}^nI_j$ at the j^{th} stage, for $j = 1, \cdots, 2^n$, before J_1 selects his control $y_j \in {}^n\mathcal{M}_1^j$ on ${}^nI_j$

i.e. J_1 plays a varying control $y(t)$



and J_2 plays a varying control $z(t)$



The difference between the game H_n^+ and the game E_n^+ is that in the game H_n^+ the dynamics equation is defined only at the discrete time points $t \in \mathbf{T}_n$. The dynamics for the game H_n^+ are given by

$$\begin{aligned} x_H^n(0) &= x(0) \\ x_H^n(t_j) &= x_H^n(t_{j-1}) + \int_{t_{j-1}}^{t_j} f(t_{j-1}, x_H^n(t_{j-1}), y(t), z(t))dt \end{aligned} \tag{3.1}$$

where $t_j = j\Delta_n$ for $j = 0, \cdots, 2^n$.

The payoff is given by

$$p_H^n(y,z) = \sum_{j=1}^{2^n} \int_{t_{j-1}}^{t_j} h(t_{j-1}, x_H^n(t_{j-1}), y(t), z(t))dt + g(x_H^n(1)). \tag{3.2}$$

3.1.1 Strategies for the game H_n^+

The sets of strategies for each player in the game H_n^+ are exactly the same as for the game E_n^+ i.e. J_1 uses strategies $\Sigma \in \Gamma^n$ and similarly, J_2 uses strategies $\Pi \in \Upsilon^n$ for each integer n.

3.1.2 Value for the game H_n^+

By the general theory of alternate move games, in exactly the same way as for the existence of the value V_n^+ for the game E_n^+, it can be shown that H_n^+ has a value which is denoted by S_n^+.

Note: Similarly, a game H_n^- (based on E_n^-) can be defined and this game has value denoted by S_n^-.

We now consider the nonstandard version of the game H_n^+.

Proposition 3.1.2

For each fixed infinite $N \in {}^*\mathbb{N}$, given a pair of controls $Y \in {}^{N*}\mathcal{M}_1$ and $Z \in {}^{N*}\mathcal{M}_2$, for each initial state, $x(0)$, $X_H^N : \mathbf{T}_N \to {}^*\mathbb{R}^d$ is $\mathcal{S}$-continuous.

Proof: Take $t_j > t_k$ with $t_j \approx t_k$ where t_j and $t_k \in \mathbf{T}_N$ then,

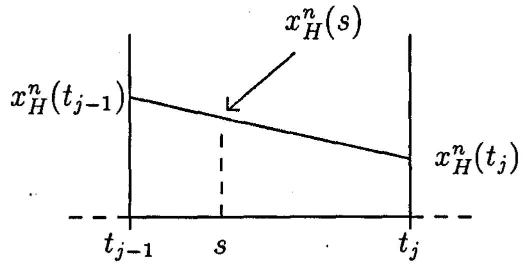
$$\begin{aligned}
|X_H^N(t_j) - X_H^N(t_k)| &= |x(0) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} {}^*f(t_{i-1}, X_H^N(t_{i-1}), Y(\sigma), Z(\sigma))d\sigma \\
&\quad - x(0) - \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} {}^*f(t_{i-1}, X_H^N(t_{i-1}), Y(\sigma), Z(\sigma))d\sigma| \\
&\leq \sum_{i=k+1}^{j} \int_{t_{i-1}}^{t_i} |{}^*f(t_{i-1}, X_H^N(t_{i-1}), Y(\sigma), Z(\sigma))|d\sigma \\
&\leq R(t_j - t_k) \qquad \text{by (2.5)} \\
&\approx 0.
\end{aligned}$$

□

We would like to be able to define $X_H^N(\sigma)$ for $\sigma \notin \mathbf{T}_N$ therefore we make the following definition.

Definition 3.1.3

The function x^n_H is extended so that $x^n_H : [0,1] \to \mathbb{R}^d$ with the following definition. For $s \in]t_{j-1}, t_j]$, we define $x^n_H(s)$ by linearly joining up $x^n_H(t_{j-1})$ and $x^n_H(t_j)$



i.e. if $s \in]t_{j-1}, t_j]$, $x^n_H(s)$ is given by

$$x^n_H(s) = x^n_H(t_j)(1 - j + \frac{s}{\Delta_n}) + x^n_H(t_{j-1})(j - \frac{s}{\Delta_n}).$$

Remarks 3.1.4

(i) In the nonstandard setting with the above definition of $X^N_H(\tau)$ for $\tau \notin \mathbf{T}_N$, we see that if $N \in {}^*\mathbb{N}$ is infinite then we have a function $X^N_H : {}^*[0,1] \to {}^*\mathbb{R}^d$ which is $\mathcal{S}$-continuous.

From here onwards when we refer to the function X^N_H we will mean the extended version $X^N_H : {}^*[0,1] \to {}^*\mathbb{R}^d$ i.e. X^N_H is defined for **all** $\tau \in {}^*[0,1]$.

(ii) Using the above $\mathcal{S}$-continuity property and the continuity of h, we see that given a pair of controls $Y \in {}^N{}^*\mathcal{M}_1$ and $Z \in {}^N{}^*\mathcal{M}_2$, when $N \in {}^*\mathbb{N}$ is infinite, the payoff in the game H^+_N satisfies

$$\begin{aligned} P^N_H(Y,Z) &= \sum_{j=1}^{2^N} \int_{t_{j-1}}^{t_j} {}^*h(t_{j-1}, X^N_H(t_{j-1}), Y(\tau), Z(\tau))d\tau + {}^*g(X^N_H(1)) \quad \text{by (3.2)} \\ &\approx \int_0^1 {}^*h(\tau, X^N_H(\tau), Y(\tau), Z(\tau))d\tau + {}^*g(X^N_H(1)) \end{aligned}$$

since, for each $j = 1, \cdots, 2^N$, if $\tau \in]t_{j-1}, t_j]$ where $t_j = j\Delta_N$ then, $\tau \approx t_{j-1} \approx t_j$ and $X_H^N(\tau) \approx X_H^N(t_{j-1}) \approx X_H^N(t_j)$.

We now show that ${}^\circ X_H^N$ solves equation (3.3); this will then be used in section 3.2 to show that the games E_N^+ and H_N^+ have the same value.

Proposition 3.1.5

For a fixed infinite $N \in {}^*\mathbb{N}$, given a fixed pair of controls Y and Z of the form

$$Y : {}^*[0,1] \to {}^*\mathcal{Y} \quad \text{and} \quad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

the trajectory ${}^\circ X_H^N(\tau)$ solves the following equation

$$x(\tau) = x(0) + \int_0^\tau f({}^\circ\sigma, x(\sigma), {}^\circ Y(\sigma), {}^\circ Z(\sigma))d\sigma_L. \tag{3.3}$$

Proof: For $\tau \in]t_{j-1}, t_j]$

$$\begin{aligned}
{}^\circ X_H^N(\tau) &= {}^\circ X_H^N(t_j) \qquad \text{(since } X_H^N \text{ is } \mathcal{S}\text{-continuous)} \\
&= {}^\circ\Big(x(0) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} {}^*f(t_{i-1}, X_H^N(t_{i-1}), Y(\sigma), Z(\sigma))d\sigma\Big) \\
&= {}^\circ\Big(x(0) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} {}^*f(\sigma, X_H^N(\sigma), Y(\sigma), Z(\sigma))d\sigma\Big) \\
&\qquad \text{(by } \mathcal{S}\text{-continuity of } X_H^N \text{ and continuity of } f) \\
&= {}^\circ\Big(x(0) + \int_0^{t_j} {}^*f(\sigma, X_H^N(\sigma), Y(\sigma), Z(\sigma))d\sigma\Big) \\
&= x(0) + \int_0^\tau {}^\circ({}^*f(\sigma, X_H^N(\sigma), Y(\sigma), Z(\sigma)))d\sigma_L \\
&\qquad \text{(by Loeb Theory)} \\
&= x(0) + \int_0^\tau f({}^\circ\sigma, {}^\circ X_H^N(\sigma), {}^\circ Y(\sigma), {}^\circ Z(\sigma))d\sigma_L \\
&\qquad \text{(by Anderson's Lusin Theorem)}
\end{aligned}$$

and so we see that ${}^\circ X_H^N$ solves equation (3.3).

□

Here we show that if we play the H game over two different discrete time lines $\mathbf{T}_n$ and $\mathbf{T}_m$ i.e. if we play the games H_n^+ and H_m^+, then the trajectories and payoffs are equal in both games if both n and m are infinite. This result will be use in Chapter 5.

Corollary 3.1.6

If $N,\ M \in {}^*\mathbb{N}$ are both infinite then for any pair of controls of the form

$$Y : {}^*[0,1] \to {}^*\mathcal{Y} \quad \text{and} \quad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

the corresponding trajectories in the games H_N^+ and H_M^+ are infinitely close in the sense of the uniform topology i.e.

$$X_H^N(\tau) \approx X_H^M(\tau) \qquad \text{for a.a. } \tau \in {}^*[0,1].$$

Proof: By Proposition 3.1.5 we see that since N and M are both infinite, ${}^\circ X_H^N$ and ${}^\circ X_H^M$ both solve equation (3.3) and it was shown in the proof of Proposition 1.3.7 that this equation has a unique solution.

□

Proposition 3.1.7

If $N,\ M \in {}^*\mathbb{N}$ are both infinite then for **any** pair of controls of the form

$$Y : {}^*[0,1] \to {}^*\mathcal{Y} \quad \text{and} \quad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

the payoffs in the games H_N^+ and H_M^+ are infinitely close i.e.

$$P_H^N(Y,Z) \approx P_H^M(Y,Z).$$

Proof: By Remarks 3.1.4 (ii) we see that

$$\begin{aligned} P_H^N(Y,Z) &\approx \int_0^1 {}^*h(\tau, X_H^N(\tau), Y(\tau), Z(\tau))d\tau + {}^*g(X_H^N(1)) \\ &\approx \int_0^1 {}^*h(\tau, X_H^M(\tau), Y(\tau, Z(\tau))d\tau + {}^*g(X_H^M(1)) \\ &\qquad \text{(by Corollary 1.3.6 and continuity of } h \text{ and } g) \\ &\approx P_H^M(Y,Z). \end{aligned}$$

□

This result will be used in the proof of Theorem 5.2.1.

3.2 The game H_n^+ compared to the game E_n^+

In this section we compare the two games E_n^+ and H_n^+. We show that for a fixed pair of controls and a fixed infinite $N \in {}^*\mathbb{N}$, the trajectories and payoffs in the two games are infinitely close. We then show that the values are infinitely close when $N \in {}^*\mathbb{N}$ is infinite.

Recall that in the nonstandard setting, $X_E^N(\tau)$, a solution in the game E_N^+ corresponding to fixed controls $Y \in {}^N{}^*\mathcal{M}_1$ and $Z \in {}^N{}^*\mathcal{M}_2$, is given by

$$X_E^N(\tau) = x(0) + \int_0^\tau {}^*f(\sigma, X_E^N(\sigma), Y(\sigma), Z(\sigma))d\sigma. \tag{3.4}$$

We know that since this is the same as equation (1.31), for each fixed infinite $N \in {}^*\mathbb{N}$, given a pair of controls $Y \in {}^N{}^*\mathcal{M}_1$ and $Z \in {}^N{}^*\mathcal{M}_2$, for each initial state, $x(0)$, $X_E^N : {}^*[0,1] \to {}^*\mathbb{R}^d$ is $\mathcal{S}$-continuous.

Proposition 3.2.1

For a fixed infinite $N \in {}^*\mathbb{N}$, given a fixed pair of controls Y and Z of the form

$$Y : {}^*[0,1] \to {}^*\mathcal{Y} \quad \text{and} \quad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

the corresponding trajectories in the games H_N^+ and E_N^+ are infinitely close in the sense of the uniform topology i.e.

$$X_H^N(\tau) \approx X_E^N(\tau) \quad \text{for a.a. } \tau \in {}^*[0,1].$$

Proof: Given a fixed infinite $N \in {}^*\mathbb{N}$,

$$\begin{aligned} {}^\circ X_E^N(\tau) &= {}^\circ\Big(x(0) + \int_0^\tau {}^*f(\sigma, X_E^N(\sigma), Y(\sigma), Z(\sigma))d\sigma\Big) \\ &= x(0) + \int_0^\tau f({}^\circ\sigma, {}^\circ X_E^N(\sigma), {}^\circ Y(\sigma), {}^\circ Z(\sigma))d\sigma_L \\ &\quad \text{(by Anderson's Lusin Theorem and the continuity of } f) \end{aligned}$$

and so we see that ${}^\circ X_E^N$ solves (3.3). By Proposition 3.1.5 we know that ${}^\circ X_H^N$ also solves (3.3) and so the result follows since equation (3.3) has unique solution.

□

So we have shown that if $N \in {}^*\mathbb{N}$ is infinite then given any controls $Y \in {}^*\mathcal{M}_1$ and $Z \in {}^*\mathcal{M}_2$ the corresponding trajectories in the games E_N^+ and H_N^+ are infinitely close.

We now go on to show that for fixed controls, and a fixed infinite N, the payoffs in the two games are infinitely close.

Proposition 3.2.2

For a fixed infinite $N \in {}^*\mathbb{N}$, given a fixed pair of controls of the form

$$Y : {}^*[0,1] \to {}^*\mathcal{Y} \qquad \text{and} \qquad Z : {}^*[0,1] \to {}^*\mathcal{Z}$$

we have

$$P_H^N(Y,Z) \approx P_E^N(Y,Z).$$

Proof: Fix $N \in {}^*\mathbb{N}$ infinite. By Remarks 3.1.4 (ii), we see that

$$\begin{aligned} P_H^N(Y,Z) &\approx \int_0^1 {}^*h(\sigma, X_H^N(\sigma), Y(\sigma), Z(\sigma))d\sigma + {}^*g(X_H^N(1)) \\ &\approx \int_0^1 {}^*h(\sigma, X_E^N(\sigma), Y(\sigma), Z(\sigma))d\sigma + {}^*g(X_E^N(1)) \\ &\quad \text{(by Proposition 3.2.1 and the continuity of } h \text{ and } g\text{)} \\ &\approx P_E^N(Y,Z). \end{aligned}$$

□

Therefore, by comparing the two games H_N^+ and E_N^+ we have shown that for a fixed infinite N and a fixed pair of controls, the trajectories and payoffs in the two games are infinitely close.

We now go on to use nonstandard methods to show that the values of the two games are equal.

Theorem 3.2.3

$$\lim_{n\to\infty} S_n^+ = V^+. \tag{3.5}$$

Proof: Fix $N \in {}^*\mathbb{N}$ infinite. We need to show that

$$\inf_{\Pi \in \Upsilon^N} \sup_{\Sigma \in \Gamma^N} P_H^N(\Sigma, \Pi) \approx \inf_{\Pi \in \Upsilon^N} \sup_{\Sigma \in \Gamma^N} P_E^N(\Sigma, \Pi).$$

We know that

$$P_H^N(\Sigma, \Pi) \approx P_E^N(\Sigma, \Pi) \qquad \text{for all } \Sigma \in \Gamma^N,\ \Pi \in \Upsilon^N$$

and so since the operation inf preserves the infinite closeness (Lemma D.1.2) we have

$$\inf_{\Pi \in \Upsilon^N} P_H^N(\Sigma, \Pi) \approx \inf_{\Pi \in \Upsilon^N} P_E^N(\Sigma, \Pi)$$

which means

$$\sup_{\Sigma \in \Gamma^N} \inf_{\Pi \in \Upsilon^N} P_H^N(\Sigma, \Pi) \approx \sup_{\Sigma \in \Gamma^N} \inf_{\Pi \in \Upsilon^N} P_E^N(\Sigma, \Pi)$$

since the operation sup also preserves the infinite closeness (Lemma D.1.1) i.e. for each infinite $N \in {}^*\mathbb{N}$

$$S_N^+ \approx V_N^+,$$

and we know by definition

$$V_N^+ \approx V^+$$

for all infinite $N \in {}^*\mathbb{N}$. Therefore we have

$$S_N^+ \approx V^+$$

for all infinite $N \in {}^*\mathbb{N}$ and so

$$\lim_{n \to \infty} S_n^+ = V^+.$$

□

Note: Similarly, by comparing the games E_n^- and H_n^-, it can be seen that

(3.6) $$\lim_{n \to \infty} S_n^- = V^-.$$

Therefore we have shown that for a fixed pair of nonstandard controls, the value of the game E_N^+ where the dynamics are defined continuously on ${}^*[0,1]$ and the value of the game H_N^+ where the dynamics are only defined at discrete time points in $\mathbf{T}_N$ are equal if N is infinite.

We now give details of a variation of the game H_n^+ which is denoted by $\hat{H}_n^+$. We will use this game in a later chapter to show that, under certain circumstances, we have value in the sense of Friedman.

3.3 The game $\hat{H}_n^+$

Notation 3.3.1

Let ${}^n\tilde{\mathcal{M}}_1$ denote the class of all functions

$$y : [0,1] \to \mathcal{Y}$$

which are constant on the intervals ${}^nI_j$ for $j = 1, \cdots, 2^n$. Similarly, let ${}^n\tilde{\mathcal{M}}_2$ denote the class of all functions

$$z : [0,1] \to \mathcal{Z}$$

which are constant on the intervals ${}^nI_j$ for $j = 1, \cdots, 2^n$. Then, for each $j = 1, \cdots, 2^n$, ${}^n\tilde{\mathcal{M}}_1^j$ is the class of all constant functions $y_j : {}^nI_j \to \mathcal{Y}$ and ${}^n\tilde{\mathcal{M}}_2^j$ is the class of all constant functions $z_j : {}^nI_j \to \mathcal{Z}$.

Note:

(i) For $i = 1, 2$ we have ${}^n\tilde{\mathcal{M}}_i \subset {}^n\mathcal{M}_i$.

(ii) If $n \geq m$ then ${}^m\tilde{\mathcal{M}}_i \subseteq {}^n\tilde{\mathcal{M}}_i$ for $i = 1, 2$.

Definition 3.3.2

A control $y : [0,1] \to \mathcal{Y}$ is said to be **n-constant** if $y \in {}^n\tilde{\mathcal{M}}_1$. Similarly $z : [0,1] \to \mathcal{Z}$ is said to be n-constant if $z \in {}^n\tilde{\mathcal{M}}_2$.

A pair of controls $y : [0,1] \to \mathcal{Y}$ and $z : [0,1] \to \mathcal{Z}$ is said to be **n-constant** if both y and z are n-constant.

In subsequent chapters, when we have a varying control for one of the players against an n-constant control for the other player we would like to be able to replace the varying control by an m-constant control, for some m, without changing the outcome of the game.

With this in mind we look at a variation of the game H_n^+, the game $\hat{H}_n^+$.

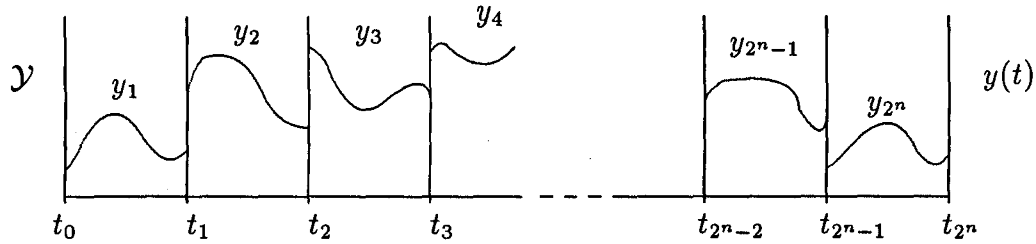
Definition 3.3.3 (The game $\hat{H}_n^+$)

The game $\hat{H}_n^+$ has the same dynamics (discrete), initial condition and payoff as H_n^+ and is played in exactly the same way except that in the game $\hat{H}_n^+$ the class of controls for J_2 is restricted.

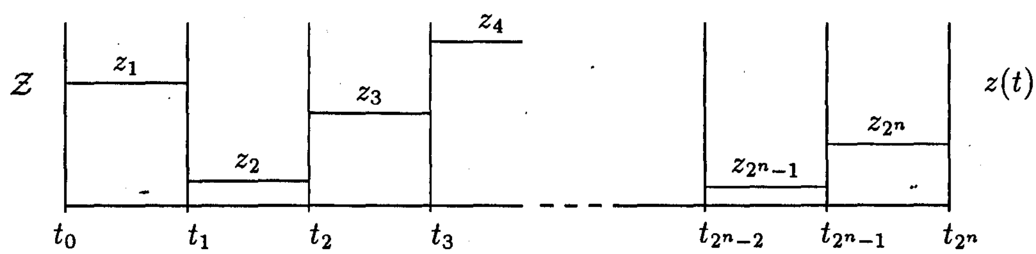
3.3.1 Controls for the game $\hat{H}_n^+$

In the game $\hat{H}_n^+$, J_1 is free to play any control $y \in {}^n\mathcal{M}_1$ while J_2 is restricted to n-constant controls, $z \in {}^n\tilde{\mathcal{M}}_2$

i.e. J_1 plays a control $y(t)$ of the form



and J_2 plays a control $z(t)$ of the form



3.3.2 Strategies for the game $\hat{H}_n^+$

A strategy for player J_1 in the game $\hat{H}_n^+$ is a collection of maps $\Sigma = (\Sigma_1, \cdots, \Sigma_{2^n})$ where for $j = 1, \cdots, 2^n$

$$\Sigma_j : {}^n\tilde{\mathcal{M}}_2^1 \times \cdots \times {}^n\tilde{\mathcal{M}}_2^j \to {}^n\mathcal{M}_1^j.$$

Similarly, a strategy for J_2 in the game $\hat{H}_n^+$ is a member of ${}^n\tilde{\mathcal{M}}_2^1$ together with a collection of maps $(\Pi_2, \cdots, \Pi_{2^n})$ where for $j = 2, \cdots, 2^n$

$$\Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\tilde{\mathcal{M}}_2^j.$$

Let $\hat{\Gamma}^n$ denote the class of all strategies for J_1 in the game $\hat{H}_n^+$ and similarly, let $\hat{\Upsilon}^n$ denote the class of all strategies for J_2 in the game $\hat{H}_n^+$.

Note: For each integer n,

$$\hat{\Upsilon}^n \subset \Upsilon^n \tag{3.7}$$

since if $\Pi \in \hat{\Upsilon}^n$ then $\quad \Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\tilde{\mathcal{M}}_2^j$

and if $\Pi \in \Upsilon^n$ then $\quad \Pi_j : {}^n\mathcal{M}_1^1 \times \cdots \times {}^n\mathcal{M}_1^{j-1} \to {}^n\mathcal{M}_2^j$

and we know that ${}^n\tilde{\mathcal{M}}_2^j \subset {}^n\mathcal{M}_2^j$ for each $j = 1, \cdots, 2^n$.

3.3.3 Value for the game $\hat{H}_n^+$

Clearly $\hat{H}_n^+$ has a value which we denote by $\hat{S}_n^+$ where

$$\hat{S}_n^+ = \inf_{\Pi \in \hat{\Pi}^n} \sup_{\Sigma \in \hat{\Gamma}^n} p_H^n(\Sigma, \Pi). \tag{3.8}$$

We now give a result which we shall use in Chapter 10.

Proposition 3.3.4

For each integer n,

$$\hat{S}_n^+ \geq S_n^+.$$

Proof: By definition and (2.15) it can be seen that

$$S_n^+ = \inf_{\Pi \in \Upsilon^n} \sup_{y \in {}^n\mathcal{M}_1} p_H^n(y, \Pi y)$$

and

$$\hat{S}_n^+ = \inf_{\Pi \in \hat{\Upsilon}^n} \sup_{y \in {}^n\mathcal{M}_1} p_H^n(y, \Pi y)$$

therefore, by (3.7) it is clear that

$$\hat{S}_n^+ \geq S_n^+.$$

□

Note: Similarly, by considering the game $\hat{H}_n^-$ which is the same as H_n^- except that at each stage J_1 is forced to play a constant control, it can be seen that

$$\hat{S}_n^- \leq S_n^-$$

for each integer n.

The game $\hat{H}_n^+$ will be used later in Chapter 4.

We now give details of Fleming's approach to the existence of value.

3.4 Value in the sense of Fleming

In this section we give details of the game K_n^+ which appears in [10]. We then use this game and go on to compare two games, each based on different discrete time lines.

3.4.1 The game K_n^+

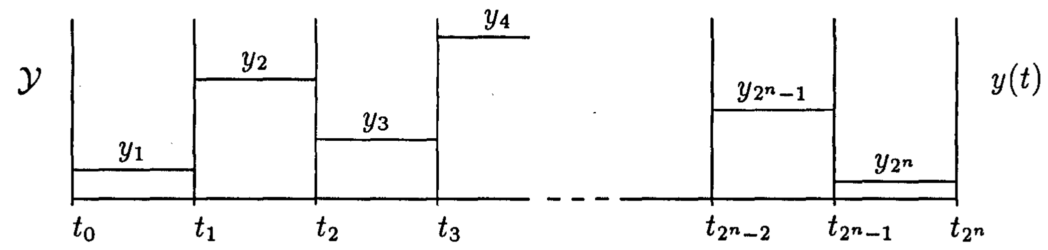
The game K_n^+ is the same as H_n^+ except that now, at each stage both players are forced to choose a constant control function.

Definition 3.4.1 (The game K_n^+)

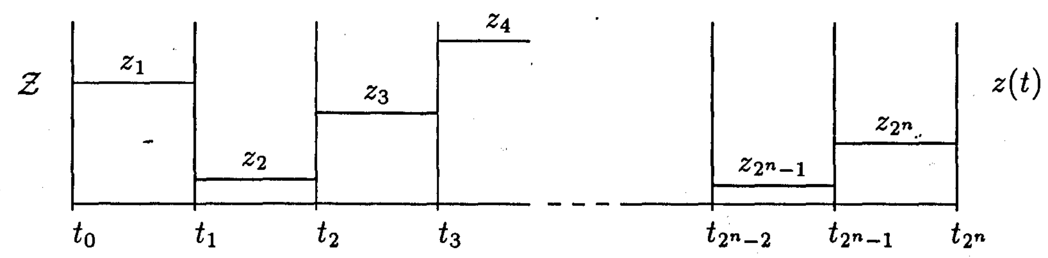
The game K_n^+ is played in exactly the same way as the games E_n^+ and H_n^+ i.e. the players play alternately with J_2 selecting his control on ${}^nI_j$ at the j^{th} stage, for $j = 1, \cdots 2^n$, before J_1 selects his control on ${}^nI_j$.

The difference between the game K_n^+ and the game H_n^+ is that in this game both players are forced to play a constant control at each stage and so at the completion of the game J_1 will have selected a sequence $y = (y_1, \cdots, y_{2^n})$ of elements of $\mathcal{Y}$ i.e. an n-constant control $y \in {}^n\tilde{\mathcal{M}}_1$ and J_2 will have selected a sequence $z = (z_1, \cdots, z_{2^n})$ of elements of $\mathcal{Z}$ i.e. an n-constant control $z \in {}^n\tilde{\mathcal{M}}_2$

i.e. J_1 plays a control $y(t)$ of the form



and J_2 plays a control $z(t)$ of the form



Remarks 3.4.2

(i) If a control is n-constant then it is also m-constant for any $m \geq n$.
(ii) If a control is to be used by a player in the game K_n^+ then this control must be n-constant, a control which is m-constant for $m > n$ but is not n-constant cannot be used in the game K_n^+.

As in the game H_n^+, the dynamics equation is only defined at the discrete time points $t \in \mathbf{T}_n$. For controls $y = (y_1, \cdots, y_{2^n}) \in {}^n\tilde{\mathcal{M}}_1$ and $z = (z_1, \cdots, z_{2^n}) \in {}^n\tilde{\mathcal{M}}_2$, the dynamics of the game K_n^+ are given by

$$\begin{aligned} x_K^n(0) &= x(0) \\ x_K^n(t_j) &= x_K^n(t_{j-1}) + \Delta_n f(t_{j-1}, x_K^n(t_{j-1}), y_j, z_j) \end{aligned} \tag{3.9}$$

where as before, $t_j = j\Delta_n$ for $j = 0, \cdots, 2^n$.

For controls $y \in {}^n\tilde{\mathcal{M}}_1$ and $z \in {}^n\tilde{\mathcal{M}}_2$, the payoff in the game K_n^+ is given by

$$p_K^n(y,z) = \Delta_n \sum_{j=1}^{2^n} h(t_{j-1}, x_K^n(t_{j-1}), y_j, z_j) + g(x_K^n(1)). \tag{3.10}$$

3.4.2 Strategies for the game K_n^+

A strategy for J_1 in the game K_n^+ is a collection of maps $\Sigma = (\Sigma_1, \cdots, \Sigma_{2^n})$ where for each $j = 1, \cdots, 2^n$

$$\Sigma_j : {}^n\tilde{\mathcal{M}}_2^1 \times \cdots \times {}^n\tilde{\mathcal{M}}_2^j \to {}^n\tilde{\mathcal{M}}_1^j.$$

Similarly, a strategy for J_2 in the game K_n^+ is a member of ${}^n\tilde{\mathcal{M}}_2^1$ together with a collection of maps $(\Pi_2, \cdots, \Pi_{2^n})$ where for each $j = 2, \cdots, 2^n$

$$\Pi_j : {}^n\tilde{\mathcal{M}}_1^1 \times \cdots \times {}^n\tilde{\mathcal{M}}_1^{j-1} \to {}^n\tilde{\mathcal{M}}_2^j.$$

Let $\tilde{\Gamma}^n$ denote the class of all strategies for J_1 in the game K_n^+ and let $\tilde{\Upsilon}^n$ denote the class of all strategies for J_2 in the game K_n^+.

Note: In the game K_n^+ a strategy only needs to be able to cope with n-constant controls whereas in the games $E_n^+$and H_n^+ they have to cope with varying controls. Therefore we have

$$\tilde{\Gamma}^n \subset \Gamma^n \quad \text{and} \quad \tilde{\Upsilon}^n \subset \Upsilon^n. \tag{3.11}$$

3.4.3 Value for the game K_n^+

Since K_n^+ is an alternate play game, by the same methods used to show that E_n^+ and H_n^+ have value, it can be seen that K_n^+ has a value which we shall denote by

W_n^+ where

$$W_n^+ = \inf_{\Pi \in \tilde{\Upsilon}^n} \sup_{\Sigma \in \tilde{\Gamma}^n} P_K^n(\Sigma, \Pi).$$

Note: We could similarly define a game K_n^- in which J_1 plays first at each stage.

Having noted that K_n^+ is actually the game H_n^+ with the classes of controls restricted to n-constant controls for both players, we make the following remarks.

Remarks 3.4.3

(i) Just as we did in the game H_n^+, by using Definition 3.1.3 to define $x_K^n(s)$ for $s \notin \mathbf{T}_n$, we can extend x_K^n to $x_K^n : [0,1] \to \mathbb{R}^d$.

From here onwards, when we refer to the function X_K^N we will mean the extended version so that $X_K^N(\tau)$ is defined for **all** $\tau \in {}^*[0,1]$.

(ii) By Proposition 3.1.2, in the nonstandard setting, for each fixed infinite $N \in {}^*\mathbb{N}$, given a pair of N-constant controls, the resulting nonstandard solution $X_K^N : {}^*[0,1] \to {}^*\mathbb{R}^d$ is $\mathcal{S}$-continuous.

(iii) By Proposition 3.1.5, for each fixed infinite $N \in {}^*\mathbb{N}$, given a pair of N-constant controls $Y \in {}^N{}^*\mathcal{M}_1$ and $Z \in {}^N{}^*\mathcal{M}_2$ the trajectory ${}^\circ X_K^N(\tau)$ solves equation (3.3).

(iv) For each fixed infinite $N \in {}^*\mathbb{N}$, by the $\mathcal{S}$-continuity of X_K^N the payoff in the game K_N^+ satisfies

$$(3.12) \qquad P_K^N(Y,Z) \approx \int_0^1 {}^*h(\tau, X_K^N(\tau), Y(\tau), Z(\tau))d\tau + {}^*g(X_K^N(1))$$

The obvious question which arises from this is whether the limit of the values W_n^+ and W_n^- as $n \to \infty$ exist. To show this Fleming used the Isaacs-Bellman equation (we will look at this in Chapter 5) , we however manage to show the existence of these limits without using the Isaacs-Bellman equation and without as many restrictions as imposed by Fleming – this forms the main content of the following chapters.

We are keen to show that these limits exist since then, we are able to show that under certain circumstances the values V^+ and V^- are equal and so the game G has value in the sense of Friedman.

Before we do this we give a brief summary on relaxed controls in the context of game theory (see section 1.2 for relaxed controls in control theory) since relaxed controls will be used in subsequent chapters.

———— oOo ————

Chapter 4

Relaxed Controls

Since the subsequent chapters involve relaxed controls, here, in this chapter, we give a brief summary of relaxed controls. Relaxed controls were first introduced into control theory in [37] (see Section 1.2) and into game theory in [11].

4.1 Relaxed play games

We gave the definition of relaxed controls in section 1.2.

Notation 4.1.1

Let ${}^n\mathcal{R}_1$ denote the class of all relaxed controls for J_1. Similarly, let ${}^n\mathcal{R}_2$ denote the class of all relaxed controls for J_2.

Similarly, we denote the class of all n-constant (see Definition (3.3.2) relaxed controls for J_1 by ${}^n\tilde{\mathcal{R}}_1$ and those for J_2 by ${}^n\tilde{\mathcal{R}}_2$.

Note: Just as in Section 1.2, we note that ${}^n\mathcal{M}_i \subset {}^n\mathcal{R}_i$ for $i = 1, 2$.

With this definition of relaxed controls, we need to extend the definitions of the functions f and h. The function $f : I \times \mathbb{R}^d \times \Lambda(\mathcal{Y}) \times \Lambda(\mathcal{Z}) \to \mathbb{R}^d$ is defined by

$$f_i(t, x, \mu, \gamma) = \int_{\mathcal{Z}} \int_{\mathcal{Y}} f_i(t, x, y, z) d\mu(y) d\gamma(z) \tag{4.1}$$

for $i = 1, 2, \cdots, d$ and the function $h : I \times \mathbb{R}^d \times \Lambda(\mathcal{Y}) \times \Lambda(\mathcal{Z}) \to \mathbb{R}$ is defined by

$$h(t, x, \mu, \gamma) = \int_{\mathcal{Z}} \int_{\mathcal{Y}} h(t, x, y, z) d\mu(y) d\gamma(z). \tag{4.2}$$

With the above definitions, it is easy to verify that the extended f and h will satisfy Lipschitz and continuity conditions of the same type as satisfied by the original f and h.

Since $\Lambda(\mathcal{Y})$ and $\Lambda(\mathcal{Z})$ are compact metric spaces, all of the results in Chapters 2 and 3 hold for relaxed controls.

Notation 4.1.2

There are four versions of the game G which can be considered. The original game where both players are using ordinary controls is denoted by G, the game where player J_1 is allowed to use relaxed controls while J_2 is still restricted to ordinary controls is denoted by G_1. Similarly, the game where J_2 is allowed to use relaxed controls and J_1 is restricted to ordinary controls is denoted by G_2 and the game where both players J_1 and J_2 are allowed to use relaxed controls is denoted by G_{12}.

All four of the games can be treated as in the preceding discussions. From here onwards a subscript 1, 2 or 12 denotes the fact that a quantity refers to the game G_1, G_2 or G_{12} respectively.

4.2 The game $K^+_{n,1}$

In this section we give a brief description of what we mean by the game $K^+_{n,1}$; it is actually the game K^+_n with player J_1 permitted to use relaxed controls of the form above. We will then, in section 4.3, compare this game with the game $\hat{H}_n^+$ (section 3.3) . By comparing these two games we are then able to show that, under certain circumstances, a varying control can be replaced by a constant relaxed control, without changing the outcome of the game.

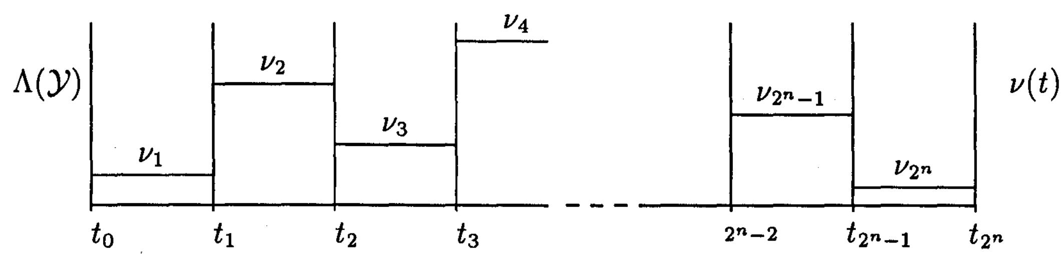
Definition 4.2.1 (The game $K_{n,1}^+$)

The game $K_{n,1}^+$ is actually the game K_n^+ except that now player J_1 is allowed to use n-constant relaxed controls of the from

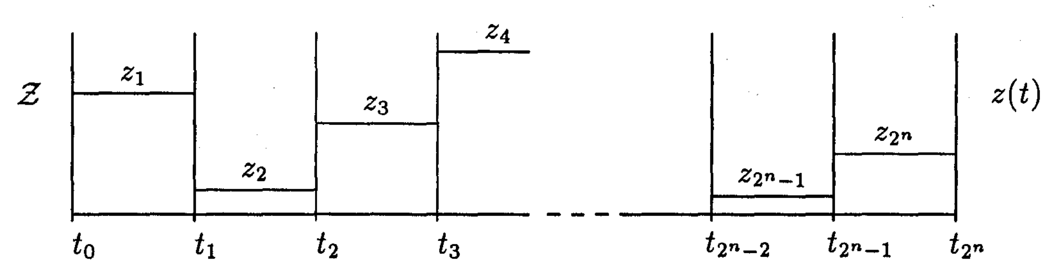
$$\nu : [0,1] \rightarrow \Lambda(\mathcal{Y}) \tag{4.3}$$

i.e. J_1 uses controls $\nu \in {}^n\tilde{\mathcal{R}}_1$ while J_2 is still using n-constant ordinary controls, $z \in {}^n\tilde{\mathcal{M}}_2$

i.e. J_1 plays a control $\nu(t)$ of the form



and J_2 plays a control $z(t)$ of the form



Recall, from (3.9), the dynamics for the game $K_{n,1}^+$ are given by

$$x_K^n(t_j) = x(0) + \sum_{i=1}^{j} \Delta_n f(t_{i-1}, x_K^n(t_{i-1}), \nu_i, z_i)$$

and the payoff is given by

$$\begin{aligned} p_K^n(\nu, z) &= \Delta_n \sum_{j=1}^{2^n} h(t_{j-1}, x_K^n(t_{j-1}), \nu_j, z_j) + g(x_K^n(1)) \\ &= \sum_{i=1}^{2^n} \int_{t_{i-1}}^{t_i} h(t_{i-1}, x_K^n(t_{i-1}), \nu(t), z(t))dt + g(x_K^n(1)). \end{aligned}$$

where f and h are now the extended versions so that they can cope with relaxed controls (see (4.1)).

Since $\Lambda(\mathcal{Y})$ is a metric space we know the game has value. We denote the value of the game $K_{n,1}^+$ by $W_{n,1}^+$.

We now go on to compare this game to the game $\hat{H}_n^+$ (section 3.3).

4.3 The game $K_{n,1}^+$ compared to the game $\hat{H}_n^+$

Here we compare the two games $K_{n,1}^+$ and $\hat{H}_n^+$ and show that under certain circumstances a varying control can be replaced by a constant control without changing the outcome of the game.

Elliott and Kalton ([10, page 41]) showed that, over each interval I_j for $j = 1, \cdots, 2^n$, from a varying ordinary control $y(t)$ for player J_1, a constant relaxed control can be defined which has the same effect against a constant control for J_2. We give a more general result; we show that from a varying relaxed control $\nu(t)$ for J_1 we can define a constant relaxed control which has the same effect against a constant control for J_2 as the original varying control.

Lemma 4.3.1

Given a varying control $\nu \in {}^n\mathcal{R}_1$ for J_1, we can define an n-constant relaxed control $\bar{\nu} = (\bar{\nu}_1, \cdots, \bar{\nu}_{2^n}) \in {}^n\tilde{\mathcal{R}}_1$ by

$$\int_{\mathcal{Y}} \varphi(y)d\bar{\nu}_j(y) = \frac{1}{\Delta_n} \int_{t_{j-1}}^{t_j} \varphi(\nu(t))dt. \tag{4.4}$$

for all continuous functions $\varphi : \mathcal{Y} \to \mathbb{R}^d$. Clearly $\bar{\nu}_j \in \Lambda(\mathcal{Y})$ for each $j = 1, \cdots, 2^n$.

Proof: Since every compact metric space is a locally compact Hausdorff space, for each $j = 1, \cdots, 2^n$, given a control $\nu(t)$ on the interval I_j, by defining a linear functional on $\mathcal{C}(\mathcal{Y})$ by

$$\begin{aligned} \theta\varphi &= \frac{1}{\Delta_n} \int_{t_{j-1}}^{t_j} \varphi(\nu(t))dt \\ &= \frac{1}{\Delta_n} \int_{t_{j-1}}^{t_j} \int_{\mathcal{Y}} \varphi(y) d\nu(t)(y) dt \end{aligned}$$

for each $\varphi \in \mathcal{C}(\mathcal{Y})$, the result follows from the Riesz Representation Theorem (see [32, page 42]) i.e.

$$\int_{\mathcal{Y}} \varphi(y) d\bar{\nu}_j(y) = \frac{1}{\Delta_n} \int_{t_{j-1}}^{t_j} \varphi(\nu(t))dt.$$

□

Clearly, Elliott and Kalton's result (given below) is a Corollary to ours:

Corollary 4.3.2 *(Elliott and Kalton ([10, page 41]))*

Given a varying control $y \in {}^n\mathcal{M}_1$ for J_1, we can define an n-constant relaxed control $\bar{\nu} = (\bar{\nu}_1, \cdots, \bar{\nu}_{2^n}) \in {}^n\tilde{\mathcal{R}}_1$ by

$$\int_{\mathcal{Y}} \varphi(y) d\bar{\nu}_j(y) = \frac{1}{\Delta_n} \int_{t_{j-1}}^{t_j} \varphi(y(t))dt \tag{4.5}$$

for all continuous functions $\varphi : \mathcal{Y} \to \mathbb{R}^d$. Clearly $\bar{\nu}_j \in \Lambda(\mathcal{Y})$ for each $j = 1, \cdots, 2^n$.

Proof: This follows directly from Lemma 4.3.1 since ${}^n\mathcal{M}_1 \subset {}^n\mathcal{R}_1$.

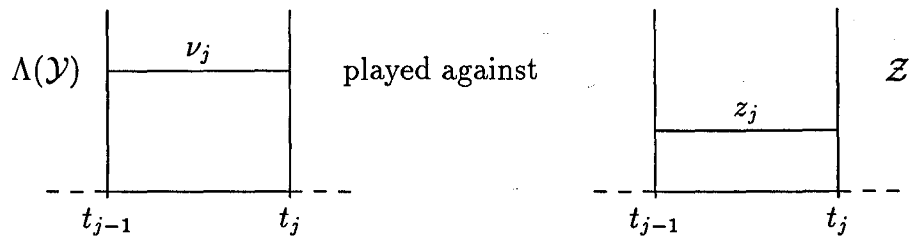
□

From Lemma 4.3.1 we obtain the following result.

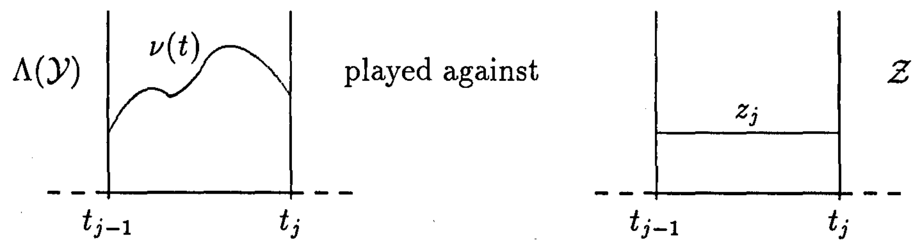
Proposition 4.3.3

If in the game $\hat{H}^+_{n,1}$, J_2 plays an n-constant control $z = (z_1, \cdots, z_{2^n}) \in {}^n\tilde{\mathcal{M}}_2$ and J_1 responds with a varying control $\nu \in {}^n\mathcal{R}_1$ then J_1 could achieve the same result by playing the relaxed n-constant control $\bar{\nu} = (\bar{\nu}_1, \cdots, \bar{\nu}_{2^n}) \in {}^n\tilde{\mathcal{R}}_1$ where $\bar{\nu}_j \in {}^n\tilde{\mathcal{R}}_1^j$ is given by (4.4) for each $j = 1, \cdots, 2^n$ against J_2's control z in the game $K_{n,1}^+$

i.e. for each $j = 1, \cdots, 2^n$, given a varying control $\nu \in {}^n\mathcal{R}_1^j$ over the interval ${}^nI_j$ there exists a constant relaxed control $\bar{\nu}_j \in {}^n\tilde{\mathcal{R}}_1^j$ acting on ${}^nI_j$ such that given a constant control $z_j \in {}^n\tilde{\mathcal{M}}_2^j$ for J_2 acting on ${}^nI_j$



is exactly the same as



Proof: Since the dynamics equation for both of the games $\hat{H}_{n,1}^+$ and $K_{n,1}^+$ is the same we shall, for simplicity, just denote the trajectories by x in this proof.

$$
\begin{aligned}
x(t_j) &= x(0) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} f(t_{i-1}, x(t_{i-1}), \nu(t), z_i)dt \\
&= x(0) + \sum_{i=1}^{j} \Delta_n \int_{\mathcal{Y}} f(t_{i-1}, x(t_{i-1}), y, z_i)d\bar{\nu}_i(y) && \text{by (4.4)} \\
&= x(0) + \sum_{i=1}^{j} \Delta_n f(t_{i-1}, x(t_{i-1}), \bar{\nu}_i, z_i). && \text{by (4.1)}
\end{aligned}
$$

Similarly, by (4.2) and (4.4), we have

$$
\sum_{j=1}^{2^n} \int_{t_{j-1}}^{t_j} h(t_{j-1}, x_H^n(t_{j-1}), \nu(t), z_j)dt = \sum_{j=1}^{2^n} \Delta_n h(t_{j-1}, x_K^n(t_{j-1}), \bar{\nu}_j, z_j)
$$

$$
\Longrightarrow p_H^n(\nu, z) = p_K^n(\bar{\nu}, z).
$$

Thus J_1 can exactly duplicate the effect of any control function (even a varying control) in $\hat{H}_{n,1}^+$ by a constant control function in $K_{n,1}^+$.

□

Note: From Proposition 4.3.3, for each $j = 1, \cdots, 2^n$, since ${}^n\mathcal{M}_1 \subset {}^n\mathcal{R}_1$, we could equally well replace a varying ordinary control $y(t)$ for J_1 over the interval I_j by a constant relaxed control. With this in mind we obtain the following Corollary to Proposition 4.3.3 which we will use in Chapter 10.

Corollary 4.3.4

For any integer n,

$$\hat{S}_n^+ \leq W_{n,1}^+.$$

Proof: We have shown (Proposition 4.3.3) that whatever J_1 can achieve by using a control $y \in {}^n\mathcal{M}_1$ in the game $\hat{H}_n^+$ he can equally well achieve by using the n-constant relaxed control $\nu \in {}^n\tilde{\mathcal{R}}_1$ as given by (4.5) in the game $K_{n,1}^+$. Note, this comes from the fact that ${}^n\mathcal{M}_1 \subset {}^n\mathcal{R}_1$. Therefore, K_n^+ is at least as favourable to J_1 as $\hat{H}_n^+$.

□

Note: Similarly, by comparing the game $K_{n,2}^-$ against the game $\hat{H}_n^-$ we could obtain

$$W_{n,2}^- \leq \hat{S}_n^-.$$

———— oOo ————

Chapter 5

The game $H^+_{m,n}$

At this stage we have described the games K^+_n and their values W^+_n but unlike for the games E^+_n and H^+_n it is not clear that these values actually tend to a limit. To show this Fleming found it necessary to solve the Isaacs-Bellman equation (details of which will be given later) and he also had to impose certain conditions on the functions f, g and h. We, trying to avoid having to do this, investigate what happens if we allow the constant governing the length of the intervals of time and the constant which governs the time points at which the dynamics are defined to be different in the game H^+_n. This gave us a new game of our own which we denote by $H^+_{m,n}$. Using this new game we are able to show some of the same results as Fleming without using the Isaacs-Bellman equation or all of the restrictions on the functions.

5.1 The game $H^+_{m,n}$

Here we give details of our new game $H^+_{m,n}$.

Definition 5.1.1 (The game $H^+_{m,n}$)

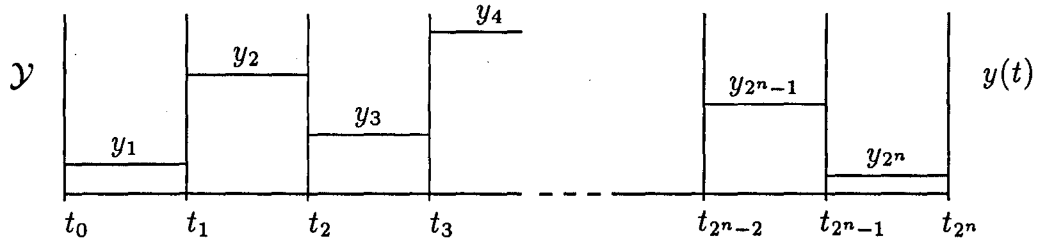
The game $H^+_{m,n}$ is played in exactly the same way as the game H^+_n i.e. the players play alternately on the intervals ${}^nI_j$ for $j = 1, \cdots, 2^n$ with J_2 selecting his control at the j^{th} stage before J_1 selects his.

The difference between the game $H^+_{m,n}$ and the game H^+_n is that in the game $H^+_{m,n}$ the players are restricted in their choice of controls, and the dynamics are defined at the time points $t \in \mathbf{T}_m$ (not $\mathbf{T}_n$ as in the game H^+_n).

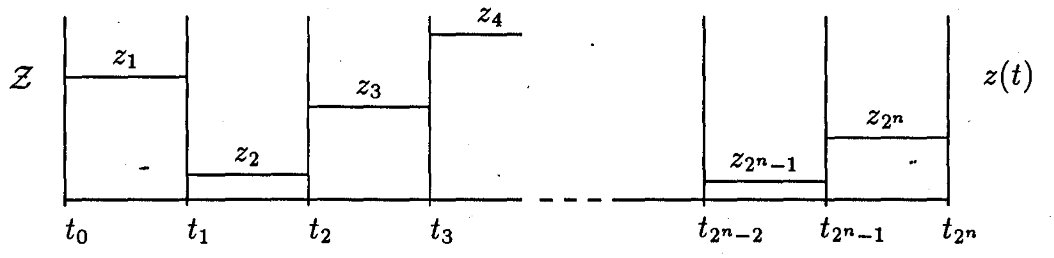
5.1.1 Controls for the game $H^+_{m,n}$

The class of controls for each player in the game $H^+_{m,n}$ is exactly the same as in the game K^+_n i.e. J_1 uses n-constant controls $y \in {}^n\tilde{\mathcal{M}}_1$ and J_2 uses n-constant controls $z \in {}^n\tilde{\mathcal{M}}_2$

i.e. J_1 plays a control $y(t)$ of the form



and J_2 plays a control $z(t)$ of the form



where $|t_j - t_{j-1}| = \Delta_n$ for $j = 1, \cdots, 2^n$.

5.1.2 Dynamics and payoff for the game $H^+_{m,n}$

Given a pair of n-constant controls, $y \in {}^n\tilde{\mathcal{M}}_1$ and $z \in {}^n\tilde{\mathcal{M}}_2$, for the game $H^+_{m,n}$ the dynamics equation is the same as for the game H^+_m and is given by

$$\begin{aligned} x_H^{m,n}(0) &= x(0) \\ x_H^{m,n}(t'_j) &= x_H^{m,n}(t'_{j-1}) + \int_{t'_{j-1}}^{t'_j} f(t'_{j-1}, x_H^{m,n}(t'_{j-1}), y(\sigma), z(\sigma))d\sigma \end{aligned} \tag{5.1}$$

where $t'_j = j\Delta_m$ for $j = 0, 1, \cdots, 2^m$.

The payoff is given by

$$p_H^m(y,z) = \Delta_m \sum_{j=1}^{2^m} h(t'_{j-1}, x_H^{m,n}(t'_{j-1}), y_j, z_j) + g(x_H^{m,n}(1)) \tag{5.2}$$

where $t'_j = j\Delta_m$ for $j = 0, 1, \cdots, 2^m$.

5.1.3 Strategies for the game $H^+_{m,n}$

A strategy for player J_1 in the game $H^+_{m,n}$ is a collection of maps $\Sigma = (\Sigma_1, \cdots, \Sigma_{2^n})$ where

$$\Sigma_j : {}^n\tilde{\mathcal{M}}_2^1 \times \cdots \times {}^n\tilde{\mathcal{M}}_2^j \to {}^n\tilde{\mathcal{M}}_1^j \qquad \text{for } j = 1, \cdots, 2^n$$

while a strategy for player J_2 is a member $z_1 \in {}^n\tilde{\mathcal{M}}_2^1$ along with a collection of maps $(\Pi_2, \cdots, \Pi_{2^n})$ where

$$\Pi_j : {}^n\tilde{\mathcal{M}}_1^1 \times \cdots \times {}^n\tilde{\mathcal{M}}_1^{j-1} \to {}^n\tilde{\mathcal{M}}_2^j \qquad \text{for } j = 2, \cdots, 2^n$$

i.e. Player J_1 uses strategies $\Sigma \in \tilde{\Gamma}^n$ and player J_2 uses strategies $\Pi \in \tilde{\Upsilon}^n$ and so we see that the set of strategies for both players in the game $H^+_{m,n}$ is exactly the same as for the game K^+_n.

5.1.4 Value for the game $H^+_{m,n}$

The game $H^+_{m,n}$ has value which we shall denote by $S^+_{m,n}$. This value is given by

$$S^+_{m,n} = \inf_{\Pi \in \tilde{\Upsilon}^n} \sup_{\Sigma \in \tilde{\Gamma}^n} p_H^m(\Sigma, \Pi). \tag{5.3}$$

Notation 5.1.2

In the game $H^+_{m,n}$, we refer to m as the **dynamics mesh**, this denotes the fact that the dynamics equation is defined at the time points $t \in \mathbf{T}_m$. We refer to n as the **play mesh**, since it denotes the fact that the players play alternately on intervals of length Δ_n using n-constant controls.

5.2 Varying the dynamics and play mesh

We now investigate what happens when we allow the dynamics mesh and/or the play mesh to vary.

5.2.1 Varying the dynamics mesh

Here we consider what happens if we keep the play mesh fixed but allow the dynamics mesh to vary i.e. fix n in the game $H^+_{m,n}$ but allow m to vary.

Theorem 5.2.1

For infinite N, $M \in {}^*\mathbb{N}$

$$S^+_{M,L} \approx S^+_{N,L}$$

for any constant L.

Proof: By Proposition 3.1.7, when N and M are both infinite and L is a constant

$$P^M_H(\Sigma, \Pi) \approx P^N_H(\Sigma, \Pi)$$

for each $\Sigma \in \tilde{\Gamma}^L$ and $\Pi \in \tilde{\Upsilon}^L$ and so, since the operations sup and inf preserve the infinite closeness (Lemmas D.1.1 and D.1.2) we have

$$\inf_{\Pi \in \tilde{\Upsilon}^L} \sup_{\Sigma \in \tilde{\Gamma}^L} P^M_H(\Sigma, \Pi) \approx \inf_{\Pi \in \tilde{\Upsilon}^L} \sup_{\Sigma \in \tilde{\Gamma}^L} P^N_H(\Sigma, \Pi)$$

when N and M are both infinite i.e. we have

$$S^{+}_{M,L} \approx S^{+}_{N,L}$$

when N, $M \in {}^{*}\mathbb{N}$ are infinite and L is a constant.

□

5.2.2 Varying the play mesh

Here we consider what happens when we keep the dynamics mesh constant but allow the play mesh to vary.

Theorem 5.2.2

For infinite N, $M \in {}^{*}\mathbb{N}$

$$S^{+}_{M,N,1} \leq S^{+}_{M,M,1}$$

if $N \geq M$.

Proof: By the definition of $S^{+}_{M,M,1}$ we have

$$\forall\epsilon > 0 \exists\Pi \in \tilde{\Upsilon}^{M} \forall\Sigma \in \tilde{\Gamma}^{M} P^{M}_{H}(\Sigma, \Pi) < S^{+}_{M,M,1} + \epsilon$$

i.e. $\forall\epsilon > 0$

$$\exists Z_1 \forall\nu_1 \cdots \exists Z_{2M} \forall\nu_{2M} P^{M}_{H}(Z_1, \nu_1, \cdots, Z_{2M}, \nu_{2M}) < S^{+}_{M,M,1} + \epsilon \tag{5.4}$$

where $Z_j \in {}^{M*}\tilde{\mathcal{M}}^{j}_{2}$ and $\nu_j \in {}^{M*}\tilde{\mathcal{R}}^{j}_{1}$ for each $j = 1, \cdots, 2^M$. Therefore,

$$\exists Z_1 \forall(\nu^1_1 \cdots \nu^{2^L}_1) \cdots \exists Z_{2M} \forall(\nu^1_{2M} \cdots \nu^{2^L}_{2M}) P^{M}_{H}(Z_1, \nu^1_1 \cdots \nu^{2^L}_1, \cdots, Z_{2M}, \nu^1_{2M} \cdots \nu^{2^L}_{2M}) < S^{+}_{M,M,1} + \epsilon$$

where L is a constant, this works because we know, by Proposition 4.3.3, that if we have varying control $\nu^1_j \nu^2_j \cdots \nu^{2^L}_j$ against a constant control Z_j on the interval ${}^{M}I_j$ then we can replace the varying control by a constant control ν_j on the interval ${}^{M}I_j$ without changing the outcome of the game and then, we know that for all such $\nu_j \in {}^{M*}\mathcal{R}^{j}_{1}$, there exists a $Z_j \in {}^{M*}\mathcal{M}^{j}_{2}$ satisfying (5.4) which means

$$\exists Z_1 \forall\nu^1_1 \cdots \forall\nu^{2^L}_1 \cdots \exists Z_{2M} \forall\nu^1_{2M} \cdots \forall\nu^{2^L}_{2M} P^{M}_{H}(Z_1, \nu^1_1, \cdots, \nu^{2^L}_1 Z_2, \cdots, \nu^1_{2M}, \cdots, \nu^{2^L}_{2M}) < S^{+}_{M,M,1} + \epsilon$$

$$\Longrightarrow \exists Z_1^1 \cdots \exists Z_1^{2^L} \forall \nu_1^1 \cdots \forall \nu_1^{2^L} \cdots \exists Z_{2^M}^1 \cdots \exists Z_{2^M}^{2^L} \forall \nu_{2^M}^1 \cdots \forall \nu_{2^M}^{2^L} P_H^{M+L}(Z_1^1, \cdots, \nu_{2^M}^{2^L}) < S_{M,M,1}^+ + \epsilon$$

(Since for each $j = 1, \cdots, 2^M$ we can split Z_j into $Z_j^1, Z_j^2, \cdots, Z_j^{2^L}$ over the interval ${}^M I_j$ where Z_j^i acts on ${}^{M+L} I_{2^L(j-1)+i}$ for each $j = 1, \cdots, 2^M$ and $i = 1, \cdots, 2^L$.)

$$\Longrightarrow \exists Z_1^1 \forall \nu_1^1 \exists Z_1^2 \forall \nu_1^2 \cdots \exists Z_{2^M}^{2^L} \forall \nu_{2^M}^{2^L} P_H^{M+L}(Z_1^1, \nu_1^1, \cdots, Z_{2^M}^{2^L}, \nu_{2^M}^{2^L}) < S_{M,M,1}^+ + \epsilon.$$

Therefore, for each $\epsilon > 0$ we have

$$\exists \Pi \in \tilde{\Upsilon}^{M+L} \forall \Sigma \in \tilde{\Gamma}^{M+L} P_H^{M+L}(\Sigma, \Pi) < S_{M,M,1}^+ + \epsilon$$

$$\Longrightarrow \exists \Pi \in \tilde{\Upsilon}^{M+L} \sup_{\Sigma \in \tilde{\Gamma}^{M+L}} P_H^{M+L}(\Sigma, \Pi) \leq S_{M,M,1}^+ + \epsilon$$

$$\Longrightarrow \inf_{\Pi \in \tilde{\Upsilon}^{M+L}} \sup_{\Sigma \in \tilde{\Gamma}^{M+L}} P_H^{M+L}(\Sigma, \Pi) \leq S_{M,M,1}^+ + \epsilon$$

$$\Longrightarrow S_{M,(M+L),1}^+ \leq S_{M,M,1}^+$$

i.e.

$$S_{M,N,1}^+ \leq S_{M,M,1}^+$$

when $N \geq M$.

□

Remarks 5.2.3

The same argument shows in fact that

$$S_{M,(N+1),1}^+ \leq S_{M,N,1}^+$$

for $N \geq M$ and hence,

$$S_{M,L,1}^+ \leq S_{M,N,1}^+$$

when $L \geq N \geq M$.

5.3 The game $H_{m,n}^+$ compared to the game K_n^+

If we consider what happens when we make the play mesh and dynamics mesh equal in the game $H_{m,n}^+$ i.e. if we look at the game $H_{n,n}^+$ then we see that this is in fact exactly the same as the game K_n^+. We can now go on to use Theorems 5.2.1 and 5.2.2 to show that the limit of the values $W_{n,12}^+$ as $n \to \infty$ exists.

Theorem 5.3.1

Given $N,\ M \in {}^*\mathbb{N}$ infinite,

$$W_{M,1}^+ \gtrsim W_{N,1}^+$$

if $N \geq M$.

Proof: For an infinite N, by Theorems 5.2.1 and 5.2.2, we have

$$W_{M,1}^+ = S_{M,M,1}^+ \geq S_{M,N,1}^+ \approx S_{N,N,1}^+ = W_{N,1}^+$$

when $N \geq M$ i.e.

$$W_{M,1}^+ \gtrsim W_{N,1}^+$$

when N and M both infinite and $N \geq M$.

□

5.4 The existence of W_{12}^+, W_{12}^-, W_1^+ and W_2^-

We are now in a position where we can show the existence of the limits of the values $W_{n,1}^+$ and $W_{n,12}^+$ as $n \to \infty$ exist.

Theorem 5.4.1

The values $W_{n,1}^+$ of the games $K_{n,1}^+$ tend to a limit denoted by W_1^+ i.e. the limit

$$\lim_{n\to\infty} W_{n,1}^+ = W_1^+$$

exists.

Proof: Let $l = \inf\{{}^\circ W_{N,1}^+ : N \text{ infinite }\}$. Then given any $\epsilon > 0\ \exists$ infinite $M \in {}^*\mathbb{N}$ such that

$$l \leq {}^\circ W_{M,1}^+ < l + \epsilon$$

and we know, by Theorem 5.3.1, that for any $N \geq M$

$$l \leq {}^{\circ}W^{+}_{N,1} \leq {}^{\circ}W^{+}_{M,1} < l + \epsilon$$

i.e. for all $N \geq M$ we have

$$l - \epsilon < W^{+}_{N,1} < l + \epsilon$$

so by transfer we have

$$\forall(\epsilon > 0)\exists m \forall(n \geq m)|W^{+}_{n,1} - l| < \epsilon$$

i.e.

$$W^{+}_{n,1} \to l \qquad \text{as } n \to \infty.$$

□

Corollary 5.4.2

The values $W^{+}_{n,12}$ of the games $K^{+}_{n,12}$ tend to a limit denoted by W^{+}_{12} i.e. the limit

$$\lim_{n\to\infty} W^{+}_{n,12} = W^{+}_{12}$$

exists.

Proof: By considering the game G_{12}, this follows directly from Theorem 5.4.1.

□

Note: By considering the games $K^{-}_{n,2}$ and $K^{-}_{n,12}$ we can, by analogous methods to those used above, show the existence of the limits of the values $W^{-}_{n,2}$ and $W^{-}_{n,12}$ as $n \to \infty$ i.e. we can also show (without using the Isaacs-Bellman equation as Fleming found necessary) that

$$\lim_{n\to\infty} W^{-}_{n,2} = W^{-}_{2} \qquad \text{and} \qquad \lim_{n\to\infty} W^{-}_{n,12} = W^{-}_{12}$$

exist.

Remarks 5.4.3

At this stage we have the existence of the limits W_{12}^+, W_1^+, W_{12}^- and W_2^- without using the Isaacs-Bellman equation which Fleming found necessary and we have only assumed one of the five conditions on the functions f, g and h that Fleming required. For details of the Isaacs-Bellman equation and Fleming's conditions, see section 6.2. We have, at this stage, only assumed Fleming's condition (F1) - however this method fails to provide us with the existence of the remaining values W_2^+, W_1^- and the ones we require most of all, W^+ and W^-.

We therefore go on to give another method, using the ideas of Fleming (still not using the Isaacs-Bellman equation or assuming as many restrictions as he found necessary), which when combined with the above results provides the existence of the remaining limits and also provides a way of showing that some of these limits are in fact equal.

oOo

Chapter 6

The Isaacs-Bellman equation

In this chapter we give a method which provides the existence of the remaining limits $W^+(W_2^+)$ and W^- (W_1^-). This method also provides a way of showing that some of these values are in fact equal, it is based on the game $K_n^+(t_j,\zeta)$, as used by Fleming, and an adaption of this game which we shall denote by $K_{n,\lambda}^+(t_j,\zeta)$.

We first give details of the game $K_n^+(t_j,\zeta)$ and then a brief summary of the work done by Fleming using the Isaacs-Bellman equation. We then go on to give details of the new game $K_{n,\lambda}^+(t_j,\zeta)$ and use this new game to show, still without the Isaacs-Bellman equation, the existence of the limits W^+ and W^-. To do this we have to impose some of the restrictions on the functions that Fleming found necessary but not all of them.

6.1 The game $K_n^+(t_j,\zeta)$

Here we give details of the game $K_n^+(t_j,\zeta)$ which appears in [10].

Definition 6.1.1 (The game $K_n^+(t_j,\zeta)$)

The game $K_n^+(t_j,\zeta)$ is the same as the game K_n^+ except that in the game $K_n^+(t_j,\zeta)$ the play begins at time $t_j = j\Delta_n$ with initial value $x(t_j) = \zeta$.

6.1.1 Controls for the game $K_n^+(t_j,\zeta)$

After a complete play of the game $K_n^+(t_j,\zeta)$ player J_1 will have a selected a sequence $y=(y_{j+1},\cdots,y_{2^n})$ where for each $i=(j+1),\cdots,2^n$ the control $y_i \in {}^n\tilde{\mathcal{M}}_1^i$ i.e. y_i is a constant function on the interval ${}^nI_i$. Similarly, J_2 will have selected a sequence $z=(z_{j+1},\cdots,z_{2^n})$ where for each $i=(j+1),\cdots,2^n$ the control $z_i \in {}^n\tilde{\mathcal{M}}_2^i$.

6.1.2 Dynamics and payoff for the game $K_n^+(t_j,\zeta)$

Since it is clear that here we are working in the game $K_n^+(t_j,\zeta)$ with n fixed, we drop our usual notation x_K^n etc... to make the necessary notation less cumbersome. In the game $K_n^+(t_j,\zeta)$ we denote the trajectory corresponding to a pair of controls $y=(y_{j+1},\cdots,y_{2^n})$ and $z=(z_{j+1},\cdots,z_{2^n})$ by x_j^ζ where

$$\begin{aligned} x_j^\zeta(t_j) &= \zeta \\ x_j^\zeta(t_i) &= x_j^\zeta(t_{i-1})+\Delta_n f(t_{i-1},x_j^\zeta(t_{i-1}),y_i,z_i). \end{aligned} \tag{6.1}$$

Remarks 6.1.2

We see that the trajectory $x_{j+1}^{\zeta'}$ is such that

$$x_{j+1}^{\zeta'}(t_i)=x_j^\zeta(t_i) \qquad \text{for all } i\geq(j+1).$$

where

$$\zeta'=\zeta+\Delta_n f(t_j,\zeta,y_{j+1},z_{j+1}).$$

The payoff in $K_n^+(t_j,\zeta)$ corresponding to such a pair of controls is given by

$$p_j^\zeta(z_{j+1},y_{j+1},\cdots,z_{2^n},y_{2^n})=g(x_j^\zeta(1))+\sum_{i=j+1}^{2^n}\Delta_n h(t_{i-1},x_j^\zeta(t_{i-1}),y_i,z_i). \tag{6.2}$$

Note: The payoff for the game $K_n^+(t_{2^n},\zeta)$ is just given by

$$p_{2^n}^\zeta=g(x_{2^n}^\zeta(1))=g(\zeta).$$

The game $K_n^+(t_j,\zeta)$ has value denoted by $W_n^+(t_j,\zeta)$ where

(6.3)

$$W_n^+(t_j,\zeta) = \min_{z_{j+1}\in {}^n\tilde{\mathcal{M}}_2^{j+1}} \max_{y_{j+1}\in {}^n\tilde{\mathcal{M}}_1^{j+1}} \cdots \min_{z_{2^n}\in {}^n\tilde{\mathcal{M}}_2^{2^n}} \max_{y_{2^n}\in {}^n\tilde{\mathcal{M}}_1^{2^n}} p_j^\zeta(z_{j+1},y_{j+1},\cdots,z_{2^n},y_{2^n}).$$

Note: For the game $K_n^+(t_{2^n},\zeta)$ the value is just given by

$$W_n^+(t_{2^n},\zeta) = p_{2^n}^\zeta.$$

Lemma 6.1.3

For $j = 0,1,\cdots,(2^n-1)$, the value $W_n^+(t_j,\zeta)$ can be expressed in the following way

$$W_n^+(t_j,\zeta) = \min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{W_n^+(t_{j+1},\zeta') + \Delta_n h(t_j,\zeta,y,z)\} \tag{6.4}$$

where

$$\zeta' = \zeta + \Delta_n f(t_j,\zeta,y,z) \tag{6.5}$$

and

$$W_n^+(t_{2^n},\zeta) = g(\zeta). \tag{6.6}$$

Proof: For $j = 2^n$ by (6.3) we have

$$W_n^+(t_{2^n},\zeta) = g(x_{2^n}^\zeta(t_{2^n})) = g(x_{2^n}^\zeta(1)) = g(\zeta).$$

Now consider the game $K_n^+(t_j,\zeta)$ for some j where $0\le j\le(2^n-1)$, the controls $z_{j+1}\in {}^n\tilde{\mathcal{M}}_2^{j+1}$, $y_{j+1}\in {}^n\tilde{\mathcal{M}}_1^{j+1},\cdots,z_{2^n}\in {}^n\tilde{\mathcal{M}}_2^{2^n}$ and $y_{2^n}\in {}^n\tilde{\mathcal{M}}_1^{2^n}$ are still to be chosen and we know by (6.3) that

$$\begin{aligned}
&W_n^+(t_j,\zeta)\\
=\ &\min_{z_{j+1}}\max_{y_{j+1}}\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{p_j^\zeta(z_{j+1},y_{j+1},\cdots,z_{2^n},y_{2^n})\}\\
=\ &\min_{z_{j+1}}\max_{y_{j+1}}\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{g(x_j^\zeta(1)) + \sum_{i=j+1}^{2^n}\Delta_n h(t_{i-1},x_j^\zeta(t_{i-1}),y_i,z_i)\}\\
&\quad(\text{by } (6.2)\ )\\
=\ &\min_{z_{j+1}}\max_{y_{j+1}}\{\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{g(x_j^\zeta(1)) + \sum_{i=j+2}^{2^n}\Delta_n h(t_{i-1},x_j^\zeta(t_{i-1}),y_i,z_i)\}\\
&\qquad\qquad+\Delta_n h(t_j,\zeta,y_{j+1},z_{j+1})\}
\end{aligned}$$

then by Remarks 6.1.2 we see that this is the same as

$$= \min_{z_{j+1}}\max_{y_{j+1}}\{\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{g(x_{j+1}^{\zeta'}(1)) + \sum_{i=j+2}^{2^n}\Delta_n h(t_{i-1}, x_{j+1}^{\zeta'}(t_{i-1}), y_i, z_i)\}$$
$$+\Delta_n h(t_j, \zeta, y_{j+1}, z_{j+1})\}$$

where $\zeta' = x_j^{\zeta}(t_{j+1})$ i.e. $\zeta' = \zeta + \Delta_n f(t_j, \zeta, y_{j+1}, z_{j+1})$ by (6.1)

$$= \min_{z_{j+1}}\max_{y_{j+1}}\{\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{p_{j+1}^{\zeta'}(z_{j+2}, y_{j+2}, \cdots, z_{2^n}, y_{2^n})\}$$
$$+\Delta_n h(t_j, \zeta, y_{j+1}, z_{j+1})\}$$

(by (6.2))

$$= \min_{z_{j+1}}\max_{y_{j+1}}\{W_n^+(t_{j+1}, \zeta') + \Delta_n h(t_j, \zeta, y_{j+1}, z_{j+1})\} \qquad \text{by (6.3).}$$

This can simply be written as

$$W_n^+(t_j, \zeta) = \min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{W_n^+(t_{j+1}, \zeta') + \Delta_n h(t_j, \zeta, y, z)\}$$

where

$$\zeta' = \zeta + \Delta_n f(t_j, \zeta, y, z).$$

So we have shown that (6.7) holds for all $j = 0, 1, \cdots, (2^n - 1)$.

□

Note: If we take the game $K_n^+(0,0)$ this is exactly the same as the game K_n^+ as described in Chapter 3 i.e.

$$x_K^n(t) = x_0^0(t) \qquad \text{for all } t \in \mathbf{T}_n$$

and

$$p_K^n(y,z) = p_0^0(y,z) \qquad \text{for all } n\text{-constant controls } y \text{ and } z$$

and so

$$W_n^+(0,0) = W_n^+.$$

6.2 The Isaacs-Bellman equation

Considering games of this kind led Isaacs to derive heuristically the Isaacs-Bellman differential equation for the upper value $R(t, \zeta)$ of the game G, subject to the initial condition $x(t) = \zeta$ and with payoff given by

$$p(y,z) = \int_t^1 h(s, x(s), y(s), z(s))ds + g(x(1)). \tag{6.7}$$

Note: The value R is in fact the value W^+ but we don't have the existence of this yet.

The Isaacs-Bellman equation is the following partial differential equation

$$\frac{\partial R}{\partial t} + F^+(t, \zeta, \nabla R) = 0 \tag{6.8}$$

where

$$\begin{aligned} F^+(t,\zeta,p) &= \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} \{\sum_{i=1}^{d} p_i f_i(t,\zeta,y,z) + h(t,\zeta,y,z)\} \\ &= \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} \{p.f + h\} \end{aligned} \tag{6.9}$$

for $p = (p_i) \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$ and $t \in I$.

However, there are no theorems guaranteeing the existence or uniqueness of solutions to (6.8). Fleming developed an approach ([14], [15], [16]) to avoid this difficulty and produce a reasonable solution to (6.8). To do this he had to impose certain restrictions, (F1)-(F5) which are as follows

(F1) For $t \in I$, x_1, $x_2 \in \mathbb{R}^d$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$

$$|f(t,x_1,y,z) - f(t,x_2,y,z)| \leq \kappa |x_1 - x_2|$$

(Note, we already assume this.)

(F2) For $t \in I$, x_1, $x_2 \in \mathbb{R}^d$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$

$$|h(t,x_1,y,z) - h(t,x_2,y,z)| \leq D|x_1 - x_2|$$

(F3) For x_1, $x_2 \in \mathbb{R}^d$

$$|g(x_1) - g(x_2)| \leq Q|x_1 - x_2|$$

(F4) For t_1, $t_2 \in I$, $x \in \mathbb{R}^d$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ there exists a constant $A > 0$ such that

$$|f(t_1,x,y,z) - f(t_2,x,y,z)| \leq A|t_1 - t_2|$$

$$|h(t_1,x,y,z) - h(t_2,x,y,z)| \leq A|t_1 - t_2|$$

(F5) The function g is twice continuously differentiable and its derivatives

$$\frac{\partial g}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 g}{\partial x_i \partial x_j}$$

each satisfy Lipschitz conditions in x.

Games satisfying these conditions (F1)-(F5) are said to be of type F.

For games of type F Fleming considered the parabolic equation

$$\frac{\lambda^2}{2}\nabla^2 R + \frac{\partial R}{\partial t} + F^+(t,\zeta,\nabla R) = 0 \tag{6.10}$$

subject to $R(1,\zeta) = g(\zeta)$.

Quoting the results of Friedman ([19]) or Oleinik and Kruzhkov ([29]), he observes that this equation has a unique solution, $W_\lambda^+(t,\zeta)$ for $\lambda > 0$, and that W_λ^+ is continuously differentiable in t and twice continuously differentiable in the space variable x. Furthermore, W_λ^+ and its derivatives $\frac{\partial W_\lambda^+}{\partial t}$, $\frac{\partial W_\lambda^+}{\partial \zeta_i}$ and $\frac{\partial^2 W_\lambda^+}{\partial \zeta_i \partial \zeta_j}$ each satisfy Hölder conditions of the form

$$|\psi(t,\zeta) - \psi(t',\zeta')| \leq Q'[|t-t'|^{\frac{1}{2}} + |x - x'|].$$

For $\lambda > 0$ and $\delta = 2^{-n}$ with n an integer, Fleming considers a stochastic difference equation related to (6.4)

$$W_{n,\lambda}^+(t_j,\zeta) = \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} \{\mathbb{E}[W_{n,\lambda}^+(t_{j+1},\zeta') + \Delta_n h(t_k,\zeta,y,z)]\} \tag{6.11}$$

where

$$\zeta' = \zeta + \Delta_n f(t_j,\zeta,y,z) + (\Delta_n)^{\frac{1}{2}}\lambda\eta_j. \tag{6.12}$$

Here $(\eta_0,\cdots,\eta_{2^n-1})$ is a sequence of normalised mutually independent Gaussian random variables (and $\mathbb{E}$ denotes the expectation). $W_{n,\lambda}^+$ is determined for $t_j = j\Delta_n$, $j = 0,1,\cdots,2^n$ by the boundary condition

$$W_{n,\lambda}^+(1,\zeta) = W_{n,\lambda}^+(t_{2^n},\zeta) = g(\zeta). \tag{6.13}$$

Using this difference equation Fleming obtains the following Theorems (see [16]) .

Theorem A

$$\lim_{n\to\infty} W^+_{n,\lambda}(t,\zeta) = W^+_\lambda(t,\zeta)$$

for $\lambda > 0$ and dyadically rational t, uniformly on compacta.

Theorem B

$$\lim_{\lambda\to 0} W^+_{n,\lambda}(t,\zeta) = W^+_n(t,\zeta)$$

uniformly in n for each dyadically rational t, and n such that $t = p2^{-n}$ with p an integer.

From these he deduces:

Theorem C

$$\lim_{\lambda\to 0} W^+_\lambda(t,\zeta) = \lim_{n\to\infty} W^+_n(t,\zeta)$$

for dyadically rational t.

In particular,

$$W^+ = \lim_{n\to\infty} W^+_n.$$

It can also be deduced that

$$\lim_{\lambda\to 0} W^+_\lambda(t,\zeta) = W^+(t,\zeta)$$

exists for all $t \in I$ and all $\zeta \in \mathbb{R}^d$.

Fleming shows that the function W^+ is a generalised solution of the Isaacs-Bellman equation (6.11), this is known as the Fleming solution of the Isaacs-Bellman equation.

Elliott and Kalton ([10]) observed that the Fleming solution depends only on the function $F^+(t,\zeta,p)$ (see (6.12)) (and the boundary condition $g(\zeta)$).

The same analysis can be applied to the values W^-_n of the games K^-_n and the existence of the limit

$$W^- = \lim_{n\to\infty} W^-_n$$

can be deduced, where $W^- = W^-(0,0)$ and the function $W^-(t,\zeta)$ is the Fleming solution of the equation

$$\frac{\partial R}{\partial t} + F^-(t,\zeta,\nabla R) = 0$$

where

$$\begin{aligned} F^-(t,\zeta,p) &= \max_{y\in\mathcal{Y}} \min_{z\in\mathcal{Z}} \{\sum_{i=1}^{d} p_i f_i(t,\zeta,y,z) + h(t,\zeta,y,z)\} \\ &= \max_{y\in\mathcal{Y}} \min_{z\in\mathcal{Z}} \{p.f + h\} \end{aligned} \tag{6.14}$$

for $p = (p_i) \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$ and $t \in I$. Again, R must also satisfy the boundary condition

$$R(1,\zeta) = g(\zeta).$$

Definition 6.2.1

The game G is said to satisfy the Isaacs condition if for each $t \in [0,1]$, $\zeta \in \mathbb{R}^d$ and $p \in \mathbb{R}^d$ the following holds

$$F^+(t,\zeta,p) = F^-(t,\zeta,p), \tag{6.15}$$

where F^+ and F^- are given by (g) and (h)

Elliott and Kalton ([10]) go on to use the work done by Fleming to show that if G is of type F and satisfies the Isaacs condition then $W^+ = W^-$.

(For more details on this see [10]).

We however, avoid the Isaacs-Bellman equation completely; we simply define a game which has (6.14) as its value and then show, using this new game, that the values W_n^+ of the games K_n^+ tend to a limit as $n \to \infty$. Our method also requires fewer restrictions on the functions f, g and h.

We now give details of this game.

6.3 The game $K^+_{n,\lambda}(t_j,\zeta)$

We are still trying to show that the limits W^+ and W^- exist. We want to avoid using the Isaacs-Bellman equation – to do this we develop a game, $K^+_{n,\lambda}(t_j,\zeta)$, whose value is given by equation (6.14). We then compare $K^+_{n,\lambda}(t_j,\zeta)$ to the original game $K^+_n(t_j,\zeta)$.

First we give some notation which we will use in the definition of the game $K^+_{n,\lambda}(t_j,\zeta)$.

Notation 6.3.1

Given a collection $(\eta_0,\cdots,\eta_{2^n-1})$ of normalised mutually independent Gaussian random variables, we denote the expectation with respect to the single variable η_j by

$$\mathbb{E}_j$$

and we denote the expectation with respect to the variables $\eta_j,\cdots,\eta_{2^n-1}$ (i.e. η_j upwards) by

$$\hat{\mathbb{E}}_j \quad .$$

Definition 6.3.2 (The game $K^+_{n,\lambda}(t_j,\zeta)$)

The game $K^+_{n,\lambda}(t_j,\zeta)$ is played in exactly the same way as the game $K^+_n(t_j,\zeta)$ i.e. play starts at time $t_j = j\Delta_n$ with initial value $x(t_j) = \zeta$. Play continues alternately with J_2 playing first at each stage.

Player J_1 plays a sequence $y = (y_{j+1},\cdots,y_{2^n})$ where $y_i \in {}^n\tilde{\mathcal{M}}_1^i$ for each $i = (j+1),\cdots,2^n$. Similarly, J_2 plays a sequence $z = (z_{j+1},\cdots,z_{2^n})$ where $z_i \in {}^n\tilde{\mathcal{M}}_2^i$ for each $i = (j+1),\cdots,2^n$.

6.3.1 Dynamics and payoff for the game $K^+_{n,\lambda}(t_j,\zeta)$

The difference between the game $K^+_{n,\lambda}(t_j,\zeta)$ and the game $K^+_n(t_j,\zeta)$ is that in the game $K^+_{n,\lambda}(t_j,\zeta)$ the dynamics are given by

$$
\begin{aligned}
x^\zeta_{j,\lambda}(t_j) &= \zeta \\
x^\zeta_{j,\lambda}(t_i) &= x^\zeta_{j,\lambda}(t_{i-1}) + \Delta_n f(t_{i-1}, x^\zeta_{j,\lambda}(t_{i-1}), y_i, z_i) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_{i-1}
\end{aligned}
\tag{6.16}
$$

where $(\eta_0, \cdots, \eta_{2^n-1})$ is a sequence of normalised mutually independent Gaussian random variables.

Note: As for the game $K^+_n(t_j,\zeta)$ we have dropped the usual notation of x^n_K etc...

Remarks 6.3.3

The trajectory $x^{\zeta'}_{j+1,\lambda}$ is such that

$$x^{\zeta'}_{j+1,\lambda}(t_i) = x^\zeta_{j,\lambda}(t_i) \qquad \text{for all } i \geq (j+1)$$

where

$$\zeta' = \zeta + \Delta_n f(t_j, y_{j+1}, z_{j+1}) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_j.$$

The payoff in $K^+_{n,\lambda}(t_j,\zeta)$ is given by

$$p^\zeta_{j,\lambda}(z_{j+1}, y_{j+1}, \cdots, z_{2^n}, y_{2^n}) = \hat{\mathbb{E}}_j[g(x^\zeta_{j,\lambda}(1)) + \sum_{i=j+1}^{2^n} \Delta_n h(t_{i-1}, x^\zeta_{j,\lambda}(t_{i-1}), y_i, z_i)]$$

Note: For the game $K^+_{n,\lambda}(t_{2^n},\zeta)$ the payoff is just given by

$$p^\zeta_{2^n,\lambda} = g(x^\zeta_{2^n,\lambda}(1)) = g(\zeta) \quad .$$

The game $K^+_{n,\lambda}(t_j,\zeta)$ has value denoted by $W^+_{n,\lambda}(t_j,\zeta)$ where

(6.17)

$$W^+_{n,\lambda}(t_j,\zeta) = \min_{z_{j+1}\in {}^n\bar{\mathcal{M}}_2^{j+1}} \max_{y_{j+1}\in {}^n\bar{\mathcal{M}}_1^{j+1}} \cdots \min_{z_{2^n}\in {}^n\bar{\mathcal{M}}_2^{2^n}} \max_{y_{2^n}\in {}^n\bar{\mathcal{M}}_1^{2^n}} \{p^\zeta_{j,\lambda}(z_{j+1}, y_{j+1}, \cdots, z_{2^n}, y_{2^n})\}$$

Note: For the game $K^+_{n,\lambda}(t_{2^n},\zeta)$ the value is just given by

$$W^+_{n,\lambda}(t_{2^n},\zeta) = p^\zeta_{2^n,\lambda}.$$

Lemma 6.3.4

For $j = 0, 1, \cdots, (2^n - 1)$, the value $W^+_{n,\lambda}(t_j, \zeta)$ can be expressed in the following way

$$(6.18) \qquad W^+_{n,\lambda}(t_j,\zeta) = \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} \{\mathbb{E}_j[W^+_{n,\lambda}(t_{j+1},\zeta') + \Delta_n h(t_j,\zeta,y,z)]\},$$

where

$$(6.19) \qquad \zeta' = \zeta + \Delta_n f(t_j,\zeta,y,z) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_j$$

and

$$(6.20) \qquad W^+_{n,\lambda}(t_{2^n},\zeta) = g(\zeta).$$

Proof: For $j = 2^n$ by (6.20) we have

$$W^+_{n,\lambda}(t_{2^n},\zeta) = g(x^{\zeta}_{2^n,\lambda}(1)) = g(\zeta).$$

Now consider the game $K^+_{n,\lambda}(t_j, x)$ for some j, where $0 \le j \le (2^n - 1)$, the controls $z_{j+1} \in {}^n\tilde{\mathcal{M}}_2^{j+1}$, $y_{j+1} \in {}^n\tilde{\mathcal{M}}_1^{j+1}, \cdots, z_{2^n} \in {}^n\tilde{\mathcal{M}}_2^{2^n}$ and $y_{2^n} \in {}^n\tilde{\mathcal{M}}_1^{2^n}$ are still to be chosen and so by (6.17) we have

$$\begin{aligned}
&W^+_{n,\lambda}(t_j,\zeta)\\
=&\min_{z_{j+1}}\max_{y_{j+1}}\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{p^{\zeta}_{j,\lambda}(z_{j+1},y_{j+1},\cdots,z_{2^n},y_{2^n})\}\\
=&\min_{z_{j+1}}\max_{y_{j+1}}\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{\hat{\mathbb{E}}_j[g(x^{\zeta}_{j,\lambda}(1))+\sum_{i=j+1}^{2^n}\Delta_n h(t_{i-1},x^{\zeta}_{j,\lambda}(t_{i-1}),y_i,z_i)]\}\\
=&\min_{z_{j+1}}\max_{y_{j+1}}\{\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{\hat{\mathbb{E}}_j[g(x^{\zeta}_{j,\lambda}(1))+\sum_{i=j+2}^{2^n}\Delta_n h(t_{i-1},x^{\zeta}_{j,\lambda}(t_{i-1}),y_i,z_i)]\}\\
&\qquad\qquad +\Delta_n h(t_j,\zeta,y_{j+1},z_{j+1})\}\\
=&\min_{z_{j+1}}\max_{y_{j+1}}\{\mathbb{E}_j[\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{\hat{\mathbb{E}}_{j+1}[g(x^{\zeta}_{j,\lambda}(1))+\sum_{i=j+2}^{2^n}\Delta_n h(t_{i-1},x^{\zeta}_{j,\lambda}(t_{i-1}),y_i,z_i)]\}\\
&\qquad\qquad +\Delta_n h(t_j,\zeta,y_{j+1},z_{j+1})]\}
\end{aligned}$$

which by Remarks 6.3.3 we see is the same as

$$\begin{aligned}
=&\min_{z_{j+1}}\max_{y_{j+1}}\{\mathbb{E}_j[\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{\hat{\mathbb{E}}_{j+1}[g(x^{\zeta'}_{j+1,\lambda}(1))+\sum_{i=j+2}^{2^n}\Delta_n h(t_{i-1},x^{\zeta'}_{j+1,\lambda}(t_{i-1}),y_i,z_i)]\}\\
&\qquad\qquad +\Delta_n h(t_j,\zeta,y_{j+1},z_{j+1})]\}
\end{aligned}$$

where $\zeta' = x^{\zeta}_{j,\lambda}(t_{j+1})$ i.e. $\zeta' = \zeta + \Delta_n f(t_j, \zeta, y_{j+1}, z_{j+1}) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_j$

$$=\min_{z_{j+1}}\max_{y_{j+1}}\{\mathbb{E}_j[\min_{z_{j+2}}\max_{y_{j+2}}\cdots\min_{z_{2^n}}\max_{y_{2^n}}\{p^{\zeta'}_{j+1,\lambda}(z_{j+2}, y_{j+2}, \cdots, z_{2^n}, y_{2^n})\} + \Delta_n h(t_j, \zeta, y_{j+1}, z_{j+1})]\}$$

$$=\min_{z_{j+1}}\max_{y_{j+1}}\{\mathbb{E}_j[W^+_n(t_{j+1}, \zeta') + \Delta_n h(t_j, \zeta, y_{j+1}, z_{j+1})]\}$$

(by (6.17)).

This can simply be written as

$$W^+_{n,\lambda}(t_j, \zeta) = \min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_j[W^+_{n,\lambda}(t_{j+1}, \zeta') + \Delta_n h(t_j, \zeta, y, z)]\}$$

where

$$\zeta' = \zeta + \Delta_n f(t_j, \zeta, y, z) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_j.$$

So we have shown that (6.18) holds for all $j = 0, 1, \cdots, (2^n - 1)$.

□

Now, just as we noted that the game $K^+_n(0,0)$ is actually the game K^+_n, using the above game, $K^+_{n,\lambda}(t_j, \zeta)$, we can define a game $K^+_{n,\lambda} = K^+_{n,\lambda}(0,0)$ as follows.

6.4 The game $K^+_{n,\lambda}$

Definition 6.4.1 (The game $K^+_{n,\lambda}$)

Given a pair of controls $y = (y_1, \cdots, y_{2^n}) \in {}^n\tilde{\mathcal{M}}_1$ and $z = (z_1, \cdots, z_{2^n}) \in {}^n\tilde{\mathcal{M}}_2$ the corresponding trajectory is given by

$$\begin{aligned} x^n_{K,\lambda}(t_0) &= x(0) \\ x^n_{K,\lambda}(t_j) &= x^n_{K,\lambda}(t_{j-1}) + \Delta_n f(t_{j-1}, x^n_{K,\lambda}(t_{j-1}), y_j, z_j) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_{j-1} \quad (6.21) \end{aligned}$$

where $\eta_0, \cdots, \eta_{2^n-1}$ is a sequence of normalised mutually independent Gaussian random variables.

Note: For this game we have gone back to our original notation for the dynamics and payoff – this is because later we shall be comparing this game with the game K^+_n.

The payoff corresponding to such a pair of controls is given by

$$(6.22)\qquad p^n_{K,\lambda}(y,z) = \hat{\mathbb{E}}_0[g(x^n_{K,\lambda}(1)) + \Delta_n \sum_{j=1}^{2^n} h(t_{j-1}, x^n_{K,\lambda}(t_{j-1}), y_j, z_j)].$$

Note: The sets of strategies for both players in the game $K^+_{n,\lambda}$ are exactly the same as for the game K^+_n i.e. J_1 uses strategies $\Sigma \in \tilde{\Gamma}^n$ and J_2 uses strategies $\Pi \in \tilde{\Upsilon}^n$.

The game $K^+_{n,\lambda}$ has value which we denote by $W^+_{n,\lambda}$ given by

$$W^+_{n,\lambda} = \min_{z_1 \in {}^n\tilde{\mathcal{M}}^1_2} \max_{y_1 \in {}^n\tilde{\mathcal{M}}^1_1} \cdots \min_{z_{2^n} \in {}^n\tilde{\mathcal{M}}^{2^n}_2} \max_{y_{2^n} \in {}^n\tilde{\mathcal{M}}^{2^n}_1} \{p^n_{K,\lambda}(z_1, y_1, \cdots, z_{2^n}, y_{2^n})\}$$

by considering strategies we see that this is equivalent to

$$(6.23)\qquad W^+_{n,\lambda} = \inf_{\Pi \in \tilde{\Upsilon}^n} \sup_{\Sigma \in \tilde{\Gamma}^n} p^n_{K,\lambda}(\Sigma, \Pi).$$

□

Note: Similarly, we could define a game $K^-_{n,\lambda}$ in which J_1 plays first at each stage, this game has value denote by $W^-_{n,\lambda}$.

6.4.1 The dynamics for the game $K^+_{n,\lambda}$

If we go back to the dynamics for the game $K^+_{n,\lambda}$ we see that

$$x^n_{K,\lambda}(t_j) = x(0) + \sum_{i=1}^{j} \Delta_n f(t_{i-1}, x^n_{K,\lambda}(t_{i-1}), y_i, z_i) + \lambda \sum_{i=1}^{j} (\Delta_n)^{\frac{1}{2}} \eta_{i-1}.$$

Now we change the notation. For each $j = 0, \cdots, (2^n - 1)$ let

$$(6.24)\qquad \Delta B(t_j) = (\Delta_n)^{\frac{1}{2}} \eta_j$$

then we see that this gives

$$x^n_{K,\lambda}(t_j) = x(0) + \sum_{i=1}^{j} \Delta_n f(t_{i-1}, x^n_{K,\lambda}(t_{i-1}), y_i, z_i) + \lambda \sum_{i=1}^{j} \Delta B(t_{i-1})$$

and if we let

$$(6.25)\qquad B(\omega, t_j) = \sum_{i=1}^{j} \Delta B(t_{i-1}) \qquad \text{with} \qquad B(\omega, 0) = 0$$

then, for each $B(\omega, t_j)$ is defined for $j = 0, 1, \cdots, 2^n$ and we have

$$x^n_{K,\lambda} = x(0) + \sum_{i=1}^{j} \Delta_n f(t_{i-1}, x^n_{K,\lambda}(t_{i-1}), y_i, z_i) + \lambda B(\omega, t_j). \tag{6.26}$$

Now, using the information on the Gaussian distribution given in Appendix A and the fact that

$$\eta_j \sim \mathcal{N}(0,1) \tag{6.27}$$

for each $j = 0, 1, \cdots, (2^n - 1)$ we get the following results:

Lemma 6.4.2

(i) $\Delta B(t_j) \sim \mathcal{N}(0, \Delta_n)$ for all $j = 0, 1, \cdots, (2^n - 1)$.
(ii) $B(\omega, t_j) \sim \mathcal{N}(0, t_j)$ for all $j = 0, 1, \cdots, 2^n$ and each fixed ω.
(iii) $\mathbb{E}[B(\omega, \frac{1}{2^n})^4] = \frac{3}{2^{2n}}$ for each fixed ω

Proof (i): Fix j and ω.

$$\mathbb{E}[\Delta B(t_j)] = \mathbb{E}[(\Delta_n)^{\frac{1}{2}} \eta_j] = (\Delta_n)^{\frac{1}{2}} \mathbb{E}[\eta_j] = 0$$

$$\begin{aligned}
\text{var}[\Delta B(t_j)] &= \mathbb{E}[(\Delta B(t_j) - \mathbb{E}[\Delta B(t_j)])^2] = \mathbb{E}[(\Delta B(t_j))^2] \\
&= \mathbb{E}[((\Delta_n)^{\frac{1}{2}} \eta_j)^2] && \text{by (6.24)} \\
&= \Delta_n \mathbb{E}[\eta_j^2] \\
&= \Delta_n && \text{by (6.27).}
\end{aligned}$$

Proof (ii): Fix j and ω.

$$\mathbb{E}[B(\omega, t_j)] = \mathbb{E}[\sum_{i=1}^{j} \Delta B(t_{i-1})] = \sum_{i=1}^{j} \mathbb{E}[\Delta B(t_{i-1})] = 0 \qquad \text{by (i)}$$

$$\text{var}[B(\omega, t_j)] = \sum_{i=1}^{j} \text{var}[\Delta B(t_{i-1})] = j\Delta_n = t_j \qquad \text{by (6.25).}$$

Proof (iii): Fix ω.

$$\begin{aligned}
\mathbb{E}[B(\omega, \frac{1}{2^n})^4] &= \mathbb{E}[(\Delta B(t_0))^4] && \text{by (6.25) since } \tfrac{1}{2^n} = t_1 \\
&= \mathbb{E}[((\Delta_n)^{\frac{1}{2}} \eta_0)^4] && \text{by (6.24)} \\
&= (\Delta_n)^2 \mathbb{E}[\eta_0^4] \\
&= 3(\Delta_n)^2 && \text{by (A.5)} \\
&= \frac{3}{2^{2n}} && \text{since } \Delta_n = \frac{1}{2^n}.
\end{aligned}$$

□

We now give, for completeness, a result which can be found in [7] and [9] and is essentially Anderson's construction of Brownian motion ([2]).

Proposition 6.4.3

For each infinite $N \in {}^*\mathbb{N}$, $B(\omega, \bullet)$ is $\mathcal{S}$-continuous for a.a. ω with respect to the Loeb measure.

Proof: Let

$$\Omega_{m,n} = \{\omega : \exists i : 0 \leq i \leq 2^{(m-1)} \exists t \in [\frac{i}{2^m}, \frac{i+1}{2^m}] : |B(\omega, \frac{i}{2^m}) - B(\omega, t)| \geq \frac{1}{n}\},$$

then $B(\omega, \bullet)$ is $\mathcal{S}$-continuous $\Leftrightarrow \forall n \exists m (\omega \in \Omega^c_{m,n})$. Therefore to show that $B(\omega, \bullet)$ is $\mathcal{S}$-continuous for a.a. ω with respect to the Loeb measure, we have to show that

$$\mu_L(\cap_n \cup_m \Omega^c_{m,n}) = 1$$

i.e.

$$\mu_L(\cup_n \cap_m \Omega_{m,n}) = 0.$$

This is equivalent to showing that

$$\forall n \mu_L(\Omega_{m,n}) \to 0 \qquad \text{as } m \to \infty.$$

$$\begin{aligned}
\mu(\Omega_{m,n}) &\leq \sum_{i=0}^{2^m-1} \mu(\{\omega : \exists t \in [\frac{i}{2^m}, \frac{i+1}{2^m}] : |B(\omega, \frac{i}{2^m}) - B(\omega, t)| \geq \frac{1}{n}\}) \\
&= 2^m \mu(\{\omega : \exists t \in [0, \frac{1}{2^m}] : |B(\omega, 0) - B(\omega, t)| \geq \frac{1}{n}\}) \\
&\leq 2 \times 2^m \mu(\{\omega : |B(\omega, \frac{1}{2^m})| \geq \frac{1}{n}\}) \\
&\doteq 2^{m+1} \mu(\{\omega : |B(\omega, \frac{1}{2^m})^2| \geq \frac{1^2}{n}\}) \\
&\leq 2^{m+1} n^4 \mathbb{E}(B(\omega, \frac{1}{2^m})^4) \qquad \text{by Chebychev's inequality (see (A.6))} \\
&= 2^{m+1} n^4 \frac{3}{2^{2m}} \qquad \text{by Lemma 6.4.2 (iii)} \\
&= \frac{6n^4}{2^m} \to 0 \qquad \text{as } m \to \infty.
\end{aligned}$$

Therefore $B(\omega, \bullet)$ is $\mathcal{S}$-continuous for a.a. ω.

□

The above Proposition give us the following result.

Corollary 6.4.4

If $\lambda \approx 0$ then for a.a. ω, $\lambda B(\omega, t_j) \approx 0$ for all t_j where $j = 0, 1, \cdots, 2^N$ where $N \in *\mathbb{N} \backslash \mathbb{N}$.

□

Proposition 6.4.5

For each fixed infinite $N \in {}^*\mathbb{N}$, given a pair of N-constant controls $Y \in {}^{N*}\tilde{\mathcal{M}}_1$ and $Z \in {}^{N*}\tilde{\mathcal{M}}_2$ for each initial state, $x(0)$, the trajectory $X^N_{K,\lambda} : \mathbf{T}_N \to {}^*\mathbb{R}^d$ is $\mathcal{S}$-continuous for a.a ω.

Proof: Take $t_j > t_k$ with $t_j \approx t_k$ where t_j and $t_k \in \mathbf{T}_N$ then,

$$\begin{aligned}
& |X^N_{K,\lambda}(t_j) - X^N_{K,\lambda}(t_k)| \\
= \ & |x(0) + \Delta_n \sum_{i=1}^{j} {}^*f(t_{i-1}, X^N_{K,\lambda}(t_{i-1}), Y_i, Z_i) + \lambda B(\omega, t_j) \\
& \quad -x(0) - \Delta_n \sum_{i=1}^{k} {}^*f(t_{i-1}, X^N_{K,\lambda}(t_{i-1}), Y_i, Z_i) - \lambda B(\omega, t_k)| \\
\leq \ & \Delta_n \sum_{i=k+1}^{j} |{}^*f(t_{i-1}, X^N_{K,\lambda}(t_{i-1}), Y_i, Z_i)| + |\lambda B(\omega, t_j) - \lambda B(\omega, t_k)| \\
\leq \ & R(t_j - t_k) + \lambda |B(\omega, t_j) - B(\omega, t_k)| \qquad \text{by (2.5)} \\
\approx \ & R(t_j - t_k) \qquad \text{by Proposition 6.4.3} \\
\approx \ & 0 \qquad \text{since } t_j \approx t_k.
\end{aligned}$$

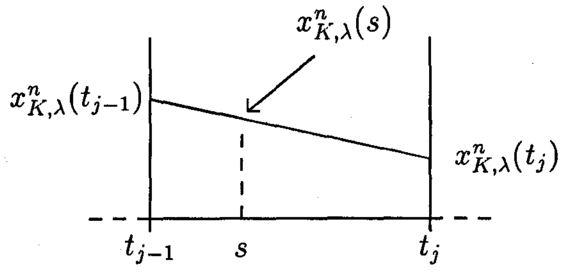
□

We would like to be able to define $X^N_{K,\lambda}(\sigma)$ for $\sigma \notin \mathbf{T}_N$ therefore we make the following definition.

Definition 6.4.6

The function $x^n_{K,\lambda}$ is extended so that $x^n_{K,\lambda} : [0,1] \to \mathbb{R}^d$ with the following definition.

For $s \in]t_{j-1}, t_j]$, we define $x^n_{K,\lambda}(s)$ by linearly joining up $x^n_{K,\lambda}(t_{j-1})$ and $x^n_{K,\lambda}(t_j)$



i.e. if $s \in]t_{j-1}, t_j]$ then $x^n_{K,\lambda}(s)$ is given by

$$x^n_{K,\lambda}(s) = x^n_{K,\lambda}(t_j)(1 - j + \frac{s}{\Delta_n}) + x^n_{K,\lambda}(t_{j-1})(j - \frac{s}{\Delta_n}).$$

Remarks 6.4.7

(i) In the nonstandard setting with the above definition of $X^N_{K,\lambda}(\tau)$ for $\tau \notin \mathbf{T}_N$, we see that if $N \in {}^*\mathbb{N}$ is infinite then we have a function $X^N_{K,\lambda} : {}^*[0,1] \to {}^*\mathbb{R}^d$ which is $\mathcal{S}$-continuous a.s.

From here onwards when we refer to the function $X^N_{K,\lambda}$ we will mean the extended version $X^N_{K,\lambda} : {}^*[0,1] \to {}^*\mathbb{R}^d$ i.e. $X^N_{K,\lambda}$ is defined for **all** $\tau \in {}^*[0,1]$.

Proposition 6.4.8

For a fixed infinite $N \in {}^*\mathbb{N}$, given a pair of N-controls $Y \in {}^{N*}\tilde{\mathcal{M}}_1$ and $Z \in {}^{N*}\tilde{\mathcal{M}}_2$ then if $\lambda \approx 0$ then for a.a. ω, ${}^\circ X^N_{K,\lambda}$ is a solution to equation (3.3).

Proof: For a.a. ω, for $\tau \in]t_{j-1}, t_j]$

$$\begin{aligned}
{}^\circ X^N_{K,\lambda}(\tau) &= {}^\circ X^N_{K,\lambda}(t_j) \\
&\qquad \text{(since } X^N_{K,\lambda} \text{ is } \mathcal{S}\text{-continuous)} \\
&= {}^\circ\Big(x(0) + \Delta_n \sum_{i=1}^{j} {}^*f(t_{i-1}, X^N_{K,\lambda}(t_{i-1}), Y_i, Z_i) + \lambda B(\omega, t_j)\Big) \\
&= {}^\circ\Big(x(0) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} {}^*f(t_{i-1}, X^N_{K,\lambda}(t_{i-1}), Y_i, Z_i)) d\sigma + \lambda B(\omega, t_j)\Big) \\
&= {}^\circ\Big(x(0) + \int_0^{\tau} {}^*f(\sigma, X^N_{K,\lambda}(\sigma), Y(\sigma), Z(\sigma)) d\sigma + \lambda B(\omega, t_j)\Big) \\
&\qquad \text{(by } \mathcal{S}\text{-continuity of } X^N_{K,\lambda} \text{ and continuity of } f) \\
&= x(0) + \int_0^{\tau} f({}^\circ\sigma, {}^\circ X^N_{K,\lambda}(\sigma), {}^\circ Y(\sigma), {}^\circ Z(\sigma)) d\sigma_L + {}^\circ(\lambda B(\omega, t_j)) \\
&= x(0) + \int_0^{\tau} f({}^\circ\sigma, {}^\circ X^N_{K,\lambda}(\sigma), {}^\circ Y(\sigma), {}^\circ Z(\sigma)) d\sigma_L \\
&\qquad \text{(by Corollary 6.4.4)}
\end{aligned}$$

and so we see that ${}^\circ X^N_{K,\lambda}$ solves (3.3) when $N \in {}^*\mathbb{N}$ is infinite and $\lambda \approx 0$.

□

6.5 Comparing the games K_n^+ and $K_{n,\lambda}^+$

We now show that in the nonstandard setting, for a fixed pair of N-constant controls where N is infinite, the trajectories in the games K_N^+ and $K_{N,\lambda}^+$ are infinitely close when λ is infinitesimal.

Theorem 6.5.1

For a fixed pair of N-constant controls $Y \in {}^{N*}\tilde{\mathcal{M}}_1$ and $Z \in {}^{N*}\tilde{\mathcal{M}}_2$ if N is infinite and $\lambda \approx 0$, the trajectories $X^N_K(\tau)$ and $X^N_{K,\lambda}(\tau)$ are close in the sense of the uniform topology i.e. for a.a. ω

$$\sup_{\tau \in {}^*[0,1]} |X^N_K(\tau) - X^N_{K,\lambda}(\tau)| \approx 0.$$

Proof: By Remarks 3.4.2 (iii) we know that if $N \in {}^*\mathbb{N}$ is infinite then ${}^\circ X^N_K(\tau)$ solves equation (3.3) and by Proposition 6.4.8 we know that if $N \in {}^*\mathbb{N}$ is infinite and $\lambda \approx 0$ then ${}^\circ X^N_{K,\lambda}(\tau)$ also solves equation (3.3) for a.a. ω. Now, it was shown in Proposition 3.2.3 that (3.3) has a unique solution and so, for a.a. ω

$$ {}^\circ X^N_{K,\lambda}(\tau) = {}^\circ X^N_K(\tau) \qquad \text{for all } \tau \in {}^*[0,1]. $$

□

We have shown that when $N \in {}^*\mathbb{N}$ is infinite and $\lambda \approx 0$, the trajectories in the two games $K^+_{N,\lambda}$ and K^+_N are infinitely close. We now show that if $N \in {}^*\mathbb{N}$ is infinite and $\lambda \approx 0$ then the payoffs in the two games are also infinitely close.

Proposition 6.5.2

Given an infinite N, for each fixed pair of N-constant controls $Y \in {}^{N*}\tilde{\mathcal{M}}_1$ and $Z \in {}^{N*}\tilde{\mathcal{M}}_2$

$$ P^N_{K,\lambda}(Y,Z) \approx P^N_K(Y,Z) $$

if N is infinite and $\lambda \approx 0$.

Proof: Fix the controls and fix $\lambda \approx 0$ then we have the following

$$
\begin{aligned}
P^N_{K,\lambda}(Y,Z) &= \hat{\mathbb{E}}_0[\sum_{j=1}^{2^N}\int_{t_{j-1}}^{t_j} {}^*h(t_{j-1}, X^N_{K,\lambda}(t_{j-1}), Y_j, Z_j)d\sigma + {}^*g(X^N_{K,\lambda}(1))] \\
&\approx \hat{\mathbb{E}}_0[\int_0^1 {}^*h(\sigma, X^N_{K,\lambda}(\sigma), Y(\sigma), Z(\sigma))d\sigma + {}^*g(X^N_{K,\lambda}(1))] \\
&= \int_0^1 {}^*h(\sigma, X^N_K(\sigma), Y(\sigma), Z(\sigma))d\sigma + {}^*g(X^N_K(1)) \\
&\quad \text{(by Theorem 6.5.1 and continuity of } h \text{ and } g) \\
&\approx P^N_K(Y,Z) \qquad \text{(by Remarks 3.4.2 (v)).}
\end{aligned}
$$

□

Using the above we go on to show that the values of the games K^+_N and $K^+_{N,\lambda}$ are infinitely close when $N \in {}^*\mathbb{N}$ is infinite and $\lambda \approx 0$.

Theorem 6.5.3

For a fixed infinite $N \in {}^*\mathbb{N}$,

$$ W^+_{N,\lambda} \approx W^+_N $$

for all $\lambda > 0$, $\lambda \approx 0$.

Proof: Fix $N \in {}^*\mathbb{N}$ infinite and $\lambda \approx 0$, $\lambda > 0$.

$$W_N^+ = \inf_{\Pi \in \tilde{\Upsilon}^N} \sup_{\Sigma \in \tilde{\Gamma}^N} P_K^N(\Sigma, \Pi)$$

$$W_{N,\lambda}^+ = \inf_{\Pi \in \tilde{\Upsilon}^N} \sup_{\Sigma \in \tilde{\Gamma}^N} P_{K,\lambda}^N(\Sigma, \Pi)$$

By Proposition 6.5.2

$$P_K^N(\Sigma, \Pi) \approx P_{K,\lambda}^N(\Sigma, \Pi)$$

for all $\Sigma \in \tilde{\Gamma}^N$ and $\Pi \in \tilde{\Upsilon}^N$. Therefore, since the operations sup and inf preserve the infinite closeness (Lemmas D.1.1 and D.1.2) the result follows.

□

Similarly, by comparing the nonstandard versions of the games K_n^- and $K_{n,\lambda}^-$, it can be seen that

$$W_{N,\lambda}^- \approx W_N^-$$

for all infinite $N \in {}^*\mathbb{N}$ and all $\lambda > 0$, $\lambda \approx 0$.

□

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Chapter 7

Lipschitz conditions and $W^+_{n,\lambda}(t_j, \zeta)$

7.1 The Lipschitz condition in ζ

In this chapter we show that the values $W^+_{n,\lambda}$ and $W^-_{n,\lambda}$ satisfy a uniform Lipschitz condition in x. To do this we have to place further constraints on our functions, namely Flemings conditions (F1)-(F3) but still find it unnecessary to assume the remaining two conditions, (F4) and (F5).

Proposition 7.1.1

If G satisfies Fleming's conditions (F1)-(F3) then for all $j = 0, 1, \cdots, 2^n$ there exists a constant c_j such that

$$|W^+_{n,\lambda}(t_j, \zeta_1) - W^+_{n,\lambda}(t_j, \zeta_2)| \leq c_j|\zeta_1 - \zeta_2| \tag{7.1}$$

whenever $\zeta_1,\ \zeta_2 \in \mathbb{R}^d$.

Proof: Fix an integer n and fix $\lambda > 0$.

Consider $j = 2^n$

$$|W^+_{n,\lambda}(t_{2^n}, \zeta_1) - W^+_{n,\lambda}(t_{2^n}, \zeta_2)| = |g(\zeta_1) - g(\zeta_2)| \leq Q|\zeta_1 - \zeta_2|$$

where Q is the Lipschitz constant on g, so here we use Fleming's condition (F3). Therefore if we let $c_{2^n} = Q$ then (7.1) holds when $j = 2^n$.

Now assume (7.1) holds when $j = (i+1)$ i.e.

$$|W^+_{n,\lambda}(t_{i+1},\zeta_1) - W^+_{n,\lambda}(t_{i+1},\zeta_2)| \le c_{i+1}|\zeta_1 - \zeta_2| \tag{7.2}$$

whenever ζ_1, $\zeta_2 \in \mathbb{R}^d$. We show assuming (7.2) that (7.1) holds when $j = i$.

Now recall, from Lemma 6.3.4

$$W^+_{n,\lambda}(t_i,\zeta) = \min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta') + \Delta_n h(t_i,\zeta,y,z)]\}$$

where

$$\zeta' = \zeta + \Delta_n f(t_i,x,y,z) + \lambda(\Delta_n)^{\frac{1}{2}}\eta_i$$

and

$$W^+_{n,\lambda}(t_{2^n},\zeta) = g(\zeta).$$

For each fixed $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ consider the following

$$\begin{aligned}
&|\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_1{+}\Delta_n f(t_i,\zeta_1,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i){+}\Delta_n h(t_i,\zeta_1,y,z)] \\
&\quad -\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_2{+}\Delta_n f(t_i,\zeta_2,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i){+}\Delta_n h(t_i,\zeta_2,y,z)]| \\
\le\ &|\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_1{+}\Delta_n f(t_i,\zeta_1,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)] \\
&\quad -\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_2{+}\Delta_n f(t_i,\zeta_2,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)]{+}\Delta_n|h(t_i,\zeta_1,y,z){-}h(t_i,\zeta_2,y,z)| \\
\le\ &\mathbb{E}_i[|W^+_{n,\lambda}(t_{i+1},\zeta_1{+}\Delta_n f(t_i,\zeta_1,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i) \\
&\quad -W^+_{n,\lambda}(t_{i+1},\zeta_2{+}\Delta_n f(t_i,\zeta_2,y,z){+}(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)|]{+}\Delta_n|h(t_i,\zeta_1,y,z){-}h(t_i,\zeta_2,y,z)| \\
\le\ &c_{i+1}|\zeta_1 + \Delta_n f(t_i,\zeta_1,y,z) - \zeta_2 - \Delta_n f(t_i,\zeta_2,y,z)| + \Delta_n D|\zeta_1 - \zeta_2|
\end{aligned}$$

by the assumption (7.2), where D is the Lipschitz constant on h, so we have used Fleming's condition (F2). This gives us

$$\begin{aligned}
&\le c_{i+1}(|\zeta_1 - \zeta_2| + \Delta_n\kappa|\zeta_1 - \zeta_2|) + \Delta_n D|\zeta_1 - \zeta_2| \qquad \text{by (2.4)} \\
&\le (c_{i+1}(1 + \Delta_n\kappa) + \Delta_n D)|\zeta_1 - \zeta_2| \\
&= c_i|\zeta_1 - \zeta_2|
\end{aligned}$$

where

$$\begin{aligned}
c_i &= c_{i+1}(1 + \Delta_n\kappa) + \Delta_n D \\
\text{and} \quad c_{2^n} &= Q
\end{aligned}$$

κ is the Lipschitz constant for f, so we have used Fleming's condition (F1) , Q is the Lipschitz constant for g and D is the Lipschitz constant for h.

Now, by Lemma C.1.1 and the above, we have

$$\begin{aligned} &|\min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_1+\Delta_n f(t_i,\zeta_1,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_1,y,z)]\} \\ &\qquad -\min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1},\zeta_2+\Delta_n f(t_i,\zeta_2,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_2,y,z)]| \\ \leq\ & c_i|\zeta_1-\zeta_2| \end{aligned}$$

i.e. by Lemma 6.3.4 we see that this is the same as

$$|W^+_{n,\lambda}(t_i,\zeta_1)-W^+_{n,\lambda}(t_i,\zeta_2)|\leq c_i|\zeta_1-\zeta_2|$$

therefore equation (7.1) holds when $j=i$.

Hence equation (7.1) holds for all $j=0,1,\cdots,2^n$ by induction.

Note: To get to this result we have only used Fleming's conditions (F1)-(F3) and not (F4) or (F5).

□

Note: Similarly, by considering the game $K^-_{n,\lambda}$ it can be seen that for each $j=0,1,\cdots,2^n$ there exists a constant c_j such that

$$|W^-_{n,\lambda}(t_j,\zeta_1)-W^-_{n,\lambda}(t_j,\zeta_2)|\leq c_j|\zeta_1-\zeta_2| \tag{7.3}$$

where ζ_1 and $\zeta_2\in\mathbb{R}^d$.

Similarly, this result only requires Fleming's conditions (F1)-(F3).

□

We now show that in the nonstandard setting, if $N\in{}^*\mathbb{N}$ is infinite then we actually have a uniform Lipschitz condition in ζ.

Theorem 7.1.2

If G satisfies conditions (F1)-(F3) then, for each fixed infinite $N \in {}^*\mathbb{N}$, there exists a finite constant c such that

$$|W^+_{N,\lambda}(t_j,\zeta_1) - W^+_{N,\lambda}(t_j,\zeta_2)| \leq c|\zeta_1 - \zeta_2|$$

for all $j = 0, 1, \cdots, 2^N$, when $\zeta_1, \zeta_2 \in {}^*\mathbb{R}^d$.

Proof: By Theorem 7.1.1 we have

$$|W^+_{N,\lambda}(t_j,\zeta_1) - W^+_{N,\lambda}(t_j,\zeta_2)| \leq c_j|\zeta_1 - \zeta_2|$$

for each $j = 0, 1, \cdots, 2^N$ where

$$\begin{aligned} c_i &= c_{i+1}(1 + \Delta_N\kappa) + \Delta_N D \qquad \text{for } i = 0, 1, \cdots, (2^n - 1) \\ \text{and} \qquad c_{2^N} &= Q \end{aligned}$$

where κ, Q and D are the Lipschitz constants on f, g and h respectively as in Fleming's conditions (F1)-(F3).

Now, we consider a function defined by

$$\begin{aligned} \theta(0) &= Q \\ \theta(t + \Delta_N) &= \theta(t) + (\theta(t)\kappa + D)\Delta_N \end{aligned}$$

i.e. $\theta(t_j) = c_{2^n - j}$ for each $j = 0, 1, \cdots, 2^n$.

Note, $\theta(0) > 0$ which implies $\theta(t)$ increases with t.

$$\begin{aligned} \theta(t)\kappa + D &= \theta(t)\kappa + \frac{D}{\theta(0)}\theta(0) \\ &\leq \theta(t)\kappa + \frac{D}{\theta(0)}\theta(t) \\ &= E\theta(t) \qquad \text{where } E = (\kappa + \tfrac{D}{Q}) \text{ is a constant.} \end{aligned}$$

Therefore we have

$$\begin{aligned} \theta(t + \Delta_N) &\leq \theta(t) + E\theta(t)\Delta_N \\ &= (1 + E\Delta_N)\theta(t) \end{aligned}$$

and so

$$\begin{aligned}\theta(t_j) &\leq (1+E\Delta_N)^j\theta(0)\\ &= (1+\frac{1}{j}Ej\Delta_N)^j\theta(0)\\ &= (1+\frac{1}{j}Et_j)^j\theta(0)\\ &\approx \theta(0)e^{Et_j}\end{aligned}$$

for all $j = 0, 1, \cdots, 2^N$. So

$$\theta(t_j) \lesssim c \qquad \text{where } c = Qe^E.$$

Thus there exists a c such that

$$c_j \leq c < \infty \qquad \text{for all } j = 0, 1, \cdots, 2^N. \tag{7.4}$$

Therefore by (7.4) and Proposition 7.1.1 we have the required result.

□

Note: Similarly, by considering the game $K^-_{N,\lambda}$ with $N \in {}^*\mathbb{N}$ infinite, it can be seen that there exists a constant c such that

$$|W^-_{N,\lambda}(t_j, \zeta_1) - W^-_{N,\lambda}(t_j, \zeta_2)| \leq c|\zeta_1 - \zeta_2|$$

for all $j = 0, 1, \cdots, 2^N$, ζ_1, $\zeta_2 \in {}^*\mathbb{R}^d$.

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Chapter 8

The existence of the values W^+ and W^-

In this chapter we show that the value W^+ exists and that $W^+ = W_1^+$ and similarly W^- exists and $W^- = W_2^-$.

To do this we consider the game $K_{n,\lambda}^+$ with one player using relaxed controls and compare this to when both players are using ordinary controls. Therefore we give details of this first and then go on to show the existence of W^+ and W^-.

8.1 The game $K_{n,\lambda}^+(t_j, \zeta)$ compared to $K_{n,\lambda,1}^+(t_j, \zeta)$

The game $K_{n,\lambda,1}^+(t_j, \zeta)$ is played in exactly the same way as the game $K_{n,\lambda}^+(t_j, \zeta)$ except that now player J_1 is allowed to use relaxed controls while J_2 is still restricted to the ordinary controls.

Theorem 8.1.1

If G satisfies (F1)-(F3) then, for each infinite $N \in {}^*\mathbb{N}$, there exists $\lambda > 0$, $\lambda \approx 0$ such that

$$W_{N,\lambda}^+(t_j, \zeta) \approx W_{N,\lambda,1}^+(t_j, \zeta) \tag{8.1}$$

for all $j = 0, 1, \cdots, 2^N$ and all $\zeta \in {}^*\mathbb{R}^d$.

Proof: Here, for ease of notation we set $d = 1$; all of the results hold for $\mathbb{R}^d$ with only notational changes.

Fix an infinite $N \in {}^*\mathbb{N}$, fix $\lambda > 0$ and $\zeta \in {}^*\mathbb{R}^d$.

Take $j = 2^N$

$$W^+_{N,\lambda}(t_{2^N}, \zeta) = {}^*g(\zeta) = W^+_{N,\lambda,1}(t_{2^N}, \zeta)$$

so (8.1) holds when $j = 2^N$.

Now we assume that (8.1) holds when $j = (i+1)$, i.e. assume that for each $\zeta \in {}^*\mathbb{R}^d$

$$(8.2) \qquad |W^+_{N,\lambda}(t_{i+1}, \zeta) - W^+_{N,\lambda,1}(t_{i+1}, \zeta)| \leq \epsilon_{i+1}$$

where $\epsilon_{i+1} \approx 0$.

Now consider $j = i$, let

$$\psi(\zeta) = \mathbb{E}_i[W^+_{N,\lambda,1}(t_{i+1}, \zeta + (\Delta_N)^{\frac{1}{2}}\lambda\eta_i)]$$

and let

$$\theta(\nu, Z) = {}^*f(t_i, \zeta, \nu, Z) \quad \text{and} \quad \bar{\theta}(\nu, Z) = {}^*h(t_i, \zeta, \nu, Z) \quad \text{for each } \nu \in {}^*\Lambda(\mathcal{Y})$$

and

$$\psi(\zeta + \Delta_N\theta(\nu, Z)) + \Delta_N\bar{\theta}(\nu, Z) = \Theta(\nu, Z).$$

By Appendix B, Lemma B.1.1 we know that ψ is twice differentiable and so we can use Taylor's series to obtain the following:
For each fixed $Z \in {}^*\mathcal{Z}$ and $\nu \in {}^*\Lambda(\mathcal{Y})$

$$\begin{aligned}
& \psi(\zeta + \Delta_N\theta(\nu, Z)) \\
= \; & \psi(\zeta) + \Delta_N\theta(\nu, Z)\psi'(\zeta) + \frac{1}{2}(\Delta_N)^2\theta(\nu, Z)^2\psi''(\bar{\zeta})
\end{aligned}$$

for some $\bar{\zeta}$ between 0 and ζ and so by Lemma B.2.1 (ii) we have

$$= \; \psi(\zeta) + \Delta_N\theta(\nu, Z)\psi'(\zeta) + \epsilon$$

where $|\epsilon| \leq \frac{1}{2}(\Delta_N)^{\frac{3}{2}}R^2\frac{c}{\lambda}\sqrt{\frac{2}{\pi}}$ (here c is the Lipschitz constant as in Chpt 7 and R is the bound on f)

and so for each fixed $Z \in {}^*\mathcal{Z}$ we have

$$\begin{aligned}
& \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) \\
= \quad & \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \psi(\zeta) + \Delta_N \theta(\nu, Z)\psi'(\zeta) + \Delta_N \bar{\theta}(\nu, Z) + \epsilon' \\
& \text{where } |\epsilon'| \leq \tfrac{1}{2}(\Delta_N)^{\frac{3}{2}} R^2 \tfrac{c}{\lambda}\sqrt{\tfrac{2}{\pi}} \text{ and so by Lemma D.2.1 we have} \\
= \quad & \max_{Y \in {}^*\mathcal{Y}} \psi(\zeta) + \Delta_N \theta(Y, Z)\psi'(\zeta) + \Delta_N \bar{\theta}(Y, Z) + \epsilon' \\
& \text{which by Taylor's series is} \\
= \quad & \max_{Y \in {}^*\mathcal{Y}} \Theta(\nu, Z) + \epsilon' + \epsilon'' \\
& \text{where by Lemma B.2.1 (ii) } |\epsilon''| \leq \tfrac{1}{2}(\Delta_N)^{\frac{3}{2}} R^2 \tfrac{c}{\lambda}\sqrt{\tfrac{2}{\pi}}
\end{aligned}$$

i.e. for each fixed $Z \in {}^*\mathcal{Z}$ we have

$$|\max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) - \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z)| \leq (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}.$$

Now by Lemma D.2.2 we see that

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z)| \leq (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}.$$

From this, since

$$\psi(\zeta + \Delta_N \theta(\nu, Z)) = \mathbb{E}_i[W^+_{N,\lambda,1}(t_{i+1}, \zeta + \Delta_N \theta(\nu, Z) + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)]$$

and $\theta(\nu, Z) = {}^*f(t_i, \zeta, \nu, Z)$ we see that

$$\begin{aligned}
& |\min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \{\mathbb{E}_i[W^+_{N,\lambda,1}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i, \zeta, \nu, Z) + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)] + \Delta_N {}^*h(t_i, \zeta, \nu, Z)\} \\
& - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \{\mathbb{E}_i[W^+_{N,\lambda,1}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i, \zeta, Y, Z) + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)] + \Delta_N {}^*h(t_i, \zeta, Y, Z)\}| \\
& \leq (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}
\end{aligned}$$

and so by the assumption, (8.2), this means

$$\begin{aligned}
& |\min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \{\mathbb{E}_i[W^+_{N,\lambda,1}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i, \zeta, \nu, Z) + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)] + \Delta_N {}^*h(t_i, \zeta, \nu, Z)\} \\
& - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \{\mathbb{E}_i[(W^+_{N,\lambda}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i, \zeta, Y, Z) + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)] + \Delta_N {}^*h(t_i, \zeta, Y, Z)\}| \\
& \leq (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} + \epsilon_{i+1}.
\end{aligned}$$

By Lemma 6.3.4 this is the same as

$$|W^+_{N,\lambda,1}(t_i, \zeta) - W^+_{N,\lambda}(t_i, \zeta)| \leq (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} + \epsilon_{i+1}.$$

Now, by definition $W^+_{N,\lambda}(t_{2^n},\zeta) = W^+_{N,\lambda,1}(t_{2^n},\zeta) = g(\zeta)$ therefore $\epsilon_{2^N} = 0$ which means for each $i = 0, 1, \cdots, 2^N$

$$|W^+_{N,\lambda,1}(1 - i\Delta_N,\zeta) - W^+_{N,\lambda}(1 - i\Delta_N,\zeta)| \leq i(\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}.$$

Now if we require

$$W^+_{N,\lambda,1}(t_j,\zeta) \approx W^+_{N,\lambda}(t_j,\zeta) \tag{8.3}$$

for all $j = 0, 1, \cdots, 2^N$ then we require

$$2^N (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} \approx 0$$

i.e. we require

$$\frac{(\Delta_N)^{\frac{1}{2}}}{\lambda} \approx 0. \tag{8.4}$$

Such λ exist, one such λ is given by $\lambda = (\Delta_N)^{\frac{1}{4}}$.

□

Corollary 8.1.2

If G satisfies (F1)-(F3) then, for each fixed infinite $N \in {}^*\mathbb{N}$ there exists $\lambda \approx 0$ such that

$$W^+_{N,\lambda,1} \approx W^+_{N,\lambda}. \tag{8.5}$$

Proof:

$$W^+_{N,\lambda,1} = W^+_{N,\lambda,1}(0,0) \quad \text{and} \quad W^+_{N,\lambda}(0,0) = W^+_{N,\lambda} \tag{8.6}$$

therefore this result follows directly from Theorem 8.1.1.

□

Similarly, by considering the game $K^-_{N,\lambda}$ it can be seen that if G satisfies (F1)-(F3) then, given any infinite $N \in {}^*\mathbb{N}$ there exists $\lambda \approx 0$ such that

$$W^-_{N,\lambda,2} \approx W^-_{N,\lambda}. \tag{8.7}$$

Note: The set of λ's satisfying (8.5) is the same as the set of λ's satisfying (8.7) – both are given by (8.4).

8.2 The existence of W^+ and W^-

We can now go on to show the existence of the values $W^+$and W^- without having assumed all of the conditions which Fleming found necessary.

Theorem 8.2.1

If G satisfies (F1)-(F3) then, the values W_n^+ of the games K_n^+ tend to a limit denoted by W^+ i.e. the limit

$$W^+ = \lim_{n\to\infty} W_n^+$$

exists and $W^+ = W_1^+$.

Proof: By Theorem 6.4.3 and Corollary 8.1.2 given an infinite $N \in {}^*\mathbb{N}$ there exists $\lambda \approx 0$, $\lambda > 0$ such that

$$W_{N,1}^+ \approx W_{N,\lambda,1}^+ \approx W_{N,\lambda}^+ \approx W_N^+$$

so for infinite $N \in {}^*\mathbb{N}$ we have

$$W_{N,1}^+ \approx W_N^+.$$

By Theorem 5.4.2 we know that $W_{N,1}^+ \approx W_1^+$ for infinite N therefore

$$\lim_{n\to\infty} W_n^+ = W^+ \qquad \text{exists}$$

and

$$W^+ = W_1^+.$$

□

From the above Theorem applied to the game G_2, we clearly have the existence of the limit

$$W_2^+ = \lim_{n\to\infty} W_{n,2}^+$$

if G satisfies (F1)-(F3) and $W_2^+ = W_{12}^+$.

Theorem 8.2.2

If G satisfies (F1)-(F3) then, the values W_n^- of the games K_n^- tend to a limit denoted by W^- i.e. the limit

$$W^- = \lim_{n\to\infty} W_n^-$$

exists and $W^- = W_2^-$.

Proof: An analogous proof to that of Theorem 8.2.1 above gives the required result.

□

From the above Theorem applied to the game G_1, we clearly have the existence of the limit

$$W_1^- = \lim_{n\to\infty} W_{n,1}^-$$

if G satisfies (F1)-(F3) and $W_1^- = W_{12}^-$.

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Chapter 9

The values W^+, W^- and the Isaacs condition

In this chapter we show that if the Isaacs condition holds then we can guarantee the existence of value.

First we give a proposition which we will then use to prove the result which leads to the proof of the above claim.

9.1 $W^+ = W^-$ given the Isaacs condition

Proposition 9.1.1

For a fixed infinite $N \in {}^*\mathbb{N}$, if G satisfies (F1)-(F3) and the Isaacs condition (6.15) holds then $\exists \lambda > 0$, $\lambda \approx 0$ such that

$$W^+_{N,\lambda}(t_j, \zeta) \approx W^-_{N,\lambda}(t_j, \zeta) \tag{9.1}$$

for all $j = 0, 1, \cdots, 2^N$, $\zeta \in {}^*\mathbb{R}^d$.

Proof: As in the proof of Theorem 8.1.1, for ease of notation, we set $d = 1$ in this proof however it generalises to $\mathbb{R}^d$ with only notational changes.

For $j = 2^N$ we see that

$$W^+_{N,\lambda}(t_{2^N}, \zeta) = {}^*g(\zeta) = W^-_{N,\lambda}(t_{2^N}, \zeta)$$

and so we see that equation (9.1) holds for $j = 2^N$.

Now assume that equation (9.1) holds for $j = (i+1)$ i.e. assume

$$|W^+_{N,\lambda}(t_{i+1}, \zeta) - W^-_{N,\lambda}(t_{i+1}, \zeta)| \leq \epsilon_{i+1} \approx 0 \quad \text{for all } \zeta \in {}^*\mathbb{R}^d.$$

Now consider $j = i$ where $0 \leq i < 2^N$. Let

$$\theta(Y, Z) = {}^*f(t_i, \zeta, Y, Z) \quad \text{and} \quad \bar{\theta}(Y, Z) = {}^*h(t_i, \zeta, Y, Z)$$

and let

$$\varphi_1(\zeta) = W^+_{N,\lambda}(t_{i+1}, \zeta) \quad \text{and} \quad \varphi_2(\zeta) = W^-_{N,\lambda}(t_{i+1}, \zeta)$$

and for $k = 1, 2$ let

$$\psi_k(\zeta) = \mathbb{E}_i[\varphi_k(\zeta + (\Delta_N)^{\frac{1}{2}} \lambda \eta_i)]$$

and

$$\psi_k(\zeta + \Delta_N \theta(Y, Z)) + \Delta_N \bar{\theta}(Y, Z) = \Theta_k(Y, Z)$$

By Taylor's Series we see that (as before) for $k = 1, 2$ and each $Y \in {}^*\mathcal{Y}$, $Z \in {}^*\mathcal{Z}$,

$$\begin{aligned}
& \psi_k(\zeta + \Delta_N \theta(Y, Z)) \\
= \; & \psi_k(\zeta) + \Delta_N \theta(Y, Z) \psi_k'(\zeta) + \frac{1}{2} (\Delta_N)^2 \theta(Y, Z)^2 \psi_k''(\bar{\zeta}) \\
& \text{for some } \bar{\zeta} \text{ between 0 and } \zeta \\
= \; & \psi_k(\zeta) + \Delta_N \theta(Y, Z) \psi_k'(\zeta) + \epsilon \\
& \text{where } |\epsilon| \leq \tfrac{1}{2} (\Delta_N)^{\frac{3}{2}} R^2 \tfrac{c}{\lambda} \sqrt{\tfrac{2}{\pi}}\,.
\end{aligned} \tag{9.2}$$

Now, for k =1 and 2, let

$$\phi_k(Y, Z) = \psi_k(\zeta) + \Delta_N \theta(Y, Z) \psi_k'(\zeta) + \Delta_N \bar{\theta}(Y, Z) \tag{9.3}$$

then by Lemma D.2.3 we see that

$$\min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) = \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z) + \epsilon' \tag{9.4}$$

and by Lemma D.2.4

$$\max_{Y \in {}^*\mathcal{Y}} \min_{Z \in {}^*\mathcal{Z}} \Theta_k(Y, Z) = \max_{Y \in {}^*\mathcal{Y}} \min_{Z \in {}^*\mathcal{Z}} \phi_k(Y, Z) + \epsilon'' \tag{9.5}$$

where

$$(9.6)\qquad |\epsilon'|,|\epsilon''| \le \frac{1}{2}(\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}.$$

Now, the Isaacs condition (6.14) states that for $k = 1, 2$

$$(9.7)\qquad \min_{Z\in{}^*\mathcal{Z}} \max_{Y\in{}^*\mathcal{Y}} \phi_k(Y,Z) = \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \phi_k(Y,Z)$$

and so for $k = 1$ and 2, by (9.4) we have

$$(9.8)\qquad |\min_{Z\in{}^*\mathcal{Z}} \max_{Y\in{}^*\mathcal{Y}} \Theta_k(Y,Z) - \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \phi_k(Y,Z)| \le |\epsilon'|$$

and so by (9.5)

$$(9.9)\qquad |\min_{Z\in{}^*\mathcal{Z}} \max_{Y\in{}^*\mathcal{Y}} \Theta_k(Y,Z) - \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \Theta_k(Y,Z)| \le |\epsilon'| + |\epsilon''|.$$

Our assumption is that

$$|\varphi_1(\zeta) - \varphi_2(\zeta)| \le \epsilon_{i+1} \approx 0 \quad \text{for all } \zeta \in {}^*\mathbb{R}^d$$

therefore, by Lemma A.1.1, we have

$$|\psi_1(\zeta) - \psi_2(\zeta)| \le \epsilon_{i+1} \quad \text{for all } \zeta \in {}^*\mathbb{R}^d$$

and so

$$|\max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \psi_1(\zeta) - \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \psi_2(\zeta)| \le \bar{\epsilon} \quad \text{for all } \zeta \in {}^*\mathbb{R}^d$$

where

$$|\bar{\epsilon}| \le \epsilon_{i+1}.$$

Therefore, by (9.9) we have

$$|\min_{Z\in{}^*\mathcal{Z}} \max_{Y\in{}^*\mathcal{Y}} \Theta_1(Y,Z) - \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \Theta_2(Y,Z)| \le |\epsilon'| + |\epsilon''| + |\bar{\epsilon}|$$

i.e.

$$\begin{aligned}
&|\min_{Z\in{}^*\mathcal{Z}} \max_{Y\in{}^*\mathcal{Y}} \{\mathbb{E}_i[W^+_{N,\lambda}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i,\zeta,Y,Z) + \lambda(\Delta_N)^{\frac{1}{2}}\eta_i)] + \Delta_N {}^*h(t_i,\zeta,Y,Z)\} \\
&\quad - \max_{Y\in{}^*\mathcal{Y}} \min_{Z\in{}^*\mathcal{Z}} \{\mathbb{E}_i[W^-_{N,\lambda}(t_{i+1}, \zeta + \Delta_N {}^*f(t_i,\zeta,Y,Z) + \lambda(\Delta_N)^{\frac{1}{2}}\eta_i)] + \Delta_N {}^*h(t_i,\zeta,Y,Z)\}| \\
&\le (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} + \epsilon_{i+1}.
\end{aligned}$$

By Lemma 6.3.4 we see that this is the same as

$$|W^{+}_{N,\lambda}(t_i,\zeta) - W^{-}_{N,\lambda}(t_i,\zeta)| \le (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} + \epsilon_{i+1}$$

for each $i = 0, 1, \cdots, 2^N$.

So assuming that

$$|W^{+}_{N,\lambda}(t_{i+1},\zeta) - W^{-}_{N,\lambda}(t_{i+1},\zeta)| \le \epsilon_{i+1} \approx 0 \quad \text{for all } \zeta \in {}^*\mathbb{R}^d$$

$$\Longrightarrow |W^{+}_{N,\lambda}(t_i,\zeta) - W^{-}_{N,\lambda}(t_i,\zeta)| \le (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} + \epsilon_{i+1}.$$

Now, our aim is to show that

$$W^{+}_{N,\lambda}(t_j,\zeta) \approx W^{-}_{N,\lambda}(t_j,\zeta) \qquad \text{for all} \quad j = 0, 1, \cdots, 2^N. \tag{9.10}$$

Note, $\epsilon_{2^N} = 0$ since,

$$W^{+}_{N,\lambda}(t_{2^N},\zeta) = {}^*g(\zeta) = W^{-}_{N,\lambda}(t_{2^N},\zeta)$$

therefore, we see that

$$|W^{+}_{N,\lambda}(1 - j\Delta_N,\zeta) - W^{-}_{N,\lambda}(1 - j\Delta_N,\zeta)| \le j(\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}}$$

for each $j = 0, 1, \cdots, 2^N$ and so for (9.10) to hold we need

$$2^N (\Delta_N)^{\frac{3}{2}} R^2 \frac{c}{\lambda}\sqrt{\frac{2}{\pi}} \approx 0 \tag{9.11}$$

i.e. we need

$$\frac{(\Delta_N)^{\frac{1}{2}}}{\lambda} \approx 0. \tag{9.12}$$

Such λ exist. One such λ is given by $\lambda = (\Delta_N)^{\frac{1}{4}}$.

□

Corollary 9.1.2

For each fixed infinite $N \in {}^*\mathbb{N}$, if G satisfies (F1)-(F3) and the Isaacs condition holds then, there exists $\lambda \approx 0$ such that

$$W^{+}_{N,\lambda} \approx W^{-}_{N,\lambda}. \tag{9.13}$$

Proof:

$$W^+_{N,\lambda} = W^+_{N,\lambda}(0,0) \quad \text{and} \quad W^-_{N,\lambda} = W^-_{N,\lambda}(0,0)$$

and so this follows directly from Proposition 9.1.1.

□

Note: The set of λ's satisfying (9.12) is exactly the same as the set of λ's which satisfy (8.5) and (8.7) and is given by (9.11).

From Proposition 9.1.1, by considering limits we obtain the following results.

Theorem 9.1.3

If G satisfies (F1)-(F3) and the Isaacs condition (6.18) holds then we have

$$W^+ = W^-.$$

Proof: If the Isaacs condition holds, by Theorem 6.4.3 and Proposition 9.1.1, given an infinite $N \in {}^*\mathbb{N}$, there exists $\lambda \approx 0$ such that

$$W^+_N \approx W^+_{N,\lambda} \approx W^-_{N,\lambda} \approx W^-_N$$

and so for an infinite $N \in {}^*\mathbb{N}$ provided the Isaacs condition holds we have

$$W^+_N \approx W^-_N.$$

Therefore, by considering limits, which we know exist by Theorems 8.2.1 and 8.2.2, we have

$$W^+ = W^-.$$

□

The following results now follow from the above.

Corollary 9.1.4

If G satisfies (F1)-(F3) and the Isaacs condition (6.18) holds then we have

$$W^+_1 = W^-_2.$$

Proof: If the Isaacs condition holds, by Theorem 9.1.3, $W^+ = W^-$. Therefore, since by Theorem 8.2.1 $W_1^+ = W^+$ and by Theorem 8.2.2 $W_2^- = W^-$, we have the required result.

□

9.2 $W_{12}^+ = W_{12}^-$ "without" the Isaacs condition

In this section we show that we always have $W_{12}^+ = W_{12}^-$.

Theorem 9.2.1

If G satisfies (F1)-(F3) then $W_{12}^+ = W_{12}^-$.

Proof: Since, by Wald's Theorem (see Appendix E, Theorem E.2.1 or [36]) with relaxed controls the Isaacs condition is always satisfied, this follows directly from Theorem 9.1.3 applied to the game G_{12}.

□

This in fact gives us the following Corollary.

Corollary 9.2.2

If G satisfies (F1)-(F3) then

$$W_{12}^+ = W_2^+ = W_1^- = W_{12}^-.$$

Proof: This follows from Proposition 8.2.1, Proposition 8.2.2 and Theorem 9.2.1.

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Chapter 10

The existence of value

10.1 Value and the Isaacs condition

Proposition 10.1.1

If G satisfies (F1)-(F3) then

$$W^+ \geq V^+ \qquad \text{and} \qquad W^- \leq V^-.$$

Proof: By Theorem 8.2.1, Corollary 4.3.4, Proposition 3.3.4 and Theorem 3.2.3 for a fixed infinite $N \in {}^*\mathbb{N}$ we have

$$W_N^+ \approx W_{N,1}^+ \geq \hat{S}_N^+ \geq S_N^+ \approx V_N^+$$

so by taking limits (which exist by Theorems 8.2.1 and 2.3.7) we have

$$W^+ \geq V^+.$$

Similarly, by Theorem 8.2.2, Corollary 4.3.4, Proposition 3.3.4 and Theorem 3.2.3 for each fixed infinite $N \in {}^*\mathbb{N}$ we have

$$W_N^- \approx W_{N,2}^- \leq \hat{S}_N^- \leq S_N^- \approx V_N^-$$

and so by taking limits (which exist by Theorems 8.2.2 and 3.2.3) we have

$$W^- \leq V^-.$$

□

At this stage we have achieved the same results as Elliott and Kalton ([10]) and can now go on to show by their method, that if the Isaacs condition holds then value for the game G exists in the sense of Friedman.

Theorem 10.1.2

If G satisfies (F1)-(F3) and the Isaacs condition holds then G has value in the sense of Friedman i.e.

$$V^- = V^+.$$

Proof: By Proposition 10.1.1 and Theorem 9.1.3 if the Isaacs condition holds then

$$W^- \leq V^- \leq V^+ \leq W^+ = W^-$$

i.e. if the Isaacs condition hold then

$$W^- = V^- = V^+ = W^+.$$

□

Therefore we have shown that if the game G satisfies (F1)-(F3) and the Isaacs condition then the game has value.

10.2 Existence of value for relaxed controls

Here we give the main and final result, that is we show that there is always value for relaxed controls.

Theorem 10.2.1

If G satisfies (F1)-(F3) then there exists value for relaxed controls i.e.

$$V_{12}^+ = V_{12}^-.$$

Proof: We know by Wald's Theorem that relaxed controls always satisfy the Isaacs condition, therefore this result follows directly from Theorem 10.1.2 applied to the game G_{12}.

□

Therefore it follows that even if the game G does not satisfy the Isaacs condition we can introduce relaxed controls and obtain a value

$$V_{12} = V_{12}^{-} = V_{12}^{+}$$

for the game.

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Appendix A

Preliminaries and notation for the Gaussian distribution

In this Appendix we give details of the Gaussian distribution.

A.1 The Gaussian distribution

The Gaussian Distribution with mean 0 and variance 1 is referred to as the Normal Distribution. If η is a normalised Gaussian random variable we denote this by $\eta \sim \mathcal{N}(0,1)$.

The Normal Distribution has density function given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{A.1}$$

and distribution function given by

$$p\{\omega : \eta(\omega) \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

i.e. given a set A

$$p\{\omega : \eta(\omega) \in A\} = \frac{1}{\sqrt{2\pi}} \int_{A} e^{\frac{-u^2}{2}} du. \tag{A.2}$$

If $\eta \sim \mathcal{N}(0,1)$ then the expectation of a function of η is given by

$$\mathbb{E}[f(\eta)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{\frac{-y^2}{2}} dy. \tag{A.3}$$

If $\eta_1, \cdots, \eta_n$ is a sequence of normalised mutually independent Gaussian random variables then

$$p\{\omega : \eta_1(\omega) \in A_1, \cdots, \eta_n(\omega) \in A_n\} = \Pi_{j=1}^{n} p\{\omega : \eta_j(\omega) \in A_j\}. \tag{A.4}$$

The sum of normalised mutually independent random variables is a random variable.

If $\eta \sim \mathcal{N}(0,1)$ then

$$\mathbb{E}[\eta^4] = 3 \tag{A.5}$$

and

$$\mathbb{E}[\exp(\alpha\eta - \frac{\alpha^2}{2})] = 1.$$

For $\eta \sim \mathcal{N}(0,1)$ Chebychev's inequality states:

$$P(|\eta| \geq \alpha) \leq \frac{\mathbb{E}[\eta^2]}{\alpha^2} \quad \text{if} \quad \alpha > 0. \tag{A.6}$$

Therefore, if η is $\mathcal{N}(0,1)$, then we have for $\alpha > 0$ the well-known estimates

$$P(|\eta| \geq \alpha) \leq \frac{3}{\alpha^4} \tag{A.7}$$

and

$$P(\eta \geq \alpha) \leq \exp(\frac{-\alpha^2}{2}). \tag{A.8}$$

Lemma A.1.1

If functions φ_1 and φ_2 are such that

$$|\varphi_1(x) - \varphi_2(x)| \leq \epsilon$$

and $\eta \sim \mathcal{N}(0,1)$ then we have

$$|\mathbb{E}[\varphi_1(x + \alpha\eta)] - \mathbb{E}[\varphi_2(x + \alpha\eta)]| \leq \epsilon.$$

Proof:

$$\begin{aligned}
& |\mathbb{E}[\varphi_1(x+\alpha\eta)] - \mathbb{E}[\varphi_2(x+\alpha\eta)]| \\
= & \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} \varphi_1(x+\alpha y) e^{\frac{-y^2}{2}} dy - \int_{-\infty}^{\infty} \varphi_2(x+\alpha y) e^{\frac{-y^2}{2}} dy \right| \\
\leq & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\varphi_1(x+\alpha y) - \varphi_2(x+\alpha y)| e^{\frac{-y^2}{2}} dy \\
\leq & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \epsilon e^{\frac{-y^2}{2}} dy \\
= & \epsilon.
\end{aligned}$$

□

For more information on probability and random processes see [5] and [24].

Appendix B

Expectation functions and derivatives

In this Appendix we give some expressions which we use in Chapter 8.

B.1 Derivatives

Lemma B.1.1

We are now working with $d = 1$.

If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz with constant c i.e.

$$|\varphi(x_1) - \varphi(x_2)| \leq c|x_1 - x_2| \tag{B.1}$$

where x_1 and $x_2 \in \mathbb{R}^d$, and

$$\psi(x) = \mathbb{E}[\varphi(x + \alpha\eta)] \tag{B.2}$$

where α is a constant and $\eta \sim \mathcal{N}(0, 1)$ then ψ is differentiable and

$$\psi'(x) = \frac{1}{\alpha}\mathbb{E}[\varphi(x + \alpha\eta)\eta]$$

and

$$\psi''(x) = \tfrac{1}{\alpha}^2\mathbb{E}[\varphi(x + \alpha\eta)(\eta^2 - 1)] \cdot$$

Proof: Note that from (A.3) for a random variable $\eta \sim \mathcal{N}(0,1)$ we have

$$\text{(B.3)} \qquad \mathbb{E}[f(\eta)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{\frac{-y^2}{2}} dy.$$

Therefore we have

$$\begin{aligned}
\psi(x) &= \mathbb{E}[\varphi(x+\alpha\eta)] \quad \text{by (B.2)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\alpha y).e^{\frac{-y^2}{2}} dy \qquad \text{by (B.3)} \\
\psi(x+\delta) &= \mathbb{E}[x+\delta+\alpha\eta)] \quad \text{by(B.2)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\delta+\alpha y) e^{\frac{-y^2}{2}} dy \quad \text{by (B.3)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\alpha y) e^{\frac{-(y-\frac{\delta}{\alpha})^2}{2}} dy.
\end{aligned}$$

Now,

$$\begin{aligned}
\psi'(x) &= \lim_{\delta\to 0} \frac{\psi(x+\delta)-\psi(x)}{\delta} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\delta\to 0} \left(\int_{-\infty}^{\infty} \varphi(x+\alpha y) [\frac{e^{\frac{-(y-\frac{\delta}{\alpha})^2}{2}} - e^{\frac{-y^2}{2}}}{\delta}] dy \right) \\
&= \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\alpha y) y e^{\frac{-y^2}{2}} dy \\
&= \frac{1}{\alpha} \mathbb{E}[\varphi(x+\alpha\eta)\eta]
\end{aligned}$$

and

$$\begin{aligned}
\psi'(x+\delta) &= \frac{1}{\alpha} \mathbb{E}[\varphi(x+\delta+\alpha\eta)\eta] \\
&= \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\delta+\alpha y) y e^{\frac{-y^2}{2}} dy \\
&= \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\alpha y)(y-\frac{\delta}{\alpha}) e^{\frac{-(y-\frac{\delta}{\alpha})^2}{2}} dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi''(x) &= \lim_{\delta\to 0} \frac{\psi'(x+\delta)-\psi'(x)}{\delta} \\
&= \frac{1}{\alpha\sqrt{2\pi}} \lim_{\delta\to 0} \left(\int_{-\infty}^{\infty} \varphi(x+\alpha y) [\frac{(y-\frac{\delta}{\alpha}) e^{\frac{-(y-\frac{\delta}{\alpha})^2}{2}} - y e^{\frac{-y^2}{2}}}{\delta}] dy \right) \\
&= \frac{1}{\alpha^2\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x+\alpha y)(y^2-1) e^{\frac{-y^2}{2}} dy \\
&= \frac{1}{\alpha^2} \mathbb{E}[\varphi(x+\alpha\eta)(\eta^2-1)].
\end{aligned}$$

□

B.2 Bounds

Now we look at the bounds on ψ' and ψ''.

Proposition B.2.1

$$\text{(i)}|\psi'(x)| \leq c \quad \text{and} \quad \text{(ii)}|\psi''(x)| \leq \frac{c}{\alpha}\sqrt{\frac{2}{\pi}}$$

where c is the Lipschitz constant on φ.

Proof: (i)

$$\begin{aligned}
|\psi'(x)| &= |\frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x+\alpha y)ye^{\frac{-y^2}{2}}dy| \\
&= \frac{1}{\alpha\sqrt{2\pi}}|\int_{-\infty}^{\infty}(\varphi(x+\alpha y)-\varphi(x)+\varphi(x))ye^{\frac{-y^2}{2}}dy| \\
&= \frac{1}{\alpha\sqrt{2\pi}}|\int_{-\infty}^{\infty}(\varphi(x+\alpha y)-\varphi(x))ye^{\frac{-y^2}{2}}dy+\varphi(x)\int_{-\infty}^{\infty}ye^{\frac{-y^2}{2}}dy| \\
&= \frac{1}{\alpha\sqrt{2\pi}}|\int_{-\infty}^{\infty}(\varphi(x+\alpha y)-\varphi(x))ye^{\frac{-y^2}{2}}dy| \\
&\leq \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}|\varphi(x+\alpha y)-\varphi(x)y|e^{\frac{-y^2}{2}}dy \\
&\leq \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}c\alpha|y|^2e^{\frac{-y^2}{2}}dy \\
&= \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^2e^{\frac{-y^2}{2}}dy \\
&= c.
\end{aligned}$$

□

(ii)

$$\begin{aligned}
|\psi''(x)| &= \left|\frac{1}{\alpha^2\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x+\alpha y)(y^2-1)e^{\frac{-y^2}{2}}dy\right| \\
&= \frac{1}{\alpha^2\sqrt{2\pi}}\left|\int_{-\infty}^{\infty}(\varphi(x+\alpha y)-\varphi(x))(y^2-1)e^{\frac{-y^2}{2}}dy+\varphi(x)\int_{-\infty}^{\infty}(y^2-1)e^{\frac{-y^2}{2}}dy\right| \\
&= \frac{1}{\alpha^2\sqrt{2\pi}}\left|\int_{-\infty}^{\infty}(\varphi(x+\alpha y)-\varphi(x))(y^2-1)e^{\frac{-y^2}{2}}dy\right| \\
&\le \frac{1}{\alpha^2\sqrt{2\pi}}\int_{-\infty}^{\infty}|(\varphi(x+\alpha y)-\varphi(x))(y^2-1)|e^{\frac{-y^2}{2}}dy \\
&\le \frac{1}{\alpha^2\sqrt{2\pi}}\int_{-\infty}^{\infty}c\alpha|y||y^2-1|e^{\frac{-y^2}{2}}dy \\
&= \frac{c}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}|y(y^2-1)|e^{\frac{-y^2}{2}}dy \\
&= \frac{c}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\infty}|y^3-y|e^{\frac{-y^2}{2}}dy.
\end{aligned}$$

Now since $|y^3-y|$ is an even function we have

$$\begin{aligned}
|\psi''(x)| &\le \frac{2c}{\alpha\sqrt{2\pi}}\int_0^{\infty}(y^3-y)e^{\frac{-y^2}{2}}dy \\
&= \frac{2c}{\alpha\sqrt{2\pi}}\left(\int_0^{\infty}(-y^2)(-y)e^{\frac{-y^2}{2}}dy+\int_0^{\infty}(-y)e^{\frac{-y^2}{2}}dy\right) \\
&= \frac{2c}{\alpha\sqrt{2\pi}}\left([(-y^2)e^{\frac{-y^2}{2}}]_o^{\infty}-\int_0^{\infty}(-2y)e^{\frac{-y^2}{2}}dy+\int_0^{\infty}(-y)e^{\frac{-y^2}{2}}dy\right) \\
&= \frac{-2c}{\alpha\sqrt{2\pi}}\int_0^{\infty}(-y)e^{\frac{-y^2}{2}}dy \\
&= \frac{-2c}{\alpha\sqrt{2\pi}}\left[e^{\frac{-y^2}{2}}\right]_0^{\infty} \\
&= \frac{c}{\alpha}\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

□

Lemma B.2.2 ***Gronwall's Lemma***

If f is a continuous function on [0,1] such that

$$f(t)\le C+K\int_0^t f(s)ds$$

for some positive constants C (C may be 0) and K then

$$f(t)\le Ce^{Kt}.$$

Proof:

$$
\begin{aligned}
& \frac{d}{dt}\left\{e^{-Kt}\int_0^t f(s)ds\right\} \\
= \; & -Ke^{-Kt}\int_0^t f(s)ds + e^{-Kt}f(t) \\
= \; & e^{-Kt}\left(f(t) - K\int_0^t f(s)ds\right) \\
\leq \; & e^{-Kt}\left(K\int_0^t f(s)ds + C - K\int_0^t f(s)ds\right) \\
= \; & Ce^{-Kt}.
\end{aligned}
$$

Therefore,

$$
\int_0^t \left(\frac{d}{dt}\left\{e^{-K\tau}\int_0^\tau f(s)ds\right\}\right)dt \leq \int_0^t Ce^{-K\tau}dt
$$

i.e.

$$
\begin{aligned}
e^{-Kt}\int_0^t f(s)ds &\leq -\frac{1}{K}Ce^{-Kt} + \frac{C}{K} \qquad (K > 0) \\
&= \frac{C}{K}(1 - e^{-Kt})
\end{aligned}
$$

and so

$$
\begin{aligned}
K\int_0^t f(s)ds &\leq Ce^{Kt}(1 - e^{-Kt}) \\
&= Ce^{Kt} - C
\end{aligned}
$$

i.e.

$$
C + K\int_0^t f(s)ds \leq Ce^{Kt}
$$

this means

$$
f(t) \leq C + K\int_0^t f(s)ds \leq Ce^{Kt}.
$$

□

Appendix C

Lipschitz inequalities

This Lemma is used in Chapter 7 as part of the proof that $W^{+}_{N,\lambda}$ and $W^{-}_{N,\lambda}$ satisfy Lipschitz conditions in x.

C.1 Lipschitz conditions

We assume the notation of Chapter 7 for this Appendix.

Lemma C.1.1

If for each fixed i we have

$$\begin{aligned} &|\mathbb{E}_i[W^{+}_{n,\lambda}(t_{i+1},\zeta_1+\Delta_n f(t_i,\zeta_1,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_1,y,z)] \\ &\quad -\mathbb{E}_i[W^{+}_{n,\lambda}(t_{i+1},\zeta_2+\Delta_n f(t_i,\zeta_2,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_2,y,z)]| \\ \leq\ & c_i|\zeta_1-\zeta_2| \end{aligned}$$

then for each fixed i we have

$$\begin{aligned} &|\min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^{+}_{n,\lambda}(t_{i+1},\zeta_1+\Delta_n f(t_i,\zeta_1,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_1,y,z)]\} \\ &\quad -\min_{z\in\mathcal{Z}}\max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^{+}_{n,\lambda}(t_{i+1},\zeta_2+\Delta_n f(t_i,\zeta_2,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_2,y,z)]| \\ \leq\ & c_i|\zeta_1-\zeta_2|. \end{aligned}$$

Proof: Fix i and let

$$A(\zeta_j, y, z) = \mathbb{E}_i[W^+_{n,\lambda}(t_{i+1}, \zeta_j + \Delta_n f(t_i, \zeta_j, y, z) + (\Delta_n)^{\frac{1}{2}}\lambda\eta_i) + \Delta_n h(t_i, \zeta_j, y, z)]$$

for $j = 1$ and 2 then we know that

$$|A(\zeta_1, y, z) - A(\zeta_2, y, z)| \leq c_i|\zeta_1 - \zeta_2|$$

for each $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ and $\zeta_1, \zeta_2 \in \mathbb{R}^d$.

Let

$$\max_{y\in\mathcal{Y}} A(\zeta_j, y, z) = A(\zeta_j, y_j, z) \qquad \text{for } j = 1, 2$$

then for all $z \in \mathcal{Z}$ we have

(a) $|A(\zeta_1, y_1, z) - A(\zeta_2, y_1, z)| \leq c_i|\zeta_1 - \zeta_2|$

(b) $A(\zeta_2, y_2, z) \geq A(\zeta_2, y_1, z)$

(c) $|A(\zeta_2, y_2, z) - A(\zeta_1, y_2, z)| \leq c_i|\zeta_1 - \zeta_2|$

(d) $A(\zeta_1, y_1, z) \geq A(\zeta_1, y_2, z)$

and so $A(\zeta_1, y_1, z) - c_i|\zeta_1 - \zeta_2| \leq A(\zeta_2, y_1, z) \leq A(\zeta_2, y_2, z) \leq A(\zeta_1, y_1, z) + c_i|\zeta_1 - \zeta_2|$ hence

$$|\max_{y\in\mathcal{Y}} A(\zeta_1, y, z) - \max_{y\in\mathcal{Y}} A(\zeta_2, y, z)| \leq c_i|\zeta_1 - \zeta_2|.$$

the dual of the above result gives the fact that

$$|\min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} A(\zeta_1, y, z) - \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}} A(\zeta_2, y, z)| \leq c_i|\zeta_1 - \zeta_2|$$

which in the notation of Chapter 7 is

$$\begin{aligned} &|\min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1}, \zeta_1 + \Delta_n f(t_i, \zeta_1, y, z) + (\Delta_n)^{\frac{1}{2}}\lambda\eta_i) + \Delta_n h(t_i, \zeta_1, y, z)]\} \\ &\qquad - \min_{z\in\mathcal{Z}} \max_{y\in\mathcal{Y}}\{\mathbb{E}_i[W^+_{n,\lambda}(t_{i+1}, \zeta_2 + \Delta_n f(t_i, \zeta_2, y, z) + (\Delta_n)^{\frac{1}{2}}\lambda\eta_i) + \Delta_n h(t_i, \zeta_2, y, z)]| \\ \leq\ & c_i|\zeta_1 - \zeta_2|. \end{aligned}$$

□

Lemma C.1.2

If for each fixed i we have

$$\begin{aligned} & |\mathbb{E}_i[W^-_{n,\lambda}(t_{i+1},\zeta_1+\Delta_n f(t_i,\zeta_1,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_1,y,z)] \\ & \quad -\mathbb{E}_i[W^-_{n,\lambda}(t_{i+1},\zeta_2+\Delta_n f(t_i,\zeta_2,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_2,y,z)]| \\ \leq\ & c_i|\zeta_1-\zeta_2| \end{aligned}$$

then for each fixed i we have

$$\begin{aligned} & |\max_{y\in\mathcal{Y}}\min_{z\in\mathcal{Z}}\{\mathbb{E}_i[W^-_{n,\lambda}(t_{i+1},\zeta_1+\Delta_n f(t_i,\zeta_1,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_1,y,z)]\} \\ & \quad -\max_{y\in\mathcal{Y}}\min_{z\in\mathcal{Z}}\{\mathbb{E}_i[W^-_{n,\lambda}(t_{i+1},\zeta_2+\Delta_n f(t_i,\zeta_2,y,z)+(\Delta_n)^{\frac{1}{2}}\lambda\eta_i)+\Delta_n h(t_i,\zeta_2,y,z)]| \\ \leq\ & c_i|\zeta_1-\zeta_2|. \end{aligned}$$

Proof: The proof of this is analogous to that of Lemma C.1.1.

□

Appendix D

Min and max inequalities

In this Appendix we give some useful inequalities which we use in Chapters 8 and 9.

D.1 Preservation of closeness

The following Lemma shows that the operations of supremum and infimum preserve infinite closeness.

Lemma D.1.1

Given internal functions A and B such that ${}^*\mathcal{Y} \times {}^*\mathcal{Z} \to {}^*\mathbb{R}$ where $\mathcal{Y}$ and $\mathcal{Z}$ are compact metric spaces, if

$$A(Y,Z) \approx B(Y,Z)$$

for each $Y \in {}^*\mathcal{Y}$ and $Z \in {}^*\mathcal{Z}$ then

$$\sup_{Y \in {}^*\mathcal{Y}} A(Y,Z) \approx \sup_{Y \in {}^*\mathcal{Y}} B(Y,Z)$$

for each fixed $Z \in {}^*\mathcal{Z}$.

Proof: Fix $Z \in {}^*\mathcal{Z}$ then we know that

$$A(Y,Z) \approx B(Y,Z) \leq \sup_{Y \in {}^*\mathcal{Y}} B(Y,Z)$$

for each $Y \in {}^*\mathcal{Y}$. This gives that for fixed $Z \in {}^*\mathcal{Z}$

$$\sup_{Y \in {}^*\mathcal{Y}} A(Y,Z) \lesssim \sup_{Y \in {}^*\mathcal{Y}} B(Y,Z).$$

Similarly by exchanging the roles of A and B we have

$$\sup_{Y \in {}^*\mathcal{Y}} B(Y,Z) \lesssim \sup_{Y \in {}^*\mathcal{Y}} A(Y,Z)$$

which gives

$$\sup_{Y \in {}^*\mathcal{Y}} A(Y,Z) \approx \sup_{Y \in {}^*\mathcal{Y}} B(Y,Z)$$

□

Lemma D.1.2

Given continuous functions A and B such that ${}^*\mathcal{Y} \times {}^*\mathcal{Z} \to {}^*\mathbb{R}$ where $\mathcal{Y}$ and $\mathcal{Z}$ are compact metric spaces, if

$$A(Y,Z) \approx B(Y,Z)$$

for each $Y \in {}^*\mathcal{Y}$ and $Z \in {}^*\mathcal{Z}$ then

$$\inf_{Z \in {}^*\mathcal{Z}} A(Y,Z) \approx \inf_{Z \in {}^*\mathcal{Z}} B(Y,Z)$$

for each fixed $Y \in {}^*\mathcal{Y}$.

Proof: This follows as a dual result of Lemma D.1.1.

□

D.2 Min and max inequalities

Lemma D.2.1

Using the notation of Chapter 8, for each fixed $Z \in {}^*\mathcal{Z}$ we have

$$\max_{\nu \in {}^*\Lambda(\mathcal{Y})} \psi(\zeta)+\Delta_N\theta(\nu,Z)\psi'(\zeta)+\Delta_N\bar{\theta}(\nu,Z) = \max_{Y \in {}^*\mathcal{Y}} \psi(\zeta)+\Delta_N\theta(Y,Z)\psi'(\zeta)+\Delta_N\bar{\theta}(Y,Z)$$

Proof: For each fixed $Z \in {}^*\mathcal{Z}$, let

$$A(\zeta, \nu, Z) = \psi(\zeta) + \Delta_N \theta(\nu, Z)\psi'(\zeta) + \Delta_N \bar{\theta}(\nu, Z) \qquad \text{for } \nu \in {}^*\Lambda(\mathcal{Y})$$

and

$$A(\zeta, Y, Z) = \psi(\zeta) + \Delta_N \theta(Y, Z)\psi'(\zeta) + \Delta_N \bar{\theta}(Y, Z) \qquad \text{for } Y \in {}^*\mathcal{Y}$$

then for each fixed $Z \in {}^*\mathcal{Z}$

$$A(\zeta, \nu, Z) = \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\nu(Y)$$

for each $\nu \in {}^*\Lambda(\mathcal{Y})$. Now we see that for each $Z \in {}^*\mathcal{Z}$ we have

$$A(\zeta, \nu, Z) = \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\nu(Y) \leq \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z) \int_{{}^*\mathcal{Y}} d\nu(Y) = \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z)$$

$$\Longrightarrow \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\nu(Y) = \max_{\nu \in {}^*\Lambda(\mathcal{Y})} A(\zeta, \nu, Z) \leq \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z)$$

For the other direction,

$$A(\zeta, Y, Z) = \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\delta_Y(Y)$$

where δ_Y satisfies

$$\delta_Y(a) = \begin{cases} 1 & \text{if } a = Y \\ 0 & \text{if } a \neq Y \end{cases}$$

Therefore we have

$$\max_{\nu \in {}^*\Lambda(\mathcal{Y})} \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\nu(Y) \geq \max_{Y \in {}^*\mathcal{Y}} \int_{{}^*\mathcal{Y}} A(\zeta, Y, Z) d\delta_Y(Y) = \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z)$$

i.e.

$$\max_{\nu \in {}^*\Lambda(\mathcal{Y})} A(\zeta, \nu, Z) \geq \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z).$$

Therefore for each $Z \in {}^*\mathcal{Z}$ we have

(D.1) $$\max_{\nu \in {}^*\Lambda(\mathcal{Y})} A(\zeta, \nu, Z) = \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z)$$

which implies

(D.2) $$\min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} A(\zeta, \nu, Z) = \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} A(\zeta, Y, Z).$$

By an analogous method it can also be shown that for each fixed $Y \in {}^*\mathcal{Y}$,

$$\min_{\rho \in {}^*\Lambda(\mathcal{Z})} \psi(\zeta)+\Delta_N\theta(Y,\rho)\psi'(\zeta)+\Delta_N\bar{\theta}(Y,\rho) = \min_{Z \in {}^*\mathcal{Z}} \psi(\zeta)+\Delta_N\theta(Y,Z)\psi'(\zeta)+\Delta_N\bar{\theta}(Y,Z).$$

□

Lemma D.2.2

Using the notation of Chapter 8 we show that for each fixed $Z \in {}^*\mathcal{Z}$ if

$$|\max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) - \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z)| \leq K$$

where $K < \infty$ then

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z)| \leq K$$

Proof: For each fixed $Z \in {}^*\mathcal{Z}$ let

$$\max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z) = A(Z) \qquad \text{and} \qquad \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z) = B(Z).$$

then we know that for each $Z \in {}^*\mathcal{Z}$

$$|A(Z) - B(Z)| \leq K \tag{D.3}$$

and we want to show that

$$|\min_{Z \in {}^*\mathcal{Z}} A(Z) - \min_{Z \in {}^*\mathcal{Z}} B(Z)| \leq K.$$

Let

$$\min_{Z \in {}^*\mathcal{Z}} A(Z) = A(\hat{Z}) \qquad \text{and} \qquad \min_{Z \in {}^*\mathcal{Z}} B(Z) = B(\bar{Z}).$$

then we have

$$|A(\hat{Z}) - B(\hat{Z})| \leq K \quad \text{by (D.3)}$$

$$B(\hat{Z}) \leq B(\bar{Z})$$

$$|B(\bar{Z}) - A(\bar{Z})| \leq K \quad \text{by (D.3)}$$

$$A(\bar{Z}) \leq A(\hat{Z}).$$

Therefore,

$$A(\hat{Z}) - K \leq B(\hat{Z}) \leq B(\bar{Z}) \leq A(\bar{Z}) + K \leq A(\hat{Z}) + K$$

so

$$|A(\hat{Z}) - B(\bar{Z})| \leq K$$

i.e

$$|\min_{Z \in {}^*\mathcal{Z}} A(Z) - \min_{Z \in {}^*\mathcal{Z}} B(Z)| \leq K$$

which in the notation of Chapter 8 is

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta(Y, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{\nu \in {}^*\Lambda(\mathcal{Y})} \Theta(\nu, Z)| \leq K.$$

□

Lemma D.2.3

Using the notation of Chapter 9, if for $k = 1$ and 2

$$|\Theta_k(Y, Z) - \phi_k(Y, Z)| \leq K$$

for each $Y \in^* \mathcal{Y}$ and $Z \in {}^*\mathcal{Z}$, where $K < \infty$ then for $k = 1$ and 2 we have

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z)| \leq K.$$

Proof: We will do this in two stages, first we will show that

$$|\max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) - \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z)| \leq K.$$

and then use the proof of Lemma D.2.2 to show that

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z)| \leq K.$$

We know that for each $Z \in {}^*\mathcal{Z}$ and $Y \in {}^*\mathcal{Y}$

(D.4) $$|\Theta_k(Y, Z) - \phi_k(Y, Z)| \leq K$$

and we want to show that

$$|\max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) - \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z)| \leq K.$$

Fix $Z \in {}^*\mathcal{Z}$ and let

$$\max_{Y\in {}^*\mathcal{Y}} \Theta_k(Y,Z) = \Theta_k(\hat{Y},Z) \qquad \text{and} \qquad \max_{Y\in {}^*\mathcal{Y}} \phi_k(Y,Z) = \phi_k(\bar{Y},Z).$$

then we have

$$|\Theta_k(\hat{Y},Z) - \phi_k(\hat{Y},Z)| \le K \quad \text{by(D.4)}$$

$$\Theta_k(\bar{Y},Z) \le \Theta_k(\hat{Y},Z)$$

$$|\Theta_k(\bar{Y},Z) - \phi_k(\bar{Y},Z)| \le K \quad \text{by(D.4)}$$

$$\phi_k(\hat{Y},Z) \le \phi_k(\bar{Y},Z).$$

Therefore,

$$\Theta_k(\hat{Y},Z) - K \le \phi_k(\hat{Y},Z) \le \phi_k(\bar{Y},Z) \le \Theta_k(\bar{Y},Z) + K \le \Theta_k(\hat{Y},Z) + K$$

so

$$|\Theta_k(\hat{Y},Z) - \phi_k(\bar{Y},Z)| \le K$$

i.e for each fixed $Z \in {}^*\mathcal{Z}$

$$|\max_{Y\in {}^*\mathcal{Y}} \Theta_k(Y,Z) - \max_{Y\in {}^*\mathcal{Y}} \phi_k(Y,Z)| \le K.$$

Now, for each fixed $Z \in {}^*\mathcal{Z}$ let

$$\max_{Y\in {}^*\mathcal{Y}} \Theta_k(Y,Z) = A_k(Z) \qquad \text{and} \qquad \max_{Y\in {}^*\mathcal{Y}} \phi_k(Y,Z) = B_k(Z).$$

then we have just shown that for each $Z \in {}^*\mathcal{Z}$

(D.5) $$|A_k(Z) - B_k(Z)| \le K$$

and we want to show that

$$|\min_{Z\in {}^*\mathcal{Z}} A_k(Z) - \min_{Z\in {}^*\mathcal{Z}} B_k(Z)| \le K.$$

The proof of this is identical to that of Lemma D.2.2 i.e. let

$$\min_{Z\in {}^*\mathcal{Z}} A_k(Z) = A_k(\hat{Z}) \qquad \text{and} \qquad \min_{Z\in {}^*\mathcal{Z}} B_k(Z) = B_k(\bar{Z}).$$

then we have

$$|A_k(\hat{Z}) - B_k(\hat{Z})| \le K \quad \text{by (D.5)}$$

$$B_k(\hat{Z}) \leq B_k(\bar{Z})$$

$$|B_k(\bar{Z}) - A_k(\bar{Z})| \leq K \quad \text{by (D.5)}$$

$$A_k(\bar{Z}) \leq A_k(\hat{Z}).$$

Therefore,

$$A_k(\hat{Z}) - K \leq B_k(\hat{Z}) \leq B_k(\bar{Z}) \leq A_k(\bar{Z}) + K \leq A_k(\hat{Z}) + K$$

so

$$|A_k(\hat{Z}) - B_k(\bar{Z})| \leq K$$

i.e

$$|\min_{Z \in {}^*\mathcal{Z}} A_k(Z) - \min_{Z \in {}^*\mathcal{Z}} B_k(Z)| \leq K$$

which in the notation of Chapter 8 is

$$|\min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \Theta_k(Y, Z) - \min_{Z \in {}^*\mathcal{Z}} \max_{Y \in {}^*\mathcal{Y}} \phi_k(Y, Z)| \leq K.$$

Lemma D.2.4

Also using the notation of Chapter 9, if for $k = 1$ and 2

$$|\Theta_k(Y, Z) - \phi_k(Y, Z)| \leq K$$

for each $Y \in^* \mathcal{Y}$ and $Z \in {}^*\mathcal{Z}$, where $K < \infty$ then for $k = 1$ and 2 we have

$$|\max_{Y \in {}^*\mathcal{Y}} \min_{Z \in {}^*\mathcal{Z}} \Theta_k(Y, Z) - \max_{Y \in {}^*\mathcal{Y}} \min_{Z \in {}^*\mathcal{Z}} \phi_k(Y, Z)| \leq K.$$

Proof: The proof of this is analogous to that of Lemma D.2.3.

□

Appendix E

Wald's Theorem

In this Appendix we give a nonstandard proof of Wald's Theorem which we use in Chapter 9. First we give, without proof, some well-known Lemmas which we need when proving Wald's Theorem. For a standard proof of this Theorem we refer the reader to [36].

E.1 Preliminaries for Wald's Theorem

Lemma E.1.1

Given a compact metric space $\mathcal{Y}$ with metric d, there exists an infinite $N \in {}^*\mathbb{N}$ and elements ${}^*a_1, \cdots, {}^*a_N \in {}^*\mathcal{Y}$ such that $\mathcal{Y} = \{{}^\circ({}^*a_1), \cdots, {}^\circ({}^*a_N)\}$.

Proof: The compact metric space $\mathcal{Y}$ is separable so has a countable dense subset i.e. $\exists a : \mathbb{N} \rightarrow \mathcal{Y}$ therefore, by transfer ${}^*a : {}^*\mathbb{N} \rightarrow {}^*\mathcal{Y}$.
Let $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ and let $\hat{\mathcal{Y}}$ denote the internal set $\{{}^*a_1, \cdots, {}^*a_N\} \subseteq {}^*\mathcal{Y}$. Since $\mathcal{Y}$ is compact, every point in ${}^*\mathcal{Y}$ is nearstandard, and so $\{{}^\circ({}^*a_1), \cdots, {}^\circ({}^*a_N)\} \subseteq \mathcal{Y}$. For the other direction, we have to show that given any $y \in \mathcal{Y}$, there is an element ${}^*a_i \in \hat{\mathcal{Y}}$ satisfying ${}^*a_i \approx y$. Consider the closed balls

$$ {}^*B({}^*y, \frac{1}{k}) = \{b \in {}^*\mathcal{Y} : {}^*d({}^*y, b) \leq \frac{1}{k}\}. $$

By density, for each $k \in \mathbb{N}$

$$^*B(^*y, \frac{1}{k}) \cap \hat{\mathcal{Y}} \neq \emptyset$$

therefore, by overflow, there exists $K \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that

$$^*B(^*y, \frac{1}{K}) \cap \hat{\mathcal{Y}} \neq \emptyset$$

and so

$$y \approx {}^*a_i \qquad \text{for some } a_i \in \hat{\mathcal{Y}}$$

hence $\mathcal{Y} \subseteq \{{}^\circ({}^*a_1), \cdots, {}^\circ({}^*a_N)\}$.

□

Lemma E.1.2

Given a compact metric space $\mathcal{Y}$, we have shown in lemma E.1.1 that there exists an $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ and an internal set $\hat{\mathcal{Y}} = \{{}^*a_1, \cdots {}^*a_N\} \subseteq {}^*\mathcal{Y}$ satisfying $\mathcal{Y} = \{{}^\circ({}^*a_1), \cdots, {}^\circ({}^*a_N)\}$. We now show that we can find an internal sequence, $A_1, \cdots, A_N$ of disjoint *Borel subsets of ${}^*\mathcal{Y}$ satisfying

$$\cup_{i=1}^N A_i = {}^*\mathcal{Y} \quad \text{and} \quad A_i \subseteq \text{monad}({}^\circ({}^*a_i)) \qquad \text{for each } i = 1, \cdots, N.$$

Having got these sets, we then show that given any Borel probability measure ν on $\mathcal{Y}$ i.e. $\nu : \mathcal{B}(\mathcal{Y}) \to [0,1]$, and any continuous function $g : \mathcal{Y} \to \mathbb{R}$

$$\text{(E.1)} \qquad \int_{\mathcal{Y}} g(y) d\nu(y) = {}^\circ(\sum_{i=1}^N {}^*g({}^*a_i) p_i^\nu)$$

where $p_i^\nu = {}^*\nu(A_i)$ for each $i = 1, \cdots, N$.

Proof: Let $A_1, \cdots, A_N$ be defined by

$$\begin{aligned} A_1 &= {}^*B({}^*a_1, \delta) \\ A_{n+1} &= {}^*B({}^*a_{n+1}, \delta) \backslash \cup_{i=1}^n A_n \end{aligned}$$

where

$$^*B({}^*a_i, \delta) = \{b \in^* \mathcal{Y} : {}^*d(b, {}^*a_i) \leq \delta\} \quad \text{and} \quad \delta = \sup_{b \in {}^*\mathcal{Y}} \min_{i=1}^N {}^*d(b, {}^*a_i) \approx 0.$$

Now, given a continuous function g, for each $i = 1, \cdots, N$

$$|{}^*g(b) - {}^*g({}^*a_i)| \approx 0 \qquad \text{for all } b \in A_i$$

and if we let

$$\epsilon = \max_{i=1}^{N} \sup_{b \in A_i} |{}^*g(b) - {}^*g({}^*a_i)| \approx 0$$

then

$$\begin{aligned}
0 &\approx \epsilon \sum_{i=1}^{N} {}^*\nu(A_i) \qquad \text{since } \textstyle\sum_{i=1}^{N} {}^*\nu(A_i) = {}^*\nu(\mathcal{Y}) = 1 \\
&= \epsilon \sum_{i=1}^{N} \int_{A_i} d^*\nu(b) \\
&\geq \sum_{i=1}^{N} \int_{A_i} |{}^*g(b) - {}^*g({}^*a_i)| d^*\nu(b) \\
&\geq |\sum_{i=1}^{N} \int_{A_i} ({}^*g(b) - {}^*g({}^*a_i)) d^*\nu(b)| \\
&= |\int_{\mathcal{Y}} {}^*g(b) d^*\nu(b) - \sum_{i=1}^{N} {}^*g({}^*a_i) \int_{A_i} d^*\nu(b)| \\
&= |\int_{\mathcal{Y}} g(y) d\nu(y) - \sum_{i=1}^{N} {}^*g({}^*a_i) {}^*\nu(A_i)|
\end{aligned}$$

so

$$\int_{\mathcal{Y}} g(y) d\nu(y) \approx \sum_{i=1}^{N} {}^*g({}^*a_i) {}^*\nu(A_i).$$

Hence, since for each $i = 1, \cdots, N$ we have $p_i^\nu = {}^*\nu(A_i)$ this gives

$$\int_{\mathcal{Y}} g(y) d\nu(y) = {}^\circ(\sum_{i=1}^{N} {}^*g({}^*a_i) p_i^\nu).$$

□

Definition E.1.3

Given a compact metric space $\mathcal{Y}$ we have show n in lemma E.1.1 that there exists $N \in {}^*\mathbb{N} \backslash \mathbb{N}$ and a set $\hat{\mathcal{Y}} = \{{}^*a_1, \cdots, {}^*a_N\} \subseteq {}^*\mathcal{Y}$ such that $\mathcal{Y} = \{{}^\circ({}^*a_1), \cdots, {}^\circ({}^*a_N)\}$. With this notation we now go on to show that if $p = (p_1, \cdots, p_N)$ is an internal sequence of elements of ${}^*[0,1]$ with $\sum_{i=1}^{N} p_i = 1$ and $p_i \geq 0$, and we define $\nu_p : \mathcal{B}(\mathcal{Y}) \to [0,1]$ by

(E.2) $$\nu_p(A) = \mu_L(st^{-1}(A) \cap \hat{\mathcal{Y}})$$

for all $A \in \mathcal{B}(\mathcal{Y})$ where for each $i = 1, \cdots, N$, $\mu({}^*a_i) = p_i$ then

$$\int_{\mathcal{Y}} g(y) d\nu(y) = {}^\circ(\sum_{i=1}^{N} {}^*g({}^*a_i)p_i).$$

Proof: The function ${}^*g$ satisfies ${}^\circ({}^*g(a_i)) = g({}^\circ a_i)$ for all $i = 1, \cdots, N$.

If we define $f : \hat{\mathcal{Y}} \to \mathbb{R}$ by $f({}^*a_i) = g({}^\circ({}^*a_i))$ for each $i = 1, \cdots, N$ then ${}^\circ({}^*g({}^*a_i)) = f({}^*a_i)$ for all $i = 1, \cdots, N$ therefore, by Loeb Theory,

$$\int_{\mathcal{Y}} g(y) d\nu_p(y) = \int_{\hat{\mathcal{Y}}} f(b) d\mu_L(b) = {}^\circ(\sum_{i=1}^{N} {}^*g({}^*a_i)p_i).$$

□

Lemma E.1.4

If $\mathcal{Y}$ is a compact metric space, then given any Borel probability measure ν on $\mathcal{Y}$ i.e. $\nu : \mathcal{B}(\mathcal{Y}) \to [0,1]$, ν is of the form ν_p for some p, namely $p = p^\nu$ (where ν_p and p_ν are as defined in (E.1.2) and (E.1.3) respectively.)

Proof: Given ν, if we define p^ν as in Lemma E.1.2 then

$$\int_{\mathcal{Y}} g(y) d\nu(y) = {}^\circ(\sum_{i=1}^{N} {}^*g({}^*a_i)p_i^\nu) \tag{E.3}$$

and if we apply Lemma E.1.3 to this p^ν then we have

$$\int_{\mathcal{Y}} g(y) d\nu_{p^\nu}(y) = {}^\circ(\sum_{i=1}^{N} {}^*g({}^*a_i)p_i^\nu) \tag{E.4}$$

and so by equating (E.3) and (E.4) we have

$$\int_{\mathcal{Y}} g(y) d\nu(y) = \int_{\mathcal{Y}} g(y) d\nu_{p^\nu}(y)$$

for each $g \in \mathcal{C}(\mathcal{Y})$ and so $\nu \equiv \nu_{p^\nu}$.

□

E.2 Wald's Theorem

In this section we give a nonstandard proof of Wald's Theorem using the results of the previous section.

Theorem E.2.1 (Wald's Theorem)

If $\mathcal{Y}$, $\mathcal{Z}$ are two compact metric spaces and g is a continuous function $g : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ then

$$\sup_{\nu \in \Lambda(\mathcal{Y})} \inf_{\rho \in \Lambda(\mathcal{Z})} \int_{\mathcal{Z}} \int_{\mathcal{Y}} g(y,z) d\nu(y) d\rho(z) = \inf_{\rho \in \Lambda(\mathcal{Z})} \sup_{\nu \in \Lambda(\mathcal{Y})} \int_{\mathcal{Z}} \int_{\mathcal{Y}} g(y,z) d\nu(y) d\rho(z).$$

Proof: By Lemma E.1.1 there exists $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and elements $y_1, \cdots, y_N \in {}^*\mathcal{Y}$ such that $\mathcal{Y} = \{{}^\circ y_1, \cdots, {}^\circ y_N\}$ and by Lemma E.1.2 there exists an internal sequence $A_1, \cdots, A_N$ of *Borel subsets of ${}^*\mathcal{Y}$ such that given a Borel probability measure ν on $\mathcal{Y}$ i.e. $\nu : \mathcal{B}(\mathcal{Y}) \to [0,1]$ if we define ${}^*\nu(A_i) = p_i^\nu$ for each $i = 1, \cdots, N$ then $\sum_{i=1}^N p_i^\nu = 1$ and for any continuous function $g : \mathcal{Y} \to \mathbb{R}$

$$\text{(E.5)} \qquad \int_{\mathcal{Y}} g(y) d\nu(y) = {}^\circ(\sum_{i=1}^{N} {}^*g(y_i) p_i^\nu).$$

If we do the same thing for a similar compact metric space $\mathcal{Z}$ and a Borel probability measure $\rho : \mathcal{B}(\mathcal{Z}) \to [0,1]$ then there exists an $M \in {}^*\mathbb{N} \setminus \mathbb{N}$ and a sequence $z_1, \cdots, z_N \in {}^*\mathcal{Z}$ such that $\mathcal{Z} = \{{}^\circ z_1, \cdots, {}^\circ z_M\}$ and we can find an internal sequence $B_1, \cdots, B_M$ of disjoint *Borel subsets of ${}^*\mathcal{Z}$ such that

$$\text{(E.6)} \qquad \int_{\mathcal{Z}} g(z) d\rho(z) = {}^\circ(\sum_{j=1}^{M} {}^*g(z_j) q_j^\rho)$$

where $q^\rho = (q_1^\rho, \cdots, q_M^\rho)$ is defined from ρ as in Lemma E.1.2.

Now we consider the product space $\mathcal{Y} \times \mathcal{Z}$, we have

$$\int_{\mathcal{Z}} \int_{\mathcal{Y}} g(y,z) d\nu(y) d\rho(z) = {}^\circ(\sum_{j=1}^{M} \sum_{i=1}^{N} {}^*g(y_i, z_j) p_i^\nu q_j^\rho)$$

for all $g \in \mathcal{C}(\mathcal{Y} \times \mathcal{Z})$.

Now, if we are given an internal sequence of elements of ${}^*[0,1]$, $p = (p_1, \cdots, p_N)$ satisfying $\sum_{i=1}^{N} p_i = 1$, $p_i \geq 0$ and define ν_p as in Lemma E.1.3 then we have

$$\text{(E.7)} \qquad \int_{\mathcal{Y}} g(y) d\nu_p(y) = {}^\circ \sum_{i=1}^{N} {}^* g(y_i) p_i.$$

If we do the same thing for $\mathcal{Z}$ and a given internal sequence of elements of ${}^*[0,1]$, $q = (q_1, \cdots, q_M)$, then we have

$$\text{(E.8)} \qquad \int_{\mathcal{Z}} g(z) d\rho(z) = {}^\circ (\sum_{j=1}^{M} {}^* g(z_j) q_j^\rho)$$

where ρ_q is constructed from q as in Lemma E.1.3.

Now considering the product space $\mathcal{Y} \times \mathcal{Z}$, we have

$$\text{(E.9)} \qquad \int_{\mathcal{Z}} \int_{\mathcal{Y}} g(y,z) d\nu_p(y) d\rho_q(z) = {}^\circ (\sum_{j=1}^{M} \sum_{i=1}^{N} {}^* g(y_i, z_j) p_i q_j)$$

for all $g \in \mathcal{C}(\mathcal{Y} \times \mathcal{Z})$.

If we let the left hand side of (E.9) be denoted by $g(\nu_p, \rho_q)$ and the right hand side by ${}^\circ G(p,q)$ then

$$\text{(E.10)} \qquad g(\nu_p, \rho_q) = {}^\circ G(p,q).$$

Using the standard Minimax Theorem (see [30]) we know that a saddle point exists, suppose it occurs at $\hat{p}$, $\hat{q}$ then for all p, q we have

$$ {}^\circ G(p, \hat{q}) \leq {}^\circ G(\hat{p}, \hat{q}) \leq {}^\circ G(\hat{p}, q)$$

$$\Longrightarrow g(\nu_p, \rho_{\hat{q}}) \leq g(\nu_{\hat{p}}, \rho_{\hat{q}}) \leq g(\nu_{\hat{p}}, \rho_q)$$

Let $\nu_{\hat{p}} = \nu'$ and $\rho_{\hat{q}} = \rho'$ then, since by Lemma E.1.4 every ν is of the form ν_p for some p and every ρ is of the form ρ_q for some q, we have

$$g(\nu, \rho') \leq g(\nu', \rho') \leq g(\nu', \rho)$$

hence

$$\sup_{\nu \in \Lambda(\mathcal{Y})} g(\nu, \rho') = g(\nu', \rho') = \inf_{\rho \in \Lambda(\mathcal{Z})} g(\nu', \rho)$$

therefore

$$\inf_{\rho \in \Lambda(\mathcal{Z})} \sup_{\nu \in \Lambda(\mathcal{Y})} g(\nu, \rho) = \inf_{\rho \in \Lambda(\mathcal{Z})} g(\nu', \rho) = \sup_{\nu \in \Lambda(\mathcal{Y})} g(\nu, \rho') = \sup_{\nu \in \Lambda(\mathcal{Y})} \inf_{\rho \in \Lambda(\mathcal{Z})} g(\nu, \rho)$$

i.e.

$$\inf_{\rho \in \Lambda(\mathcal{Z})} \sup_{\nu \in \Lambda(\mathcal{Y})} g(\nu, \rho) = \sup_{\nu \in \Lambda(\mathcal{Y})} \inf_{\rho \in \Lambda(\mathcal{Z})} g(\nu, \rho).$$

□

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