THE UNIVERSITY OF HULL

The Stochastic Nonlinear Heat Equation

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Abstract

This thesis considers a stochastic partial differential equation which may be viewed as a stochastic version of the nonlinear heat equation studied by Eells and Sampson. The special case of loops on a compact Riemannian manifold M is studied, where the loop is parametrised by the unit circle. Using ideas of Eells and Sampson and the theory of stochastic evolution equations on infinite dimensional M-type 2 Banach spaces, existence and uniqueness of an $\mathcal{M}$-valued solution is shown, where $\mathcal{M}$ is a certain Sobolev-Slobodetski space of loops on the manifold M. In particular $\mathcal{M}$ is an infinite dimensional manifold modelled on an M-type 2 Banach space.

Finally, an approximation result of the Wong-Zakai type for Stratonovich integrals in M-type 2 Banach spaces is given.

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> ' ... the music of evil, of the enemy sounded... he fought it, singing softly the Song of the Family, of the safety and warmth and wholeness of the family. '
>
> *John Steinbeck, The Pearl*

To Mum, Dad, Sarah and Matthew

Contents

Chapter 1

Introduction

This thesis is concerned with stochastic partial differential equations, SPDEs, of the following general form

$$du(t) = \underline{\Delta} u(t) + \text{"noise"}. \tag{1.0.1}$$

$\underline{\Delta}$ is the nonlinear Laplacian acting on smooth maps $u : N \to M$, where N, $M \subset \mathbb{R}^d$ are Riemannian manifolds. The noise term involves a suitable space-time white noise.

Equations of the form (1.0.1) are motivated by physical literature where they appear in the kinetic theory of phase transitions and in the theory of stochastic quantisation, see [Fu,92] and references therein. Even so, for the case $N = S^1$, S^1 the unit circle, equation (1.0.1) is of interest as it defines a diffusive motion of loops on the manifold M, where the loop is parametrised by $\sigma \in S^1$.

Equation (1.0.1) may be considered as a stochastic version of the nonlinear heat equation studied by Eells and Sampson, [Ee/Sa,64]. Eells and Sampson proved that, given $\tilde{f} \in C^\infty(N, M)$, where N is a general compact manifold and $M \subset \mathbb{R}^d$ is a compact Riemannian manifold with nonpositive sectional curvature, there exists a unique $f : [0, \infty) \times N \to M$ satisfying

$$\frac{\partial f_s}{\partial s}(x) = \underline{\Delta} f_s(x), \quad s > 0, \quad x \in N, \tag{1.0.2}$$

with $f_0 = \tilde{f}$. Here we have written $f_s(\cdot) := f(s, \cdot)$. Hamilton, [Ha,75], extended this result to include the case where M has a boundary. Ottarsson, [Ot,84], considered the special case of loops on the manifold M. Indeed, in this simpler case, it was shown that one may drop the curvature restriction on M.

We now make our problem more explicit. Let w_t be an E-valued Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where E is a suitable Banach space of loops on $\mathbb{R}^m$, i.e. $\eta : S^1 \to \mathbb{R}^m$. For $u : [0, \infty) \times S^1 \times \Omega \to M$, consider the following quasilinear SPDE

$$du_t(\sigma) = \underline{\Delta} u_t(\sigma) dt + v(u_t(\sigma)) dt + h(u_t(\sigma)) \circ dw_t(\sigma), \quad t > 0, \quad \sigma \in S^1, \tag{1.0.3}$$

where we write $u_t(\sigma) := u(t, \sigma)$ and we have suppressed the dependence on $\omega \in \Omega$. We explain the meaning of the terms in the equation (1.0.3) :

(a) v is a smooth vector field on M

(b) h is a smooth section of a bundle $\mathbb{F}$ over M, whose fibres are $\mathbb{F}_x = L(\mathbb{R}^m, T_x M)$, $x \in M$.

(c) $\circ dw_t$ denotes the Stratonovich differential.

Equation (1.0.3) will often be referred to as the stochastic nonlinear heat equation, SNHE.

Existence of a solution to (1.0.3) (in terms of generalised functions) was proved by Funaki, see [Fu,92]. (Strictly speaking, the Wiener process used by Funaki was of a different form to the one in (1.0.3), but the idea of proof is still the same.) Using an ad hoc version of Trotter's product formula, Funaki constructs a solution to (1.0.3), as an infinitesimal composition of a solution to (1.0.2) and a solution to the following stochastic differential equation

$$du_t(\sigma) = v(u_t(\sigma))dt + h(u_t(\sigma)) \circ dw_t(\sigma), \quad t > 0, \sigma \in S^1, \tag{1.0.4}$$

(The idea of composing two different solutions is not dissimilar to the Fractional Step Method used by Kotelenz, see [Kot,92], [Go/Kot,96] and references therein.) More precisely, corresponding to each partition π of the interval $[0,T]$, Funaki constructs the process f_π as a composition of the solutions to (1.0.2) and (1.0.4). Existence of a solution to (1.0.3) is then a consequence of the following three theorems

(i) The family of distributions $\{\mathcal{P}_\pi\}$ corresponding to $\{f_\pi\}$ is tight in the space $C(0,T;\mathcal{M})$, where $\mathcal{M}$ is a certain Sobolev-Slobodetski space of loops on M.

(ii) Every limit $\mathcal{P}$ of $\{\mathcal{P}_\pi\}$, as mesh$\pi \to 0$, solves the Martingale problem corresponding to the SPDE (1.0.3).

(iii) The SPDE (1.0.3) and the Martingale problem are equivalent.

The main technicality arises in proving (i), where Funaki calculates deep (kinetic and potential) energy estimates for the processes f_π. This method, although quite ingenius, is very probabilistic in nature, and nontrivial to say the least.

In this thesis we propose a different method of solving (1.0.3). This method is more direct and more in the spirit of the ideas used in the deterministic case, in particular, those of Hamilton. We briefly describe Hamilton's method. Imbedding the target manifold M in some Euclidean space $\mathbb{R}^d$ and extending the metric on M to $\mathbb{R}^d$, one first solves the problem (1.0.2) uniquely in Euclidean coordinates. The extension of the metric is carried out to ensure existence of an involutive isometry i on the tubular neighbourhood U of M. In particular i has M as its fixed point set. By showing that $i \circ f$ also solves (1.0.2) on some short time interval, where f is the original solution, Hamilton deduces that if f starts on the manifold then it must remain there for a short time period. To prove this is true on the half time-line, he employs the method of energy estimates used by Eells and Sampson.

We now give a description of our work. We first consider (1.0.3) as an SPDE in Euclidean coordinates. This requires extending the metric (as in Hamilton), but also the maps v and h, suitably to $\mathbb{R}^d$. In these Euclidean coordinates the nonlinear Laplacian then takes the form

$$\underline{\Delta} = -A + F \tag{1.0.5}$$

where $-A = \frac{d^2}{d\sigma^2}$ is the standard Laplacian and F is a nonlinear term. We then reformulate the SPDE (1.0.3) as a stochastic evolution equation, SEE, on a suitable function space, i.e.

$$du(t) + Au(t)dt = F(u(t))dt + V(u(t))dt + H(u(t)) \circ dw(t), \tag{1.0.6}$$

where V and H are the Nemytski maps corresponding to the extensions of v and h.

Equation (1.0.6) is an example of an SPDE with multiplicative noise. The theory for such equations in Hilbert spaces began with the works of Curtain and Falb, [Cu/Fa,71], Pardoux, [Par,75], [Par,79] and in Banach spaces, Krylov and Rozovskii, [Kr/Ro,79], but to name a few. Many of the ideas and techniques used to solve SPDEs are generalisations of those used for deterministic partial differential equations, PDEs. For example, Bensoussan, Teman and Pardoux, [Ben/Te,72], [Par,75], [Par,79], developed a theory for SPDEs for monotone coercive operators by generalising the method of monotone operators developed by Lions, [Li,69], to solve nonlinear PDEs. Another example is in the case of stochastic Navier-Stokes equations, see, for example, the works of Bensoussan and Teman, [Ben/Te,73].

For our problem we use the semigroup approach to SPDEs. Although the operator $-A$ is unbounded, it is well-known that it is the generator of an analytic semigroup. In such cases, it is a standard technique in PDEs to look for a solution in terms of the semigroup, see, for example, [Fr,69]. Such a solution is often referred to as a mild solution. Dawson, [Da,75], first considered this approach in the case of SPDEs on Hilbert spaces. The theory was essentially developed by Da Prato in collaboration with authors such as Iannelli, Tubaro and Zabczyk. We refer the reader to the book [DP/Z,92] which gives an extensive treatment of this theory in the Hilbert space case. See also the papers by Ichikawa, [Ic,78] and Flandoli, [Fl,92]. In the papers [Br,95] and [Br,97], Brzeźniak continued this line of research by considering stochastic evolution equations on M-type 2 Banach spaces. M-type 2 (also known as 2-uniformly smooth) Banach spaces are a class of Banach spaces, on which one can define Itô integration, see [Ne,78], [De,91] and references therein.

We consider equation (1.0.6) as an SEE on the Sobolev-Slobodetski spaces $W^{\varrho,p}(S^1,\mathbb{R}^d)$, $\varrho \in (0,\infty)\setminus\mathbb{N}$, $p \geq 2$, which are defined by the real interpolation method, see [Tr,78]. The choice of function space is essentially at our disposal, provided that the equation (1.0.6) is well posed. The spaces $W^{\varrho,p}(S^1,\mathbb{R}^d)$ are particularly well suited to our problem for two reasons. For $p \geq 2$, they are examples of M-type 2 Banach spaces. Secondly, the Nemytski maps F, V and H satisfy nice regularity properties on these spaces. In particular, F, V and H are smooth and satisfy a local Lipschitz condition, see [Br/El,98]. (Indeed, if v and h are extended to functions of compact support, then V and H are also of linear growth.)

Using the theory developed by Brzeźniak, we prove existence of a local (and maximal) solution in the space $W^{s,p}(S^1,\mathbb{R}^d)$, $\frac{3}{2} > s > 1+\frac{1}{p}$, $p > 2$. This is the best we can hope for when using such general methods, considering that the term F is not of linear growth. Even in the case of ordinary differential equations, without the linear growth condition, there are well known examples of solutions which are not global. Although the general procedure follows that in [Br,97], we prove stronger estimates on the solutions and moreover we consider SEEs on real interpolation spaces, whereas in [Br,97], the author considers the complex interpolation spaces.

We explain briefly why we work in different spaces. The complex interpolation method gives rise to a different class of Sobolev spaces, denoted $H^{\varrho,p}(S^1,\mathbb{R}^d)$, $\varrho \in (0,\infty)\setminus\mathbb{N}$, $p \geq 2$. Although these spaces are M-type 2, the regularity results for the Nemystki maps, mentioned above, may not hold. The reason for this is that the Lipschitzian properties of F, V and H depend on a specific characterisation of the spaces $W^{\varrho,p}(S^1,\mathbb{R}^d)$, see [Tr,78]. Such a characterisation is unknown for the $H^{\varrho,p}(S^1,\mathbb{R}^d)$ spaces.

The second step is to prove that our solution lies on the loop manifold $\mathcal{M} = W^{s,p}(S^1, M)$, which is a closed submanifold of the infinite dimensional Banach space $W^{s,p}(S^1, \mathbb{R}^d)$. The Nemytski map I corresponding to i is an involution on the open set $W^{s,p}(S^1, U)$, where U is the tubular neighbourhood of M. Moreover I has $\mathcal{M}$ as its set of fixed points. By considering the notion of a weak solution, we show that both u and $I(u)$ are solutions to the problem (1.0.6). Using the fixed point properties of I and the uniqueness of solution, we deduce that u lies on the manifold $\mathcal{M}$ for the time it is defined. The use of Stratonovich integrals (instead of Itô integrals) plays an important rôle in this step, along with the following crucial identities:

$$I'(\cdot)V(\cdot) = V(I(\cdot)), \tag{1.0.7}$$

$$I'(\cdot)H(\cdot) = H(I(\cdot)), \tag{1.0.8}$$

$$I'(\cdot)\underline{\Delta}(\cdot) = \underline{\Delta}(I(\cdot)), \tag{1.0.9}$$

where I' is the Frechet derivative of I. The identities (1.0.7) and (1.0.8) are particular to our choice of extensions of v and h. The identity (1.0.9) just follows from the works of Hamilton. To make use of the above identities we need to approximate our mild solution (written in terms of the semigroup) by strict solutions, which is the main difficultly for this part.

Finally, to prove that our solution is global, we calculate energy estimates for the maximal solution, which gives us a bound for the nonlinear term F. Thus, on any finite time interval, we can show that the norm of the solution does not 'explode', i.e. the solution is global. For this step we will need certain results on energy estimates found in [Ee/Sa,64]. Moreover we will again need to use an approximation proceedure similar to above.

We now briefly descibe the layout of this thesis. In Chapter 2 we present the necessary material needed for our work. This should make the thesis fairly self contained. We omit nearly all the proofs and just give references. We present a proof of the Stochastic Fubini Theorem in M-type 2 Banach spaces. Although such a result is well known and well used, the author does not know of any proof of this (in the M-type 2 case) in the literature.

In Chapter 3 we discuss the problem (1.0.3) in more detail and state a precise definition of a solution. The remainder of this chapter focuses on the extensions of the maps v and h and the regularity properties of the Nemytski maps.

Existence of a maximal solution is proved in the second part of Chapter 4. The first being dedicated to studying the regularity properties of the generalised stochastic convolution process. This uses the Da Prato-Kwapien-Zabczyk Factorisation method, see [DP/K/Z,87].

In Chapter 5 we prove that our solution lies on the manifold $\mathcal{M}$ and then we prove globality of solution in Chapter 6. This will conclude our work on this particular problem. At the end of Chapter 6 we briefly discuss ideas concerning further research relating to this problem.

In Chapter 7 we prove an approximation result of the Wong-Zakai type for Stratonovich integrals in M-type 2 Banach spaces. This is a generalisation of a result proved in the PhD thesis by Dowell, [Dow,80], who considered the Hilbert space case. There are certain implications of this result regarding the equation (1.0.3). One may be tempted to think that we used the Stratonovich differential in (1.0.3) (as opposed to the Itô differential) just because it 'works'. This result suggests that when dealing with stochastic equations on manifolds, the Stratonovich differential is the most natural choice.

In the appendix we prove that the factorisation operator, used in the Da Prato-Kwapien-Zabczyk Factorisation method, is the fractional power of a certain abstract parabolic operator. Although this result was essentially proved in [Br,97], we present here all the necessary details and proofs not included there.

♡

Chapter 2

Preliminary Material

2.1 C_0-Semigroups, Fractional Powers of Operators, Analtyic Semigroups

C_0-semigroups

The definitions and results of this subsection are standard and we refer the reader to [Paz,83] for more details and proofs.
Let X be a complex Banach space with norm $|\cdot|_X$. (If X is a real Banach space then we take its complexification). Let $L(X) := L(X,X)$ be the space of all bounded linear operators from X into X endowed with the supremum norm, denoted $|\cdot|_{L(X)}$. When it is clear from the context which norm we are using we will often write $|\cdot|$ instead of $|\cdot|_X$ or $|\cdot|_{L(X)}$.

Definition 2.1.1 *A C_0-semigroup on X, $\{T_t\}_{t\geq 0}$, is a family of bounded linear operators on X such that*
(a) $T_{t+s} = T_t T_s$, $\forall t, s \geq 0$ and $T_0 = I$, where I is the identity operator on X,
(b) $\lim_{t\downarrow 0} T_t x = x$, for every $x \in X$.

For a C_0-semigroup $\{T_t\}_{t\geq 0}$ there exist constants $\rho \geq 0$ and $M \geq 1$ such that

$$|T_t| \leq Me^{\rho t} \text{ for } t \geq 0. \tag{2.1.1}$$

If $\rho = 0$ then $\{T_t\}_{t\geq 0}$ is said to be uniformly bounded. If in addition $M = 1$, then $\{T_t\}_{t\geq 0}$ is called a contraction C_0-semigroup. Using (2.1.1) one can show that for every $x \in X$, the function $T(\cdot)x : [0,\infty) \to X$, $t \to T_t x$, is continuous.

The linear operator A defined by

$$D(A) := \left\{x \in X : \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ exists}\right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ for } x \in D(A)$$

is called the infinitesimal generator of the semigroup $\{T_t\}_{t\geq 0}$. A is a closed operator and $D(A)$, the domain of A, is dense in X. Moreover $D(A)$ is a Banach space with respect to the graph norm $|\cdot|_{D(A)}$, where

$$|x|_{D(A)} := |x| + |Ax|, \quad x \in D(A).$$

The following properties hold for the infinitesimal generator A.
For $x \in X$, $\int_0^t T_s x ds \in D(A)$ and

$$A\left(\int_0^t T_s x ds\right) = T_t x - x. \tag{2.1.2}$$

For $x \in D(A)$, $T_t x \in D(A)$ and

$$\frac{d}{dt} T_t x = A T_t x = T_t A x. \tag{2.1.3}$$

It follows from (2.1.3) that

$$T_t x - T_s x = \int_s^t A T_r x dr = \int_s^t T_r A x dr. \tag{2.1.4}$$

The resolvent set $\rho(A)$ of a linear operator A is the set of all complex numbers λ for which $(\lambda I - A)^{-1}$, (the resolvent of A), is a bounded linear operator in X. The following theorem characterises the generators of C_0-semigroups.

Theorem 2.1.2 (Hille-Yosida) *Let $A : D(A) \subset X \to X$ be a closed operator. Then A is the generator of a C_0-semigroup $\{T_t\}_{t \geq 0}$ that satisfies* (2.1.1) *if and only if $D(A)$ is dense in X, $\rho(A)$ contains the set $\{\lambda \in \mathbb{C} : Re\lambda > \rho\}$ and*

$$|\,(\lambda - A)^{-n}\,| \leq \frac{M}{(\lambda - \rho)^n}, \quad Re\lambda > \rho, n \in \mathbb{N}. \tag{2.1.5}$$

Remark 2.1.3 It is a consequence Theorem 2.1.2 that for $Re\lambda > \rho$,

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T_t dt. \tag{2.1.6}$$

Remark 2.1.4 Using the identity

$$A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I, \quad Re\lambda > \rho, \tag{2.1.7}$$

it is straightforward to show that, for $Re\lambda > \rho$,

$$(\lambda - A)^{-1} : X \to D(A) \tag{2.1.8}$$

is linear and bounded, where $D(A)$ is endowed with the graph norm.

◇

Let A be the generator of a C_0-semigroup. For each $n \in \mathbb{N}$ one defines the Yosida Approximations, A_n, of A by

$$A_n = nA(n - A)^{-1} = n^2(n - A)^{-1} - nI. \tag{2.1.9}$$

One can show that

$$\lim_{n \to \infty} n(n - A)^{-1} x = x, \quad \forall x \in X \tag{2.1.10}$$

and

$$\lim_{n \to \infty} A_n x = Ax, \quad \forall x \in D(A). \tag{2.1.11}$$

Positive Operators and Fractional Powers

The following two definitions are taken from [Tr,78].

Definition 2.1.5 *A closed and densely defined linear operator A on X is said to be positive if $(-\infty, 0] \subset \rho(A)$ and there exists $C \geq 1$ such that*

$$| (\mu + A)^{-1} | \leq \frac{C}{1+\mu}, \quad \mu \in [0, \infty). \tag{2.1.12}$$

Definition 2.1.6 *Let A be a positive operator on X. For $\alpha \in (0,1)$, the fractional power of A, $A^{-\alpha}$, is defined through the formula*

$$A^{-\alpha}x = \frac{\sin \pi \alpha}{\pi} \int_0^\infty t^{-\alpha}(t+A)^{-1}x dt, \quad x \in D(A). \tag{2.1.13}$$

Note that if A is positive, then $A^{-\alpha}$, $\alpha \in (0,1)$, is a well defined bounded linear operator.
The proofs of the following assertions can be found in [Paz,83] or [Tr,78].
Let A be a positive operator on X. Then

(i) there exists a constant $C > 0$ such that

$$| A^{-\alpha} | \leq C \text{ for } \alpha \in [0,1]. \tag{2.1.14}$$

(ii) For $\alpha, \beta, \alpha+\beta \in [0,1]$ we have

$$A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta}. \tag{2.1.15}$$

(iii) For each $\alpha \in (0,1]$, $A^{-\alpha}$ is one-to-one.

In view of (iii) one defines

$$A^{\alpha} := (A^{-\alpha})^{-1} \text{ with } A^0 = I. \tag{2.1.16}$$

One then sets $D(A^\alpha) := R(A^{-\alpha})$, where $R(A^{-\alpha})$ is the range of $A^{-\alpha}$. A^α is a closed densely defined operator and $D(A^\alpha)$ is a Banach space endowed with the norm $| \cdot |_{D(A^\alpha)}$, where

$$| x |_{D(A^\alpha)} = | A^\alpha x |.$$

Finally, we have the following properties:

(iv) For $\alpha, \beta, \alpha+\beta \in (0,1]$,

$$A^{\alpha+\beta} = A^\alpha A^\beta, \tag{2.1.17}$$

(v) if $\alpha \geq \beta > 0$ then

$$D(A^\alpha) \subset D(A^\beta). \tag{2.1.18}$$

Analytic Semigroups

One can extend the notion of a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X to that of an analytic semigroup $\{T_z\}_{z\in\Gamma}$ on X, where the index set Γ is some sector of the complex plane. Clearly to preserve the semigroup structure, see Definition 2.1.1, this sector must be an additive semigroup of complex numbers. For our definition we restrict ourselves to angles around the positive real axis.

Definition 2.1.7 *Let $\Gamma = \{z \in \mathbb{C} : \phi_1 < \arg z < \phi_2, \phi_1 < 0 < \phi_2\}$, where $\arg z$ is the argument of the complex number z and $\phi_1, \phi_2 \in \mathbb{R}$. An analytic semigroup, $\{T_z\}_{z\in\Gamma}$ on X, is a family of bounded linear operators on X, such that* *(a)* $z \longrightarrow T_z$ *is analytic in* Γ,

(b) $T_{z_1+z_2} = T_{z_1}T_{z_2}$, *for* $z_1, z_2 \in \Gamma$ *and* $T_0 = I$, *where* I *is the identity operator on* X,

(c) $\lim_{z\to 0} T_z x = x$, *for every* $x \in X$ *and* $z \in \Gamma$.

From the definition one can see that the restriction of an analytic semigroup to the real axis is a C_0-semigroup. If the analytic semigroup restricts to a contraction (respectively, uniformly bounded) semigroup then we call it a contraction (respectively, uniformly bounded) analytic semigroup. There are many advantages of using analytic semigroups over C_0-semigroups and these are highlighted in the following Theorem.

Theorem 2.1.8 *Let* A *be a positive operator on* X *such that* $-A$ *is the generator of an uniformly bounded analytic semigroup. We denote its restriction to the real axis by* $\left\{e^{-tA}\right\}_{t\geq 0}$. *Then for* $\alpha \in (0,1]$,

(a) $e^{-tA} : X \longrightarrow D(A^\alpha)$ *for every* $t > 0$.

(b) For every $x \in D(A^\alpha)$, $e^{-tA}A^\alpha x = A^\alpha e^{-tA} x$.

(c) For $t > 0$, $A^\alpha e^{-tA}$ *is bounded and there exists a constant* $C_1(\alpha) > 0$ *such that*

$$| A^\alpha e^{-tA} | \leq C_1(\alpha) t^{-\alpha} \tag{2.1.19}$$

(d) There exists a constant $C_2(\alpha) > 0$, *such that, for* $x \in D(A^\alpha)$,

$$| e^{-tA}x - x | \leq C_2(\alpha) t^\alpha \mid A^\alpha x \mid, \quad t \geq 0. \tag{2.1.20}$$

♡

2.2 Real Interpolation Spaces

The Real Interpolation Spaces $(X, D(A))_{\theta,p}$

Suppose three Banach spaces X, Y and D satisfy

$$Y \subset D \subset X,$$

where $\subset$ denotes continuous imbedding, then D is called an intermediate space between X and Y. If, in addition, for every linear operator $T \in L(X)$, such that $T_{|Y} \in L(Y)$, where $T_{|Y}$ denotes the restriction of T to Y, one has $T_{|D} \in L(D)$, then D is called an interpolation space between X and Y. There are various ways of constructing interpolation spaces between X and Y and this theory is covered extensively in, for example, [Be/Bu,67], [Ber,Lö,76] and [Tr,78], see also [Lu,95] for a concise yet sophisticated presentation. We are interested in those spaces defined using the real interpolation method with exponents $\theta \in (0,1)$, $p \in [1,\infty)$ and they are denoted $(X,Y)_{\theta,p}$. A deep understanding of this theory is not necessary for reading our work. Indeed, we are only interested in the special case when $Y = D(A)$, the domain of an operator A. In particular, we assume that A is a positive operator with $-A$ the generator of an analytic semigroup $\left\{e^{-tA}\right\}_{t\geq 0}$. In this case, see [Tr,78], the spaces $(X, D(A))_{\theta,p}$ can be characterised as follows. For $\theta \in (0,1)$ and $p \in [1,\infty)$,

$$(X, D(A))_{\theta,p} = \left\{x \in X : | x |_{\theta,p} := \int_0^1 | t^{1-\theta-\frac{1}{p}} A e^{-tA} x |^p \, dt < \infty\right\}. \tag{2.2.21}$$

The spaces $(X, D(A))_{\theta,p}$ are Banach spaces with the norm $| \cdot |_{\theta,p}$ and, for $0 < \delta < \theta$, we have the following inclusions which are continuous and dense

$$D(A) \subset (X, D(A))_{\theta,p} \subset (X, D(A))_{\delta,p} \subset X. \tag{2.2.22}$$

In particular, the spaces $(X, D(A))_{\theta,p}$ satisfy the interpolation property mentioned above, i.e. if a linear operator $T : X \to X$ is such that $T \in L(X)$ and $T \in L(D(A))$ then $T \in L((X, D(A))_{\theta,p})$ for each $\theta \in (0,1)$, $p \geq 1$. Moreover, there exists a constant $C(\theta, p) > 0$ such that

$$| T |_{L((X,D(A))_{\theta,p})} \leq C(\theta, p) \, | T |_{L(D(A))}^{\theta} | T |_{L(X)}^{1-\theta}. \tag{2.2.23}$$

The spaces $(X, D(A))_{\theta,p}$ also satisfy the following:

(i) For $\mu \in (0,1)$, $p > 1$,

$$(X, D(A))_{\mu,p} = (X, D(A^2))_{\frac{\mu}{2},p} \tag{2.2.24}$$

(ii) For s, $\mu, \mu_1, \mu_2 \in (0,1)$, $p > 1$,

$$((X, D(A))_{\mu_1,p}, (X, D(A))_{\mu_2,p})_{s,p} = (X, D(A))_{(1-s)\mu_1+s\mu_2,p} \tag{2.2.25}$$

with equivalence of the respective norms.

(iii) For $\alpha > \theta$,

$$D(A^\alpha) \subset (X, D(A))_{\theta,p}. \tag{2.2.26}$$

Note that (2.2.22), (2.2.23) and (ii) hold for any real interpolation space $(X, Y)_{\theta,p}$, where $Y \subset X$, not just in the case $Y = D(A)$. Note that (ii) is a version of the so-called Reiteration Theorem, see [Tr,78].

The Sobolev-Slobodetski Spaces

The contents of this subsection can be found in [Tr,78]. For $p \in [1,\infty)$, $n \in \mathbb{N}$ and $\mathcal{O}$ an open bounded interval of $\mathbb{R}$, the Sobolev space $W^{n,p}(\mathcal{O}, \mathbb{R})$ is defined as the the vector space of all functions $u : \mathcal{O} \to \mathbb{R}$ whose weak derivatives $D^\alpha u$, $\alpha \in \mathbb{N}$, $\alpha \leq n$, belong to $L^p(\mathcal{O}, \mathbb{R})$, the space of Lebesgue p-integrable functions. Recall that $v \in L^p(\mathcal{O}, \mathbb{R})$ is the n^{th} weak derivative of $u \in L^p(\mathcal{O}, \mathbb{R})$ if

$$\int_{\mathcal{O}} u(x)\phi^{(n)}(x)dx = (-1)^n \int_{\mathcal{O}} v(x)\phi(x)dx$$

for every test function ϕ. v is then denoted $D^n u$. For $n \in \mathbb{N}$, $p \in [1,\infty)$, the Sobolev spaces $W^{n,p}(\mathcal{O}, \mathbb{R})$ are Banach spaces with the norm

$$| u |_{n,p} := \sum_{i=0}^{n} | D^i u |_{L^p} \tag{2.2.27}$$

where $|\cdot|_{L^p}$ denotes the norm on $L^p(\mathcal{O},\mathbb{R})$.
For $s\in(0,\infty)\setminus\mathbb{N}$, the Sobolev-Slobodetski space $W^{s,p}(\mathcal{O},\mathbb{R})$ is defined via the real interpolation method, i.e.

$$W^{s,p}(\mathcal{O},\mathbb{R}):=\left(L^p(\mathcal{O},\mathbb{R}^d),W^{n,p}(\mathcal{O},\mathbb{R}^d)\right)_{\theta,p}, \tag{2.2.28}$$

where $n\in\mathbb{N}$ is such that $\frac{s}{n}=\theta\in(0,1)$.
Different Sobolev spaces are obtained using other interpolation methods, yet one reason for the choice of these particular Sobolev spaces is the following useful characterisation:
$u\in W^{s,p}(\mathcal{O},\mathbb{R})$ if and only if $u\in W^{n,p}(\mathcal{O},\mathbb{R})$, where $n=[s]$, the integer part of s, and

$$\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|D^nu(x_1)-D^nu(x_2)|^p}{|x_1-x_2|^{1+sp}}dx_1dx_2<\infty. \tag{2.2.29}$$

For $s\in(0,\infty)\setminus\mathbb{N}$, the spaces $W^{s,p}(\mathcal{O},\mathbb{R})$ are Banach spaces with the norm

$$|u|_{s,p}:=|u|_{n,p}+\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|D^nu(x_1)-D^nu(x_2)|^p}{|x_1-x_2|^{1+sp}}dx_1dx_2, \tag{2.2.30}$$

where $n=[s]$ and $|\cdot|_{n,p}$ is given by (2.2.27). Note that we have used the same notation for the norms of the spaces $(X,D(A))_{s,p}$ and $W^{s,p}$. It will always be clear from the context to which space they refer. Finally we say that $u\in W^{s,p}(\mathcal{O},\mathbb{R}^d)$ if and only if each of the real-valued coordinate functions of u belong to $W^{s,p}(\mathcal{O},\mathbb{R})$.
We will need the following theorem:

Theorem 2.2.1 (The Sobolev Imbedding Theorem) *Suppose $s>n+\frac{1}{p}$, then the imbedding map*

$$W^{s,p}(\mathcal{O},\mathbb{R}^d)\hookrightarrow C^n(\overline{\mathcal{O}},\mathbb{R}^d)$$

is well defined and bounded.

Here $C^n(\overline{\mathcal{O}},\mathbb{R}^d)$ is the space of continuous functions on $\overline{\mathcal{O}}$, the closure of $\mathcal{O}$, whose derivatives up to order n exist on $\mathcal{O}$ and have continuous extensions to $\overline{\mathcal{O}}$.
We end this subsection with an example of an analytic generator of a semigroup on $L^p(\mathcal{O},\mathbb{R}^d)$, where $\mathcal{O}=(0,2\pi)$. Theorem 2.2.1 implies that

$$W^{2,p}(0,2\pi;\mathbb{R}^d)\hookrightarrow C^1([0,2\pi];\mathbb{R}^d).$$

Thus, each $u\in W^{2,p}(0,2\pi;\mathbb{R}^d)$ can be identified with a continuously differentiable function $\tilde{u}$, where $\tilde{u}$ and its (classical) derivative, $\tilde{u}'$, have continuous extensions to the whole of the closed interval $[0,2\pi]$. We denote $\tilde{u}$ by u. Consider the operator $Q:=D^2$ with the boundary conditions $u(0)=u(2\pi)$ and $u'(0)=u'(2\pi)$. One can show, see [Tr,78], that Q is the generator of a contraction analytic semigroup on $L^p(0,2\pi;\mathbb{R}^d)$ with domain

$$D(Q)=\left\{u\in W^{2,p}(0,2\pi;\mathbb{R}^d):u(0)=u(2\pi)\text{ and }u'(0)=u'(2\pi)\right\}. \tag{2.2.31}$$

The space defined by the RHS of (2.2.31) will be denoted $W^{2,p}_{per}(0,2\pi;\mathbb{R}^d)$.

♡

2.3 Manifold Theory

We present the basic theory required for our work. There are many classical texts where our presentation can be found, e.g. [Ko/No,63] and [Sp,75]. The author refers the beginner to the book [O'Ne,83], which is an excellent introduction to this theory.

By a smooth m-dimensional manifold M we mean a topological second countable Hausdorff space along with a complete atlas of dimension m. For each $p \in M$ we have a tangent space T_pM which is an m-dimensional vector space. Using the topology of M one can glue these spaces together to obtain the Tangent Bundle TM of M, i.e.

$$TM = \bigcup_{p \in M} T_pM. \tag{2.3.32}$$

TM has the structure of a $2m$-dimensional smooth manifold. A smooth vector field v on M is a smooth map $v : M \to TM$, such that $v(p) \in T_pM$ for $p \in M$. We denote the vector space of these maps by $C^\infty(M, TM)$. For each $p \in M$, let $\mathcal{L}_2^s(T_pM; \mathbb{R})$ denote the space of bilinear, symmetric and nondegenerate forms on T_pM. A smooth map g on M, which assigns to each $p \in M$ an element $g(p) = g(p)(\cdot,\cdot) \in \mathcal{L}_2^s(T_pM; \mathbb{R})$, is called a metric on M. In particular, for each $p \in M$, $g(p)$ is an inner product on the tangent space T_pM. Sometimes we will write g_p for $g(p)$, $p \in M$. As an example, the Euclidean metric $<,>_{(\cdot)}$ on $\mathbb{R}^d$ is defined by

$$< v_p, u_p >_p := \sum_{i=1}^{d} u_i v_i, \quad p \in \mathbb{R}^d \tag{2.3.33}$$

where, for example, $v_p = (p, v) \in T_p\mathbb{R}^d$ with $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$. Thus, the Euclidean metric just assigns the standard inner product on $\mathbb{R}^d$ to each tangent space $T_p\mathbb{R}^d$, $p \in \mathbb{R}^d$.

Let W_i, $i = 1, 2$ be open sets in M and $i : W_1 \to W_2$ a diffeomorphism. Then i is said to be an isometry if

$$g(p)(u, v) = g(i(p))(i'(p)u, i'(p)v), \quad p \in M, \quad u, v \in T_pM, \tag{2.3.34}$$

where i' is the derivative of i and for each $p \in M$, $i'(p) : T_pM \to T_{i(p)}M$ is a linear map. Let $\alpha : I \to M$ be a smooth curve in M, where I is an open interval on the real line. Suppose that $\alpha(a) = p \in M$ for some $a \in I$, then there exists a map P, associated with the Levi-Civita connection on M, such that for any $b \in I$

$$P_p^{\alpha(b)} : T_pM \to T_{\alpha(b)}M.$$

P is called Parallel Translation along α. In particular, for the isometry i described above, let $\tilde{P}$ be translation, along the curve $i \circ \alpha$, from $T_{i(p)}M$ to $T_{i(q)}M$ where $q = \alpha(b) \in M$. One can prove, see [O'Ne,83],

$$i'(q) \circ P_p^q = \tilde{P}_{i(p)}^{i(q)} \circ i'(p). \tag{2.3.35}$$

Note that Parallel Translation depends on the smooth curve α.

We may imbed M smoothly into some Euclidean space, $\mathbb{R}^d$, $d > m$, and thus view (the image of) M as a submanifold of $\mathbb{R}^d$ and (the image of) each T_pM as a linear subspace of $T_p\mathbb{R}^d$. As a result, for each $p \in M$, there is a direct sum decomposition of the tangent space $T_p\mathbb{R}^d$, i.e.

$$T_p\mathbb{R}^d = T_pM \oplus T_pM^{\perp} \tag{2.3.36}$$

where

$$T_pM^{\perp} := \left\{ v \in T_p\mathbb{R}^d :< v, u >_p = 0, \forall u \in T_pM \right\}. \tag{2.3.37}$$

Here $<,>_{(\cdot)}$ is the Euclidean metric on $\mathbb{R}^d$ given by (2.3.33).
Given the decomposition (2.3.36) one defines the Normal Bundle of M by

$$NM := \bigcup_{p\in M} T_pM^{\perp}. \tag{2.3.38}$$

The Normal Bundle has the structure of a d-dimensional smooth manifold. There exists a smooth projection map $\pi_N : NM \to M$ given by

$$\pi_N(v) = p \text{ if } v \in T_pM^{\perp}, \;\; p \in M. \tag{2.3.39}$$

We now introduce the notion of a tubular neighbourhood, which will play an important rôle in our work. Using the above notation, there exists an open neighbourhood U of M in $\mathbb{R}^d$ and a diffeomorphism φ from an open set V in NM onto U. This U is called the tubular (or Normal) neighbourhood of M in $\mathbb{R}^d$. The existence of a tubular neighbourhood for M is a nontrivial result. Most texts prove the result for the case M is compact, which will suffice for us. See [O'Ne,83] which deals also with the noncompact case. On an intuitive level, the tubular neighbourhood can be described as follows, see [Bo/Tu,82]. Suppose M is a curved length of string in $\mathbb{R}^3$, where M is imbedded in a tube U. U can be thought of as being made up of cross sectional discs each of which is perpendicular to the string at the center.

♡

2.4 Stochastic Analysis and Stochastic Integration in M-type 2 Banach Spaces

Stochastic Analysis

We assume basic knowledge of stochastic processes and present here certain definitions and results for completeness. The following notation will be used throughout. Let X be a metric space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with given increasing right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$ with $\mathcal{F}_0$ complete.

Definition 2.4.1 *A stopping time τ is a random function $\tau : \Omega \to [0, \infty]$ such that*

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \textit{ for each } t \geq 0. \tag{2.4.40}$$

Proposition 2.4.2 *If τ and σ are stopping times then so are $min\{\tau, \sigma\} := \tau \wedge \sigma$, $max\{\tau, \sigma\} := \tau \vee \sigma$ and $\tau \pm \sigma$. Furthermore if $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times then $\sup_{n\in\mathbb{N}} \tau_n$ is also a stopping time.*

Proposition 2.4.3 *Let $\xi(t)$, $t \geq 0$, be a X-valued stochastic process with continuous paths and U an open set in X. Define*

$$\tau_U := inf\{t \in [0,\infty) : \xi(t) \notin U\},$$

with the convention that $\tau_U := \infty$ if ξ never leaves the set U. Then τ_U is a stopping time.

The proofs of the above Propositions can be found in [We,81]. Following [Br/El,98], see also [Kun,90], we call a stopping time τ accessible if and only if there exists a sequence of stopping times $\{\tau_n\}$ such that

$$\tau_n < \tau \text{ a.s. and } \lim_{n\to\infty} \tau_n = \tau \text{ a.s..} \tag{2.4.41}$$

For a stopping time τ we set

$$\Omega_t(\tau) := \{\omega \in \Omega : t < \tau(\omega)\}, \tag{2.4.42}$$

$$[0,\tau) \times \Omega := \{(t,\omega) \in [0,\infty) \times \Omega : 0 \leq t < \tau(\omega)\}. \tag{2.4.43}$$

A process $\xi : [0,\tau) \times \Omega \to X$, written $\xi(t)$, $t < \tau$, is said to be admissible if and only if

(i) $\xi_{|\Omega_t(\tau)} : \Omega_t(\tau) \to X$ is $\mathcal{F}_t$-measurable for any $t \geq 0$, i.e. ξ is adapted,

(ii) for almost all $\omega \in \Omega$, $[0,\tau(\omega)) \ni t \mapsto \xi(t,\omega) \in X$ is continuous.

Stochastic Integration in M-type 2 Banach Spaces

The following definition is fundamental for our work.

Definition 2.4.4 *A Banach space X is called M-type 2 if and only if there exists a constant $C(X) > 0$ such that for any X-valued martingale $\{M_k\}$ the following inequality holds*

$$\sup_k \mathbb{E} \mid M_k \mid^2 \leq C(X) \sum_k \mathbb{E} \mid M_k - M_{k-1} \mid^2. \tag{2.4.44}$$

For the definition of a Banach space valued Martingale see [Me,82]. Using the properties of the conditional expectation, see [DP/Z,92], one can prove that any Hilbert space is M-type 2. Furthermore for $2 \leq p < \infty$, $L^p(\mathcal{O},\mathbb{R}^d)$ are M-type 2, see [Br,95], and using this fact one can show that for $2 \leq p < \infty$ and $\theta \in (0,1)$ the spaces $W^{\theta,p}(\mathcal{O},\mathbb{R}^d)$ are also M-type 2, see [Br/El,98].

The theory of stochastic integration in infinite dimensional Hilbert spaces has been developed and is well understood, see [DP/Z,92] and [Ic,83] for a summary of this theory. It is known that for general separable Banach spaces there are difficulties even in the finite-dimensional case. In an unpublished thesis by Neidhardt, [Ne,78], a theory of stochastic integration was developed for a certain class of Banach spaces known as 2-uniformly smooth Banach spaces, which are Banach spaces which satisfy

$$\mid x+y \mid^2 + \mid x-y \mid^2 \leq \mid x \mid^2 + A \mid y \mid^2, \text{ for } x,y \in X \tag{2.4.45}$$

for some constant $A > 0$. Similar work was carried out independently by Dettweiler, see [De,91] and references therein. It is known, see [Pi,76], that a Banach space is 2-uniformly smooth if and only if it is M-type 2. What we present here is taken from [Br/El,98]. Some results in [Br/El,98], which have been proved using the inequality (2.4.44) instead of (2.4.45), are stronger than those in [Ne,78].

Definition 2.4.5 *For separable Hilbert and Banach spaces H and X we set*

$$M(H,X) := \{T : H \to X : T \in L(H,X) \quad \text{and } T \text{ is } \gamma\text{-radonifying}\} \tag{2.4.46}$$

By γ-radonifying we mean the image $T(\gamma_H)$ of the canonical finitely additive Gaussian measure γ_H on H is σ-additive on the algebra of cylindrical sets in X.

Remark 2.4.6 The algebra of cylindrical sets in X generates the Borel σ-algebra, $\mathcal{B}(X)$ on X, see [Kuo,75]. Thus $T(\gamma_H)$ extends to a Borel measure on $\mathcal{B}(X)$ which we denote by ν_T. In particular, ν_T is a Gaussian measure on $\mathcal{B}(X)$, i.e. the image measure $\lambda(\nu_T)$ is a Gaussian measure on $\mathcal{B}(\mathbb{R})$ for each $\lambda \in X^*$, the dual of X. If $\lambda(\nu_T)$ has mean value 0 for each $\lambda \in X^*$, then ν_T is called a centered/symmetric Gaussian measure. $\diamondsuit$

For $T \in M(H,X)$ we put

$$| T |^2_{M(H,X)} := \int_X | x |^2 \, d\nu_T(x). \tag{2.4.47}$$

As ν_T is Gaussian, then by the Fernique-Landau-Shepp Theorem, see [Kuo,75], $| T |_{M(H,X)}$ is finite. Furthermore, see [Ne,78], $M(H,X)$ is a separable Banach space endowed with the norm (2.4.47).

Definition 2.4.7 *Let E be a separable Banach space. We say that $i : H \hookrightarrow E$ is an Abstract Wiener Space, AWS, if and only if i is a linear, one-to-one map and $i \in M(H,E)$. If $i : H \hookrightarrow E$ is an AWS, then the Gaussian measure ν_i on E will be denoted by μ and called the canonical Gaussian measure on E.*

Remark 2.4.8 The notion of an AWS was introduced by Gross, [Gr,65], who named it thus since the classical Wiener space is the most familiar example. There is a vast literature on the theory of AWS's, yet the above definition will suffice for our needs. We refer the interested reader to [Kuo,75] and [Rö,93] for more details and references. We point out that many authors require $i(H)$ to be dense in E in the definition of an AWS, in alignment with the work of Gross. This is an unnecessary restriction for us. Indeed, Sato, [Sa,69], proved that given a separable Banach space with Gaussian measure μ, then there always exists a Hilbert subspace $H \subset E$ such that $i : H \hookrightarrow E$ is an AWS, with $\mu = \nu_i$, where i is the inclusion mapping. $\diamondsuit$

Remark 2.4.9 The Hilbert space H appearing in the above definition is often referred to as the reproducing kernel Hilbert space, RKHS, of (E, μ).

As an example relevant to our work, let $\mathcal{O}$ be an interval and $H^{1,2}(\mathcal{O}, \mathbb{R}^d)$ be the Hilbert space of functions f such that f and its weak derivative Df both belong to $L^2(\mathcal{O}, \mathbb{R}^d)$. Then, see [Br,96], $i : H^{1,2}(\mathcal{O}, \mathbb{R}^d) \hookrightarrow W^{\theta,p}(\mathcal{O}, \mathbb{R}^d)$ is an AWS, where $\theta \in (0, \frac{1}{2})$, $p \geq 2$. Here $H^{1,2}(\mathcal{O}, \mathbb{R}^d)$ is the Hilbert space of functions $u \in L^2(\mathcal{O}, \mathbb{R}^d)$ whose weak derivative u' also belongs to $L^2(\mathcal{O}, \mathbb{R}^d)$. Moreover, the range of $H^{1,2}(\mathcal{O}, \mathbb{R}^d)$ is dense in $W^{\theta,p}(\mathcal{O}, \mathbb{R}^d)$. Another example is given by

$$H^{1,2}_{per}(0, 2\pi; \mathbb{R}^d) \hookrightarrow W^{\theta,p}_{per}(0, 2\pi; \mathbb{R}^d),$$

with θ as above, where

$$H^{1,2}_{per}(0, 2\pi; \mathbb{R}^d) = \{u \in H^{1,2}(0, 2\pi; \mathbb{R}^d) : u(0) = u(2\pi)\}$$

and

$$W^{\theta,p}_{per}(0,2\pi;\mathbb{R}^d) = \{u \in W^{\theta,p}(0,2\pi;\mathbb{R}^d) : u(0) = u(2\pi)\}.$$

◇

Suppose that a triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and let $i : H \hookrightarrow E$ be an AWS. Let $w(t)$, $t \geq 0$, denote the canonical E-valued Wiener process, i.e. a continuous process on E such that

(i) $w(0) = 0$ a.s.;

(ii) the law of the random function $t^{-\frac{1}{2}}w(t) : \Omega \to E$ equals μ, for any $t > 0$;

(iii) if $\mathcal{F}_s$ is the σ-algebra generated by $w(r)$, $r \in [0,s]$, then $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for any $t \geq s \geq 0$.

Remark 2.4.10 We explain what we mean by *canonical* Wiener process. Given any separable Banach space E with Gaussian measure μ then there may exist a variety of Wiener processes related to μ. Due to the result of Sato, there exists H and i such that $i : H \hookrightarrow E$ is an AWS. Let $\{e_k\}_{k\geq 1}$ be an orthonormal basis of H and $\{\beta_k(t)\}_{k\geq 1}$ a sequence of independent, identically distributed real-valued Brownian motions. For each $t \geq 0$, the series

$$W(t) = \sum_{k}^{\infty} \beta_k(t) i(e_k) \tag{2.4.48}$$

converges almost surely in E and is an E-valued Wiener process, as described above. We refer to this Wiener process as the *canonical* Wiener process. Although we will not make use of the representation (2.4.48), we will always make this canonical choice of Wiener process. ◇

Let S be a Banach space and $T \in (0,\infty]$. Let $\mathcal{N}(0,T;S)$ be the space of (equivalence classes of) functions $\xi : [0,T) \times \Omega \to S$ which are progressively measurable, i.e.

$$[0,t] \times \Omega \ni (s,\omega) \mapsto \xi(s,\omega) \in S$$

is $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$ measurable for each $t \in [0,T)$, where $\mathcal{B}_{[0,t]}$ is the class of Borel subsets of $[0,t]$. Let $\mathcal{N}_{step}(0,T;S)$ be the space of all $\xi \in \mathcal{N}(0,T;S)$ for which there exists a partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that $\xi(t) = \xi(t_k)$ for $t \in [t_k, t_{k+1})$, $0 \leq k \leq n-1$, $k \in \mathbb{N}$.
For $p \in [1,\infty)$ we define

$$M^p(0,T;S) := \left\{\xi \in \mathcal{N}(0,T;S) : \mid \xi \mid_{M^p} := \mathbb{E}\int_0^T \mid \xi(s) \mid^p ds < \infty\right\}. \tag{2.4.49}$$

$M^p(0,T;S)$ is a closed subspace of $L^p([0,T] \times \Omega; S)$ and is thus a Banach space. Set $M^p_{step}(0,T;S) := M^p \cap \mathcal{N}_{step}$. For $\xi \in M^p(0,T;L(E,X))$ define a measurable map $I(\xi) : \Omega \to X$ by

$$I(\xi) := \sum_{j=1}^{n-1} \xi(t_k)(w(t_{k+1}) - w(t_k)). \tag{2.4.50}$$

Lemma 2.4.11 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, X is an M-type 2 Banach space and $T \in (0,\infty]$. Then for $\xi \in M^p_{step}(0,T;L(E,X))$, $I(\xi) \in L^2(\Omega;X)$, $\mathbb{E}I(\xi) = 0$ and*

$$\mathbb{E} \mid I(\xi) \mid_X^2 \leq C \int_0^T \mathbb{E} \mid \xi(t) \circ i \mid^2_{M(H,X)} dt. \tag{2.4.51}$$

The proof of Lemma 2.4.11 uses the M-type 2 property defined earlier. Furthermore it uses the fact that if $i \in M(H,E)$ and $A \in L(E,X)$ then $A \circ i \in M(H,X)$ with $\mid A \circ i \mid_{M(H,X)} \leq C \mid A \mid_{L(E,X)}$, see [Br/El,98].
The fundamental property of the map I is that it extends uniquely to a bounded linear map from $M^2(0,T;M(H,X))$ into $L^2(\Omega;X)$. This is a consequence of (2.4.51) and the fact, proven in [Ne,78], that $M^p_{step}(0,T;L(E,X))$ is dense in $M^p(0,T;M(H,X))$. For $\xi \in M^p(0,T;M(H,X))$, the value of this extension will be denoted by $\int_0^T \xi(s)dw(s)$.
Let τ be a finite stopping time with respect to the filtration $\{\mathcal{F}_t\}$, i.e. $\tau < \infty$ a.s.. For $\xi \in M^p(0,\infty;M(H,X))$ we define

$$\int_0^\tau \xi(s)dw(s) := \int_0^\infty 1_{[0,\tau)}\xi(s)dw(s) \tag{2.4.52}$$

where $1_{[0,\tau)}$ is the characteristic function of the stochastic interval $[0,\tau)$. We have, see [Br,97] and [Br/El,98],

Theorem 2.4.12 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Assume that $\xi \in M^2(0,\infty;L(E,X))$ and let $I(t) := \int_0^t \xi(s)dw(s)$ for $t > 0$. Then, $I(t)$ is a continuous X-valued martingale and for any $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that, for any finite stopping time $\tau > 0$,*

$$\mathbb{E} \sup_{0\leq s\leq\tau} \mid I(s) \mid_X^p \leq C_p \left\{ \int_0^\tau \mathbb{E} \mid \xi(s) \mid^2_{M(H,X)} ds \right\}^{\frac{p}{2}}. \tag{2.4.53}$$

The inequality (2.4.53) is the Burkholder inequality. The case $p = 2$ was proved in [Ne,78] and later, using the M-type 2 inequality, (2.4.44), was proved in [De,91] for $p \geq 2$.

Remark 2.4.13 In the above we may replace $M(H,X)$ by $L(E,X)$. In particular, $\int_0^T \xi(s)dw(s)$ exists for any $\xi \in M^2(0,T;L(E,X))$ and satisfies

$$\mathbb{E} \sup_{0\leq s\leq\tau} \mid \int_0^s \xi(r)dw(r) \mid_X^p \leq C_p \left\{ \int_0^\tau \mathbb{E} \mid \xi(s) \mid^2_{L(E,X)} ds \right\}^{\frac{p}{2}}. \tag{2.4.54}$$

◇

Henceforth we will work with processes $\xi \in M^2(0,T;L(E,X))$. The following localization property of the Itô integral, defined in (2.4.52), will be of some importance.

Theorem 2.4.14 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. For $k = 1, 2$, let $\xi_k \in M^2(0, \infty; L(E, X))$. Assume that we have a stopping time τ and $\Omega_0 \in \mathcal{F}$, such that $\mathbb{P}(\mathcal{F}) > 0$ and $\tau > 0$ on Ω_0. Suppose further that, for each $t > 0$,*

$$\xi_1(t) = \xi_2(t) \text{ a.s. on } \{\omega \in \Omega_0 : t < \tau(\omega)\}.$$

Then, for any stopping time σ, satisfying $\sigma \leq \tau$ a.s. on Ω_0, it follows

$$\int_0^\sigma \xi_1(s)dw(s) = \int_0^\sigma \xi_2(s)dw(s) \text{ a.s. on } \Omega_0 . \tag{2.4.55}$$

Before we state the Itô formula we need to introduce some additional notation. By $L_2(E; X)$ we denote the space of bounded bilinear maps, $\Lambda : E \times E \to X$. Let $i : H \to E$ be an AWS. We define the map $tr : L_2(E; X) \to X$ by

$$tr\Lambda := \int_E \Lambda(e, e)d\mu(e), \tag{2.4.56}$$

where μ is the canonical Gaussian measure on E. In view of the Fernique-Landau-Shepp Theorem, *tr* is a bounded linear map. Note also that the *tr* map depends on the choice of AWS.

Theorem 2.4.15 (Ito Formula) *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X and Y are M-type 2 Banach spaces. Assume that a function $f : [0, T) \times X \to Y$ is of $C^{1,2}$-class, i.e. $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous on $[0, T) \times X$ with values in the appropriate space. Suppose we have a process $\xi(t)$, $t \in [0, T]$, given by*

$$\xi(t) = \xi(0) + \int_0^t a(s)ds + \int_0^t b(s)dw(s), \tag{2.4.57}$$

where $a \in M^1(0, T; X)$ and $b \in M^2(0, T; L(E, X))$. Then, for all $t \in [0, T)$, the Y-valued process $f(t, \xi(t))$ is given by

$$\begin{aligned} f(t, \xi(t)) - f(0, \xi(0)) &= \int_0^t \frac{\partial f}{\partial s}(s, \xi(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, \xi(s))a(s)ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, \xi(s))b(s)dw(s) \\ &\quad + \frac{1}{2}\int_0^t tr\left\{\frac{\partial^2 f}{\partial x^2}(s, \xi(s)) \circ (b(s), b(s))\right\} ds \end{aligned} \tag{2.4.58}$$

The following Theorem and Proposition will be needed. The author does not know of any proofs in the literature which cover the M-type 2 Banach space case and so we present them here. The proofs are not dissimilar to the Hilbert space case, see [Cu/Pr,78], but we present them for completeness.

Theorem 2.4.16 (Stochastic Fubini Theorem) *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Let $T \in (0, \infty]$. Suppose*

$$h \in L^2([0, T) \times [0, T) \times \Omega; L(E, X))$$

is such that, for almost all $t \in [0, T)$, $h(\cdot, t) \in M^2(0, T; L(E, X))$. Then, for all $t \geq 0$

$$\int_0^t \int_0^t h(s, r)dw(s)dr = \int_0^t \int_0^t h(s, r)drdw(s) \text{ a.s.}. \tag{2.4.59}$$

Proof : Let $h \in L^2([0,T) \times [0,T) \times \Omega; L(E,X))$ such that, for almost all, a.a., $t \in [0,T)$, $h(\cdot,t) \in M^2_{step}(0,T; L(E,X))$. Then

$$\begin{aligned}
\int_0^t \int_0^t h(s,r)dw(s)dr &= \int_0^t \left(\sum_{k=0}^{n-1} h(t_k,r)(w(t_{k+1}) - w(t_k)) \right) dr \\
&= \sum_{k=0}^{n-1} \left(\int_0^t h(t_k,r)dr \right) (w(t_{k+1}) - w(t_k)) \\
&= \int_0^t \int_0^t h(s,r)drdw(s) \text{ a.s..} \qquad (2.4.60)
\end{aligned}$$

Let $h \in L^2([0,T) \times [0,T) \times \Omega; L(E,X))$ such that a.a. $t \in [0,T)$, $h(\cdot,t) \in M^2(0,T; L(E,X))$. There exists a sequence

$$\{h_n\}_{n\in\mathbb{N}} \subset L^2([0,T) \times [0,T) \times \Omega; L(E,X))$$

which are step functions in the first variable and

$$\mathbb{E} \int_0^t \int_0^t | h_n(s,r) - h(s,r) |_X^2 \, dsdr \to 0, \text{ as } n \to \infty, \qquad (2.4.61)$$

see [Ne,78]. Using (2.4.60), the Hölder and Burkholder inequalities, we have the following sequence of inequalities

$$\begin{aligned}
\mathbb{E} \left| \int_0^t \int_0^t h(s,r)dw(s)dr \right. &- \left. \int_0^t \int_0^t h(s,r)drdw(s) \right|_X^2 \\
&\leq \mathbb{E} \left| \int_0^t \int_0^t h(s,r) - h_n(s,r)dw(s)dr \right|_X^2 \\
&\quad + \mathbb{E} \left| \int_0^t \int_0^t h_n(s,r) - h(s,r)drdw(s) \right|_X^2 \\
&\leq t^{\frac{1}{2}} \mathbb{E} \int_0^t \left| \int_0^t h(s,r) - h_n(s,r)dw(s) \right|_X^2 dr \\
&\quad + C\mathbb{E} \int_0^t \left| \int_0^t h_n(s,r) - h(s,r)dr \right|_X^2 ds \\
&\leq C(t)\mathbb{E} \int_0^t \int_0^t | h(s,r) - h_n(s,r) |_X^2 \, dsdr \\
&\quad + C(t)\mathbb{E} \int_0^t \int_0^t | h_n(s,r) - h(s,r) |_X^2 \, drds.
\end{aligned}$$

Let n tend to infinity, then (2.4.61) and the standard Fubini Theorem imply (2.4.59).

♠

Proposition 2.4.17 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Let $T \in (0,\infty]$. Suppose $h \in M^2(0,T; L(E, D(A))$ where $D(A)$ is the domain of a closed densely defined linear operator A on X. Then for each $t \geq 0$*

$$\int_0^t h(s)dw(s) \in D(A) \text{ a.s.} \qquad (2.4.62)$$

and

$$A \int_0^t h(s)dw(s) = \int_0^t Ah(s)dw(s) \text{ a.s..} \qquad (2.4.63)$$

Proof : Note that $M^2_{step}(0,T;L(E,D(A)))$ is dense in $M^2(0,T;L(E,D(A)))$ where $D(A)$ is endowed with the graph norm. For $h \in M^2(0,T;L(E,D(A)))$ there exists a sequence

$$\{h_n\}_{n\in\mathbb{N}} \subset M^2(0,T;L(E,D(A)))$$

such that for each $t \geq 0$

$$\mathbb{E}\int_0^t \mid h_n(s) - h(s) \mid_X^2 ds + \mathbb{E}\int_0^t \mid Ah_n(s) - Ah(s) \mid_X^2 ds \to 0, \tag{2.4.64}$$

as $n \to \infty$. It is straightforward to show that for each $t \geq 0$ and $n \in \mathbb{N}$,

$$A\int_0^t h_n(s)dw(s) = \int_0^t Ah_n(s)dw(s) \text{ a.s..} \tag{2.4.65}$$

Denote

$$x(t) := \int_0^t h(s)dw(s), \;\; x_n(t) := \int_0^t h_n(s)dw(s) \text{ and } y(t) := \int_0^t Ah(s)dw(s).$$

Note that as $A \in L(D(A),X)$ then $Ah \in M^2(0,T;L(E,X))$ and so y is well defined. Moreover the definition of the stochastic integral for $Ah \in M^2(0,T;L(E,X)$ is independent of the approximating sequence. As $\{Ah_n\}$ approximates Ah, we have, for each $t \geq 0$,

$$\lim_{n\to\infty} Ax_n(t) = y(t) \text{ in } L^2(\Omega, X), \tag{2.4.66}$$

It then follows that, for each $t \geq 0$,

$$\lim_{n\to\infty} (x_n(t), Ax_n(t)) = (x(t), y(t)) \text{ in } L^2(\Omega, X \times X). \tag{2.4.67}$$

(2.4.67) then implies that there exists a subsequence of $\{(x_n(t), Ax_n(t))\}$, denoted $\{(x_n(t), Ax_n(t))\}$ again, such that

$$(x_n(t), Ax_n(t)) \to (x(t), y(t)) \text{ a.s. in } \Omega,\ t \geq 0. \tag{2.4.68}$$

(2.4.62) and (2.4.63) now follow from the closedness of A.

♠

Stratonovich Integrals

The following two definitions are taken from [Br/El,98].

Definition 2.4.18 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Let $T \in (0,\infty]$ and $\xi(t)$, $t \in [0,T]$ be a stochastic process such that for any $t \geq 0$*

$$\xi(t) = \xi(0) + \int_0^t a(s)ds + \int_0^t b(s)dw(s) \text{ a.s.} \tag{2.4.69}$$

where $a \in M^1(0,T;X)$ and $b \in M^2(0,T;L(E,X))$. Then for a C^1 map $h : X \to L(E,X)$ we define the Stratonovich Integral of $h(\xi(t))$ as

$$\int_0^t h(\xi(s)) \circ dw(s) := \int_0^t h(\xi(s))dw(s) + \frac{1}{2}\int_0^t tr[h'(\xi(s))b(s)]ds. \tag{2.4.70}$$

♠

Definition 2.4.19 *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Let $T \in (0, \infty]$. Let h be as above and let $f : X \to X$ be a continuous function. We say that a process $\xi(t)$, $t \in [0, T]$, is a solution to the Stratonovich equation*

$$d\xi(t) = f(\xi(t))dt + h(\xi(t)) \circ dw(t) \tag{2.4.71}$$

if and only if it is a solution to the Itô equation

$$d\xi(t) = \left\{ f(\xi(t)) + \frac{1}{2} tr\{h'(\xi(t))h(\xi(t))\} \right\} dt + h(\xi(t))dw(t). \tag{2.4.72}$$

Thus $\xi(t)$ is a solution to (2.4.71) if and only if it satisfies for each $t \geq 0$

$$\begin{aligned} \xi(t) = \xi(0) \quad &+ \quad \int_0^t f(\xi(s)) + \frac{1}{2} tr\{h'(\xi(s))h(\xi(s))\} ds \\ &+ \int_0^t h(\xi(s))dw(s) \ \ a.s.. \end{aligned} \tag{2.4.73}$$

♡

2.5 Some Inequalities

Lemma 2.5.1 (Gronwall Inequality) *Let $u : [0, \alpha] \to \mathbb{R}$ be continuous and non-negative. suppose $C \geq 0$, $K \geq 0$ are such that*

$$u(t) \leq C + K \int_0^t u(s)ds$$

for all $t \in [0, \alpha]$. Then

$$u(t) \leq Ce^{Kt} \tag{2.5.74}$$

for all $t \in [0, \alpha]$.

Lemma 2.5.2 (Young Inequality) *Assume p, q and $r \in [1, \infty]$ satisfy*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{2.5.75}$$

If $f \in L^p(\mathbb{R}^d, \mathbb{R})$ and $g \in L^q(\mathbb{R}^d, \mathbb{R})$, the convolution $f * g$, given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy, \tag{2.5.76}$$

exists almost everywhere, belongs to $L^r(\mathbb{R}^d, \mathbb{R})$ and satisfies

$$| f * g |_{L^r} \leq | f |_{L^p} | g |_{L^q} . \tag{2.5.77}$$

If p and q are conjugate exponents then $f * g$ is bounded and uniformly continuous on $\mathbb{R}^d$.

♡

Chapter 3

The Stochastic Nonlinear Heat Equation

3.1 The Stochastic Nonlinear Heat Equation

Let M be a smooth compact m-dimensional Riemannian manifold with metric g and let S^1 be the unit circle. Let w_t be an E-valued Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where E is a suitable Banach space of loops on $\mathbb{R}^m$, i.e. $\eta : S^1 \to \mathbb{R}^m$. For $u : [0, \infty) \times S^1 \times \Omega \to M$, we will consider the following stochastic partial differential equation, SPDE,

$$du_t(\sigma) = \underline{\Delta} u_t(\sigma)dt + v(u_t(\sigma))dt + h(u_t(\sigma)) \circ dw_t(\sigma), \;\; t > 0, \;\; \sigma \in S^1, \tag{3.1.1}$$

where we write $u_t(\sigma) := u(t, \sigma)$ and we have suppressed the dependence on $\omega \in \Omega$. We explain the notation used in (3.1.1).

(i) $v \in C^\infty(M, TM)$ i.e. v is a smooth vector field on M.

(ii) $h \in C^\infty(M, L(\mathbb{R}^m, TM))$ i.e. h is a smooth section of a bundle $\mathbb{F}$ over M, whose fibres are $\mathbb{F}_x = L(\mathbb{R}^m, T_xM), x \in M$.

(iii) $\circ dw_t$ denotes the Stratonovich differential.

(iv) $\underline{\Delta}$ is the nonlinear Laplacian, i.e. for a C^2 map $u : N \to M$, where N is a smooth manifold, $\underline{\Delta} u$ is the trace of the second derivative of u. For our case, i.e. $N = S^1$, as S^1 is one dimensional, $\underline{\Delta} u$ is just the second derivative of u.

Let $\sigma \in S^1$ and $(\tilde{U}, \varphi)$ be a chart covering the point $u(\sigma) \in M$, i.e. $\tilde{U}$ is an open set in M with $u(\sigma) \in \tilde{U}$ and $\varphi : \tilde{U} \to \mathbb{R}^m$ is a diffeomorphism onto some open set in $\mathbb{R}^m$. The local expression for $\underline{\Delta} u(\sigma) \in T_{u(\sigma)}M$ with respect to this chart, where $T_{u(\sigma)}M$ is the tangent space of M at the point $u(\sigma)$, is given by

$$\underline{\Delta} u(\sigma) = \left\{ \frac{d^2 u^k}{d\sigma^2}(\sigma) + \sum_{i,j=1}^{m} \Gamma^k_{ij}(u(\sigma)) \frac{du^i}{d\sigma}(\sigma) \frac{du^j}{d\sigma}(\sigma) \right\}_{k=1}^{m}, \tag{3.1.2}$$

see [Ha,75]. For $i, j, k = 1, \ldots, m$, $\Gamma^k_{ij} : U \to \mathbb{R}$ are the Christoffel symbols on M, corresponding to the metric g. In the case of $M = \mathbb{R}^d$ with g taken as the usual Euclidean metric, the Christoffel symbols vanish and we are left with the standard Laplacian.

The following discussion will motivate a satisfactory definition of a solution to (3.1.1). Given (M, g) we may imbed M smoothly into some Euclidean space $\mathbb{R}^d$. Furthermore the metric on M can be extended to a metric on $\mathbb{R}^d$, which we denote by g again. Thus M is an imbedded submanifold of $(\mathbb{R}^d, g)$. Moreover as $(\mathbb{R}^d, g)$ is covered by a single chart, then for a C^2 function $u : S^1 \to \mathbb{R}^d$, the nonlinear Laplacian of u takes the form

$$\underline{\Delta} u = -Au + F(u), \tag{3.1.3}$$

where $-A = \frac{d^2}{d\sigma^2}$ and for $\sigma \in S^1$

$$F(u)(\sigma) := \sum_{i,j=1}^{d} \left\{ \Gamma_{ij}^k(u(\sigma)) \frac{du^i}{d\sigma}(\sigma) \frac{du^j}{d\sigma}(\sigma) \right\}_{k=1}^{d}. \tag{3.1.4}$$

For $i, j, k = 1, \ldots, d$, $\Gamma_{ij}^k : \mathbb{R}^d \to \mathbb{R}$ are now the Christoffel symbols relating to the metric g on $\mathbb{R}^d$. Note that these functions need not vanish, as in the case of the Euclidean metric. If we extend v and h to $\tilde{v}$ and $\tilde{h}$, which are defined on the whole of $\mathbb{R}^d$, then we may then consider the SPDE (3.1.1) as an SPDE in the Euclidean space $\mathbb{R}^d$, i.e. for $u : [0, \infty) \times S^1 \times \Omega \to \mathbb{R}^d$,

$$\begin{aligned} du_t(\sigma) = \frac{d^2 u_t}{d\sigma^2}(\sigma) dt \quad &+ \left\{ \sum_{i,j=1}^{d} \Gamma_{ij}^k(u(\sigma)) \frac{du^i}{d\sigma}(\sigma) \frac{du^j}{d\sigma}(\sigma) \right\}_{k=1}^{d} dt \\ &+ \tilde{v}(u_t(\sigma)) dt + \tilde{h}(u_t(\sigma)) \circ dw_t(\sigma). \end{aligned} \tag{3.1.5}$$

We may reformulate (3.1.5) as a stochastic evolution equation, (SEE), on a suitable function space, i.e.

$$du(t) + Au(t)dt = F(u(t))dt + V(u(t))dt + H(u(t)) \circ dw(t). \tag{3.1.6}$$

Here $F(u)$ has the same meaning as in (3.1.4). V and H are the Nemytski maps corresponding to $\tilde{v}$ and $\tilde{h}$, i.e. for a map $u : S^1 \to \mathbb{R}^d$, V and H act through the following formulas

$$V(u)(\sigma) = \tilde{v}(u(\sigma)), \tag{3.1.7}$$

$$H(u)(\sigma) = \tilde{h}(u(\sigma)), \tag{3.1.8}$$

where $\sigma \in S^1$. If, for each $\sigma \in S^1$, $u(\sigma) \in M$, we have

$$V(u)(\sigma) = v(u(\sigma)), \tag{3.1.9}$$

$$H(u)(\sigma) = h(u(\sigma)). \tag{3.1.10}$$

By a solution to (3.1.6) we would ideally want a $D(A)$-valued process $u(t)$, where $D(A)$ is the domain of the operator A, satisfying the following integral equation

$$\begin{aligned} u(t) + \int_0^t Au(s)ds \quad &= \quad u(0) + \int_0^t (F(u(s)) + V(u(s)))\, ds \\ &\quad + \int_0^t H(u(s)) \circ dw(s). \end{aligned} \tag{3.1.11}$$

Such a solution is often referred to as a strict solution. To prove the existence of such a solution is nontrivial to say the least. The difficulty lies in the fact that A is an unbounded operator. It is known that $-A$ is the generator of an analytic semigroup, $\left\{e^{-tA}\right\}_{t\geq 0}$, on $L^p(S^1, \mathbb{R}^d)$ with domain $D(-A) = W^{2,p}(S^1, \mathbb{R}^d)$. Furthermore, see [Ic,83], for example, if a strict solution exists then the solution also satisfies a mild version of (3.1.11), i.e.

$$\begin{aligned} u(t) &= e^{-tA}u(0) + \int_0^t e^{-(t-s)A}\left(F(u(s)) + V(u(s))\right) ds \\ &\quad + \int_0^t e^{-(t-s)A} H(u(s)) \circ dw(s). \end{aligned} \tag{3.1.12}$$

To prove existence of a process satisfying (3.1.12) is easier because, for each $t \geq 0$, e^{-tA} is a bounded linear operator. Such a solution is called a mild solution. After the statement of our Theorem we discuss the possibility of a mild solution also being a strict solution.
The function spaces we will be working in will be the Sobolev-Slobodetski spaces, $W^{\varrho,p}(S^1, \mathbb{R}^d)$, $\varrho > 0$, $p \geq 1$, whose precise definition we will give later. We define for $s > \frac{1}{p}$, $p > 1$,

$$W^{s,p}(S^1, M) := \left\{u \in W^{s,p}(S^1, \mathbb{R}^d) : u(\sigma) \in M, \ \forall \sigma \in S^1\right\}. \tag{3.1.13}$$

By the Sobolev Imbedding Theorem, $W^{s,p}(S^1, \mathbb{R}^d) \hookrightarrow C(S^1, \mathbb{R}^d)$, for $s > \frac{1}{p}$, where $C(S^1, \mathbb{R}^d)$ is the space of continuous functions, and so (3.1.13) does make sense. We are now in a position to define what we mean by a solution to problem (3.1.1).

Definition 3.1.1 *Let $w(t)$, $t \geq 0$, be an E-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where E is a Banach space of loops on $\mathbb{R}^m$, i.e. $\eta : S^1 \to \mathbb{R}^m$. Let $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$ be a right continuous filtration such that $w(t)$ is adapted to this filtration and the increment $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for each $t, s \geq 0$.*
A stochastic process $u(t)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a mild solution to (3.1.1) *if*

(i) for some $s > 1 + \frac{1}{p}$, $p > 1$, $u(t)$, $t \geq 0$, is a continuous progressively measurable $W^{s,p}(S^1, M)$-valued process, on $[0, T]$, for each $T > 0$, and

(ii) for each $t \in [0, T]$, $u(t)$ satisfies the following mild stochastic integral equation

$$\begin{aligned} u(t) &= e^{-tA}u(0) + \int_0^t e^{-(t-s)A}\left(F(u(s)) + V(u(s))\right) ds \\ &\quad + \int_0^t e^{-(t-s)A} H(u(s)) \circ dw(s) \ \ a.s., \end{aligned} \tag{3.1.14}$$

where F, V and H are defined as in (3.1.4), (3.1.7) *and* (3.1.8).

Remark 3.1.2 The requirement that $s > 1 + \frac{1}{p}$ is to guarantee that the term involving the nonlinear map F, given by (3.1.4), makes sense classically. Indeed if $s > 1 + \frac{1}{p}$ then by the Sobolev Imbedding Theorem, $W^{s,p}(S^1, \mathbb{R}^d) \hookrightarrow C^1(S^1, \mathbb{R}^d)$, where $C^1(S^1, \mathbb{R}^d)$ is the space of continuously differentiable functions from S^1 to $\mathbb{R}^d$.

$\diamondsuit$

Before we state our result we first explain how to extend the metric g from M to $\mathbb{R}^d$. We follow Hamilton's construction, see [Ha,75]. We first imbed M smoothly into some Euclidean space $\mathbb{R}^d$. In doing so, we identify M with its image, which we denote by M again, and the tangent spaces T_pM, $p \in M$, with linear subspaces of $T_p\mathbb{R}^d$, which we denote T_pM again. Let U be the tubular neighbourhood of M in $\mathbb{R}^d$. As M is compact, there exists $\varepsilon > 0$ such that $M \subset U_\varepsilon \subset U$ where

$$U_\varepsilon := \left\{x \in \mathbb{R}^d : d(x, M) < \varepsilon\right\}. \tag{3.1.15}$$

Here d is the distance function on $\mathbb{R}^d$. It is worth noting that if $0 < \varepsilon_1 < \varepsilon$ then defining U_{ε_1}, as in (3.1.15), we have $M \subset U_{\varepsilon_1} \subset U_\varepsilon$. Moreover U_ε and U_{ε_1} are also tubular neighbourhoods of M. As M is compact then there exists $R > 0$ such that $U \subset B(0, R)$, where $B(0, R)$ is a ball of radius R in $\mathbb{R}^d$.

From the definition and properties of a tubular neighbourhood there exists a smooth map $i : \mathbb{R}^d \to \mathbb{R}^d$ which vanishes outside the ball $B(0, R)$, maps U_ε to itself and i^2 = identity on U_ε. This also holds when we replace U_ε with U_{ε_1} for any $0 < \varepsilon_1 < \varepsilon$. Furthermore i satisfies

$$i(m) = m \Leftrightarrow m \in M. \tag{3.1.16}$$

We wish to obtain a metric on $\mathbb{R}^d$ so that i becomes an isometry on U_ε. The metric g on M can be extended smoothly to U_ε giving a map $g^1 : U_\varepsilon \to \mathcal{L}_s^2(\mathbb{R}^d; \mathbb{R})$. Now define $g^2 : U_\varepsilon \to \mathcal{L}_s^2(\mathbb{R}^d; \mathbb{R})$ by averaging the metric g^1 under i, i.e.

$$g_p^2(u, v) := \frac{1}{2}\left\{g_p^1(u, v) + g_{i(p)}^1(i'(p)u, i'(p)v)\right\}$$

where $p \in U_\varepsilon$. In particular, this makes $i : U_\varepsilon \to U_\varepsilon$ an isometry and indeed $i : U_{\varepsilon_1} \to U_{\varepsilon_1}$ an isometry for any $0 < \varepsilon_1 < \varepsilon$. Now extend g^2 to the whole of $\mathbb{R}^d$ so that outside the ball $B(0, R)$ it coincides with the usual Euclidean metric. This new metric on $\mathbb{R}^d$ we denote by g again and we consider $\mathbb{R}^d$ as a Riemmanian manifold with this metric. By construction, the metric induced on M as a submanifold of $\mathbb{R}^d$ coincides with the original metric on M. For this reason, along with the isometric properties of $i : U_\varepsilon \to U_\varepsilon$, we have the following results, whoose proofs can be found in [Ha,75]:

Proposition 3.1.3 *Suppose that $u : S^1 \to M \subset B(0, R)$ is of C^2 class. Then $\underline{\Delta} u$ may be calculated in two ways. We may calculate the nonlinear Laplacian of u treating u as a map into $B(0, R)$, where we consider $B(0, R)$ with the metric g we have just constructed. This we denote $\underline{\Delta}_B u$. Or, treating u as a map into M, we may calculate $\underline{\Delta}_M u$. It follows that*

$$\underline{\Delta}_B u = \underline{\Delta}_M u. \tag{3.1.17}$$

♠

Proposition 3.1.4 *Suppose that $u : [0, \infty) \times S^1 \to U_\varepsilon$ is a solution to the deterministic nonlinear heat equation*

$$\frac{\partial u_s(\sigma)}{\partial s} = \underline{\Delta} u_s(\sigma), \ \ s > 0, \ \ \sigma \in S^1, \tag{3.1.18}$$

$$u_0(\sigma) = v(\sigma), \;\; v \in C^\infty(S^1, U_\epsilon), \tag{3.1.19}$$

where we have denoted $u_s(\cdot) := u(s,\cdot)$ *and we consider* U_ϵ *with the constructed metric* g. *Then* $i \circ u$ *is also a solution to* (3.1.18) *with initial value* $i \circ v$. *Moreover we have the following identity*

$$i'(u)\underline{\triangle} u = \underline{\triangle}(i \circ u). \tag{3.1.20}$$

Finally, note that as g coincides with the Euclidean metric outside the ball $B(0,R)$, then the Christoffel symbols appearing in (3.1.4) will vanish outside $B(0,R)$. Thus we may deduce that these functions are of compact support.
We now give a statement of our main Theorem.

Theorem 3.1.5 *Let* $w(t)$ *be a* $W^{\theta,p}(S^1,\mathbb{R}^d)$*-valued Wiener process, where* $\theta \in (\frac{1}{p},\frac{1}{2})$, $p > 2$. *Let* $u_0 \in L^q(\Omega,\mathcal{F}_0;W^{s,p}(S^1,M))$ *where the numbers* q *and* s *satisfy* $q > p$ *and* $\frac{3}{2}-\frac{2}{q} > s > 1+\frac{1}{p}$. *Then there exists a continuous, progressively measurable* $W^{s,p}(S^1,M)$*-valued process,* $u(t)$, $t \geq 0$, *with* $u(0) = u_0$, *such that* u *is the unique mild solution to* (3.1.1) *with initial value* u_0. *In particular, for each* $t \geq 0$, $u(t)$ *satisfies*

$$\begin{aligned} u(t) &= e^{-tA}u(0) + \int_0^t e^{-(t-s)A}\left(F(u(s)) + V(u(s))\right) ds \\ &\quad + \int_0^t e^{-(t-s)A} H(u(s)) \circ dw(s) \;\; a.s.. \end{aligned} \tag{3.1.21}$$

Remark 3.1.6 We can choose $u_0 \in L^p(\Omega,\mathcal{F}_0;W^{s,p}(S^1,M))$. Note though that

$$\frac{3}{2}-\frac{2}{p} > 1+\frac{1}{p} \Leftrightarrow p > 6,$$

i.e. we have a solution only if $p > 6$, see Chapter 4. ◇

Remark 3.1.7 The above Theorem gives the existence of a mild $W^{s,p}(S^1,\mathbb{R}^d)$-valued solution, $s > 1+\frac{1}{p}$. We do not say whether this process takes values in $D(A) := W^{2,p}(S^1,\mathbb{R}^d)$ or not. It will become clear from the proof of our Theorem that the low regularity of the space variable of $u(t)$ is due to the choice of Wiener process. We have chosen $w(t)$ as a Wiener process in the space $W^{\theta,p}(S^1,\mathbb{R}^d)$, $\frac{1}{p} < \theta < \frac{1}{2}$, which guarantees only that for each $t \geq 0$, $w(t) \in C(S^1,\mathbb{R}^d)$. If $w(t)$ were $W^{\theta+1,p}(S^1,\mathbb{R}^d)$-valued, then it turns out, using a simplification (!) of our methods, that the mild solution, denoted $\tilde{u}(t)$, would be a $W^{s+1,p}(S^1,M)$-valued solution, with s as above. In particular, $\tilde{u}(t) \in D(A)$ and so, see [Br,95] or [Ic,83], $\tilde{u}(t)$ would satisfy

$$\begin{aligned} \tilde{u}(t) &= \tilde{u}(0) + \int_0^t -A\tilde{u}(s) + F(\tilde{u}(s))ds + \int_0^t V(\tilde{u}(s))ds \\ &\quad + \int_0^t H(\tilde{u}(s)) \circ dw(s). \end{aligned} \tag{3.1.22}$$

Furthermore as $s+1 > 2+\frac{1}{p}$, then $\tilde{u}(t) : S^1 \to M$ is C^2 class for each $t \geq 0$. (3.1.3), (3.1.17) and (3.1.22) then imply

$$\begin{aligned} \tilde{u}(t) &= \tilde{u}(0) + \int_0^t \underline{\triangle}_M \tilde{u}(s)ds + \int_0^t V(\tilde{u}(s))ds \\ &\quad + \int_0^t H(\tilde{u}(s)) \circ dw(s). \end{aligned} \tag{3.1.23}$$

This observation reinforces Definition 3.1.1 as a satisfactory definition of a mild solution to (3.1.1). ◇

♡

3.2 The Extensions of v and h

Before proceeding with constructing the extensions of v and h, we need to look more closely at how the map $i : \mathbb{R}^d \to \mathbb{R}^d$ is constructed. There exists a diffeomorphism $\varphi : V \to U$, where V is an open set in NM, the normal bundle of M, and U is the tubular neighbourhood of M. We define $\tau : V \to V$ by

$$\tau(v) := -v, \;\; v \in V. \tag{3.2.24}$$

τ corresponds to multiplication by -1 in the fibres $T_pM^{\perp}$, $p \in M$. Note that τ is an involution, i.e. $\tau^2(v) = v$, for each $v \in V$. Furthermore there exists a smooth map $\psi : \mathbb{R}^d \to \mathbb{R}$, such that $0 \leq \psi \leq 1$, with $\psi \equiv 1$ on U_ε and $\psi \equiv 0$ on $\overline{U}$, where $\varepsilon > 0$ is chosen as in (3.1.15). The map $i : \mathbb{R}^d \to \mathbb{R}^d$ is then defined as

$$i(p) := \psi(p)\{\varphi \circ \tau \circ \varphi^{-1}(p)\}, \;\; p \in \mathbb{R}^d. \tag{3.2.25}$$

Using the canonical projection map $\pi_N : NM \to M$, define the smooth map $\eta : U \to M$, by

$$\eta := \pi_N \circ \varphi^{-1}. \tag{3.2.26}$$

Lemma 3.2.1 *On U_ε we have*

$$\eta(i(p)) = \eta(p) = i(\eta(p)), \;\; p \in U_\varepsilon. \tag{3.2.27}$$

Proof : The second equality is trivial as $i(m) = m \Leftrightarrow m \in M$. For the first equality, note that if $\eta(i(p)) \neq \eta(p)$ then by the definition (3.2.26), there exists $r, s \in M$, $r \neq s$ such that

$$\varphi^{-1}(i(p)) \in T_rM^{\perp} \text{ and } \varphi^{-1}(p) \in T_sM^{\perp}.$$

Using (3.2.25) and the fact that $p \in U_\varepsilon$, then

$$\varphi^{-1}(i(p)) = \varphi^{-1} \circ \varphi \circ \tau \circ \varphi^{-1}(p) = \tau \circ \varphi^{-1}(p) \in T_rM^{\perp}.$$

As τ is an involution,

$$\tau \circ \tau \circ \varphi^{-1}(p) = \varphi^{-1}(p) \in T_rM^{\perp} \neq T_sM^{\perp},$$

which is a contradiction. It follows that $\eta(i(p)) = \eta(p)$.

♠

For a, $b \in U_\varepsilon$ let $P_a^b : T_a\mathbb{R}^d \to T_b\mathbb{R}^d$ be parallel translation with respect to the constructed metric g. (Note that we need to specify the choice of curve, but we simply take the straight line from a to b). Then as i is an isometry on U_ε, we have for $p,\, q \in U_\varepsilon$

$$i'(p) \circ P_q^p = P_{i(q)}^{i(p)} \circ i'(q). \tag{3.2.28}$$

Finally, see [Ha,75], we have for $m \in M$,

$$i'(m)v = v \Leftrightarrow v \in T_m M. \tag{3.2.29}$$

We are now in a position to define our extension $\tilde{v}$. For $p \in \mathbb{R}^d$, define

$$\tilde{v}(p) := \psi(p)\left\{P_{\eta(p)}^p \circ v(\eta(p))\right\}. \tag{3.2.30}$$

$\tilde{v}$ is a well-defined smooth vector field on $\mathbb{R}^d$. It is an extension of v and it is of compact support. Furthermore it has the following property:

Lemma 3.2.2 *For each $p \in U_\varepsilon$,*

$$i'(p)(\tilde{v}(p)) = \tilde{v}(i(p)). \tag{3.2.31}$$

Proof : Let $p \in U_\varepsilon$. Then, using the definition (3.2.30), and the properties (3.2.27), (3.2.28) and (3.2.29), we have

$$\begin{aligned} i'(p)\tilde{v}(p) &= i'(p)P_{\eta(p)}^p \circ v(\eta(p)) \\ &= P_{i(\eta(p))}^{i(p)} \circ i'(\eta(p))v(\eta(p)) \\ &= P_{\eta(i(p))}^{i(p)} \circ i'(\eta(p))v(\eta(p)) \\ &= P_{\eta(i(p))}^{i(p)} \circ v(\eta(p)) \\ &= P_{\eta(i(p))}^{i(p)} \circ v(\eta(i(p))) \\ &= \tilde{v}(i(p)). \end{aligned}$$

♠

Similarly, we define the extension $\tilde{h}$ of h by

$$\tilde{h}(p) = \psi(p)\left\{P_{\eta(p)}^p \circ h(\eta(p))\right\}, \quad p \in \mathbb{R}^d. \tag{3.2.32}$$

$\tilde{h}$ is well defined, smooth and has compact support. Furthermore for each $p \in U_\varepsilon$

$$i'(p)(\tilde{h}(p)(e)) = \tilde{h}(i(p))(e), \quad e \in \mathbb{R}^m. \tag{3.2.33}$$

This is proved in an identical manner to Lemma 3.2.2. The extensions of v and h are complete. Before considering their corresponding Nemytski maps, we will give a precise definition of the Sobolev-Slobodetski spaces of loops on $\mathbb{R}^d$.

3.3 Sobolev-Slobodetski Spaces Of Loops

The Spaces $W^{\theta,p}(S^1, \mathbb{R})$

Throughout this section we will write $L^p(S^1)$ for $L^p(S^1, \mathbb{R})$ and $L^p(0, 2\pi)$ for $L^p(0, 2\pi; \mathbb{R})$, with analogous abbreviations for the spaces $W^{\theta,p}(S^1, \mathbb{R})$.

Let S^1 be the unit circle and $\int_{S^1} .d\sigma$ the integral with respect to Haar measure on S^1. In particular, using the map

$$[0, 2\pi) \ni t \mapsto e^{it} \in S^1, \tag{3.3.34}$$

we may write

$$\int_{S^1} u(\sigma)d\sigma = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt. \tag{3.3.35}$$

The space $L^p(S^1)$, $p \geq 2$, is defined in an obvious manner, i.e. as the space of measurable functions $u : S^1 \to \mathbb{R}$ such that

$$\int_{S^1} | u(\sigma) |^p \, d\sigma < \infty. \tag{3.3.36}$$

The Sobolev spaces $W^{k,p}(S^1)$, $k \in \mathbb{N}$, are then defined as the space of loops on $\mathbb{R}$ whose weak derivatives up to and including order $k \in \mathbb{N}$ belong to $L^p(S^1)$.
Define the operator $\mathcal{U}$, acting on functions $g : [0, 2\pi) \to \mathbb{R}$, through the following formula

$$(\mathcal{U}g)(e^{it}) = g(t), \;\; e^{it} \in S^1. \tag{3.3.37}$$

By direct calculation, one can show that:

(i) $\mathcal{U} : L^p(0, 2\pi) \to L^p(S^1)$ is an isomorphism,

(ii) $\mathcal{U} : W^{1,p}_{per}(0, 2\pi) \to W^{1,p}(S^1)$ is an isomorphism,

where

$$W^{1,p}_{per}(0, 2\pi) = \left\{u \in W^{1,p}(0, 2\pi) : u(0) = u(2\pi)\right\}. \tag{3.3.38}$$

Note that the latter is well defined because by the Sobolev Imbedding Theorem, every $u \in W^{1,p}(0, 2\pi)$ has a representative which is a continuous function on $[0, 2\pi]$. In what follows, we restrict ourselves to choosing $\theta \in (\frac{1}{p}, 1)$, $p \geq 2$. Using real interpolation, the properties (i) and (ii) imply that

$$\mathcal{U} : \left(L^p(0, 2\pi), W^{1,p}_{per}(0, 2\pi)\right)_{\theta,p} \to \left(L^p(S^1), W^{1,p}(S^1)\right)_{\theta,p}, \tag{3.3.39}$$

is an isomorphism, where $(\cdot, \cdot)_{\theta,p}$ are real interpolation spaces. It is known, see [Tr,78],

$$\left(L^p(0, 2\pi), W^{1,p}_{per}(0, 2\pi)\right)_{\theta,p} = W^{\theta,p}_{per}(0, 2\pi), \tag{3.3.40}$$

where

$$W^{\theta,p}_{per}(0, 2\pi) = \left\{u \in W^{\theta,p}(0, 2\pi) : u(0) = u(2\pi)\right\}. \tag{3.3.41}$$

Denoting

$$W^{\theta,p}(S^1) := \left\{u \in L^p(S^1) : \int_{S^1}\int_{S^1} \frac{| u(\sigma_1) - u(\sigma_2) |^p}{| \sigma_1 - \sigma_2 |^{1+\theta p}} d\sigma_1 d\sigma_2 < \infty\right\}, \tag{3.3.42}$$

see Chapter 2, one can show that

$$\mathcal{U} : W^{\theta,p}_{per}(0, 2\pi) \to W^{\theta,p}(S^1), \tag{3.3.43}$$

is an isomorphism. We deduce that

$$W^{\theta,p}(S^1) = \left(L^p(S^1), W^{1,p}(S^1)\right)_{\theta,p}. \tag{3.3.44}$$

One can easily extend the above ideas to cover the case when $\theta \in (1,\infty)\setminus \mathbb{N}$. The starting point is showing that

$$\mathcal{U} : W^{k,p}_{per}(0,2\pi) \to W^{k,p}(S^1), \;\; k \in \mathbb{N}, \tag{3.3.45}$$

is an isomorphism. Here, $W^{k,p}_{per}(0,2\pi)$ is the space of functions $u \in W^{k,p}(0,2\pi)$ such that

$$u^{(n)}(0) = u^{(n)}(2\pi), \;\text{ for each } 0 \leq n \leq k-1, \;\; n \in \mathbb{N}, \tag{3.3.46}$$

where $u^{(n)}$ denotes the n^{th} weak derivative of u.

The Contraction Semigroup On $L^p(S^1,\mathbb{R})$

Let $Q := \frac{d^2}{ds^2}$ be the Laplacian acting on functions $u \in W^{2,p}_{per}(0,2\pi)$. Then, see Chapter 2, Q generates a contraction analytic semigroup $\{R_t\}_{t\geq 0}$ on $L^p(0,2\pi)$. Define a family of operators $\{T_t\}_{t\geq 0}$

$$T_t : L^p(S^1) \to L^p(S^1), \;\; t \geq 0, \tag{3.3.47}$$

by

$$T_t u = \left(\mathcal{U} \circ R_t \circ \mathcal{U}^{-1}\right) u, \;\; u \in L^p(S^1). \tag{3.3.48}$$

One can show, see [Tr,78],

(i) $\{T_t\}_{t\geq 0}$ is a contraction analytic semigroup on $L^p(S^1)$.

(ii) If B is the generator of $\{T_t\}_{t\geq 0}$, then

$$D(B) = \left\{u \in L^p(S^1) : \mathcal{U}^{-1}u \in D(Q) = W^{2,p}_{per}(0,2\pi)\right\} = W^{2,p}(S^1), \tag{3.3.49}$$

with $Bu = \mathcal{U}Q\mathcal{U}^{-1}u$ for $u \in D(B)$.

(iii) $Bu = D^2 u$, where D^2 denotes the second weak derivative, i.e. the Laplacian acting on functions $u \in W^{2,p}(S^1)$.

Remark 3.3.1 In the case of real-valued loops, i.e. $u : S^1 \to \mathbb{R}$, we henceforth denote B by Δ.

Remark 3.3.2 In a similar fashion, one can show that $-A := \frac{d^2}{d\sigma^2}$, where $\frac{d^2}{d\sigma^2}$ is the Laplacian acting on functions $u \in W^{2,p}(S^1,\mathbb{R}^d)$, is the generator of a contraction analytic semigroup on $L^p(S^1,\mathbb{R}^d)$, which we henceforth denote $\left\{e^{-tA}\right\}_{t\geq 0}$.

♡

3.4 The Regularity Properties of the Nemytski Maps

We now construct the Nemytski maps corresponding to $\tilde{v}$ and $\tilde{h}$ as maps on the Sobolev-Slobodetski spaces $W^{\varrho,p}(S^1,\mathbb{R}^d)$, $\varrho > \frac{1}{p}, p \geq 2$. Note first that, for $\theta \in (\frac{1}{p},\frac{1}{2})$, $p > 2$, $i : H^{1,2}(S^1,\mathbb{R}^d) \hookrightarrow W^{\theta,p}(S^1,\mathbb{R}^d)$ is an abstract Wiener space, see [Br/El,98]. We thus have a canonical $W^{\theta,p}(S^1,\mathbb{R}^d)$-valued Wiener process, denoted $w(t)$, $t \geq 0$,

corresponding to this AWS. We choose $\theta \in (\frac{1}{p}, \frac{1}{2})$, $p > 2$ as this guarantees that $W^{\theta,p}(S^1, \mathbb{R}^d) \hookrightarrow C(S^1, \mathbb{R}^d)$, and, as a result, for each $\sigma \in S^1$, $w_t(\sigma) := w(t)(\sigma)$ is a $\mathbb{R}^d$-valued Brownian motion.
The Nemytski maps corresponding to $\tilde{v}$ and $\tilde{h}$ are defined as follows

$$V(u)(\sigma) := \tilde{v}(u(\sigma)), \; u \in W^{\varrho,p}(S^1, \mathbb{R}^d), \; \sigma \in S^1, \tag{3.4.50}$$

$$H(u)(\eta)(\sigma) := \tilde{h}(u(\sigma))\eta(\sigma), \; u, \eta \in W^{\varrho,p}(S^1, \mathbb{R}^d), \; \sigma \in S^1. \tag{3.4.51}$$

Recall that $\tilde{v}$ and $\tilde{h}$ are smooth with compact support. The proof of the following result, which can be found in [Br/El,98], relies heavily on the characterisation of the spaces $W^{\varrho,p}(S^1, \mathbb{R}^d)$, see 3.3.42 and Chapter 2.

Proposition 3.4.1 *V and H are smooth maps satisfying*

$$\begin{aligned} V : W^{\varrho,p}(S^1, \mathbb{R}^d) &\rightarrow W^{\varrho,p}(S^1, \mathbb{R}^d) \\ H : W^{\varrho,p}(S^1, \mathbb{R}^d) &\rightarrow L(W^{\varrho,p}(S^1, \mathbb{R}^d), W^{\varrho,p}(S^1, \mathbb{R}^d)). \end{aligned}$$

for any $\varrho > \frac{1}{p}$ and $p \geq 2$. They and all their derivatives are Lipschitz continuous on each ball in $W^{\varrho,p}(S^1, \mathbb{R}^d)$. Furthermore V and H are both of linear growth.

We now consider the nonlinear term F given by (3.1.4). We first quote a result that we need and which can be found in [Am,91]:

Proposition 3.4.2 *Let $\beta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be given by $\beta(u, v) = (u_1 v_1, \ldots, u_d v_d)$. Now if $p \in (1, \infty)$ and $s \in (0, \infty)$ are such that $s > \frac{1}{p}$ then the bilinear mapping $\tilde{\beta} : W^{s,p}(S^1, \mathbb{R}^d) \times W^{s,p}(S^1, \mathbb{R}^d) \rightarrow W^{s,p}(S^1, \mathbb{R}^d)$ given by*

$$\tilde{\beta}(u, v) := \beta \circ (u, v)$$

is continuous and hence smooth.

The next proposition is fundamental in our work.

Proposition 3.4.3 *The nonlinear map F is a smooth locally Lipschitz map from $W^{\varrho+1,p}(S^1, \mathbb{R}^d)$ to $W^{\varrho,p}(S^1, \mathbb{R}^d)$, for any $\varrho > \frac{1}{p}$, $p \geq 2$.*

Remark 3.4.4 As $\varrho > \frac{1}{p}$ then $W^{\varrho+1,p}(S^1, \mathbb{R}^d) \hookrightarrow C^1(S^1, \mathbb{R}^d)$ and so $F(u)$ makes sense classically for $u \in W^{\varrho+1,p}$. $\diamondsuit$

Proof : Throughout this proof we will denote $W^{\varrho,p}(S^1, \mathbb{R}^d)$ by $W^{\varrho,p}$.
Recall that the norm on $W^{\varrho+1,p}$, $\varrho \in (0, 1)$, is given by

$$| u |_{\varrho+1,p} = | u |_{1,p} + \left\{ \int_{S^1 \times S^1} \frac{| \nabla u(\sigma_1) - \nabla u(\sigma_2) |^p}{| \sigma_1 - \sigma_2 |^{1+\varrho p}} d\sigma_1 d\sigma_2 \right\}^{\frac{1}{p}}. \tag{3.4.52}$$

Using (3.4.52) it is straightforward to show that the map $\nabla : W^{\varrho+1,p} \rightarrow W^{\varrho,p}$ given by $\nabla u := \left(\frac{du^1}{d\sigma}, \ldots, \frac{du^d}{d\sigma} \right)$ is linear and bounded. For $i = 1, \ldots, d$, the map $T_i : W^{\varrho,p} \rightarrow W^{\varrho,p}$ given by $T_i(u) := (u^i, \ldots, u^i)$ is linear and bounded. For $i = 1, \ldots, d$, define $\nabla_i = T_i \circ \nabla$ Then ∇_i is a bounded linear map from $W^{\varrho+1,p}$

to $W^{\varrho,p}$, given by $\nabla_i(u) := \left(\frac{du^i}{d\sigma}, \ldots, \frac{du^i}{d\sigma}\right)$.

Using $\tilde{\beta}$, as given in Proposition 3.4.2, we define, for $i, j = 1, \ldots, d$, $\nabla_{ij} : W^{\varrho+1,p} \longrightarrow W^{\varrho,p}$ by $\nabla_{ij}(u) := \tilde{\beta}(\nabla_i u, \nabla_j u)$. Note that each ∇_{ij} is smooth as a composition of smooth maps. Moreover for $u \in W^{\varrho+1,p}$ and $\sigma \in S^1$

$$\begin{aligned}\nabla_{ij}u(\sigma) = \tilde{\beta} \circ (\nabla_i u, \nabla_j v)(\sigma) &= \beta(\nabla_i u(\sigma), \nabla_j v(\sigma)) \\ &= \left(\frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma), \ldots, \frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma)\right).\end{aligned}$$

Using Proposition 3.4.2 we now show that ∇_{ij} is locally Lipschitz. For $u, v \in W^{\varrho+1,p}$ with $| u |_{\varrho+1,p}, | v |_{\varrho+1,p} \leq R$, we have

$$\begin{aligned}| \nabla_{ij}u - \nabla_{ij}v |_{\varrho,p} &= | \tilde{\beta}(\nabla_i u, \nabla_j u) - \tilde{\beta}(\nabla_i v, \nabla_j v) |_{\varrho,p} \\ &= | \tilde{\beta}(\nabla_i u - \nabla_i v, \nabla_j u) - \tilde{\beta}(\nabla_i v, \nabla_j v - \nabla_j u) |_{\varrho,p} \\ &\leq | \tilde{\beta}(\nabla_i u - \nabla_i v, \nabla_j u) |_{\varrho,p} + | \tilde{\beta}(\nabla_i v, \nabla_j v - \nabla_j u) |_{\varrho,p} \\ &\leq K | \nabla_j u |_{\varrho,p} | \nabla_i u - \nabla_i v |_{\varrho,p} \\ &\quad + K | \nabla_i v |_{\varrho,p} | \nabla_j v - \nabla_j u |_{\varrho,p} \\ &\leq K_R (| \nabla_i u - \nabla_i v |_{\varrho,p} + | \nabla_j v - \nabla_j u |_{\varrho,p}) \\ &\leq C_R | u - v |_{\varrho+1,p} .\end{aligned}$$

Thus $\nabla_{ij} : W^{\varrho+1,p} \longrightarrow W^{\varrho,p}$ is Lipschitz on each ball in $W^{\varrho+1,p}$.
We now turn to the functions $\Gamma^k_{ij} : \mathbb{R}^d \longrightarrow \mathbb{R}$, $i, j, k = 1, \ldots, d$. Define $\Gamma_{ij} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by $\Gamma_{ij} := \left\{\Gamma^1_{ij}, \ldots, \Gamma^d_{ij}\right\}$, for $i, j = 1, \ldots, d$. Each Γ_{ij} is smooth and of compact support as its component functions are. Thus the Nemytski map $\overline{\Gamma}_{ij}$ associated with Γ_{ij} is a smooth map $\overline{\Gamma}_{ij} : W^{\varrho,p} \longrightarrow W^{\varrho,p}$, which is locally Lipschitz and of linear growth, see Proposition 3.1.7. Now define $F_{ij} : W^{\varrho+1,p} \longrightarrow W^{\varrho,p}$ by $F_{ij}(u) := \tilde{\beta}\left(\overline{\Gamma}_{ij}(u), \nabla_{ij}(u)\right)$, i.e. for each $\sigma \in S^1$

$$F_{ij}(u)(\sigma) := \left(\Gamma^1_{ij}(u(\sigma))\frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma), \ldots, \Gamma^d_{ij}(u(\sigma))\frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma)\right).$$

F_{ij} is smooth as $\tilde{\beta}$, $\overline{\Gamma}_{ij}$ and ∇_{ij} are. We now show that it is locally Lipschitz on $W^{\varrho+1,p}$. As above we have for $| u |_{\varrho+1,p}, | v |_{\varrho+1,p} \leq R$

$$\begin{aligned}| F_{ij}(u) - F_{ij}(v) |_{\varrho,p} &\leq K | \overline{\Gamma}_{ij}(u) |_{\varrho,p} | \nabla_{ij}u - \nabla_{ij}v |_{\varrho,p} \\ &\quad + K | \nabla_{ij}u |_{\varrho,p} | \overline{\Gamma}_{ij}(u) - \overline{\Gamma}_{ij}(v) |_{\varrho,p} .\end{aligned}$$

Using the local Lipschitz properties of $\overline{\Gamma}_{ij}$ and ∇_{ij} gives

$$| F_{ij}(u) - F_{ij}(v) |_{\varrho,p} \leq C_R | u - v |_{\varrho+1,p}$$

Thus for each pair (i, j), $i, j = 1, \ldots, d$, F_{ij} is a locally Lipschitz map from $W^{\varrho+1,p}$ to $W^{\varrho,p}$. Our proof is therefore complete once we notice that $F = \sum_{i,j=1}^d F_{ij}$.

♠

It is clear that our map F need not be of linear growth. We do though have the following estimate which will be crucial in our work:

Corollary 3.4.5 *For $u \in W^{\varrho+1,p}(S^1,\mathbb{R}^d)$, $\varrho > \frac{1}{p}$, $p \geq 2$, we have the following estimate*

$$| F(u) |_{\varrho,p} \leq C(p,d)\left\{| u |^2_{\varrho+1,p} + | u |^3_{\varrho+1,p}\right\} \tag{3.4.53}$$

where $C(p,d)$ is a constant independent of u.

Proof: Using the notation from the proof of Proposition 3.4.3, we have

$$\begin{aligned}
| F(u) |_{\varrho,p} &\leq C(p,d) \sum_{i,j=1}^{d} | \tilde{\beta}(\overline{\Gamma}_{ij}(u), \nabla_{ij}(u)) |_{\varrho,p} \\
&\leq C(p,d) \sum_{i,j=1}^{d} | \overline{\Gamma}_{ij}(u) |_{\varrho,p} | \nabla_{ij}(u) |_{\varrho,p} \\
&\leq C(p,d) \sum_{i,j=1}^{d} (1 + | u |_{\varrho+1,p}) | u |^2_{\varrho,p} \\
&\leq C(p,d)(| u |^2_{\varrho+1,p} + | u |^3_{\varrho+1,p}),
\end{aligned}$$

where the third inequality follows from the linear growth property of $\overline{\Gamma}_{ij}$.

♠

3.5 SNHE As A Stochastic Evolution Equation

We are thus in a position to reformulate the SPDE (3.1.1) as the following stochastic evolution equation,

$$du(t) + Au(t)dt = F(u(t))dt + V(u(t))dt + H(u(t)) \circ dw(t) \tag{3.5.54}$$

Note that (3.5.54) is a Stratonovich equation and so, see Chapter 2, $u(t)$ is a solution to (3.5.54) if and only if $u(t)$ is a solution to the following Itô SEE:

$$\begin{aligned}
du(t) + Au(t)dt = V(u(t))dt &+ F(u(t))dt + H(u(t))dw(t) \\
&+ \frac{1}{2} tr\{H'(u(t))H(u(t))\}dt.
\end{aligned} \tag{3.5.55}$$

The addition of the correction term does not pose more difficulty. In fact as with the other Nemytski maps this correction term $tr(H'H)$ is a locally Lipschitz map from $W^{\varrho,p}(S^1,\mathbb{R}^d)$ to itself and is of linear growth, as we will now show.

Proposition 3.5.1 *Let $i : H^{1,2}(S^1,\mathbb{R}^d) \hookrightarrow W^{\theta,p}(S^1,\mathbb{R}^d)$ be our AWS with $\theta \in (\frac{1}{p}, \frac{1}{2})$, $p > 2$ and $H : W^{\theta,p}(S^1,\mathbb{R}^d) \to L(W^{\theta,p}(S^1,\mathbb{R}^d), W^{\theta,p}(S^1,\mathbb{R}^d))$ be as above. Then the map $tr(H'H)$ is a well defined smooth map from $W^{\theta,p}(S^1,\mathbb{R}^d)$ to itself, where*

$$tr(H'H)(u) := tr\{H'(u)H(u)\}, \ u \in W^{\theta,p}(S^1,\mathbb{R}^d).$$

Furthermore $tr(H'H)$ is locally Lipschitz and of linear growth.

Proof : As in the proof of Proposition 3.4.3 we denote $W^{\theta,p}(S^1,\mathbb{R}^d)$ by $W^{\theta,p}$.
Recall that with respect to i , the map $tr : L_2(W^{\theta,p}; W^{\theta,p}) \to W^{\theta,p}$ is a bounded

linear map. Recall also that H and its derivative H' are locally Lipschitz and of linear growth. For $u \in W^{\theta,p}$ we have

$$H(u) \in L(W^{\theta,p}, W^{\theta,p}) \text{ and } H'(u) \in L(W^{\theta,p}, L(W^{\theta,p}, W^{\theta,p}))$$

and so

$$H'(u)H(u) \in L(W^{\theta,p}, L(W^{\theta,p}, W^{\theta,p})) \simeq L_2(W^{\theta,p}; W^{\theta,p}).$$

Thus $tr(H'H)$ is well defined and moreover it is smooth as tr and H are. To prove the local Lipschitz property let $| u |_{\theta,p}, | v |_{\theta,p} \leq R$. Let $| \cdot |_{L_1}$, $|| \cdot ||$ and $| \cdot |_{L_2}$ denote the norms on $L(W^{\theta,p}, W^{\theta,p})$, $L(W^{\theta,p}, L(W^{\theta,p}, W^{\theta,p}))$ and $L_2(W^{\theta,p}; W^{\theta,p})$ respectively. It then follows that, for some generic constant K

$$\begin{aligned} | tr(H'H)(u) - tr(H'H)(v) |_{\theta,p} &\leq | tr || H'(u)H(u) - H'(v)H(v) |_{L_2} \\ &\leq K | H'(u)[H(u) - H(v)] \\ &\qquad - [H'(v) - H'(u)]H(v) |_{L_2} \\ &\leq K || H'(u) ||| H(u) - H(v) |_{L_1} \\ &\qquad + K || H'(v) - H'(u) ||| H(v) |_{L_1} \\ &\leq C_R | u - v |_{\theta,p} . \end{aligned}$$

Finally for the linear growth condition, note first that

$$\begin{aligned} | tr(H'H)(u) |_{\theta,p} &\leq K | H'(u)H(u) |_{L_2} \\ &= K \sup_{|x|=|y|=1} | H'(u)H(u)(x)(y) |_{\theta,p}, \end{aligned}$$

where x, $y \in W^{\theta,p}$. The term

$$H'(u)H(u)(x)(y) \in W^{\theta,p} \tag{3.5.56}$$

acts through the following formula

$$(H'(u)H(u)(x)(y))(\sigma) = (h'(u(\sigma))\{h(u(\sigma))x(\sigma)\})y(\sigma), \quad \sigma \in S^1. \tag{3.5.57}$$

Clearly the $L^p(S^1, \mathbb{R}^d)$ norm of (3.5.56) is bounded by a constant. We need to consider the following term, where we write u_{σ_1} for $u(\sigma_1)$, similarly for x and y,

$$\left\{ \int_{S^1 \times S^1} \frac{| h'(u_{\sigma_1})\{h(u_{\sigma_1})x_{\sigma_1}\}y_{\sigma_1} - h'(u_{\sigma_2})\{h(u_{\sigma_2})x_{\sigma_2}\}y_{\sigma_2} |^p}{| \sigma_1 - \sigma_2 |^{1+\theta p}} d\sigma_1 d\sigma_2 \right\}^{\frac{1}{p}} . \tag{3.5.58}$$

For σ_1, $\sigma_2 \in S^1$ we denote

$$\mathcal{H}(\sigma_1, \sigma_2) := | h'(u_{\sigma_1})\{h(u_{\sigma_1})x_{\sigma_1}\}y_{\sigma_1} - h'(u_{\sigma_2})\{h(u_{\sigma_2})x_{\sigma_2}\}y_{\sigma_2} | .$$

Using the fact that h and h' are of compact support and that $\sup_{\sigma \in S^1} | x_\sigma | \leq 1$ and $\sup_{\sigma \in S^1} | y_\sigma | \leq 1$ we infer that

$$\begin{aligned} \mathcal{H}(\sigma_1, \sigma_2) &\leq | h'(u_{\sigma_1})\{h(u_{\sigma_1})x_{\sigma_1}\}y_{\sigma_1} - h'(u_{\sigma_2})\{h(u_{\sigma_2})x_{\sigma_2}\}y_{\sigma_1} | \\ &\quad + | h'(u_{\sigma_2})\{h(u_{\sigma_2})x_{\sigma_2}\} (y_{\sigma_1} - y_{\sigma_2}) | \\ &\leq | h'(u_{\sigma_1})h(u_{\sigma_1})x_{\sigma_1} - h'(u_{\sigma_2})h(u_{\sigma_2})x_{\sigma_2} |_L | y_{\sigma_1} | \\ &\quad + | h'(u_{\sigma_1})h(u_{\sigma_1})x_{\sigma_1} |_L | y_{\sigma_1} - y_{\sigma_2} | \\ &\leq | x_{\sigma_1} - x_{\sigma_2} | + | y_{\sigma_1} - y_{\sigma_2} | \\ &\leq 4 \end{aligned} \tag{3.5.59}$$

where $|\cdot|_L$ denotes the norm on $L(\mathbb{R}^d,\mathbb{R}^d)$. It follows that $tr(H'H)$ is bounded in $W^{\theta,p}$ and so in particular, it is of linear growth.

♠

3.6 Summary

To summarise, we consider the following problem

$$du(t) + Au(t)dt = F(u(t))dt + V(u(t))dt + H(u(t)) \circ dw(t), \tag{3.6.60}$$

$$u(0) = u_0, \tag{3.6.61}$$

where u_0 is some initial value, $w(t)$, $t \geq 0$, is a $W^{\theta,p}(S^1,\mathbb{R}^d)$-valued Wiener process, $\theta \in (\frac{1}{p},\frac{1}{2})$, $p > 2$. The Stratonovich differential appearing in (3.6.60) takes the form

$$H(u(t)) \circ dw(t) = H(u(t))dw(t) + \frac{1}{2}tr\{H'(u(t))H(u(t))\}dt, \tag{3.6.62}$$

where the tr map relates to the AWS $i : H^{1,2}(S^1,\mathbb{R}^d) \hookrightarrow W^{\theta,p}(S^1,\mathbb{R}^d)$. Assuming that $u_0 \in L^q(\Omega,\mathcal{F}_0,W^{s,p}(S^1,\mathbb{R}^d))$, where q and s are numbers satisfying $q > p > 2$ and $\frac{3}{2}-\frac{2}{q} > s > 1+\frac{1}{p}$, we first will show existence of a *maximal* solution taking values in the Banach space $W^{s,p}(S^1,\mathbb{R}^d)$. This is done in Chapter 4. Chapter 5 is dedicated to proving that our maximal solution lies on the loop manifold $\mathcal{M} = W^{s,p}(S^1,M)$, which is a closed submanifold of $W^{s,p}(S^1,\mathbb{R}^d)$, see [Br,99]. For this we need to assume that the initial value u_0 belongs to $L^q(\Omega,\mathcal{F}_0,W^{s,p}(S^1,M))$, where q and s are as above. Finally, in Chapter 6 we prove that our maximal solution is in fact a global solution.

We end this chapter with some important observations. Henceforth we will write L^p for $L^p(S^1,\mathbb{R}^d)$ and $W^{\varrho,p}$ for $W^{\varrho,p}(S^1,\mathbb{R}^d)$. For $\theta \in (\frac{1}{p},\frac{1}{2})$, $p > 2$, we fix the AWS $i : H^{1,2}(S^1,\mathbb{R}^d) \hookrightarrow W^{\theta,p}$ and denote $X := L^p$, $E := W^{\theta,p}$, and $\tilde{H} := H^{1,2}$. Let $-A := \frac{d^2}{d\sigma^2}$ and let $\left\{e^{-tA}\right\}_{t\geq 0}$ denote the semigroup it generates. $\left\{e^{-tA}\right\}_{t\geq 0}$ is a contraction analytic semigroup on X. Thus, in particular, $D(A) = W^{2,p}$. Recall, see 3.3.44 and Chapter 2, that for $\varrho \in (0,1)$

$$(X,D(A))_{\varrho,p} = (L^p,W^{2,p})_{\varrho,p} = W^{2\varrho,p}.$$

In terms of our 'abstract' notation the Nemytski maps appearing in (3.6.60) and (3.6.62) are locally Lipschitz maps as follows

$$\begin{aligned} V,\ \frac{1}{2}tr(H'H) &: (X,D(A))_{\gamma,p} \to X \\ F &: (X,D(A))_{\gamma,p} \to X \\ H &: (X,D(A))_{\gamma,p} \to L(E,E) \end{aligned}$$

for some suitable $\gamma > 0$.

Remark 3.6.1 Let $Y_\varrho := W^{\varrho,p}$, $\varrho \in (0,2)$.Then since $Y_\varrho \subset X$ one can define a family of operators, $\left\{e^{-tA_\varrho}\right\}_{t\geq 0}$, on the space Y_ϱ by restriction i.e

$$e^{-tA_E}u := e^{-tA}u \quad \text{for } u \in Y_\varrho. \tag{3.6.63}$$

One can show that $\left\{e^{-tA_\varrho}\right\}_{t\geq 0}$ is an analytic semigroup on Y_ϱ, whose generator $-A_\varrho$ satisfies

$$A_\varrho u = Au, \text{ for } u \in D(A_\varrho) := \{u \in D(A) : Au \in Y_\varrho\}. \tag{3.6.64}$$

In particular, we have

$$D(A_\varrho) := \left\{u \in W^{2,p} : \frac{d^2u}{d\sigma^2} \in W^{\varrho,p}\right\} = W^{2+\varrho,p}. \tag{3.6.65}$$

Furthermore, as for each $t \geq 0$, $e^{-tA} \in L(X)$ and $e^{-tA} \in L(D(A))$, with $| e^{-tA} |_{L(D(A))} \leq 1$, then using the interpolation property, see Chapter 2, we deduce that

$$| e^{-tA_\varrho} |_{L(Y_\varrho)} = | e^{-tA} |_{L(Y_\varrho)} \leq M, \tag{3.6.66}$$

for some constant $M \geq 1$. Hence $\left\{e^{-tA_\varrho}\right\}_{t\geq 0}$ is an uniformly bounded analytic semigroup on Y_ϱ. Moreover, its resolvent satisfies

$$| (\lambda + A_\varrho)^{-1} |_{L(Y_\varrho)} \leq \frac{M}{\lambda}, \ \lambda > 0. \tag{3.6.67}$$

Thus, in particular, A_ϱ is nonnegative. Furthermore, noting that $D(A^2) = W^{4,p}$, using the Reiteration Theorem and the identity

$$(X, D(A))_{\nu,p} = (X, D(A^2))_{\frac{\nu}{2},p},$$

see Chapter 2, one can show that for $\nu \in (0,1)$

$$(Y_\varrho, D(A_\varrho))_{\nu,p} = (X, D(A))_{\nu+\frac{\varrho}{2},p}, \tag{3.6.68}$$

with equivalent norms. For details of the above statements, see [Br,95] and [Tr,78].

For $E := W^{\theta,p}$, where E belongs to the AWS $i : \tilde{H} \hookrightarrow E$, we will denote $A_\theta := A_E$ and the semigroup $\left\{e^{-tA_\theta}\right\}_{t\geq 0}$ by $\left\{e^{-tA_E}\right\}_{t\geq 0}$. In particular, we have for $\nu \in (0,1)$

$$(E, D(A_E))_{\nu,p} = (X, D(A))_{\nu+\frac{\theta}{2},p}, \tag{3.6.69}$$

with equivalent norms. $\diamondsuit$

Remark 3.6.2 The above abstract notation will be fixed throughout Chapter 4.

♡

Chapter 4

Existence Of A Maximal Solution To The Stochastic Nonlinear Heat Equation

4.1 The General Assumptions

In this chapter we consider the following problem:

$$du(t) + Au(t)dt = F(u(t))dt + H(u(t))dw(t) \tag{4.1.1}$$

$$u(0) = u_0, \tag{4.1.2}$$

where F and H are Nemytski maps as in Chapter 3.

Let X be a Banach space and $\{e^{-tA}\}_{t\geq 0}$ a C_0-semigroup on X. It will be stated when we use the following additional assumptions.

(A1) X is an M-type 2 Banach space.

(A1*) E is an M-type 2 Banach space. Moreover $E \subset X$, where X is as in $(A1)$ and $\subset$ denotes continuous imbedding.

(A2) $i : \tilde{H} \to E$ is an AWS, where $\tilde{H}$ is a separable Hilbert space and E is a separable Banach space. The canonical E-valued Wiener process, defined on some complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, is denoted by $w(t)$, $t \geq 0$.

(A3) $-A$ is the generator of a contraction analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on X. Furthermore A satisfies

$$| (\lambda + A)^{-1} |_{L(X)} \leq \frac{1}{\lambda}, \ \lambda > 0,$$

i.e. A is a nonnegative operator on X. Recall that $L(X) := L(X, X)$ is the space of bounded linear operators on X.

(A3*) $-A_E$ is the generator of an uniformly bounded analytic semigroup $\{e^{-tA_E}\}_{t\geq 0}$ on E. Furthermore, for some $M \geq 1$, the resolvent of A_E satisfies

$$| (\lambda + A_E)^{-1} |_{L(E)} \leq \frac{M}{\lambda}, \ \lambda > 0.$$

Remark 4.1.1 The above assumptions relate to the set-up described at the end of Chapter 3, i.e. $X = L^p(S^1, \mathbb{R}^d)$, $E = W^{\theta,p}(S^1, \mathbb{R}^d)$, $\theta \in (\frac{1}{p}, \frac{1}{2})$, $p > 2$, $\tilde{H} = H^{1,2}(S^1, \mathbb{R}^d)$ and so on. We remark that, whenever we refer to $p > 2$ it is the corresponding p from L^p and $W^{\theta,p}$. $\diamondsuit$

Remark 4.1.2 For a positive operator A on X one may define the fractional powers $A^{-\nu}$ and A^{ν}, for $\nu \in (0,1)$. Suppose A satisfies $(A3)$, in particular, A is nonnegative. Then for each $\eta > 0$, $\eta + A$ is positive and so we may define its fractional powers, $(\eta + A)^{\nu}$ and $(\eta + A)^{-\nu}$, $\nu \in (0,1)$. The spaces $D((\eta + A)^{\nu}) := R((\eta + A)^{-\nu})$ are η independent with mutually equivalent norms. One sets $D(A^{\nu}) := D((\eta + A)^{\nu})$, and we denote the norm on this space by $|\cdot|_{\nu}$. Moreover for $\eta > 0$, $-(\eta + A)$ generates the analytic semigroup $\{e^{-\eta t}e^{-tA}\}_{t\geq 0}$ and we have for $\nu \in (0,1)$,

$$| (\eta + A)^{\nu} e^{-tA} |_{L(X)} \leq C(X) t^{-\nu} e^{\eta t} \tag{4.1.3}$$

where $C(X)$ is a constant depending on the space X only. Similarly for $\eta > 0$, $-(\eta + A_E)$ generates the analytic semigroup $\{e^{-\eta t}e^{-tA_E}\}_{t\geq 0}$ and we have for $\nu \in (0,1)$,

$$| (\eta + A_E)^{\nu} e^{-tA} |_{L(E)} \leq C(E) t^{-\nu} e^{\eta t} \tag{4.1.4}$$

where $C(E)$ is a constant depending on the space E only. For proofs of the above assertions see [Paz,83] and [He,81]. $\diamondsuit$

♡

4.2 Regularity Properties Of The Generalised S-tochastic Convolution Process

For $\lambda \geq 0$ and a suitable process ξ we define the *generalised* stochastic convolution process $x_\lambda(t)$ by

$$x_\lambda(t) = \int_0^t e^{-t\lambda} e^{-(t-s)A} \xi(s) dw(s). \tag{4.2.5}$$

For $q > 1$ and a Banach space Y we set $M^q_{loc}(0,\infty; Y)$ to be the space of all progressively measurable processes $\xi : [0,\infty) \times \Omega \to Y$ which satisfy

$$\mathbb{E} \int_0^T | \xi(s) |^q_Y \, ds < \infty, \quad \text{for each } T > 0.$$

Assume $(A3)$ holds. If $\xi \in M^q_{loc}(0,\infty; L(E,X))$, $q \geq 2$, then the integrand in (4.2.5) is progressively measurable on $\Omega \times [0,t)$, for each $t \geq 0$, and we have

$$\mathbb{E} \left| \int_0^t e^{-t\lambda} e^{-(t-s)A} \xi(s) \right|^2_{L(E,X)} \leq C e^{-\lambda t} \mathbb{E} \int_0^t | \xi(s) |^2_{L(E,X)} \, ds.$$

It follows that $e^{-t\lambda} e^{-(t-\cdot)A} \xi(\cdot) \in M^2(0,t; L(E,X))$, $t \geq 0$. Thus, along with the assumptions $(A1)$ and $(A2)$, $x_\lambda(t)$ is well defined. (Note that the above is true for a general C_0-semigroup, $\{e^{-tA}\}_{t\geq 0}$, on X.)

Remark 4.2.1 Suppose that we have a stronger assumption on ξ, i.e.

$$\xi \in M^q_{loc}(0,\infty; L(E)), \tag{4.2.6}$$

where E is as in $(A1*)$. Then for $e \in E$, $\xi(s)e \in E$ and so

$$e^{-t\lambda}e^{-(t-s)A_E}\xi(s)(e) = e^{-t\lambda}e^{-(t-s)A}\xi(s)(e).$$

It then follows that the convolution process $x_\lambda(t)$, given by (4.2.5), may also be written

$$x_\lambda(t) = \int_0^t e^{-t\lambda}e^{-(t-s)A_E}\xi(s)dw(s). \tag{4.2.7}$$

◇

To study the regularity properties of this process we use the so-called DaPrato-Kwapien-Zabczyk Factorisation Method, which is based on the following classical formula

$$\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}dr = \Gamma(\alpha)\Gamma(\alpha-1), \;\; r \leq s \leq t, \;\; \alpha \in (0,1), \tag{4.2.8}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the Euler Gamma function.

This method was first described in [DP/K/Z,87] in a Hilbert space framework and was then later generalised to the Banach space setup in [Br,97]. In both papers though only the process $x_0(t)$ was considered. We will generalise the results in [Br,97] to deal with our process $x_\lambda(t)$, $\lambda > 0$.

We begin with introducing the operator R^λ_α, which we call the generalised factorisation operator. Let $\alpha \in (0,1]$, $q \in (1,\infty)$ be fixed. For any $f \in L^q(0,\infty;X)$ define, for $\lambda \geq 0$,

$$(R^\lambda_\alpha f)(t) = \int_0^t (t-s)^{\alpha-1}e^{-(t-s)\lambda}e^{-(t-s)A}f(s)ds. \tag{4.2.9}$$

It is a straightforward consequence of the Young inequality that, for $\lambda > 0$, R^λ_α is a bounded linear map from $L^q(0,\infty;X)$ to itself. Under the assumption $(A3)$ then a stronger result is true.

Theorem 4.2.2 *Assume $(A3)$, $\alpha \in (0,1]$, $q > 1$ and $\lambda > 0$. If $\beta \in [0,\alpha)$, then R^λ_α is a bounded linear map from $L^q(0,\infty;X)$ to $L^q(0,\infty;D(A^\beta))$. If $\delta \in [0,\alpha-\frac{1}{q})$ then R^λ_α is a bounded linear map from $L^q(0,\infty;X)$ to $C(0,\infty;D(A^\delta))$. Moreover the following inequality holds*

$$\sup_{t\geq 0} \mid (R^\alpha_\lambda f)(t) \mid^q_\delta + \int_0^\infty \mid (R^\alpha_\lambda f)(t) \mid^q_\beta dt \leq C\kappa(\lambda)\int_0^\infty \mid f(t) \mid^q_X dt \tag{4.2.10}$$

where C is independent of f and λ and $\kappa(\lambda) \to 0$ as $\lambda \to \infty$.

Remark 4.2.3 The above result was first proved in a Hilbert space setting, see [DP/K/Z,87] and [Sei,93]. For the Banach space case, see [Br,97]. In these papers only the case $\lambda = 0$ was considered and as a result they were restricted to finite intervals $[0,T]$, $T > 0$. Our proof is a generalisation of those in the above cited papers. ◇

Proof : For the first assertion note that for an analytic semigroup on X, $e^{-tA}x \in D(A^{\nu})$, $\nu \in [0,1]$. Thus in particular for $t \geq 0$, $(R_{\lambda}^{\alpha}f)(t) \in D(A^{\nu})$, $\nu \in [0,1]$. For $\lambda > 0$ fixed, choose $\eta > 0$ such that $\lambda > \eta > 0$. Then, using (4.1.3), a straightforward calculation gives

$$\begin{aligned} \int_0^{\infty} |(R_{\lambda}^{\alpha}f)(t)|_{\beta}^{q}\, dt &\leq C\int_0^{\infty}\Big(\int_0^{t}(t-s)^{\alpha-1-\beta}e^{-(t-s)(\lambda-\eta)}\,|f(s)|_X\, ds\Big)^{q}dt \\ &= C\,|h_1 * h_2|_{L^q(0,\infty;\mathbb{R})}^{q} \end{aligned}$$

where $h_1 * h_2(t) = \int_0^t h_1(t-s)h_2(s)ds$ with $h_1(s) = s^{\alpha-1-\beta}e^{-s(\lambda-\eta)}$, $h_1 \in L^1(0,\infty;\mathbb{R})$ and $h_2(s) = |f(s)|_X$, $h_2 \in L^q(0,\infty;\mathbb{R})$. Note also that the constant C is independent of t. By an application of the Young inequality

$$\int_0^{\infty} |(R_{\lambda}^{\alpha}f)(t)|_{\beta}^{q}\, dt \leq C\,|h_1|_{L^1}^{q}|h_2|_{L^q}^{q} \leq C\kappa_1(\lambda)\int_0^{\infty}|f(t)|_X^q\, dt \tag{4.2.11}$$

where we have denoted $\kappa_1(\lambda) = |h_1|_{L^1(0,\mathbb{R})} = \int_0^{\infty} s^{\alpha-1-\beta}e^{-s(\lambda-\eta)}ds$. The first assertion now follows as $\kappa_1(\lambda) < \infty$. The integrand in $\kappa_1(\lambda)$ tends to zero pointwise as λ tends to infinity. With an application of the Lebesgue Dominated Convergence Theorem, (LDC), in mind, we note that for sufficiently large λ

$$s^{\alpha-1-\beta}e^{-s(\lambda-\eta)} \leq \begin{cases} s^{\alpha-1-\beta} & \text{if } s \in [0,1) \\ e^{-s} & \text{if } s \in [1,\infty). \end{cases}$$

Thus by LDC $\kappa_1(\lambda) \to 0$ as $\lambda \to \infty$.
For the second assertion we begin with proving the following estimate

$$\sup_{t\geq 0} |(R_{\lambda}^{\alpha}f)(t)|_{\delta}^{q} \leq C\kappa_2(\lambda)\int_0^{\infty}|f(t)|_X^q\, dt \tag{4.2.12}$$

where C is independent of f and λ and $\kappa_2(\lambda) \to 0$ as $\lambda \to \infty$. As in the proof of the first assertion, for $t \geq 0$, $\lambda > \eta > 0$,

$$|(R_{\lambda}^{\alpha}f)(t)|_{\delta}^{q} \leq C\left(\int_0^{t}(t-s)^{\alpha-1-\delta}e^{-(t-s)(\lambda-\eta)}\,|f(s)|_X\, ds\right)^{q}.$$

Applying the Hölder inequality gives, for $t \geq 0$,

$$\begin{aligned} |(R_{\lambda}^{\alpha}f)(t)|_{\delta}^{q} &\leq C\Big(\int_0^{\infty}(t-s)^{(\alpha-1-\delta)\frac{q}{q-1}}e^{-(t-s)(\lambda-\eta)\frac{q}{q-1}}ds\Big)^{q-1}|f|_{L^q}^{q} \\ &= C\kappa_2(\lambda)^{q-1}|f|_{L^q}^{q} \end{aligned} \tag{4.2.13}$$

where $|\cdot|_{L^q}$ denotes the norm on $L^q(0,\infty;X)$ and

$$\kappa_2(\lambda) = \int_0^{\infty}(t-s)^{(\alpha-1-\delta)\frac{q}{q-1}}e^{-(t-s)(\lambda-\eta)\frac{q}{q-1}}ds.$$

The integrand in $\kappa_2(\lambda)$ converges pointwise to zero. Noting that for sufficiently large λ

$$s^{(\alpha-1-\delta)\frac{q}{q-1}}e^{-s(\lambda-\eta)\frac{q}{q-1}} \leq \begin{cases} s^{(\alpha-1-\delta)\frac{q}{q-1}} & \text{if } s \in [0,1) \\ e^{-s} & \text{if } s \in [1,\infty) \end{cases}$$

then, by LDC, $\kappa_2(\lambda)$ tends to zero as λ tends to infinty. Note that as C and $\kappa_2(\lambda)$ are independent of t, then (4.2.12) follows from (4.2.13). The estimate (4.2.10) now

follows from (4.2.11) and (4.2.12).
To prove $R_\lambda^\alpha f \in C(0,\infty; D(A^\delta))$, suppose first that $f \in C_0^\infty(0,\infty;X)$. Then by a direct calculation one can show that $R_\lambda^\alpha f$ is differentiable, as a $D(A^\delta)$-valued function, with derivative

$$(R_\lambda^\alpha f)'(t) = \int_0^t s^{\alpha-1}e^{-\lambda s}e^{-sA}f'(t-s)ds.$$

In particular $R_\lambda^\alpha f$ is continuous. The density of $C_0^\infty(0,\infty;X)$ in $L^q(0,\infty;X)$, the closedness of $C(0,\infty;D(A^\delta))$ in $L^\infty(0,\infty;D(A^\delta))$ and the estimate (4.2.12) together imply the result for $f \in L^q(0,\infty;X)$.

♠

Using $(A3*)$ instead of $(A3)$ we deduce the following:

Corollary 4.2.4 *Assume $(A3*)$, $\alpha \in (0,1]$, $q > 1$ and $\lambda > 0$. If $\beta \in [0,\alpha)$, then R_α^λ is a bounded linear map from $L^q(0,\infty;E)$ to $L^q(0,\infty;D(A_E^\beta))$. If $\delta \in [0,\alpha-\frac{1}{q})$ then R_α^λ is a bounded linear map from $L^q(0,\infty;E)$ to $C(0,\infty;D(A_E^\delta))$. Moreover the following inequality holds*

$$\sup_{t\geq 0} |\, (R_\lambda^\alpha f)(t)\, |_{E,\delta}^q + \int_0^\infty |\, (R_\lambda^\alpha f)(t)\, |_{E,\beta}^q\, dt \leq C\kappa(\lambda)\int_0^\infty |\, f(t)\, |_X^q\, dt \qquad (4.2.14)$$

where C is independent of f and λ and $\kappa(\lambda) \to 0$ as $\lambda \to \infty$. $|\cdot|_{E,\delta}$ denotes the norm on $D(A_E^\delta)$ and $|\cdot|_{E,\beta}$ denotes the norm on $D(A_E^\beta)$.

Remark 4.2.5 In the Appendix we prove that the generalised factorisation operator R_λ^α is the fractional power $(\lambda+\Lambda)^{-\alpha}$, where Λ is a certain abstract parabolic operator. ◇.

The following theorem was proved in [Br,97] for the process x_0. We extend the proof to include the process x_λ, $\lambda > 0$. Let Y be a Banach space. For $\lambda > 0$, we define

$$M_\lambda^q(0,\infty;Y) := \left\{\xi \in M_{loc}^q : \mathbb{E}\int_0^\infty |\, e^{-\lambda t}\xi(t)\, |_Y^q\, dt < \infty\right\}.$$

Theorem 4.2.6 *Assume $(A1)$, $(A2)$, $(A3)$, $(A1*)$ and $(A3*)$ all hold. Let $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4}-\frac{1}{q})$ and $\gamma \in [0,\frac{3}{4})$. Furthermore assume that the stochastic process ξ is such that $\xi \in M_\lambda^q(0,\infty;L(E))$ for $\lambda > 0$. Let x_λ, $\lambda > 0$, be the generalised stochastic convolution process* (4.2.5)*. Then there exists a modification $\tilde{x}_\lambda(t)$ of the process $x_\lambda(t)$, i.e. a process satisfying $x_\lambda(t) = \tilde{x}_\lambda(t)$ a.s. for each $t \geq 0$, such that*

$$\tilde{x}_\lambda \in Z_{\infty,\gamma,\zeta} := M^q(0,\infty;(X,D(A))_{\gamma,p}) \cap L^q(\Omega; C(0,\infty;(X,D(A))_{\zeta,p})).$$

Moreover the following estimate holds

$$\mathbb{E}\sup_{t\geq 0} |\, \tilde{x}_\lambda(t)\, |_{\zeta,p}^q + \mathbb{E}\int_0^\infty |\, \tilde{x}_\lambda(t)\, |_{\gamma,p}^q\, dt \leq C\kappa(\lambda)\int_0^\infty |\, e^{-t\lambda}\xi(t)\, |_{L(E)}^q\, dt \qquad (4.2.15)$$

where $C = C(q,E,X)$ is independent of ξ and λ, and $\kappa(\lambda)$ tends to 0 as λ tends to ∞.

Proof : The process

$$y_\lambda(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-t\lambda} e^{-(t-s)A_E} \xi(s) dw(s) \tag{4.2.16}$$

is well defined for $\xi \in M^q_{loc}(0,\infty; L(E))$, $q \geq 2$, provided $\alpha \in (0, \frac{1}{2})$. Applying the Burkholder inequality gives

$$\mathbb{E} \mid y_\lambda(t) \mid^q_E \leq C(q,E)\mathbb{E}\{\int_0^t (t-s)^{-2\alpha} e^{-2(t-s)(\lambda)} e^{-2s\lambda} \mid \xi(s) \mid^2_{L(E)} ds\}^{\frac{q}{2}}.$$

It then follows by an application of the Young inequality, as in the previous proof, that

$$\mathbb{E}\int_0^\infty \mid y_\lambda(t) \mid^q_E dt \leq C(q,E)\kappa_3(\lambda)\mathbb{E}\int_0^\infty \mid e^{-t\lambda}\xi(t) \mid^q_{L(E)} dt \tag{4.2.17}$$

where $\kappa_3(\lambda) = \int_0^\infty s^{-2\alpha} e^{-2s(\lambda)} ds$. As $\kappa_3(\lambda) < \infty$ then $\int_0^\infty \mid y_\lambda(t) \mid^q_E dt < \infty$ a.s.. Note also that $\kappa_3(\lambda)$ tends to zero as λ tends to infinity.
We now define our modification $\tilde{x}_\lambda$, for $\lambda > 0$. For each $\omega \in \Omega$ such that $y_\lambda(\cdot,\omega) \in L^q(0,\infty;E)$ define $\tilde{x}_\lambda(\cdot,\omega) := R^\lambda_\alpha(y_\lambda(\cdot,\omega))$. In view of the definition of R^λ_α, see (4.2.9), then a.s., $\tilde{x}_\lambda : [0,\infty) \to D(A^\nu_E)$ is a continuous mapping, for any $\nu \in [0,1]$. Moreover $\tilde{x}_\lambda$ is adapted and so is a progressively measurable $D(A^\nu_E)$-valued process. Suppose that α, σ and δ are nonnegative numbers that satisfy

$$\sigma + \frac{1}{q} < \alpha < \frac{1}{2}, \text{ and } \delta < \alpha < \frac{1}{2},$$

then in view of the Corollary 4.2.4 we deduce that

$$\tilde{x}_\lambda(\cdot) \in C(0,\infty; D(A^\sigma_E)) \cap L^q(0,\infty; D(A^\delta_E)) \text{ a.s.}$$

with the following inequality holding a.s.

$$\sup_{t\geq 0} \mid \tilde{x}_\lambda(t) \mid^q_{E,\sigma} + \int_0^\infty \mid \tilde{x}_\lambda(t) \mid^q_{E,\delta} dt \leq C(q,X)\kappa(\lambda)\int_0^\infty \mid y_\lambda(t) \mid^q_E dt. \tag{4.2.18}$$

Using the fact that, for $\varrho < \nu$ and $p > 1$, $D(A^\nu_E) \hookrightarrow (E, D(A_E))_{\varrho,p}$, we deduce that, for any $\nu \in (0, \frac{1}{2} - \frac{1}{q})$ and $\mu \in [0, \frac{1}{2})$,

$$\tilde{x}_\lambda(\cdot) \in C(0,\infty; (E, D(A_E))_{\nu,p}) \cap L^q(0,\infty; (E, D(A_E))_{\mu,p}) \text{ a.s.}$$

with the following inequality holding a.s.

$$\sup_{t\geq 0} \mid \tilde{x}_\lambda(t) \mid^q_{E,\nu,p} + \int_0^\infty \mid \tilde{x}_\lambda(t) \mid^q_{E,\mu,p} dt \leq C(q,X)\kappa(\lambda)\int_0^\infty \mid y_\lambda(t) \mid^q_E dt. \tag{4.2.19}$$

where $|\cdot|_{E,\nu,p}$ denotes the norm on $(E, D(A_E))_{\nu,p}$ and $|\cdot|_{E,\mu,p}$ denotes the norm on $(E, D(A_E))_{\mu,p}$. Recall from Chapter 3 that, for $\varrho \in (0,1)$,

$$(E, D(A_E))_{\varrho,p} = (X, D(A))_{\varrho+\frac{\theta}{2},p}, \quad \theta \in (\frac{1}{p}, \frac{1}{2}), \tag{4.2.20}$$

with equivalent norms. Thus, denoting $\zeta := \nu + \frac{\theta}{2}$ and $\gamma := \mu + \frac{\theta}{2}$ it follows that

$$\tilde{x}_\lambda(\cdot) \in C(0,\infty; (X, D(A))_{\zeta,p}) \cap L^q(0,\infty; (X, D(A))_{\gamma,p}) \text{ a.s.} \tag{4.2.21}$$

with the following inequality holding a.s.

$$\sup_{t\geq 0} \mid \tilde{x}_\lambda(t) \mid_{\zeta,p}^q + \int_0^\infty \mid \tilde{x}_\lambda(t) \mid_{\gamma,p}^q dt \leq C(q,X)\kappa(\lambda) \int_0^\infty \mid y_\lambda(t) \mid_E^q dt. \quad (4.2.22)$$

We deduce that $\tilde{x}_\lambda \in Z_{\infty,\gamma,\zeta}$. Moreover taking expectations in (4.2.22) and using (4.2.17) we obtain the estimate (4.2.15). To complete the proof of Theorem 4.2.6 we need to show that $\tilde{x}_\lambda$ is a modification of x_λ. For $t \geq 0$ fixed we have

$$\begin{aligned}
\tilde{x}_\lambda(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)\lambda} e^{-(t-s)A} y_\lambda(s) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1}(s-r)^{-\alpha} e^{-t\lambda} e^{-(t-r)A} \xi(r) dw(r) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} e^{-t\lambda} e^{-(t-r)A} \xi(r) ds dw(r) \\
&= \int_0^t e^{-t\lambda} e^{-(t-r)A} \xi(r) dw(r) \quad \text{a.s.}
\end{aligned}$$

where we have used both the Stochastic Fubini Theorem and (4.2.8).

♠

We introduce some new notation. Fix $0 < T \leq \infty$ and let $\tau : \Omega \to [0,T]$ be a stopping time. Define the process $\alpha : [0,T] \times \Omega \to \{0,1\}$ by

$$\alpha(s,\omega) = \begin{cases} 1 & \text{if } s < \tau(\omega) \\ 0 & \text{if } s \geq \tau(\omega) \end{cases} \quad (4.2.23)$$

α is the characteristic function of the stochastic interval $[0,\tau)$. Note that α is a right continuous adapted process. In particular, α has a progressively measurable modification.

Theorem 4.2.7 *Assume (A1), (A2), (A3), (A1*) and (A3*) all hold. Let $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$ and $\gamma \in [0, \frac{3}{4})$. Let $0 < T < \infty$ be fixed but arbitrary. Let $\xi \in M^q(0,T;L(E))$ and x be the stochastic convolution process*

$$x(t) = \int_0^t e^{-(t-s)A} \xi(s) dw(s).$$

Then, there exists a modification $\tilde{x}$ of x such that

$$\tilde{y} \in Z_{T,\gamma,\zeta} := M^q(0,T;(X,D(A))_{\gamma,p}) \cap L^q(\Omega; C(0,T;(X,D(A))_{\zeta,p})).$$

Moreover, for any stopping time $\sigma : \Omega \to [0,\infty]$ we have

$$\mathbb{E} \sup_{0\leq t\leq T} \mid \tilde{x}(t\wedge\sigma) \mid_{\zeta,p}^q \leq C(T) \int_0^T \mathbb{E} \mid \beta(s)\xi(s) \mid_{L(E)}^q \leq C(T) \mathbb{E} \int_0^{T\wedge\sigma} \mid \xi(s) \mid_{L(E)}^q ds, \quad (4.2.24)$$

where β is the characteristic function of $[0,\sigma)$ and $C(T)$ is a constant independent of ξ and σ.

Proof: We only need to prove the estimate (4.2.24). Recall how $\tilde{x}$ is defined. First define y by

$$y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-(t-s)A} \xi(s) dw(s), \quad t \in [0,T].$$

There exists a set of full measure $\tilde{\Omega}$ such that $y(\cdot,\omega) \in L^q(0,T;E)$. For $\omega \in \tilde{\Omega}$ we set

$$\tilde{x}(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)A} y(s,\omega) ds, \quad t \in [0,T].$$

Let $\omega \in \tilde{\Omega}$ be fixed. From the proof of Theorem 4.2.6, see (4.2.22), we have for each $t \in [0,T]$

$$| \tilde{x}(t,\omega) |_{\zeta,p}^q \leq C_T \int_0^t | y(s,\omega) |_E^q \, ds,$$

where C_T is a constant independent of y (and hence ω). In particular, for each $t \in [0, T \wedge \sigma(\omega)]$ we have

$$| \tilde{x}(t,\omega) |_{\zeta,p}^q \leq C_T \int_0^{T\wedge\sigma(\omega)} | y(s,\omega) |_E^q \, ds.$$

It follows that

$$\sup_{0\leq t\leq T\wedge\sigma(\omega)} | \tilde{x}(t,\omega) |_{\zeta,p}^q \leq C_T \int_0^{T\wedge\sigma(\omega)} | y(s,\omega) |_E^q \, ds. \tag{4.2.25}$$

Indeed (4.2.25) holds for all $\omega \in \tilde{\Omega}$ with the same constant C_T. Noting that

$$\sup_{0\leq t\leq T} | \tilde{x}(t\wedge\sigma) |_{\zeta,p}^q = \sup_{0\leq t\leq T\wedge\sigma} | \tilde{x}(t) |_{\zeta,p}^q,$$

it follows that

$$\sup_{0\leq t\leq T} | \tilde{x}(t\wedge\sigma) |_{\zeta,p}^q \leq C_T \int_0^{\sigma\wedge T} | y(s) |_E^q \, ds \ \text{ a.s..}$$

Taking expectations gives

$$\mathbb{E} \sup_{0\leq t\leq T} | \tilde{x}(t\wedge\sigma) |_{\zeta,p}^q \leq C_T \mathbb{E} \int_0^{\sigma\wedge T} | y(s) |_E^q \, ds. \tag{4.2.26}$$

Note that

$$\mathbb{E} \int_0^{\sigma\wedge T} | y(s) |_E^q \, ds = \mathbb{E} \int_0^T | \beta(s) y(s) |_E^q \, ds = \int_0^T \mathbb{E} | \beta(s) y(s) |_E^q \, ds. \tag{4.2.27}$$

Define the process

$$y_\beta(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-r)^{-\alpha} e^{-(s-r)A} \beta(r) \xi(r) dw(r).$$

y_β is well defined and for $s \in [0,T]$

$$\beta(s) y(s) = \begin{cases} y_\beta(s) & \text{on } \{\beta(s) = 1\} \\ 0 & \text{on } \{\beta(s) = 0\}. \end{cases} \tag{4.2.28}$$

Thus

$$\begin{aligned}\mathbb{E} \mid \beta(s)y(s) \mid_E^q &= \int_{\{\beta(s)=1\}} \mid \beta(s,\omega)y(s,\omega) \mid_E^q d\mathbb{P}(\omega) \\ &\leq \int_{\{\beta(s)=1\}} \mid \beta(s,\omega)y(s,\omega) \mid_E^q d\mathbb{P}(\omega) + \int_{\{\beta(s)=0\}} \mid y_\beta(s,\omega) \mid_E^q d\mathbb{P}(\omega) \\ &= \mathbb{E} \mid y_\beta(s) \mid_E^q .\end{aligned}$$

Thus

$$\int_0^T \mathbb{E} \mid \beta(s)y(s) \mid_E^q ds \leq \mathbb{E}\int_0^T \mid y_\beta(s) \mid_E^q ds \leq C_T\mathbb{E}\int_0^T \mid \beta(s)\xi(s) \mid_{L(E)}^q ds.$$

The inequality follows from the proof of Theorem (4.2.6), see (4.2.17). As

$$\mathbb{E}\int_0^T \mid \beta(s)\xi(s) \mid_{L(E)}^q ds \leq \mathbb{E}\int_0^{T\wedge\sigma} \mid \xi(s) \mid_{L(E)}^q ds,$$

we have

$$\int_0^T \mathbb{E} \mid \beta(s)y(s) \mid_E^q ds \leq C_T\mathbb{E}\int_0^{T\wedge\sigma} \mid \xi(s) \mid_{L(E)}^q ds. \tag{4.2.29}$$

The estimate (4.2.24) follows now from (4.2.26), (4.2.27) and (4.2.29). This completes the proof.

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Corollary 4.2.8 *Under the assumptions of Theorem* 4.2.7, *let*

$$x(t) = \int_0^t e^{-(t-s)A}\xi(s)dw(s) \;\; and \;\; x_\beta(t) = \int_0^t e^{-(t-s)A}\beta(s)\xi(s)dw(s), \;\; t \in [0,T],$$

where β is the characteristic function of $[0,\sigma)$. *Then*

$$\mathbb{E}\sup_{0\leq t\leq T} \mid x(t\wedge\sigma) - x_\beta(t\wedge\sigma) \mid_{\zeta,p}^q = 0. \tag{4.2.30}$$

Proof: Let

$$z(t) = x(t) - x_\beta(t) = \int_0^t e^{-(t-s)A}(1-\beta(s))\xi(s)dw(s), \;\; t \in [0,T].$$

By Theorem 4.2.7 z has a continuous modification $\tilde{z}$ which satisfies

$$\mathbb{E}\sup_{0\leq t\leq T} \mid \tilde{z}(t\wedge\sigma) \mid_{\zeta,p}^q \leq C_T\mathbb{E}\int_0^{T\wedge\sigma} \mid (1-\beta(s))\xi(s) \mid_{L(E)}^q ds.$$

Noting that for $s \in [0, T\wedge\sigma(\omega))$, $1-\beta(s,\omega) = 0$, we deduce that

$$\mathbb{E}\sup_{0\leq t\leq T} \mid \tilde{z}(t\wedge\sigma) \mid_{\zeta,p}^q = 0.$$

This completes the proof.

♠

Corollary 4.2.9 *Assume (A1), (A2), (A3), (A1*) and (A3*) all hold. Let* $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$ *and* $\gamma \in [0, \frac{3}{4})$. *Let* $0 < T < \infty$ *be fixed but arbitrary and* $\tau : \Omega \to [0, T]$ *be a stopping time. Furthermore assume that* $\xi(t)$, $t < \tau$ *is an admissable* $L(E)$*-valued process with*

$$\mathbb{E} \int_0^\tau |\, \xi(s) \,|_{L(E)}^q \, ds < \infty.$$

Let α *be the characteristic function of the interval* $[0, \tau)$ *and set*

$$z_\alpha(t) = \int_0^t e^{-(t-s)A} \alpha(s) \xi(s) dw(s).$$

Then, there exists a modification $\tilde{z}_\alpha$ *of* z_α *such that*

$$\tilde{z}_\alpha \in Z_{T,\gamma,\zeta} := M^q(0, T; (X, D(A))_{\gamma,p}) \cap L^q(\Omega; C(0, T; (X, D(A))_{\zeta,p})).$$

Moreover, for each $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |\, \tilde{z}_\alpha(s \wedge \tau) \,|_{\zeta,p}^q \leq C(T) \int_0^t \mathbb{E} \,|\, \alpha(s) \xi(s) \,|_{L(E)}^q \, ds, \tag{4.2.31}$$

where $C(T)$ *is independent of* ξ *and* τ.

Proof : Define the process $\eta : [0, T] \times \Omega \to L(E)$ by

$$\eta(s, \omega) = \alpha(s, \omega) \xi(s, \omega).$$

As ξ is admissable and α is right continuous and adapted, then η right continuous and adapted. In particular, η has a progressively measurable modification. Moreover, as

$$\mathbb{E} \int_0^T |\, \eta(s) \,|_{L(E)}^q \, ds = \mathbb{E} \int_0^\tau |\, \xi(s) \,|_{L(E)}^q \, ds < \infty,$$

then $\eta \in M^q(0, T; L(E))$. It follows, by Theorem 4.2.7, that the process z_α is well defined and has a continuous modification $\tilde{z}_\alpha$ with $\tilde{z}_\alpha \in Z_{T,\gamma,\zeta}$. Furthermore, for each $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |\, \tilde{z}_\alpha(s \wedge \tau) \,|_{\zeta,p}^q \leq C(T) \mathbb{E} \int_0^{t \wedge \tau} |\, \alpha(s) \xi(s) \,|_{L(E)}^q \, ds, \tag{4.2.32}$$

where $C(T)$ is independent of ξ and τ. Note that

$$\mathbb{E} \int_0^{t \wedge \tau} |\, \alpha(s) \xi(s) \,|_{L(E)}^q \, ds = \int_0^t \mathbb{E} \,|\, \alpha(s) \xi(s) \,|_{L(E)}^q \, ds.$$

Thus for each $t \in [0, T]$

$$\mathbb{E} \sup_{0 \leq s \leq t} |\, \tilde{z}_\alpha(s \wedge \tau) \,|_{\zeta,p}^q \leq C(T) \int_0^t \mathbb{E} \,|\, \alpha(s) \xi(s) \,|_{L(E)}^q \, ds. \tag{4.2.33}$$

This completes the proof.

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4.3 The Existence Theorems

We now turn to existence of solutions to the problem (4.1.1)-(4.1.2). We first define what we mean by a solution to (4.1.1)-(4.1.2).

Definition 4.3.1 *An admissible $(X, D(A))_{\zeta,p}$-valued process $u(t)$, $t \geq 0$, $\zeta \in (0,1)$, is called a mild solution to (4.1.1)-(4.1.2) if and only if there exists numbers q, $p \in (2,\infty)$, with $q \geq p$, and $1 > \gamma \geq \zeta$ such that for each $T > 0$,*

$$u \in Z_{T,\gamma,\zeta} := M^q(0,T;(X,D(A))_{\gamma,p}) \cap L^q(\Omega; C(0,T;(X,D(A))_{\zeta,p}))$$

and $u(t)$ satisfies the following mild integral equation

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u(s))ds + \int_0^t e^{-(t-s)A}H(u(s))dw(s) \qquad (4.3.34)$$

a.s. for each $t \geq 0$.

A theory for stochastic evolution equations, SEEs, on M-type 2 Banach spaces has been developed in [Br,97], see also [Br,95], and this is applicable to our set-up. Although spaces defined via the complex interpolation method were used in [Br,97], only minor changes are necessary to obtain results for the real interpolation spaces we are dealing with. In spite of this, what we present here does differ from [Br,97] and [Br, 95]. As in [Br,97], we first prove existence of a unique solution in the linear case. The estimates (4.2.10) and (4.2.15) give us stronger estimates on the solution. The effort needed to obtain these estimates is paid off in the subsequent sections. Under the assumption that the coefficients F and H are Lipschitz we use the Banach Fixed Point Theorem, (BFP) to prove existence of a global solution. In [Br,97] and [Br,95], using the BFP Theorem they show existence of a solution on some small time interval. To obtain a solution on any finite time interval they use a *gluing* procedure to extend their original solution. Such a technique is well known, see [DP/Z,92]. Our estimates (4.2.10) and (4.2.15), in conjunction with the BFP Theorem, give us a unique process, defined on the half line $[0,\infty)$, such that when restricted to any finite time interval $[0,T]$, is the unique solution to (4.1.1)-(4.1.2) on $[0,T]$. This avoids using the technical gluing procedure and is similar to approaches by [Sei,93] and [Ic,83]. They worked in Hilbert space settings. Furthermore they use the norm $\sup_{t\geq 0}\mathbb{E} \mid u(t) \mid$ whereas we prove results in the stronger norm $\mathbb{E}\sup_{t\geq 0} \mid u(t) \mid$.
Finally given locally Lipschitz coefficients we construct approximate coefficients which are globally Lipschitz. The global solutions to the equations with the approximated coefficients are then used to construct a local solution. The proof of this theorem is similar to those in [Br,97] and [Sei,93].

Theorem 4.3.2 (The Linear Equation Theorem) *Assume that (A1), (A2), (A3), (A1*) and (A3*) hold. Let $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$, $\gamma \in [\zeta, \frac{3}{4})$. If $h \in M^q_\lambda(0,\infty; L(E))$, $f \in M^q_\lambda(0,\infty; X)$ and $u_0 \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$ then there exists a unique mild solution to the problem*

$$\begin{aligned} du(t) + Au(t)dt &= h(t)dw(t) + f(t)dt \\ u(0) &= u_0 \end{aligned}$$

i.e. for any $T > 0$, $u \in Z_{T,\gamma,\zeta}$ and $u(t)$ satisfies the following mild integral equation

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds + \int_0^t e^{-(t-s)A}h(s)dw(s)$$

a.s., for each $t \geq 0$. Moreover for $\lambda > 0$ the following estimate holds:

$$\begin{aligned}\mathbb{E}\sup_{t\geq 0} \mid e^{-t\lambda}u(t) \mid_{\zeta,p}^q &+ \mathbb{E}\int_0^\infty \mid e^{-t\lambda}u(t) \mid_{\gamma,p}^q dt \\ &\leq C\kappa(\lambda)\mathbb{E}\int_0^\infty \{\mid e^{-t\lambda}h(t) \mid_{L(E)}^q + \mid e^{-t\lambda}f(t) \mid_X^q\}dt \\ &\quad + C\left(1+\frac{1}{\lambda}\right)\mathbb{E} \mid u_0 \mid_{\zeta,p}^q \end{aligned} \tag{4.3.35}$$

where $C = C(q, E, X)$ is a constant independent of f, h, u_0, λ and $\kappa(\lambda)$ tends to zero as λ tends to infinity.

Proof : The proof is carried out in a number of steps.
Step 1 : Define the process $u_1(t) := \int_0^t e^{-(t-s)A}h(s)dw(s)$. Then $x_\lambda(t) = e^{-t\lambda}u_1(t)$ and in view of Theorem 4.2.6, for $\lambda > 0$, there exists a modification $\tilde{x}_\lambda(t)$ of $x_\lambda(t)$ satisfying $\tilde{x}_\lambda \in Z_{\infty,\gamma,\zeta}$. For $\omega \in \Omega$, set $\overline{u}_1(\cdot,\omega) = e^{\cdot\lambda}\tilde{x}_\lambda(\cdot,\omega)$. Then $\overline{u}_1$ is a modification of u_1 satisfying $e^{-\cdot\lambda}\overline{u}_1 \in Z_{\infty,\gamma,\zeta}$. Thus for any $T > 0$, $\overline{u}_1 \in Z_{T,\gamma,\zeta}$ with the following estimate holding

$$\begin{aligned}\mathbb{E}\sup_{t\geq 0} \mid e^{-t\lambda}\overline{u}_1(t) \mid_{\zeta,p}^q &+ \mathbb{E}\int_0^\infty \mid e^{-t\lambda}\overline{u}_1(t) \mid_{\gamma,p}^q dt \\ &\leq C(q,E)\kappa(\lambda)\mathbb{E}\int_0^\infty \mid e^{-t\lambda}h(t) \mid_{L(E)}^q dt \end{aligned} \tag{4.3.36}$$

where $C(q, E)$ is independent of h and λ, and $\kappa(\lambda)$ tends to zero as λ tends to infinity.
Step 2 : Given $f \in M_\lambda^q(0,\infty; X)$ then $e^{-\cdot\lambda}f(\cdot,\omega) \in L^q(0,\infty; X)$ for $\omega \in \Omega_f$, where $P(\Omega_f) = 1$. Define the process $\overline{u}_2$ by

$$\overline{u}_2(t,\omega) = \int_0^t e^{-(t-s)A}f(s,\omega)ds \quad \text{for } \omega \in \Omega_f$$

Note that $\overline{u}_2$ is a continuous $(X, D(A))_{\varrho,p}$-valued process, $\varrho \in (0,1)$, which is adapted and thus progressively measurable. Consider

$$e^{-t\lambda}\overline{u}_2(t) = \int_0^t e^{-(t-s)\lambda}e^{-(t-s)A}e^{-s\lambda}f(s)ds.$$

As $e^{-\cdot\lambda}f(\cdot) \in L^q(0,\infty; X)$ a.s., then in view of Theorem 4.2.2, for $\lambda > 0$, we may deduce that, in particular, $e^{-\cdot\lambda}\overline{u}_2 \in Z_{\infty,\gamma,\zeta}$. So for any $T > 0$, $\overline{u}_2 \in Z_{T,\gamma,\zeta}$ and for any $\lambda > 0$ we have

$$\begin{aligned}\mathbb{E}\sup_{t\geq 0} \mid e^{-t\lambda}\overline{u}_2(t) \mid_{\zeta,p}^q &+ \mathbb{E}\int_0^\infty \mid e^{-t\lambda}\overline{u}_2(t) \mid_{\gamma,p}^q dt \\ &\leq C\kappa(\lambda)\mathbb{E}\int_0^\infty \mid e^{-t\lambda}f(t) \mid_X^q \}dt \end{aligned} \tag{4.3.37}$$

where $C = C(q, X)$ is a constant independent of f, h, u_0, λ and $\kappa(\lambda)$ tends to zero as λ tends to infinity.

Step 3 : Define the process $\overline{u}_3(t) = e^{-tA}u_0$. Note that u_0 is adapted and thus so is $\overline{u}_3$. It is straightforward to show that $\overline{u}_3$ is a $(X, D(A))_{\varrho,p}$-valued continuous process, for any $\varrho \in (0,1)$. One uses the strong continuity of the semigroup $\left\{e^{-tA}\right\}_{t\geq 0}$ and the following characterisation of the space $(X, D(A))_{\varrho,p}$, see Chapter 2,

$$x \in (X, D(A))_{\varrho,p} \Leftrightarrow | x |^p_{\varrho,p} = \int_0^\infty | r^{1-\varrho-\frac{1}{p}} A e^{-rA} x |^p_X \, dr < \infty.$$

It follows that $\overline{u}_3$ is progressively measurable and for $\lambda > 0$ we have

$$\mathbb{E} \sup_{t\geq 0} | e^{-t\lambda}\overline{u}_3(t) |^q_{\zeta,p} + \mathbb{E}\int_0^\infty | e^{-t\lambda}\overline{u}_3(t) |^q_{\gamma,p} \, dt \leq \left(1 + \frac{1}{\lambda}\right) \mathbb{E} | u_0 |^q_{\zeta,p}. \qquad (4.3.38)$$

As a result for any $T > 0$, $\overline{u}_3 \in Z_{T,\gamma,\zeta}$

Step 4 : We now define our mild solution. Set $u = \overline{u}_1 + \overline{u}_2 + \overline{u}_3$. Then u is clearly unique and it satisfies the required regularity conditions. The desired estimate (4.3.35) follows from (4.3.36), (4.3.37) and (4.3.38).This completes the proof.

♠

We now consider the problem (4.1.1)-(4.1.2), under the assumption that F and H satisfy a global Lipschitz condition.

Theorem 4.3.3 (The Global Existence Theorem)*: Assume* $(A1)$, $(A2)$, $(A3)$, $(A1*)$ *and* $(A3*)$ *hold. Let* $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$ *and* $\gamma \in [\zeta, \frac{3}{4})$. *Let* $u_0 \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$. *Assume that* H *and* F *are Lipschitz maps from* $(X, D(A))_{\gamma,p}$ *to* $L(E)$ *and* X *respectively, in the following sense: there exists a constant* $K > 0$ *such that for* u, $v \in (X, D(A))_{\gamma,p}$

$$\begin{aligned} | H(u) - H(v) |_{L(E)} &\leq K | u - v |_{\zeta,p} \\ | F(u) - F(v) |_X &\leq K | u - v |_{\zeta,p} \end{aligned}$$

Then, there exists a unique process u *such that for each* $T > 0$, $u \in Z_{T,\gamma,\zeta}$ *and* u *is the unique solution to the problem* (4.1.1)-(4.1.2) *on* $[0,T]$.
Note that the above Lipschitz conditions imply the following linear growth conditions:

$$\begin{aligned} | H(u) |_{L(E)} &\leq C(1 + | u |_{\zeta,p}) \\ | F(u) |_X &\leq C(1 + | u |_{\zeta,p}) \end{aligned}$$

where $C = \max(K, | F(0) |_X, | H(0) |_{L(E)})$.

Proof : For $\lambda \geq 0$ and a Banach space Y we introduce the space $C_\lambda(0,\infty; Y)$ as the space of continuous Y-valued functions which satisfy

$$| u |_\lambda := \sup_{t\geq 0} | e^{-t\lambda} u(t) |_Y < \infty.$$

$C_\lambda(0,\infty; Y)$ is a Banach space with norm $| u |_\lambda$. Define now

$$Z^\lambda_{\infty,\gamma,\zeta} = M^q_\lambda(0,\infty; (X, D(A))_{\gamma,p}) \cap L^q(\Omega; C_\lambda(0,\infty; (X, D(A))_{\zeta,p})).$$

We endow $Z^\lambda_{\infty,\gamma,\zeta}$ with the norm

$$| u |^q_{Z^\lambda_{\infty,\gamma,\zeta}} = \mathbb{E}\int_0^\infty | e^{-t\lambda}u(t) |^q_{\gamma,p} \, dt + \mathbb{E}\sup_{t\geq 0} | e^{-t\lambda}u(t) |^q_{\zeta,p} .$$

which is the norm inherited from the Banach space

$$L^q_{\infty,\gamma,\zeta} := L^q(\Omega; L^q_\lambda(0,\infty;(X,D(A))_{\gamma,p})) \cap L^q(\Omega; C_\lambda(0,\infty;(X,D(A))_{\zeta,p})).$$

It can be shown that $Z^\lambda_{\infty,\gamma,\zeta}$ is a closed subspace of $L^q_{\infty,\gamma,\zeta}$ and is thus a Banach space. We now define a map $J^\lambda_\infty : Z^\lambda_{\infty,\gamma,\zeta} \to Z^\lambda_{\infty,\gamma,\zeta}$ by

$$v = J^\lambda_\infty(u) \quad \text{if and only if} \quad \text{a.s. for each } t \geq 0$$

$$v(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u(s))d(s) + \int_0^t e^{-(t-s)A}H(u(s))dw(s).$$

We will show that, for $\lambda > 0$, J^λ_∞ is a contraction mapping from $Z^\lambda_{\infty,\gamma,\zeta}$ to itself and then the BFP Theorem infers the existence of a unique $u \in Z^\lambda_{\infty,\gamma,\zeta}$ such that $u = J^\lambda_\infty(u)$. For each $T > 0$, u restricted to $[0,T]$ is the unique solution to the problem (4.1.1)-(4.1.2) on $[0,T]$.

Step 1 : We show that J^λ_∞ does actually map $Z^\lambda_{\infty,\gamma,\zeta}$ to itself.

- Define the map Ψ^λ_∞ on $Z^\lambda_{\infty,\gamma,\zeta}$ by

$$\Psi^\lambda_\infty(u)(\cdot) = \int_0^\cdot e^{-(\cdot-s)A}H(u(s))dw(s).$$

To show that Ψ^λ_∞ maps $Z^\lambda_{\infty,\gamma,\zeta}$ to itself it is enough to show that

$$H(u(\cdot)) \in M^q_\lambda(0,\infty;L(E)).$$

This is because if $H(u(\cdot)) \in M^q_\lambda(0,\infty;L(E))$ then by applying Theorem 4.3.2 with $f \equiv 0$, $u_0 \equiv 0$ and $h(\cdot) = H(u(\cdot))$ we have, by (4.3.36), that $\Psi^\lambda_\infty(u) \in Z^\lambda_{\infty,\gamma,\zeta}$. Note that H is a continuous mapping independent of (t,ω). If $u \in Z^\lambda_{\infty,\gamma,\zeta}$, then u is progressively measurable and hence so is $H(u)$. As H is of linear growth and $(X,D(A))_{\gamma,p} \hookrightarrow (X,D(A))_{\zeta,p}$ then

$$\begin{aligned}
\mathbb{E}\int_0^\infty | e^{-s\lambda}H(u(s)) |^q_{L(E)} \, ds &\leq C(q,\gamma,\zeta)\mathbb{E}\int_0^\infty e^{-sq\lambda}(1+ | u(s) |^q_{\gamma,p})ds \\
&\leq C(q,\gamma,\zeta,T)\left\{\frac{1}{\lambda}+ | u |^q_{M^q_\lambda(0,\infty;(X,D(A))_{\delta,p})}\right\}
\end{aligned}$$

which is finite and so $H(u) \in M^q_\lambda(0,\infty;L(E))$.

- In a similar fashion we can show that the map Φ^λ_∞, defined by

$$\Phi^\lambda_\infty(u)(\cdot) = \int_0^\cdot e^{-(\cdot-s)A}F(u(s))d(s),$$

maps $Z^\lambda_{\infty,\gamma,\zeta}$ to itself.

- The map Γ^λ_∞ where $\Gamma^\lambda_\infty(u) = e^{-(\cdot-s)A}u$ maps $L^q(\Omega,\mathcal{F}_0;(X,D(A))_{\zeta,p})$ to $Z^\lambda_{\infty,\gamma,\zeta}$ as was shown in Theorem 4.3.2, which completes Step 1.

Step 2 : We now show that the map J^λ_∞ is Lipschitz.

• Consider the map $\Psi_\infty^\lambda : Z_{\infty,\gamma,\zeta}^\lambda \to Z_{\infty,\gamma,\zeta}^\lambda$. Let $h(\cdot) = H(u(\cdot)) - H(v(\cdot))$. Using the linear growth of H one can show that for $u, v \in Z_{\infty,\gamma,\zeta}^\lambda$

$$\mathbb{E}\int_0^\infty e^{-sq\lambda} \mid h(s) \mid_{L(E)}^q ds \le K^q \mid u - v \mid_{Z_{\infty,\gamma,\zeta}^\lambda}^q \tag{4.3.39}$$

Thus $h(\cdot) \in M_\lambda^q(0,\infty; L(E))$. Applying Theorem 4.3.2, with $f \equiv 0$, $u_0 \equiv 0$, the estimates (4.3.36) and (4.3.39) imply

$$\begin{aligned} \mid \Psi_\infty^\lambda(u) - \Psi_\infty^\lambda(v) \mid_{Z_{\infty,\gamma,\zeta}^\lambda} &\le C(q,X)\kappa(\lambda)\mathbb{E}\int_0^T \mid e^{-s\lambda}h(s) \mid_{L(E)}^q \\ &\le C(q,K,X)\kappa(\lambda) \mid u - v \mid_{Z_{\infty,\gamma,\zeta}^\lambda}^q \end{aligned} \tag{4.3.40}$$

where C is a constant independent of u, v, λ. It is important(!) to note that $\kappa(\lambda) \to 0$ as $\lambda \to 0$.

• Using the same argument as above we can prove that Φ_∞^λ is Lipschitz from Z_∞^λ to itself with the following estimate holding

$$\mid \Phi_\infty^\lambda(u) - \Phi_\infty^\lambda(v) \mid_{Z_{\infty,\gamma,\zeta}^\lambda} \le \tilde{C}(q,K,X)\kappa(\lambda) \mid u - v \mid_{Z_{\infty,\gamma,\zeta}^\lambda}^q \tag{4.3.41}$$

where $\tilde{C}$ is a constant independent of u, v, λ. Again it is important(!) to note that $\kappa(\lambda) \to 0$ as $\lambda \to 0$. This completes Step 2.

Step 3 : From Step 1 and Step 2 we may deduce that for each $\lambda > 0$, J_∞^λ maps $Z_{\infty,\gamma,\zeta}^\lambda$ to itself and from (4.3.40) and (4.3.41) we have, for $u, v \in Z_{\infty,\gamma,\zeta}^\lambda$,

$$\mid J_\infty^\lambda(u) - J_\infty^\lambda(v) \mid_{Z_{\infty,\gamma,\zeta}^\lambda} \le C(q,K,X)\kappa(\lambda) \mid u - v \mid_{Z_{\infty,\gamma,\zeta}^\lambda}$$

where $\kappa(\lambda) \to 0$ as $\lambda \to \infty$. Thus for large enough λ the map J_∞^λ is a contraction and so there is a unique fixed point which is the unique solution to our problem.

♠

The proof of the following Corollary follows along the lines of similar results in [Br,95] and [Ic,83].

Corollary 4.3.4 *Let u be the unique mild solution of Theorem 4.3.3. Then, for each $\mu > 0$, the process $u^\mu(t) := \mu(\mu + A)^{-1}u(t)$ belongs to the space $M^q(0,T; D(A)) \cap L^q(\Omega; C(0,T; D(A)))$ for each $T > 0$. Moreover for each $t \ge 0$, it satisfies*

$$u^\mu(t) + \int_0^t Au^\mu(t)dt = u^\mu(0) + \int_0^t F_\mu(u(s))ds + \int_0^t H_\mu(u(s))dw(s) \quad a.s. \tag{4.3.42}$$

where $u^\mu(0) := \mu(\mu + A)^{-1}u(0)$, $F_\mu := \mu(\mu + A)^{-1}F$ and $H_\mu := \mu(\mu + A)^{-1}H$.

Proof : As $(\mu + A)^{-1} : X \to D(A)$ is a bounded linear map, then all we need to prove is that $u^\lambda(t)$ satisfies (4.3.42). u is a mild solution satisfying (4.3.34). Applying the bounded linear operator $\mu(\mu + A)^{-1}$ to both sides of (4.3.34), noting that $(\mu + A)^{-1}$ and e^{-tA} commute, we have for each $t \ge 0$

$$u^\mu(t) = e^{-tA}u_0^\mu + \int_0^t e^{-(t-s)A}F_\mu(u(s))ds + \int_0^t e^{-(t-s)A}H_\mu(u(s))dw(s) \text{ a.s..}$$

For each $s \geq 0$, $e^{-(t-s)A}F_\mu(u(s)) \in D(A)$ a.s. and

$$\int_0^T \int_0^t \mid Ae^{-(t-s)A}F_\mu(u(s)) \mid_X dsdt \quad < \infty \text{ a.s..}$$

For each $e \in E$, $e^{-(t-s)A}H_\mu(u(s))(e) \in D(A)$ a.s. for $s \geq 0$ and

$$\int_0^T \int_0^t \mid Ae^{-(t-s)A}H_\mu(u(s)) \mid_{L(E,X)} dsdt \quad < \infty \text{ a.s..}$$

It then follows, by the Fubini Theorem, that

$$\int_0^t \int_0^r Ae^{-(r-s)A}F_\mu(u(s))dsdr = \int_0^t \int_s^t Ae^{-(r-s)A}F_\mu(u(s))drds. \tag{4.3.43}$$

Furthermore, by the Stochastic Fubini Theorem

$$\int_0^t \int_0^r Ae^{-(r-s)A}H_\mu(u(s))dw(s)dr = \int_0^t \int_s^t Ae^{-(r-s)A}H_\mu(u(s))drdw(s). \tag{4.3.44}$$

Note that, as $u^\mu(s) \in D(A)$ a.s., for $s \geq 0$,

$$Au^\mu(s) = A\mu(\mu + A)^{-1}u(s) = \mu u(s) - \mu(\mu + A)^{-1}u(s) \text{ a.s.}$$

so that $Au^\mu(s)$ is integrable. It now follows from (4.3.43), (4.3.44) and the identity

$$\int_s^t Ae^{-(t-r)A}xdr = x - e^{-(t-s)A},$$

see Chapter 2, that

$$\begin{aligned}
\int_0^t Au^\mu(s)ds &= \int_0^t Ae^{-sA}u^\mu(0)ds + \int_0^t \int_0^s e^{-(r-s)A}F_\mu(u(s))dsdr \\
&\quad + \int_0^t \int_0^s Ae^{-(r-s)A}H_\mu(u(s))dw(s)dr \\
&= u^\mu(0) - e^{-tA}u^\mu(0) + \int_0^t F_\mu(u(s)) - e^{-(t-s)A}F_\mu(u(s)) \; ds \\
&\quad + \int_0^t H_\mu(u(s)) - e^{-(t-s)A}H_\mu(u(s)) \; dw(s),
\end{aligned}$$

which completes the proof.

♠

We have a stronger version of Theorem 4.3.3, where we relax the global Lipschitz condition to that of a local Lipschitz condition. These are precisely the conditions we have on the Nemytski maps from Chapter 3. We first define a local solution which is a solution defined up to a stopping time.

Definition 4.3.5 *Assume (A1), (A2), (A3), (A1*) and (A3*) hold. Let τ be an accessible stopping time and let $u(t), t < \tau$ be an admissible $(X, D(A))_{\zeta,p}$-valued process, for some $p > 2$ and $\zeta \in (0,1)$. Then $u(t), t < \tau$ is a local solution to the problem (4.1.1)-(4.1.2) if and only if there exists an increasing sequence of stopping times, $\{\tau_n\}_{n\geq 1}$, satisfying $\tau_n < \tau$ and $\tau_n \to \tau$ a.s., such that for any $t \in [0,\infty)$ and $n \in \mathbb{N}$ the following integral equation holds, a.s.*

$$u(t \wedge \tau_n) = e^{-(t\wedge\tau_n)A}u_0 + \tilde{y}_{\alpha_n}(t \wedge \tau_n) + \int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A}F(u(s))ds. \tag{4.3.45}$$

The term $\tilde{y}_{\alpha_n}$ appearing in (4.3.45) *is the continuous modification of the process y_{α_n}, defined by*

$$y_{\alpha_n}(t) = \int_0^t e^{-(t-s)A}\alpha_n(s)H(u(s))dw(s),$$

where α_n is the characteristic function of the interval $[0, \tau_n)$.
Furthermore, we require that, for some $q \geq p > 2$

$$\mathbb{E}\int_0^{\tau_n} | u(s) |_{\gamma,p}^q + \mathbb{E} \sup_{0\leq t\leq \tau_n} | u(s) |_{\zeta,p}^q < \infty,$$

for each $n \in \mathbb{N}$.

Remark 4.3.6 The term $\tilde{y}_{\alpha_n}$ appearing in (4.3.45) may be considered informally as

$$`\int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A} H(u(s))dw(s)'. \tag{4.3.46}$$

Although the stochastic convolution

$$\int_0^t e^{-(t-s)A} H(u(s))dw(s)$$

does make sense (for suitable H and u), the integrand appearing in (4.3.46) is not necessarily progressively measurable. As a result (4.3.46) does not make sense. The term $\tilde{y}_{\alpha_n}(t\wedge\tau_n)$ does make sense though. In Chapters 5 and 6, when writing the equation (4.3.45), we will always write

$$`\int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A} H(u(s))dw(s)'$$

with the understanding that it is to be interpreted as '$\tilde{y}_{\alpha_n}(t\wedge\tau_n)$', as given in the above definition.

The proof of the following theorem is analogous to those in [Br,97], and [Sei,93].

Theorem 4.3.7 (The Local Existence Theorem) *Assume* (*A*1), (*A*2), (*A*3), (*A*1∗) *and* (*A*3∗) *hold. Let $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$ and $\gamma \in [\zeta, \frac{3}{4})$. Let $u_0 \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$. Assume that H and F are local Lipschitz maps from $(X, D(A))_{\gamma,p}$ to $L(E)$ and X respectively, in the following sense: For each $R > 0$, there exists a constant $K_R > 0$ such that for $u, v \in \bar{B}_\zeta(0, R) = \{x \in (X, D(A))_{\zeta,p} : | x |_{\zeta,p} \leq R\}$*

$$\begin{aligned} | H(u) - H(v) |_{L(E)} &\leq K_R | u - v |_{\zeta,p} \\ | F(u) - F(v) |_X &\leq K_R | u - v |_{\zeta,p} \end{aligned}$$

Then there exists a $(X, D(A))_{\zeta,p}$-valued process $u(t), t < \tau$, which is a local solution to (4.1.1)-(4.1.2), *where τ is an accessible stopping time.*

For the proof of Theorem 4.3.7 we need the following lemma.

Lemma 4.3.8 (Local Uniqueness Lemma) *Assume* (*A*1), (*A*2), (*A*3), (*A*1∗) *and* (*A*3∗) *hold. Let $q \geq p > 2$, $\zeta \in (\frac{1}{2p}, \frac{3}{4} - \frac{1}{q})$ and $\gamma \in [\zeta, \frac{3}{4})$. For i = 1, 2, let $u_{0,i} \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$ and let H_i and F_i be maps from $(X, D(A))_{\gamma,p}$ to $L(E)$ and X respectively. Suppose that on some open subset $\mathcal{U}$ of $(X, D(A))_{\zeta,p}$ we have*

- *F_i and H_i are uniformly Lipschitz.*
- *$F_1 = F_2$ and $H_1 = H_2$.*
- *There exists a measurable set Ω_0 of positive measure, such that*

$$u_{0,1} = u_{0,2} \in \mathcal{U} \ \ a.s. \ on \ \Omega_0.$$

Suppose that for $i = 1, 2$, $u_i(t)$, $t \in [0, T]$, $0 < T < \infty$, are solutions to the problem

$$du_i(t) + Au_i(t)dt = F_i(u_i(t))dt + H_i(u_i)dw(t),$$

$$u_i(0) = u_{0,i}.$$

Let $\tau_i := inf\{t \in [0, T] : u_i(t) \notin \mathcal{U}\}$. Then it follows that $\tau_1 = \tau_2$ a.s. on Ω_0 and

$$\sup_{0 \le t \le \tau_1} | u_1(t) - u_2(t) |_{\zeta,p} = 0 \ \ a.s. \ on \ \Omega_0.$$

Proof : For simplicity let us assume that F_1 and F_2 are equal to zero. We may further assume that $\Omega_0 = \Omega$ since we can normalize $\mathbb{P}$ on Ω_0 such that $\mathbb{P}(\Omega_0) = 1$. Similarly we can assume that a.s.

$$u_1(0) = u_2(0) \in \mathcal{U}.$$

Let $\sigma = \tau_1 \wedge \tau_2$. Consider $z : [0, T] \times \Omega \to (X, D(A))_{\zeta,p}$ given by

$$z(t) = u_1(t) - u_2(t) = \int_0^t e^{-(t-s)A} \left(H_1(u_1(s)) - H_2(u_2(s))\right) dw(s).$$

Let $\xi(s) = H_1(u_1(s)) - H_2(u_2(s))$, then $\xi \in M^q(0, T; L(E))$. By Theorem 4.2.7, z has a continuous modification $\tilde{z}$ such that for each $t \in [0, T]$

$$\mathbb{E} \sup_{0 \le s \le t} | \tilde{z}(t \wedge \sigma) | \le C(T) \mathbb{E} \int_0^{t \wedge \sigma} | \xi(s) |_{L(E)}^q \, ds.$$

Since H_1 and H_2 coincide on $\mathcal{U}$ and are also uniformly Lipschitz, we have

$$\mathbb{E} \sup_{0 \le s \le t} | \tilde{z}(t \wedge \sigma) |_{\zeta,p}^q \le \tilde{C}(t) \int_0^t \mathbb{E} \, | \beta(s)(u_1(s) - u_2(s)) |_{\zeta,p}^q \, ds,$$

where β is the characteristic function of $[0, \sigma)$. Noting that

$$\begin{aligned}
\mathbb{E} \sup_{0 \le s \le t} | \beta(s)(u_1(s) - u_2(s)) |_{\zeta,p}^q \, ds &\le \mathbb{E} \sup_{0 \le s \le t} | (u_1(s \wedge \sigma) - u_2(s \wedge \sigma)) |_{\zeta,p}^q \, ds \\
&= \mathbb{E} \sup_{0 \le s \le t} | \tilde{z}(t \wedge \sigma) |_{\zeta,p}^q,
\end{aligned}$$

we deduce, that for each $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \le s \le t} | \beta(s)(u_1(s) - u_2(s)) |_{\zeta,p}^q \, ds \le \tilde{C}(T) \int_0^t \mathbb{E} \, | \beta(s)(u_1(s) - u_2(s)) |_{\zeta,p}^q \, ds.$$

By the Gronwall inequality

$$\mathbb{E} \sup_{0 \le s \le t} | \beta(s)(u_1(s) - u_2(s)) |_{\zeta,p}^q \, ds = 0, \ \ t \in [0, T].$$

As a result, for each $t \in [0,T]$

$$u_1(t) = u_2(t) \text{ a.s. on } \{t < \sigma\}.$$

The continuity of the processes u_1 and u_2 implies, that for a.a. ω

$$u_1(\cdot,\omega) = u_2(\cdot,\omega) \text{ on } [0,\sigma(\omega)).$$

We may further deduce, that for a.a. ω

$$u_1(\cdot,\omega) = u_2(\cdot,\omega) \text{ on } [0,\sigma(\omega)],$$

using the continuity of the paths again. It follows that

$$\mathbb{E} \sup_{0\leq s\leq\sigma} | u_1(s) - u_2(s) |_{\zeta,p}^q = 0.$$

Finally, as $\sigma = \tau_1 \wedge \tau_2$, then $u_1(\sigma(\omega),\omega) = u_2(\sigma(\omega),\omega) \notin \mathcal{U}$. By definition of τ_1 and τ_2, it follows that $\tau_1 = \tau_2$ a.s.. This completes the proof.

♠

Proof (of Theorem 4.3.7) **:** Let $0 < T < \infty$ be fixed but arbitrary. For each $n \in \mathbb{N}$ define F_n and H_n so that they are uniformly Lipschitz continuous and they coincide with F and H respectively on the set

$$\{u \in (X,D(A))_{\zeta,p} :| u |_{\zeta,p}\leq n\}.$$

Define

$$\bar{u}_n(0,\omega) = \begin{cases} u_0(\omega) & \text{if } | u_0(\omega) |_{\zeta,p}\leq n \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.47}$$

For each $n \in \mathbb{N}$ there exists a unique process $u_n \in Z_{T,\gamma,\zeta}$ which satisfies (4.3.34), with u_0, F and H replaced with $\bar{u}_n(0)$, F_n and H_n respectively. Moreover there exists a set of full measure $\bar{\Omega}$ such that each u_n is continuous on $\bar{\Omega}$. For each $n \in \mathbb{N}$ define the stopping time $\tau_n(\omega) := \inf\{t \in [0,T] :| u_n(t,\omega) |_{\zeta,p}\geq n\}$, with the convention $\inf\emptyset = T$. Using Lemma 4.3.8 one can show that $\{\tau_n\}_{n\geq1}$ is a nondecreasing sequence of stopping times. Define $\tau := \sup_{n\in\mathbb{N}} \tau_n = \lim_{n\to\infty} \tau_n$, then τ is an accessible stopping time. For each n, define $A_n := \{\omega \in \Omega :| u_0 |_{\zeta,p}\leq n\}$. Then $\{A_n\}_{n\geq1}$ is an increasing sequence of sets whose union Ω', is a set of full measure. Using Lemma 4.3.8, one can show that, for $\omega \in A_n$,

$$u_n(t,\omega) = u_m(t,\omega) \text{ for } 0 \leq t \leq \tau_n(\omega) \text{ and } m > n.$$

For each $\omega \in \bar{\Omega} \cap \Omega'$ we define a process u by

$$u(t,\omega) = u_m(t,\omega), \quad 0 \leq t \leq \tau_m(\omega),\ m \geq n,\ \omega \in A_n.$$

As each u_n is continuous then so is u. Moreover as u_0 is adapted and each u_n is $\mathcal{F}_t$-measurable on $\Omega_t(\tau_n) = \{\omega \in \Omega : t < \tau_n(\omega)\}$, then it follows that u is $\mathcal{F}_t$-measurable on $\Omega_t(\tau) = \{\omega \in \Omega : t < \tau(\omega)\}$, i.e. u is adapted. Thus $u(t), t < \tau$ is a well-defined admissible $(X,D(A))_{\zeta,p}$-valued process. For $u(t)$, $t < \tau$, to be a local solution we need to show that it satisfies (4.3.45) for each $t \in [0,T]$ and $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Then, for each $t \in [0,T]$

$$u_n(t) = e^{-tA}\bar{u}_n(0) + \int_0^t e^{-(t-s)A}H_n(u_n(s))dw(s) + \int_0^t e^{-(t-s)A}F_n(u_n(s))ds \text{ a.s..}$$

Let $\tilde{x}_n$ be the continuous modification of x_n, where

$$x_n(t) = \int_0^t e^{-(t-s)A}H_n(u_n(s))dw(s).$$

For each $t \in [0,T]$ we have

$$u_n(t) = e^{-tA}\bar{u}_n(0) + \tilde{x}_n(t) + \int_0^t e^{-(t-s)A}F_n(u_n(s))ds \text{ a.s..}$$

In particular,

$$u_n(t \wedge \tau_n) = e^{-t\wedge\tau_n A}\bar{u}_n(0) + \tilde{x}_n(t \wedge \tau_n) + \int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A}F_n(u_n(s))ds \text{ a.s..}$$

Note that, by construction,

$$\bar{u}_n(0,\omega) = u(0,\omega), \;\; u_n(s,\omega) = u(s,\omega) \text{ and } F_n(u_n(s,\omega)) = F(u(s,\omega)),$$

for $0 < s \leq \tau_n(\omega)$, $\omega \in A_n$. It follows, that for each $t \in [0,T]$

$$u(t \wedge \tau_n) = e^{-t\wedge\tau_n A}u(0) + \tilde{x}_n(t \wedge \tau_n) + \int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A}F(u(s))ds \text{ a.s. on } A_n.$$

We aim to show

$$\tilde{x}_n(t \wedge \tau_n) = \tilde{y}_{\alpha_n}(t \wedge \tau_n) \text{ a.s. on } A_n.$$

Corollary 4.2.8 implies

$$\mathbb{E} \sup_{0\leq t\leq T} \mid \tilde{x}_n(t \wedge \tau_n) - x_{\alpha_n}(t \wedge \tau_n) \mid_{\zeta,p}^{q} = 0, \tag{4.3.48}$$

where

$$x_{\alpha_n}(s) = \int_0^t e^{-(t-s)A}\alpha_n(s)H_n(u_n(s))dw(s)$$

and α_n is the characteristic function of the interval $[0,\tau_n)$.
As $H_n(u_n(r)) = H(u(r))$, $r \leq s < \tau_n(\omega)$, $\omega \in A_n$, then, it follows that for a.a. ω

$$x_{\alpha_n}(\cdot,\omega) = \tilde{y}_{\alpha_n}(\cdot,\omega) \text{ on } [0,\tau_n(\omega)).$$

Using (4.3.48) and the continuity of the processes $\tilde{x}_n$ and $\tilde{y}_{\alpha_n}$ we deduce that

$$\tilde{x}_n(\cdot,\omega) = \tilde{y}_{\alpha_n}(\cdot,\omega) \text{ on } [0,\tau_n].$$

In particular, for each $t \in [0,T]$

$$\tilde{x}_n(t \wedge \tau_n) = \tilde{y}_{\alpha_n}(t \wedge \tau_n) \text{ a.s. on } A_n.$$

It follows, that for each $t \in [0,T]$

$$u(t \wedge \tau_n) = e^{-t\wedge\tau_n A}u(0) + \tilde{y}_{\alpha_n}(t \wedge \tau_n) + \int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A}F(u(s))ds \text{ a.s. on } A_n.$$

Note that if $\omega \notin A_n$, then $\tau_n(\omega) = 0$. In particular, the integral equation holds trivially. So, for each $t \in [0,T]$

$$u(t \wedge \tau_n) = e^{-t\wedge\tau_n A}u(0) + \tilde{y}_{\alpha_n}(t \wedge \tau_n) + \int_0^{t\wedge\tau_n} e^{-(t\wedge\tau_n - s)A}F(u(s))ds \text{ a.s..}$$

The proof is complete once we note that the above argument will hold for any $n \in \mathbb{N}$.

♠

The following Corollary will be used in a later chapter. It's proof relies on the above proof of Theorem 4.3.7.

Corollary 4.3.9 *Let $u(t)$, $t < \tau$, be our local solution. Then for each $\mu > 0$ the process $u^\mu(t) := \mu(\mu + A)^{-1}u(t)$, $t < \tau$, is an admissible $D(A)$-valued process satisfying, for each $t \in [0,\infty)$ and $k \in \mathbb{N}$,*

$$\begin{aligned} u^\mu(t \wedge \tau_k) + \int_0^{t\wedge\tau_k} Au^\mu(t)dt &= u^\mu(0) + \int_0^{t\wedge\tau_k} F_\mu(u(s))ds \\ &+ \int_0^{t\wedge\tau_k} H_\mu(u(s))dw(s) \text{ a.s..} \end{aligned} \tag{4.3.49}$$

Proof : Corollary 4.3.4 implies that the approximate solutions u_n satisfy (4.3.42). To show that the constructed local solution satisfies (4.3.49) one just repeats the arguments in the proof to Theorem 4.3.7.

♠

We now give the definition of a *maximal* solution.

Definition 4.3.10 *Let τ be an accessible stopping time. A local solution $u(t), t < \tau$ is said to be maximal if for any other accessible stopping time $\tilde{\tau}$, such that $u(t)$, $t < \tilde{\tau}$ is also a local solution, then $\tilde{\tau} \leq \tau$, a.s..*

We need the following Lemma, see [El,82].

Lemma 4.3.11 (The Amalgamation Lemma) *Let $\mathcal{A}$ be a family of accessible stopping times with values in $[0,\infty]$. Assume that for each $\alpha \in \mathcal{A}$, $I_\alpha : [0,\alpha) \times \Omega \to (X, D(A))_{\zeta,p}$ is an admissible process and that for any $t < \infty$, $\alpha, \beta \in \mathcal{A}$, $I_\alpha(t) = I_\beta(t)$ a.s. on $\{t < \alpha \wedge \beta\}$. Then, there exists an admissible process $I : [0,\tau) \times \Omega \to (X, D(A))_{\zeta,p}$, where $\tau := \sup\{\alpha : \alpha \in \mathcal{A}\}$, such that*

$$I(t) = I_\alpha(t) \text{ a.s. on } \{t < \alpha\}. \tag{4.3.50}$$

Moreover, if $\tilde{I} : [0,\tau) \times \Omega \to (X, D(A))_{\theta,p}$ is any other admissible process satisfying (4.3.50), then $I(t,\omega) = \tilde{I}(t,\omega)$ a.s. on $\{\omega : t < \tau(\omega)\}$. Furthermore, τ can be chosen as the limit of some increasing sequence $\{\alpha_n\}_{n\geq 1}$, where $\alpha_n \in \mathcal{A}$ for each $n \in \mathbb{N}$.

The following Theorem and proof is taken from [Br,97], see also [Br/El,98].

Theorem 4.3.12 *Under the assumptions of Theorem 4.3.7, there exists a maximal solution $u(t), t < \tau$ to the problem (4.1.1)-(4.1.2)*

Proof : Let

$$\mathcal{LS} := \{(u,\tau) : u(t), t < \tau, \quad \text{is a local solution to (4.1.1)-(4.1.2) } \}.$$

By Theorem 4.3.7, $\mathcal{LS}$ is nonempty. By the Amalgamation Lemma, there exists a unique admissible $(X, D(A))_{\zeta,p}$-valued process $\bar{u}$ and an accessible stopping time $\bar{\tau}$, such that, for every stopping time τ with $(u,\tau) \in \mathcal{LS}$, $\bar{u}(t) = u(t)$ a.s. on $\{t < \tau\}$, $t \in (0,T]$. Moreover, $\overline{u}$ satisfies (4.3.45), which is proved as in Theorem 4.3.7, noting that $\bar{\tau}$ can be taken as the limit of an increasing sequence of stopping times $\{\tau_n\}_{n\geq 1}$, where for each n, $(u_n, \tau_n) \in \mathcal{LS}$. Thus $(\bar{u}, \bar{\tau}) \in \mathcal{LS}$. By Definition 4.3.10, $\bar{u}(t)$, $t < \bar{\tau}$, is maximal.

♠

We turn now to the question of uniqueness of maximal solutions. (Henceforth we assume that the assumptions of Theorem 4.3.7 are satisfied). We need the following lemma, see [Br/El,98].

Lemma 4.3.13 *Let $u(t)$, $t < \tau$ be a maximal solution to the the problem* (4.1.1)-(4.1.2). *Then*

$$\mathbb{P}\left\{\tau < T \text{ and } \limsup_{t\to\tau} \mid u(t) \mid_{\zeta,p} < \infty\right\} = 0. \tag{4.3.51}$$

Remark 4.3.14 Let $u(t)$, $t < \tau$ be a maximal solution to (4.1.1)-(4.1.2), with $\{\tau_n\}_{n\geq 1}$ the corresponding sequence of stopping times with $\tau_n < \tau$ and $\tau_n \to \tau$. For each $n \in \mathbb{N}$, define

$$\sigma_n = \inf\{t \in [0,\tau) : \mid u(t) \mid_{\zeta,p} \geq n\}.$$

By Lemma 4.3.13, σ_n is a well defined stopping time. Furthermore, Lemma 4.3.13 also implies that

$$\tau_n \wedge \sigma_n \to \tau \text{ a.s.}.$$

Theorem 4.3.15 (Uniqueness of Maximal Solutions) *Suppose that $u(t)$, $t < \tau$ and $v(t)$, $t < \tilde{\tau}$ are maximal solutions to the problem* (4.1.1) *with the same initial value $u_0 \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$. Then*

$$\mathbb{E} \sup_{0\leq s<\tau\wedge\tilde{\tau}} \mid u(s) - v(s) \mid^q_{\zeta,p} = 0. \tag{4.3.52}$$

Proof: From the definition of a local (maximal) solution, there exist sequences of stopping times $\{\tau_n\}_{n\geq 1}$ and $\{\tilde{\tau}_n\}_{n\geq 1}$ such that $\tau_n \to \tau$ and $\tilde{\tau}_n \to \tilde{\tau}$. Define the stopping times σ_n and $\tilde{\sigma}_n$ by

$$\sigma_n = \inf\{t \in [0,\tau) : \mid u(t) \mid_{\zeta,p} \geq n\} \wedge \tau_n,$$

$$\tilde{\sigma}_n = \inf\{t \in [0,\tau) : \mid v(t) \mid_{\zeta,p} \geq n\} \wedge \tilde{\tau}_n.$$

Using the maximality of u and v, we have $\sigma_n \to \tau$ and $\tilde{\sigma}_n \to \tilde{\tau}$. Set $\kappa_n = \sigma_n \wedge \tilde{\sigma}_n$ and $\kappa = \tau \wedge \tilde{\tau}$, then $\kappa_n \to \kappa$.
Consider the process $z(t)$, $t < \kappa$ given by

$$z(t) = u(t) - v(t), \ t < \kappa.$$

For each $t \in [0,T]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} z(t \wedge \kappa_n) &= u(t \wedge \kappa_n) - v(t \wedge \kappa_n) \\ &= \tilde{y}^u_{\beta_n}(t \wedge \kappa_n) - \tilde{y}^v_{\beta_n}(t \wedge \kappa_n), \end{aligned}$$

where $\tilde{y}^u_{\beta_n}$ and $\tilde{y}^v_{\beta_n}$ are the continuous modifications of

$$\begin{aligned} y^u_{\beta_n} &= \int_0^t e^{-(t-s)A}\beta_n(s)H(u(s))dw(s), \\ y^v_{\beta_n} &= \int_0^t e^{-(t-s)A}\beta_n(s)H(v(s))dw(s), \end{aligned}$$

with β_n denoting the characteristic function of $[0,\kappa_n)$ Thus

$$z(t \wedge \kappa_n) = \int_0^t e^{-(t-s)A}\beta_n(s)\,(H(u(s)) - H(v(s)))\,dw(s).$$

Corollary 4.2.9 implies

$$\mathbb{E} \sup_{0 \le s \le t} \mid z(s \wedge \kappa_n) \mid^q_{\zeta,p} \le C(T)\mathbb{E}\int_0^t \beta_n(s) \mid (H(u(s)) - H(v(s))) \mid_{L(E)} ds.$$

H is Lipschitz on $B(0,n) \subset (X, D(A))_{\zeta,p}$ and so

$$\mathbb{E} \sup_{0 \le s \le t} \mid z(s \wedge \kappa_n) \mid^q_{\zeta,p} \le C(n,T)\mathbb{E}\int_0^t \beta_n(s) \mid (u(s) - v(s)) \mid_{L(E)} ds.$$

Observing that

$$\mathbb{E} \sup_{0 \le s \le t} \mid \beta_n(s) z(s) \mid^q_{\zeta,p} \le \mathbb{E} \sup_{0 \le s \le t} \mid z(s \wedge \kappa_n) \mid^q_{\zeta,p},$$

we have

$$\mathbb{E} \sup_{0 \le s \le t} \mid \beta_n(s)\,(u(s) - v(s)) \mid^q_{\zeta,p} \le C(n,T)\mathbb{E}\int_0^t \beta_n(s) \mid (u(s) - v(s)) \mid_{L(E)} ds.$$

The Gronwall inequality then implies, that for each $t \in [0,T]$

$$\mathbb{E} \sup_{0 \le s \le t} \mid \beta_n(s)\,(u(s) - v(s)) \mid^q_{\zeta,p} = 0,$$

i.e. for each $t \in [0,T]$

$$u(t) = v(t) \text{ on } \{t < \kappa_n\}.$$

By the continuity of u and v, for a.a. ω we have

$$u(\cdot,\omega) = v(\cdot,\omega) \text{ on } [0,\kappa_n(\omega)).$$

This holds for each $n \in \mathbb{N}$. In particular, if we let $n \to \infty$ then for a.a. ω

$$u(\cdot,\omega) = v(\cdot,\omega) \text{ on } [0,\kappa(\omega)).$$

This completes the proof.

♠

Corollary 4.3.16 (Uniqueness of Global Solutions) *Let* $0 < T < \infty$. *Let* $u(t)$ *and* $v(t)$, $t \in [0,T]$ *be global solutions to the problem* (4.1.1) *with the same initial condition* $u_0 \in L^q(\Omega, \mathcal{F}_0; (X, D(A))_{\zeta,p})$. *Then*

$$\mathbb{E} \sup_{0 \le t \le T} | u(t) - v(t) |_{\zeta,p}^q = 0.$$

Proof: As u and v are global solutions, then $u(t)$, $t < T$ and $v(t)$, $t < T$ are both maximal solutions to the problem (4.1.1)-(4.1.2). Theorem 4.3.15 implies that

$$\mathbb{E} \sup_{0 \le t < T} | u(t) - v(t) |_{\zeta,p}^q = 0.$$

As u and v are both defined and continuous on the interval $[0, T]$ then we can deduce that

$$\mathbb{E} \sup_{0 \le t \le T} | u(t) - v(t) |_{\zeta,p}^q = 0.$$

This completes the proof.

♠

4.3.1 Existence Of A Maximal Solution To The SNHE

We consider now the stochastic nonlinear heat equation, SNHE. Our problem is

$$du(t) + Au(t)dt = V(u(t))dt + F(u(t))dt + H(u(t)) \circ dw(t) \tag{4.3.53}$$

$$u(0) = u_0 \tag{4.3.54}$$

where the term involving the Stratonovich differential may be written

$$H(u(t)) \circ dw(t) = H(u(t))dw(t) + \frac{1}{2} tr\{H'(u(t))H(u(t))\}dt. \tag{4.3.55}$$

The coefficients appearing in (4.3.53) and (4.3.55) are locally Lipschitz maps with

$$\begin{aligned} V,\ \frac{1}{2}tr(H'H) &: (X, D(A))_{\gamma,p} \to X \\ F &: (X, D(A))_{\gamma,p} \to X \\ H &: (X, D(A))_{\gamma,p} \to L(E, E) \end{aligned}$$

for any $\gamma > \frac{1}{2} + \frac{1}{p}$, $p > 2$. Note that γ is chosen as such to guarantee that $F(u)$ makes sense classically for $u \in (X, D(A))_{\gamma,p}$, see Chapter 3. In terms of the Sobolev-Slobodetski spaces, the maps satisfy

$$\begin{aligned} V, \frac{1}{2}tr(H'H) &: W^{\varrho,p}(S^1, \mathbb{R}^d) \to L^p(S^1, \mathbb{R}^d) \\ F &: W^{\varrho+1,p}(S^1, \mathbb{R}^d) \to L^p(S^1, \mathbb{R}^d) \\ H &: W^{\varrho,p}(S^1, \mathbb{R}^d) \to L(W^{\theta,p}(S^1, \mathbb{R}^d), W^{\theta,p}(S^1, \mathbb{R}^d)) \end{aligned}$$

for any $\varrho \ge \theta$, $\theta \in (\frac{1}{p}, \frac{1}{2})$, $p > 2$.
The following Theorem is a consequence of Theorem 4.3.12 above.

Theorem 4.3.17 Existence of a Maximal Solution: *Let $w(t)$, $t \geq 0$, be a $W^{\theta,p}(S^1,\mathbb{R}^d)$-valued Wiener process, $\theta \in (\frac{1}{p},\frac{1}{2})$, $p > 2$, relating to the AWS $i : H^{1,2}(S^1,\mathbb{R}^d) \hookrightarrow W^{\theta,p}(S^1,\mathbb{R}^d)$. Let $u_0 \in L^q(\Omega,\mathcal{F}_0; W^{s,p}(S^1,\mathbb{R}^d))$, where $q > p$ and $\frac{3}{2}-\frac{2}{q} > s > 1+\frac{1}{p}$. Then, there exists an accessible stopping time τ and an admissible $W^{s,p}(S^1,\mathbb{R}^d)$- valued process $u(t)$, $t < \tau$, which is the maximal solution to the problem* (4.3.53)-(4.3.54).

<u>**Proof**</u> : Let u_0, q and s be as stated. Note that

$$W^{s,p}(S^1,\mathbb{R}^d) = (L^p(S^1,\mathbb{R}^d), W^{2,p}(S^1,\mathbb{R}^d))_{\frac{s}{2},p} = (X, D(A))_{\zeta,p},$$

where $\zeta := \frac{s}{2} \in (\frac{1}{2}+\frac{1}{2p},\frac{3}{4}-\frac{1}{q})$. Thus $u_0 \in L^q(\Omega,\mathcal{F}_0;(X,D(A))_{\zeta,p})$ and since the Nemytski maps V, F, H and $\frac{1}{2}tr(H'H)$ satisfy the requirements of Theorem 4.3.12 the result follows.

♠

Remark 4.3.18 Suppose we take our initial value u_0 satisfying

$$u_0 \in L^p(\Omega,\mathcal{F}_0; W^{s,p}(S^1,\mathbb{R}^d)),$$

then we would need $s \in (1+\frac{1}{p},\frac{3}{2}-\frac{2}{p})$. Note that

$$\frac{3}{2}-\frac{2}{p} > 1+\frac{1}{p} \Leftrightarrow p > 6.$$

Thus if $u_0 \in L^p(\Omega,\mathcal{F}_0; W^{s,p}(S^1,\mathbb{R}^d))$ then our results give a solution only in the case $p > 6$.

♡

Chapter 5

Existence of a Maximal Solution on the Loop Manifold $\mathcal{M}$

5.1 Properties of the Nemytski Map I

In this chapter we are concerned with proving that our maximal solution $u(t)$, $t < \tau$, lies on the loop manifold $\mathcal{M} = W^{s,p}(S^1, M)$, $s > 1 + \frac{1}{p}$, $p > 2$, where

$$W^{s,p}(S^1, M) := \left\{u \in W^{s,p}(S^1, \mathbb{R}^d) : u(\sigma) \in M, \forall \sigma \in S^1\right\}.$$

The assumption we need to impose is, for $q \geq p$, as chosen in Chapter 4,

$$u_0 \in L^q(\Omega; W^{s,p}(S^1, M)).$$

We first recall the involution map $i : \mathbb{R}^d \to \mathbb{R}^d$ and its properties, see Chapter 3. i is a smooth map with compact support, that satisfies

$$i(m) = m \Leftrightarrow m \in M. \tag{5.1.1}$$

Moreover $i : U \to U$, where U is the tubular neighbourhood of the manifold M. There exist $\varepsilon > 0$ such that $M \subset U_\varepsilon \subset U$, where

$$U_\epsilon := \left\{x \in \mathbb{R}^d : d(x, M) < \epsilon\right\}. \tag{5.1.2}$$

Furthermore, ε can be chosen small enough so that the following properties hold:

$$i'(p)\tilde{v}(p) = \tilde{v}(i(p)), \quad p \in U_\varepsilon, \tag{5.1.3}$$

$$i'(p)\tilde{h}(p) = \tilde{h}(i(p)), \quad p \in U_\varepsilon, \tag{5.1.4}$$

where $\tilde{v} : \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{h} : \mathbb{R}^d \to L(\mathbb{R}^d; \mathbb{R}^d)$ are the smooth maps constructed in Chapter 3. Henceforth we assume that $\varepsilon > 0$ is chosen small enough so that the properties (5.1.3) and (5.1.4) hold for $p \in U_\varepsilon$.
For the open set $U \subset \mathbb{R}^d$ we define $W^{s,p}(S^1, U)$ as

$$W^{s,p}(S^1, U) := \left\{u \in W^{s,p}(S^1, \mathbb{R}^d) : u(\sigma) \in U, \forall \sigma \in S^1\right\},$$

with an analogous definition for $W^{s,p}(S^1, U_\varepsilon)$. As U is open it is straightforward to show that $W^{s,p}(S^1, U)$ is an open subset of $W^{s,p}(S^1, \mathbb{R}^d)$. We now define the Nemytski map I of i by

$$I(u) := i \circ u. \tag{5.1.5}$$

We have the following important Proposition.

Proposition 5.1.1 *Let I be the Nemytski map of i as defined in* (5.1.5). *Then I is a smooth map $I : W^{s,p}(S^1, \mathbb{R}^d) \to W^{s,p}(S^1, \mathbb{R}^d)$, which is locally Lipschitz and of linear growth. Furthermore, I satisfies the following properties:*

(i) For $u \in W^{s,p}(S^1, \mathbb{R}^d)$,

$$I(u) = u \Leftrightarrow u \in W^{s,p}(S^1, M). \tag{5.1.6}$$

(ii) I maps $W^{s,p}(S^1, U)$ to itself. Similarly, I maps $W^{s,p}(S^1, U_\varepsilon)$ to itself.

(iii) For $u \in W^{s,p}(S^1, U_\varepsilon)$,

$$I'(u)V(u) = V(I(u)), \tag{5.1.7}$$

$$I'(u)H(u) = H(I(u)), \tag{5.1.8}$$

where V and H are the Nemytski maps corresponding to $\tilde{v}$ and $\tilde{h}$ respectively, see Chapter 3, and I' is the derivative of I.

(iv) For $u \in C^\infty(S^1, U_\varepsilon)$,

$$I'(u)\underline{\triangle}(u) = \underline{\triangle}I(u), \tag{5.1.9}$$

where $\underline{\triangle}$ is the nonlinear Laplacian.

Proof : For the first statement, see Proposition 3.4.1. (i) follows from (5.1.1). (ii) follows in a similar fashion as i maps U to itself and U_ε to itself. (iii) follows directly from (5.1.3) and (5.1.4). The proof of (iv) is a little more intricate. By Proposition 3.1.4, any $f : [0, \infty) \to C^\infty(S^1, U_\varepsilon)$, which is a solution of the heat equation, satisfies (5.1.9). Moreover, see [Ee/Sa,64] or [Ot,84], given any $u \in C^\infty(S^1, U_\varepsilon)$, there exists a unique $f : [0, \infty) \to C^\infty(S^1, \mathbb{R}^d)$ which is the solution to the nonlinear heat equation with $f(0) = u$. By continuity, for some small $t > 0$, $f(s) \in C^\infty(S^1, U_\varepsilon)$, $s \in (0, t)$, and so

$$I'(f(s))\underline{\triangle}f(s) = \underline{\triangle}I(f(s)), \quad s \in (0, t). \tag{5.1.10}$$

Thus, by continuity, (5.1.10) will hold for $f(0) = u$ which proves the result.

♠

Remark 5.1.2 As I is smooth, then so is its first derivative I', where

$$I' : W^{s,p}(S^1, \mathbb{R}^d) \to L(W^{s,p}(S^1, \mathbb{R}^d), W^{s,p}(S^1, \mathbb{R}^d)).$$

Furthermore, I' acts through the following formula

$$I'(u)(v)(\sigma) = i'(u(\sigma))(v(\sigma)), \quad u, v \in W^{s,p}(S^1, \mathbb{R}^d), \sigma \in S^1. \tag{5.1.11}$$

♢

We briefly descibe the contents of this chapter. To show that $u(t)$, $t < \tau$, is an $\mathcal{M}$-valued process we follow the idea used in [Ha,75]. Given our maximal mild solution $u(t)$, $t < \tau$, we show, using Proposition 5.1.1, that, for some stopping time $\tilde{\tau} \leq \tau$, the process $v(t) := I(u(t))$, $t < \tilde{\tau}$, is a weak solution to our problem . We then show that our maximal solution is also a weak solution. In particular, under the assumption that $u_0 \in \mathcal{M}$ a.s., we have, by (5.1.6), that $v(0) = I(u(0)) = u(0)$. So v and u are both weak solutions to the same problem, up to the stopping time $\tilde{\tau}$. Using a well known result that if the coefficients are locally lipschitz and u is a unique mild solution, then it is also a unique weak solution, we deduce that $v(t) = u(t)$ for $t < \tilde{\tau}$. This then implies, by (5.1.6), that $u(t) \in \mathcal{M}$ for $t < \tilde{\tau}$. It is then straightforward to show that $\tilde{\tau} = \tau$, a.s..

♡

5.2 The Approximating Processes

Recall that our unique maximal solution is an admissible $W^{s,p}(S^1,\mathbb{R}^d)$-valued process, where $s > 1+\frac{1}{p}$. Furthermore, for $t \in [0,\infty)$ and $k \in \mathbb{N}$, u satisfies

$$u(t\wedge\sigma_k) = e^{-(t\wedge\sigma_k)A}u(0) + \int_0^{t\wedge\sigma_k} e^{-(t\wedge\sigma_k - s)A}\{F(u(s)) + V(u(s))\}\,ds + \int_0^{t\wedge\sigma_k} e^{-(t\wedge\sigma_k - s)A}H(u(s))\circ dw(s) \quad \text{a.s.},$$

where $\{\sigma_k\}_{k\in\mathbb{N}}$ is a sequence of increasing stopping times such that $\sigma_k < \tau$, for each $k \in \mathbb{N}$, and $\sigma_k \to \tau$. We have written the mild equation in the Stratonovich form but we may also write it in Itô form as:

$$\begin{aligned} u(t\wedge\sigma_k) &= e^{-(t\wedge\sigma_k)A}u(0) + \int_0^{t\wedge\sigma_k} e^{-(t\wedge\sigma_k - s)A}\{F(u(s)) + V(u(s))\}\,ds \\ &+ \int_0^{t\wedge\sigma_k} e^{-(t\wedge\sigma_k - s)A}H(u(s))dw(s) \\ &+ \frac{1}{2}\int_0^{t\wedge\sigma_k} e^{-(t\wedge\sigma_k - s)A}tr\{H'(u(s))H(u(s))\}ds. \end{aligned} \tag{5.2.12}$$

The maps V and H satisfy the properties (5.1.7) and (5.1.8), whereas this need not be true for the map F. We do have condition (5.1.9) and $\underline{\Delta}$ involves the map F. A problem arises, in that, to make use of (5.1.9), we need our maximal solution to be $W^{s+1,p}(S^1,\mathbb{R}^d)$-valued, which then implies that it is $C^2(S^1,\mathbb{R}^d)$-valued. u is only $W^{s,p}(S^1,\mathbb{R}^d)$-valued and as a result only satisfies a mild integral equation. This problem may be overcome if we approximate our maximal solution with a sequence of processes, defined up to the stopping time τ, which are of a higher regularity in the space variable than our solution. As a result, they will satisfy a strict integral equation and we may apply condition (5.1.9). We begin with defining the approximations. For simplicity of notation only, we leave out the term V.
Recall that the operator $(n+A)^{-1}$, $n \in \mathbb{N}$, may be considered as a map

$$(n+A)^{-1} : W^{s,p}(S^1,\mathbb{R}^d) \to W^{s+2,p}(S^1,\mathbb{R}^d).$$

It is linear, bounded and it commutes with e^{-rA}, $r \geq 0$. For each $n \in \mathbb{N}$ define the process $u^n(t)$, $t < \tau$, by

$$u^n(t) = n(n+A)^{-1}u(t), \;\; t < \tau.$$

It follows that $u^n(t)$, $t < \tau$, is an admissible $W^{s+2,p}(S^1,\mathbb{R}^d)$-valued process and, in particular, it is also $C^2(S^1,\mathbb{R}^d)$-valued.

Remark 5.2.1 Clearly, by the definition of u^n and the continuity of u, we have that $u^n \to u$ as $n \to \infty$, pointwise on $[0,\tau(\omega))$, for almost all $\omega \in \Omega$. Using the boundedness of $n(n+A)^{-1}$ and the Lebesgue Dominated Convergence Theorem, (LDC), we have, for each $k \in \mathbb{N}$ and $t \in [0,\infty)$, $q \geq p$,

$$\mathbb{E} \mid u(t\wedge\sigma_k) - u^n(t\wedge\sigma_k) \mid_{s,p}^q \to 0 \quad \text{as} \quad n \to \infty,$$

where $|\cdot|_{s,p}$ is the norm on $W^{s,p}(S^1,\mathbb{R}^d)$. $|\cdot|_{L^p}$ will denote the norm on $L^p(S^1,\mathbb{R}^d)$. As I is continuous, then $I(u^n)\to I(u)$ pointwise on $[0,\tau(\omega))$, for almost all $\omega\in\Omega$. In fact, as $u^n\to u$ in $L^q(\Omega;L^p(S^1,\mathbb{R}^d))$, then the following inequality

$$|\, I(u)\,|_{L^p}\leq C\,|\, u\,|_{L^p},\quad u\in W^{s,p}(S^1,\mathbb{R}^d),$$

(where C is a constant independent of u), and the LDC Theorem imply that for each $k\in\mathbb{N}$, $t\in[0,\infty)$

$$I(u^n(t\wedge\sigma_k))\to I(u(t\wedge\sigma_k))\quad\text{in}\quad L^q(\Omega;L^p(S^1,\mathbb{R}^d)).$$

◊

For each $n\in\mathbb{N}$, we apply $n(n+A)^{-1}$ to both sides of (5.2.12). We then have, for $t\in[0,\infty)$ and $k\in\mathbb{N}$, the following mild integral equation holding a.s.

$$\begin{aligned} u^n(t\wedge\sigma_k) &= e^{-(t\wedge\sigma_k)A}u^n(0)+\int_0^{t\wedge\sigma_k}e^{-(t\wedge\sigma_k-s)A}n(n+A)^{-1}F(u(s))ds \\ &+\int_0^{t\wedge\sigma_k}e^{-(t\wedge\sigma_k-s)A}n(n+A)^{-1}H(u(s))dw(s) \\ &+\frac{1}{2}\int_0^{t\wedge\sigma_k}e^{-(t\wedge\sigma_k-s)A}n(n+A)^{-1}tr\{H'(u(s))H(u(s))\}ds. \end{aligned}$$

Henceforth, we write $F_n=n(n+A)^{-1}F$, $H_n=n(n+A)^{-1}H$ and $tr_n=n(n+A)^{-1}tr$. Recall that $u^n(t)$ is $W^{2,p}(S^1,\mathbb{R}^d)$-valued and so $u^n(t)\in D(A)$ a.s.. As a result, see Corollary 4.3.9, for $t\in[0,\infty)$ and $k\in\mathbb{N}$ the following strict integral equation holds a.s. in $L^p(S^1,\mathbb{R}^d)$

$$\begin{aligned} u^n(t\wedge\sigma_k) &= u^n(0)-\int_0^{t\wedge\sigma_k}Au^n(s)ds+\int_0^{t\wedge\sigma_k}F_n(u(s))ds \\ &+ \int_0^{t\wedge\sigma_k}H_n(u(s))dw(s)+\frac{1}{2}\int_0^{t\wedge\sigma_k}tr_n\{H'(u(s))H(u(s))\}ds. \end{aligned}$$

We may rewrite this as

$$\begin{aligned} u^n(t\wedge\sigma_k) &= u^n(0)-\int_0^{t\wedge\sigma_k}Au^n(s)-F(u^n(s))ds \\ &+\int_0^{t\wedge\sigma_k}F_n(u(s))-F(u^n(s))ds \\ &+\int_0^{t\wedge\sigma_k}H_n(u(s))dw(s)+\frac{1}{2}\int_0^{t\wedge\sigma_k}tr_n\{H'(u(s))H(u(s))\}ds. \end{aligned}$$

As a result we have

$$\begin{aligned} u^n(t\wedge\sigma_k) &= u^n(0)+\int_0^{t\wedge\sigma_k}\underline{\Delta}u^n(s)ds+\int_0^{t\wedge\sigma_k}F_n(u(s))-F(u^n(s))ds \\ &+\int_0^{t\wedge\sigma_k}H_n(u(s))dw(s) \\ &+\frac{1}{2}\int_0^{t\wedge\sigma_k}tr_n\{H'(u(s))H(u(s))\}ds. \end{aligned}\tag{5.2.13}$$

Remark 5.2.2 For any sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$, such that $\tau_k<\tau$ and $\tau_k\to\tau$, the above integral equations will hold with σ_k replaced by τ_k, see [Br/El,98]. ◊

♡

5.3 The Weak Solution $I(u(t))$, $t < \tau$.

We first introduce the notion of a weak solution to the problem

$$du(t) + Au(t)dt = F(u(t))dt + H(u(t)) \circ dw(t) \tag{5.3.14}$$

$$u(0) = u_0. \tag{5.3.15}$$

We use the following notation: for u, $v \in L^2(S^1, \mathbb{R}^d)$ we set

$$< u, v >_{L^2} := \int_{S^1} < u(\sigma), v(\sigma) > d\sigma,$$

where $< \cdot, \cdot >$ is the inner product on $\mathbb{R}^d$.

Definition 5.3.1 (Weak Solution) *An admissible $W^{s,p}(S^1, \mathbb{R}^d)$-valued process $u(t)$, $t < \tau$, is a weak solution to the problem* (5.3.14)-(5.3.15), *if for each $\phi \in C^\infty(S^1, \mathbb{R}^d)$ the following weak integral equation holds, for each $k \in \mathbb{N}$ and $t \in [0, \infty)$*

$$\begin{aligned} < u(t \wedge \sigma_k), \phi >_{L^2} &= < u(0), \phi >_{L^2} + \int_0^{t\wedge\sigma_k} < \nabla u(s), \nabla\phi >_{L^2} ds \\ &+ \int_0^{t\wedge\sigma_k} < F(u(s)), \phi >_{L^2} ds \\ &+ < \int_0^{t\wedge\sigma_k} H(u(s)) \circ dw(s), \phi >_{L^2} \quad a.s., \end{aligned} \tag{5.3.16}$$

where $\{\sigma_k\}$ are a sequence of increasing bounded stopping times with $\sigma_k < \tau$ and $\sigma_k \to \tau$.

We will later show that our maximal mild solution is also a weak solution in the above sense. We will first prove the following result which is more difficult. Before stating the Theorem we define the stopping time $\tilde{\tau}$. Let $\epsilon > 0$ be sufficiently small so that properties (5.1.7) and (5.1.8) hold for any $u \in W^{s,p}(S^1, U_\epsilon)$. Define now

$$\tilde{\tau} := \left\{t \in [0, \tau] : u(t) \notin W^{s,p}(S^1, U_\epsilon)\right\}. \tag{5.3.17}$$

$\tilde{\tau} \leq \tau$ is a well-defined stopping time since the process $u(t)$ is continuous and the set $W^{s,p}(S^1, U_\epsilon)$ is open.

Theorem 5.3.2 *Let $u(t)$, $t < \tau$, be our $W^{s,p}(S^1, \mathbb{R}^d)$-valued maximal solution. The process $v(t) := I(u(t))$, $t < \tilde{\tau}$, is an admissible $W^{s,p}(S^1, \mathbb{R}^d)$-valued process which satisfies the following weak integral equation: for each $k \in \mathbb{N}$, $t \in [0, \infty)$ and $\phi \in C^\infty(S^1, \mathbb{R}^d)$ we have*

$$\begin{aligned} < v(t \wedge \tau_k), \phi >_{L^2} &= < v(0), \phi >_{L^2} + \int_0^{t\wedge\tau_k} < \nabla v(s), \nabla\phi >_{L^2} ds \\ &+ \int_0^{t\wedge\tau_k} < F(v(s)), \phi >_{L^2} ds \\ &+ < \int_0^{t\wedge\tau_k} H(v(s)) \circ dw(s), \phi >_{L^2} \quad a.s. \end{aligned} \tag{5.3.18}$$

where $\tau_k := \tilde{\tau} \wedge \sigma_k$.

Proof : We first apply the Itô formula with the map I to the integral equation (5.2.13). Thus for each $t \in [0,\infty)$ and $k \in \mathbb{N}$ the following equation holds a.s.

$$\begin{aligned} I(u^n(t\wedge\tau_k)) &= I(u^n(0)) + \int_0^{t\wedge\tau_k} I'(u^n(s))\underline{\triangle} u^n(s)ds \\ &+ \int_0^{t\wedge\tau_k} I'(u^n(s))\left(F_n(u(s)) - F(u^n(s))\right)ds \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_k} I'(u^n(s))\left(tr_n\{H'(u(s))H(u(s))\}\right)ds \\ &+ \int_0^{t\wedge\tau_k} I'(u^n(s))\left(H_n(u(s))\right)dw(s) \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_k} tr\left\{I''(u^n(s))\left(H_n(u(s)), H_n(u(s))\right)\right\}ds. \end{aligned} \tag{5.3.19}$$

Denote

$$\begin{aligned} I_1^n(t_k) &:= \int_0^{t\wedge\tau_k} I'(u^n(s))\left(F_n(u(s)) - F(u^n(s))\right)ds, \\ I_2^n(t_k) &:= \frac{1}{2}\int_0^{t\wedge\tau_k} I'(u^n(s))\left(tr_n\{H'(u(s))H(u(s))\}\right)ds, \\ I_3^n(t_k) &:= \int_0^{t\wedge\tau_k} I'(u^n(s)\left(H_n(u(s))\right)dw(s), \\ I_4^n(t_k) &:= \frac{1}{2}\int_0^{t\wedge\tau_k} tr\left\{I''(u^n(s))\left(H_n(u(s)), H_n(u(s))\right)\right\}ds. \end{aligned}$$

(5.3.19) may then be written

$$\begin{aligned} I(u^n(t\wedge\tau_k)) &= I(u^n(0)) + \int_0^{t\wedge\tau_k} I'(u^n(s))[-Au^n(s) + F(u^n(s))]ds \\ &+ I_1^n(t_k) + I_2^n(t_k) + I_3^n(t_k) + I_4^n(t_k). \end{aligned} \tag{5.3.20}$$

For any $\phi \in C^\infty(S^1, \mathbb{R}^d)$ it follows, using (5.3.20), that

$$\begin{aligned} < I(u^n(t\wedge\tau_k)), \phi >_{L^2} &= < I(u^n(0)), \phi >_{L^2} \\ &- \int_0^{t\wedge\tau_k} < I'(u^n(s))(Au^n(s)), \phi >_{L^2} ds \\ &+ \int_0^{t\wedge\tau_k} < I'(u^n(s))(F(u^n(s))), \phi >_{L^2} ds \\ &+ < I_1^n(t_k) + I_2^n(t_k), \phi >_{L^2} \\ &+ < I_3^n(t_k) + I_4^n(t_k), \phi >_{L^2}. \end{aligned} \tag{5.3.21}$$

In view of Remark 5.2.1,

$$I(u^n(t\wedge\tau_k)) \to I(u(t\wedge\tau_k))$$

in $L^q(\Omega; L^p(S^1, \mathbb{R}^d))$. Now if $\phi \in C^\infty(S^1, \mathbb{R}^d)$, then ϕ is bounded. It follows that

$$\mathbb{E} < I(u^n(t\wedge\tau_k)) - I(u(t\wedge\tau_k)), \phi >_{L^2} \leq \mathbb{E} \mid I(u^n(t\wedge\tau_k)) - I(u(t\wedge\tau_k)) \mid_{L^p}^q, \tag{5.3.22}$$

which implies

$$< I(u^n(t\wedge\tau_k)), \phi >_{L^2} \to < I(u(t\wedge\tau_k)), \phi >_{L^2}$$

in $L^q(\Omega, \mathbb{R})$.

Remark 5.3.3 For a sequence of processes $\{Z_n(t)\}$ and $Z(t)$, to prove

$$< Z_n(t), \phi >_{L^2} \to < Z(t), \phi >_{L^2}$$

in $L^q(\Omega, \mathbb{R})$, it suffices to prove, by (5.3.22), that

$$\mathbb{E} \mid Z_n(t) - Z(t) \mid^q_{L^p} \to 0. \quad \diamond$$

We wish to calculate an integral expression for $< I(u), \phi >_{L^2}$. We may not simply take limits as $n \to \infty$ in (5.3.21). This is due to the fact that $u(t) \notin D(A)$, a.s. and so the term " $< I'(u(s))(Au(s)), \phi >_{L^2}$" would not make sense. To over come this problem we have the following Lemma.

Lemma 5.3.4 *For* $u \in W^{2,p}(S^1, \mathbb{R}^d)$,

$$- < I'(u)(Au), \phi >_{L^2} = \int_{S^1} < \nabla u(\sigma), \nabla \lambda(u(\sigma))\phi(\sigma) > d\sigma \tag{5.3.23}$$

for some smooth map $\lambda : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$.

Remark 5.3.5 Note that the RHS of (5.3.23) now does make sense for $u \in W^{s,p}(S^1, \mathbb{R}^d)$ where $2 > s > 1 + \frac{1}{p}$. $\diamond$

Proof : Using (5.1.11) we have

$$- < I'(u)(Au), \phi >_{L^2} = \int_{S^1} < i'(u(\sigma)) \frac{d^2u}{d\sigma^2}(\sigma), \phi(\sigma) > d\sigma. \tag{5.3.24}$$

Recall that $i' : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ and so for $x \in \mathbb{R}^d$, $i'(x) \in L(\mathbb{R}^d, \mathbb{R}^d)$. Thus for y, $z \in \mathbb{R}^d$ it follows that

$$< i'(x)y, z > = < y, [i'(x)]^* z >$$

where $[i'(x)]^* \in L(\mathbb{R}^d, \mathbb{R}^d)$ is the adjoint of $i'(x)$. Define the map $\lambda : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ by $\lambda(x) := [i'(x)]^*$. Then $\lambda = * \circ i'$, where by $*$ we mean the operation of taking adjoints of elements of $L(\mathbb{R}^d, \mathbb{R}^d)$. $*$ is linear and continuous and hence smooth. Moreover, as i' is smooth then so is λ. We rewrite (5.3.24) as

$$- < I'(u)(Au), \phi >_{L^2} = \int_{S^1} < \frac{d^2u}{d\sigma^2}(\sigma), \lambda(u(\sigma))\phi(\sigma) > d\sigma.$$

(5.3.23) now follows using integration by parts.

♠

Denoting $u(s, \sigma)$ by $u_s(\sigma)$, we will show that for any $\phi \in C^\infty(S^1, \mathbb{R}^d)$,

$$\begin{aligned} < I(u^n(t \wedge \tau_k)), \phi >_{L^2} \quad \to \quad & \int_0^{t \wedge \tau_k} \int_{S^1} < \nabla u_s(\sigma), \nabla \lambda(u_s(\sigma))\phi(\sigma) > d\sigma ds \\ & + < \mathcal{B}(t \wedge \tau_k), \phi >_{L^2}, \end{aligned}$$

as $n \to \infty$, in $L^q(\Omega, \mathbb{R})$, where the process $\mathcal{B}(t)$, $t < \tilde{\tau}$, is given by

$$\begin{aligned}
\mathcal{B}(t \wedge \tau_k) &= I(u(0)) + \int_0^{t\wedge\tau_k} I'(u(s))F(u(s))ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_k} I'(u(s))\left(tr\{H'(u(s))H(u(s))\}\right)ds \\
&+ \int_0^{t\wedge\tau_k} I'(u(s))\left(H(u(s))\right)dw(s) \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_k} tr\left\{I''(u(s))\left(H(u(s)), H(u(s))\right)\right\}ds.
\end{aligned}$$

By uniqueness of limits we will then have, for $\phi \in C^\infty(S^1, \mathbb{R}^d)$, $t \in [0, \infty)$ and $k \in \mathbb{N}$,

$$\begin{aligned}
< I(u(t\wedge\tau_k)), \phi >_{L^2} &= \int_0^{t\wedge\tau_k}\int_{S^1} < \nabla u_s(\sigma), \nabla\lambda(u_s(\sigma)), \phi(\sigma) > d\sigma ds \\
&+ < \mathcal{B}(t\wedge\tau_k), \phi >_{L^2},
\end{aligned}$$

where equality holds a.s.. This would be the first step in proving our Theorem. Before we formulate the next Lemma let us introduce the following notation

$$\gamma^n(t_k) := \int_0^{t\wedge\tau_k}\int_{S^1} < \nabla u_s^n(\sigma), \nabla\lambda(u_s^n(\sigma))\phi(\sigma) > d\sigma ds.$$

Lemma 5.3.6 *With the above notation: for each $t \in [0, \infty)$ and $k \in \mathbb{N}$,*

$$\gamma^n(t_k) \to \int_0^{t\wedge\tau_k}\int_{S^1} < \nabla u_s(\sigma), \nabla\lambda(u_s(\sigma))\phi(\sigma) > d\sigma ds \tag{5.3.25}$$

as $n \to \infty$, in $L^q(\Omega, L^p(S^1, \mathbb{R}^d))$.

Proof : For $s \in [0, t\wedge\tau_k)$, $u^n(s) \to u(s)$ pointwise in $W^{s,p}(S^1, \mathbb{R}^d)$ a.s.. Moreover, for each $\sigma \in S^1$, we have

$$< \nabla u_s^n(\sigma), \nabla\lambda(u_s^n(\sigma))\phi(\sigma) > \to < \nabla u_s(\sigma), \nabla\lambda(u_s(\sigma))\phi(\sigma) > . \tag{5.3.26}$$

With an application of the LDC Theorem in mind, we wish to find integrable bounds for the terms $| \gamma^n(t_k) |^q$. Note that as i is of compact support, then there exists some constant C such that

$$\sup_x | \lambda(x) | = \sup_x | [i'(x)]^* | = \sup_x | i'(x) | \leq C,$$

where the supremum is taken over $x \in \mathbb{R}^d$. Moreover, $| \phi |_{L^2} \leq C_1$, for some constant C_1, and so

$$\begin{aligned}
\int_{S^1} < \nabla u_s^n(\sigma), \nabla\lambda(u_s^n(\sigma))\phi(\sigma) > d\sigma &= \int_{S^1} < \frac{d^2u_s^n}{d\sigma^2}(\sigma), \lambda(u_s^n(\sigma))\phi(\sigma) > d\sigma \\
&\leq C_2 | Au^n(s) |_{L^2} \\
&\leq C_3 | u(s) |_{L^p} .
\end{aligned}$$

The last inequality follows from

$$| Au^n |_{L^2} \leq | Au^n |_{L^p} \leq | u^n |_{2,p} \leq | u |_{L^p} .$$

In particular, we have

$$\int_0^{t\wedge\tau_k}\int_{S^1} < \nabla u^n(s,\sigma), \nabla\lambda(u^n(s,\sigma))\phi(\sigma) > d\sigma ds \le C_3 \int_0^{t\wedge\tau_k} \mid u(s) \mid_{L^p} ds.$$

Since $\mathbb{E}\left(\int_0^{t\wedge\tau_k} \mid u(s) \mid_{L^p} ds\right)^q < \infty$, the result follows by LDC.

♠

Of the remaining terms the difficult term is

$$I_1^n(t_k) = \int_0^{t\wedge\tau_k} I'(u^n(s))\,(F_n(u(s)) - F(u^n(s)))\,ds.$$

We first prove the following

Lemma 5.3.7 *For each $t \in [0,\infty)$ and $k \in \mathbb{N}$*

$$\mathbb{E}\int_0^{t\wedge\tau_k} \mid I'(u^n(s))\,\{F_n(u(s)) - F(u^n(s))\} \mid_{L^p}^q ds \to 0$$

as $n \to \infty$.

Proof : We will first prove that, for $u \in W^{s,p}(S^1,\mathbb{R}^d)$,

$$\mid I'(u^n)\,\{F_n(u) - F(u^n)\} \mid_{L^p}^q \to 0,$$

where $u^n := n(n+A)^{-1}u$ and $F_n = n(n+A)^{-1}F$, $n \in \mathbb{N}$.
Using (5.1.11) and the fact that $i : \mathbb{R}^d \to \mathbb{R}^d$ is smooth with compact support, then, for $u \in W^{s,p}(S^1,\mathbb{R}^d)$, $v \in L^p(S^1,\mathbb{R}^d)$,

$$\begin{aligned}
\mid I'(u)v \mid_{L^p}^p &= \int_{S^1} \mid i'(u(\sigma))v(\sigma) \mid^p d\sigma \\
&\le \sup_x \mid i'(x) \mid^p \int_{S^1} \mid v(\sigma) \mid^p d\sigma \\
&\le C \mid v \mid_{L^p}
\end{aligned}$$

for some constant C independent of u. (In the second step, we have taken the supremum over $x \in \mathbb{R}^d$.) If $v \in C^\infty(S^1,\mathbb{R}^d)$ approximates $u \in W^{s,p}(S^1,\mathbb{R}^d)$ then $v^n = n(n+A)^{-1}v \in C^\infty(S^1,\mathbb{R}^d)$ approximates $u^n \in W^{s,p}(S^1,\mathbb{R}^d)$. This can be seen from the inequality

$$\mid u^n - v^n \mid_{s,p} \le \mid u - v \mid_{s,p}.$$

Moreover, by the continuity of F, given $\epsilon > 0$, then

$$\mid F(u^n) - F(v^n) \mid_{L^p} + \mid F(u) - F(v) \mid_{L^p} < \epsilon.$$

for u, v sufficiently close. For such u, v it follows that

$$\begin{aligned}
\mid I'(u^n)\,\{F_n(u) - F(u^n)\} \mid_{L^p} &\le \mid F_n(u) - F(u^n) \mid_{L^p} \\
&\le \mid F_n(u) - F_n(v) \mid_{L^p} + \mid F_n(v) - F(v) \mid_{L^p} \\
&+ \mid F(v) - F(v^n) \mid_{L^p} + \mid F(v^n) - F(u^n) \mid_{L^p} \\
&\le \epsilon + \mid F_n(v) - F(v) \mid_{L^p} \\
&\quad + \mid F(v) - F(v^n) \mid_{L^p},
\end{aligned}$$

where the last inequality follows as $| n(n+A)^{-1} |_{L(L^p,L^p)} \leq 1$. Letting n tend to infinity we have

$$\limsup_{n\to\infty} | I'(u^n)\{F_n(u) - F(u^n)\} |_{L^p} \leq \epsilon \text{ for any } \epsilon > 0.$$

We deduce that

$$\lim_{n\to\infty} | I'(u^n)\{F_n(u) - F(u^n)\} |_{L^p} = 0.$$

In particular, for $s \in [0, t\wedge\tau_k)$, $t \in [0,\infty)$ and $k \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} | I'(u^n(s))\{F_n(u(s)) - F(u^n(s))\} |_{L^p}^p = 0 \text{ a.s..}$$

With the LDC Theorem in mind we need to find an integrable bound. Now, by Corollary 3.4.5, we have

$$\begin{aligned} | F_n(u) - F(u^n) |_{L^p}^p &\leq | F_n(u) |_{L^p}^p + | F(u^n) |_{L^p}^p \\ &\leq | F(u) |_{L^p}^p + | F(u^n) |_{L^p}^p \\ &\leq C(p,d)\left(| u |_{s,p}^2 + | u |_{s,p}^3\right) \\ &\quad + C(p,d)\left(| u^n |_{s,p}^2 + | u^n |_{s,p}^3\right) \\ &\leq \tilde{C}(p,d)\left(| u |_{s,p}^2 + | u |_{s,p}^3\right). \end{aligned}$$

Recall that our maximal solution satisfies, for some $q \geq p > 2$,

$$\mathbb{E}\int_0^{t\wedge\tau_k} | u(s) |_{s,p}^q \, ds < \infty.$$

If $q \geq 3$ then we may apply the LDC Theorem and this will complete the proof of Lemma 5.3.7. If $p \geq 3$, then clearly $q \geq 3$. If though $p \in (2,3)$ then, see Theorem 4.3.17, q is automatically chosen so that $q > 4p(p-2)^{-1}$, in particular $q > 3$. Thus for any $p > 2$, q always satisfies $q \geq 3$ and hence we can apply the LDC Theorem to obtain our result.

♠

In view of Remark 5.3.3 we deduce that, for $t \in [0,\infty)$ and $k \in \mathbb{N}$, $< I_1^n(t_k), \phi >_{L^2} \to 0$ as $n \to \infty$ in $L^q(\Omega, \mathbb{R})$.

Using the same line of proof along with the Lipschitz properties of F, H and H', and the continuity of I, I' and I'' one can prove

$$\int_0^{t\wedge\tau_k} I'(u^n(s))(F(u^n(s)))ds \to \int_0^{t\wedge\tau_k} I'(u(s))(F(u(s)))ds$$

$$I_2^n(t_k) \to \frac{1}{2}\int_0^{t\wedge\tau_k} I'(u(s))\left(tr\{H'(u(s))H(u(s))\}\right)ds$$

$$I_4^n(t_k) \to \frac{1}{2}\int_0^{t\wedge\tau_k} tr\left\{I''(u(s))\left(H(u(s)), H(u(s))\right)\right\}ds,$$

where the convergence is in $L^q(\Omega, L^p(S^1, \mathbb{R}^d))$. The term involving the stochastic integral possesses no difficulties. One needs to apply the Burkholder inequality and then, again following along the lines of the proof of Lemma 5.3.7 above, one may deduce that in $L^q(\Omega; L^p(S^1, \mathbb{R}^d))$

$$I_3^n(t_k) \to \int_0^{t\wedge\tau_k} I'(u(s))\,(H(u(s)))\,dw(s).$$

Consequently, see Remark 5.3.3, it will follow that for each $\phi \in C^\infty(S^1, \mathbb{R}^d)$

$$< \int_0^{t\wedge\tau_k} I'(u^n(s))(F(u^n(s)))ds, \phi >_{L^2} \to < \int_0^{t\wedge\tau_k} I'(u(s))(F(u(s)))ds, \phi >_{L^2},$$

$$< I_2^n(t_k), \phi >_{L^2} \to \frac{1}{2} < \int_0^{t\wedge\tau_k} I'(u(s))\,(trH'(u(s))H(u(s)))\,ds, \phi >_{L^2},$$

$$< I_3^n(t_k), \phi >_{L^2} \to < \int_0^{t\wedge\tau_k} I'(u(s))\,(H(u(s)))\,dw(s), \phi >_{L^2},$$

$$< I_4^n(t_k), \phi >_{L^2} \to \frac{1}{2} < \int_0^{t\wedge\tau_k} tr\,\{I''(u(s))\,(H(u(s)), H(u(s)))\}\,ds, \phi >_{L^2}.$$

The fact that $< I(u^n(0)), \phi >_{L^2} \to < I(u(0)), \phi >_{L^2}$ follows from Remarks 5.2.1 and 5.3.3. Thus, in view of the discussion earlier, uniqueness of limits implies that, for $k \in \mathbb{N}$, $t \in [0, \infty)$, the following weak integral equation holds a.s.

$$\begin{aligned} < I(u(t\wedge\tau_k)), \phi >_{L^2} &= < I(u(0)), \phi >_{L^2} \\ &+ \int_0^{t\wedge\tau_k} \int_{S^1} < \nabla u_s(\sigma), \nabla\lambda(u_s(\sigma))(\phi(\sigma)) > d\sigma ds \\ &+ < \int_0^{t\wedge\tau_k} I'(u(s))(F((u(s)))ds, \phi >_{L^2} \\ &+ \frac{1}{2} < I_2(t_k), \phi >_{L^2} \\ &+ < \int_0^{t\wedge\tau_k} I'(u(s))\,(H(u(s)))\,dw(s), \phi >_{L^2} \\ &+ \frac{1}{2} < I_4(t_k), \phi >_{L^2}, \end{aligned} \tag{5.3.27}$$

where we have denoted

$$\begin{aligned} I_2(t_k) &:= \int_0^{t\wedge\tau_k} I'(u(s))\,(tr\{H'(u(s))H(u(s))\})\,ds \\ I_4(t_k) &:= \int_0^{t\wedge\tau_k} tr\,\{I''(u(s))\,(H(u(s)), H(u(s)))\}\,ds. \end{aligned}$$

The following Lemma is crucial. We continue to use the notation of Lemma 5.3.4.

Lemma 5.3.8 *For $u \in W^{s,p}(S^1, U_\epsilon)$ the following equality holds, for any $\phi \in C^\infty(S^1, \mathbb{R}^d)$:*

$$\begin{aligned} \int_{S^1} < \nabla u(\sigma), \nabla\lambda(u(\sigma))\phi(\sigma) > d\sigma &+ < I'(u)F(u), \phi >_{L^2} \\ &= < \nabla I'(u), \nabla\phi >_{L^2} \\ &\quad + < F(I(u)), \phi >_{L^2}. \end{aligned} \tag{5.3.28}$$

Proof : Let $u \in C^\infty(S^1, U_\epsilon)$. Then, using (5.1.9), we have, for any $\phi \in C^\infty(S^1, \mathbb{R}^d)$,

$$< I'(u)\underline{\triangle} u, \phi >_{L^2} = < \underline{\triangle} I(u), \phi >_{L^2} . \tag{5.3.29}$$

Consider the RHS of (5.3.29). We have the following equalities

$$\begin{aligned} < \underline{\triangle} I(u), \phi >_{L^2} &= - < AI(u), \phi >_{L^2} + < F(I(u)), \phi >_{L^2} \\ &= < \nabla I(u), \nabla \phi >_{L^2} + < F(I(u)), \phi >_{L^2} . \end{aligned} \tag{5.3.30}$$

Considering now the LHS of (5.3.29):

$$\begin{aligned} < I'(u)\underline{\triangle} u, \phi >_{L^2} &= - < I'(u)Au, \phi >_{L^2} + < I'(u)F(u), \phi >_{L^2} \\ &= \int_{S^1} < \nabla u(\sigma), \nabla \lambda(u(\sigma))\phi(\sigma) > d\sigma \\ &\quad + < I'(u)F(u), \phi >_{L^2}, \end{aligned} \tag{5.3.31}$$

where the last equality follows from (5.3.23). Putting (5.3.30) and (5.3.31) together gives us (5.3.28) for any $u \in C^\infty(S^1, U_\epsilon)$. To extend this to functions in $W^{s,p}(S^1, U_\epsilon)$, note first that (5.3.28) does actually make sense for functions in $W^{s,p}(S^1, U_\epsilon)$. Furthermore, all the terms in the expression are continuous and so using the density of $C^\infty(S^1, U_\epsilon)$ in $W^{s,p}(S^1, U_\epsilon)$ gives us our result.

♠

As a consequence of Lemma 5.3.8, (5.3.27) now reads

$$\begin{aligned} < I(u(t \wedge \tau_k)), \phi >_{L^2} &= < I(u(0)), \phi >_{L^2} - < \int_0^{t\wedge\tau_k} \nabla I(u(s)) ds, \nabla \phi >_{L^2} \\ &\quad + < \int_0^{t\wedge\tau_k} F(I(u(s))) ds, \phi >_{L^2} \\ &\quad + \frac{1}{2} < I_2(t_k), \phi >_{L^2} \\ &\quad + < \int_0^{t\wedge\tau_k} I'(u(s)) \left(H(u(s))\right) dw(s), \phi >_{L^2} \\ &\quad + \frac{1}{2} < I_4(t_k), \phi >_{L^2}, \end{aligned} \tag{5.3.32}$$

where we recall

$$\begin{aligned} I_2(t_k) &:= \int_0^{t\wedge\tau_k} I'(u(s)) \left(tr\{H'(u(s))H(u(s))\}\right) ds \\ I_4(t_k) &:= \int_0^{t\wedge\tau_k} tr\left\{I''(u(s)) \left(H(u(s)), H(u(s))\right)\right\} ds. \end{aligned}$$

As our Wiener process is $W^{\theta,p}(S^1, \mathbb{R}^d)$-valued we consider H as a map

$$H : W^{\theta,p}(S^1, \mathbb{R}^d) \to L(W^{\theta,p}(S^1, \mathbb{R}^d), W^{\theta,p}(S^1, \mathbb{R}^d)),$$

see Chapter 3. Furthermore, I is a smooth map satisfying

$$I : W^{\theta,p}(S^1, \mathbb{R}^d) \to W^{\theta,p}(S^1, \mathbb{R}^d)$$

and so its derivative I' satisfies

$$I' : W^{\theta,p}(S^1,\mathbb{R}^d) \to L(W^{\theta,p}(S^1,\mathbb{R}^d), W^{\theta,p}(S^1,\mathbb{R}^d)).$$

Define a map $J : W^{\theta,p}(S^1,\mathbb{R}^d) \to L(W^{\theta,p}(S^1,\mathbb{R}^d), W^{\theta,p}(S^1,\mathbb{R}^d))$ by

$$J(u) := I'(u)H(u).$$

J is a well-defined smooth map. An application of the chain rule gives

$$J'(u)(\cdot) = I''(u)(\cdot)H(u) + I'(u)H'(u)(\cdot), \quad u \in W^{s,p}(S^1,\mathbb{R}^d). \tag{5.3.33}$$

If $u \in W^{s,p}(S^1, U_\epsilon)$ then $J(u) = H(I(u))$, using (5.1.8), and so in this case we have

$$J'(u)(\cdot) = H'(I(u))I'(u)(\cdot). \tag{5.3.34}$$

It then follows from (5.3.33) and (5.3.34) that

$$tr\left\{I''(u)(H(u))H(u) + I'(u)H'(u)(H(u))\right\} = tr\left\{H'(I(u))I'(u)(H(u))\right\}.$$

Using (5.1.8) again we get

$$tr\left\{I''(u)(H(u))H(u) + I'(u)H'(u)(H(u))\right\}. = tr\left\{H'(I(u))H(I(u))\right\}. \tag{5.3.35}$$

Finally, from the definition of the map tr and the fact that $I'(u)$, $u \in W^{s,p}(S^1,\mathbb{R}^d)$, is a bounded linear map, we have

$$I'(u)tr\left\{H'(u)H(u)\right\} = tr\left\{I'(u)H'(u)H(u)\right\}. \tag{5.3.36}$$

Using (5.3.32), (5.3.35) and (5.3.36), we have, for $k \in \mathbb{N}$, $t \in [0,\infty)$ and $\phi \in C^\infty(S^1,\mathbb{R}^d)$, the following integral equation holding a.s.

$$\begin{aligned} < I(u(t\wedge\tau_k)),\phi >_{L^2} &= < I(u(0)),\phi >_{L^2} - < \int_0^{t\wedge\tau_k} \nabla I(u(s))ds, \nabla\phi >_{L^2} \\ &\quad + < \int_0^{t\wedge\tau_k} F(I(u(s))ds, \phi >_{L^2} \\ &\quad + < \int_0^{t\wedge\tau_k} H(I(u(s)))dw(s), \phi >_{L^2} \\ &\quad + \frac{1}{2} < \int_0^{t\wedge\tau_k} tr\left\{H'(I(u(s)))H(I(u(s)))\right\} ds, \phi >_{L^2}. \end{aligned}$$

Writing this in Stratonovich form gives

$$\begin{aligned} < I(u(t\wedge\tau_k)),\phi >_{L^2} &= < I(u(0)),\phi >_{L^2} + < \int_0^{t\wedge\tau_k} F(I(u(s))ds, \phi >_{L^2} \\ &\quad + < \int_0^{t\wedge\tau_k} H(I(u(s))) \circ dw(s), \phi >_{L^2}, \end{aligned}$$

which is the required weak integral equation (5.3.18). This concludes the proof of Theorem 5.3.2.

♠

5.4 Existence Of An $\mathcal{M}$-valued Solution

The following Theorem states that $u(t)$, $t < \tau$, is a weak solution to the problem (5.3.14), (5.3.15). We only give a sketch proof, as this proof is similar and simpler to that of Theorem 5.3.2.

Theorem 5.4.1 *$u(t)$, $t < \tau$, is a weak solution to the problem* (5.3.14), (5.3.15).

Proof : As in the previous proof one first shows that, for each $\phi \in C^\infty(S^1, \mathbb{R}^d)$, $t \in [0, \infty)$ and $k \in \mathbb{N}$,

$$< u^n(t \wedge \sigma_k), \phi >_{L^2} \to < u(t \wedge \sigma_k), \phi >_{L^2} \quad \text{in } L^1(\Omega, \mathbb{R}).$$

Then using the fact that the approximations satisfy a strict integral equation we may write, for each $k \in \mathbb{N}$, $t \in [0, \infty)$,

$$\begin{aligned}
< u^n(t \wedge \sigma_k), \phi >_{L^2} &= < u^n(0), \phi >_{L^2} - < \int_0^{t\wedge\sigma_k} \nabla u^n(s) ds, \nabla \phi >_{L^2} \\
&\quad + < \int_0^{t\wedge\sigma_k} F(u^n(s)) ds, \phi >_{L^2} \\
&\quad + < \int_0^{t\wedge\sigma_k} \{F_n(u(s)) - F(u^n(s))\} ds, \phi >_{L^2} \\
&\quad + < \int_0^{t\wedge\sigma_k} H_n(u(s)) dw(s), \phi >_{L^2} \\
&+ \frac{1}{2} < \int_0^{t\wedge\sigma_k} tr_n\{H'(u(s))H(u(s))\} ds, \phi >_{L^2} \\
&\to < u(0), \phi >_{L^2} - < \int_0^{t\wedge\sigma_k} \nabla u(s) ds, \nabla \phi >_{L^2} \\
&\quad + < \int_0^{t\wedge\sigma_k} F((u(s)) ds, \phi >_{L^2} \\
&\quad + < \int_0^{t\wedge\sigma_k} H(u(s)) dw(s), \phi >_{L^2} \\
&\quad + \frac{1}{2} < \int_0^{t\wedge\sigma_k} tr\{H'(u(s))H(u(s))\} ds, \phi >_{L^2} .
\end{aligned}$$

The proof of the above is straightforward as we do not have to concern ourselves with the map I and so the results of Lemmas 5.3.4, 5.3.6 and 5.3.8 do not apply in this case. Note that the above convergence is in $L^1(\Omega; \mathbb{R})$ and so by uniqueness of limits we may infer that $u(t)$, $t < \tau$, satisfies the required weak integral equation, see (5.3.16), and hence is a weak solution.

♠

Corollary 5.4.2 *If $u_0 \in L^q(\Omega; W^{s,p}(S^1, M))$ then the maximal solution $u(t)$, $t < \tau$, is an admissible $\mathcal{M}$-valued process.*

Proof : Due to the conditions imposed on $u(0) = u_0$ we have $v(0) = I(u(0)) = u(0)$, see Proposition 5.1.1. Theorems 5.3.2 and 5.4.1 imply that, on $[0, \tilde{\tau})$, u and v satisfy the same weak integral equation with the same initial condition. As u is the unique mild maximal solution, then, see [Kr/Ro,79], it is also the unique weak solution. Thus we must have $u = v = I(u)$ on $[0, \tilde{\tau})$. This holds if and only if $u \in \mathcal{M}$ on $[0, \tilde{\tau})$.

It is straightforward to show that $\tilde{\tau} = \tau$. Indeed, suppose $P(\tilde{\tau} < \tau) > 0$. Then, for $\omega \in \{\tilde{\tau} < \tau\}$, $u(\tilde{\tau}(\omega))$ is well-defined and by the continuity of u, $u(\tilde{\tau}) \in W^{s,p}(S^1, U_{\frac{\epsilon}{2}})$, which lies strictly inside the set $W^{s,p}(S^1, U_\epsilon)$, see (5.1.2). This, though, contradicts the definition of $\tilde{\tau}$, see (5.3.17). So we must have $P(\tilde{\tau} < \tau) = 0$ i.e. $P(\tilde{\tau} = \tau) = 1$. We may therefore conclude that our maximal solution $u(t)$, $t < \tau$, is an $\mathcal{M}$-valued process.

♠

Chapter 6

Existence of a Global Solution on the Loop Manifold $\mathcal{M}$

6.1 Introduction

We are now concerned with proving that our maximal $\mathcal{M}$-valued solution is in fact a global solution. As previously mentioned, general results on global existence require the coefficients to satisfy both local Lipschitz and linear growth conditions, see [Br,97], for example. Recall that the term F is not of linear growth. As in the case of deterministic PDEs one needs to consider different methods depending on the particular problem studied. For example, in [Br/Ga,98], where they consider stochastic reaction-diffusion equations on Banach spaces, they prove globality of solution by using certain dissipativity properties of the drift term. As another example, see [Ca/Cu,91], [Fl/Ga,95], where they consider stochastic Navier-Stokes equations, they use the fact that the drift term satisfies some orthogonality condition to prove global existence of a solution.

The method we employ is motivated (again!) by the works of Eells and Sampson. We calculate certain energy estimates for our maximal solution. These estimates play the rôle of the linear growth condition for F and ensure that the norm of our solution does not 'explode', i.e. the solution is global.

Notation : Throughout we adopt the following notation. $W^{\varrho,p} := W^{\varrho,p}(S^1, \mathbb{R}^d)$ and we denote the norm on this space by $|\cdot|_{\varrho,p}$. We denote the norm on $L^p(S^1, \mathbb{R}^d)$ by $|\cdot|_{L^p_d}$ and that on $L^p(S^1, \mathbb{R})$ by $|\cdot|_{L^p}$. For a map u on S^1 we will sometimes write $u_\sigma := u(\sigma)$ and $\nabla_\sigma u := (\nabla u)(\sigma)$, for $\sigma \in S^1$. $\Diamond$

♡

6.2 Some Fundamental Lemmas

First we recall how F is defined: for suitable $u : S^1 \to \mathbb{R}^d$ and $\sigma \in S^1$

$$F(u)(\sigma) := \sum_{i,j=1}^{d} \left(\Gamma^1_{ij}(u(\sigma))\frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma), \ldots, \Gamma^d_{ij}(u(\sigma))\frac{du^i}{d\sigma}(\sigma)\frac{du^j}{d\sigma}(\sigma) \right).$$

We have the following fundamental Lemma.

Lemma 6.2.1 *For $u \in W^{s,p}$ with $s > 1 + \frac{1}{p}$,*

$$| F(u) |_{L^p_d}^p \leq C(p,d) | \nabla u |_{L^{2p}_d}^{2p} \tag{6.2.1}$$

for some constant $C(p,d)$ independent of u.

Proof : As $s > 1 + \frac{1}{p}$, then $\nabla u \in C(S^1, \mathbb{R}^d)$ and so in particular $\nabla u \in L^{2p}(S^1, \mathbb{R}^d)$. On $\mathbb{R}^d$ we use the norm $| x | = \sum_{i=1}^d | x_i |$ for $x = (x_1, \ldots, x_d)$. For $\sigma \in S^1$ we have

$$| F(u)(\sigma) | = \sum_{k=1}^d | \sum_{i,j=1}^d \Gamma_{ij}^k(u(\sigma)) \frac{du^i}{d\sigma}(\sigma) \frac{du^j}{d\sigma}(\sigma) | .$$

For $i, j, k = 1, \ldots, d$, $\Gamma_{ij}^k : \mathbb{R}^d \to \mathbb{R}$, are smooth with compact support and so

$$\sup_{\sigma \in S^1} | \Gamma_{ij}^k(u(\sigma)) | \leq \sup_{x \in \mathbb{R}^d} | \Gamma_{ij}^k(x) | \leq M_{ijk} \leq C$$

where $C = \max\{M_{ijk} : i, j, k = 1, \ldots, d\}$ and is independent of u. Thus

$$\begin{aligned} | F(u)(\sigma) | &\leq C(d) \sum_{i,j=1}^d | \frac{du^i}{d\sigma}(\sigma) || \frac{du^j}{d\sigma}(\sigma) | \\ &= C(d) \sum_{i=1}^d \left(\sum_{j=1}^d | \frac{du^j}{d\sigma}(\sigma) | \right) | \frac{du^i}{d\sigma}(\sigma) | \\ &= C(d) | \nabla u(\sigma) | \sum_{i=1}^d | \frac{du^i}{d\sigma}(\sigma) | = C(d) | \nabla u(\sigma) |^2 . \end{aligned}$$

Thus for some constant $C(p,d)$ independent of u we have

$$\int_{S^1} | F(u)(\sigma) |^p \, d\sigma \leq C(p,d) \int_{S^1} | \nabla u(\sigma) |^{2p} \, d\sigma.$$

♠

Definition 6.2.2 *The energy density of a function $u : S^1 \to \mathbb{R}^d$ of C^1-class, is a real valued function $e(u) : S^1 \to \mathbb{R}$ defined by*

$$e(u)(\sigma) := \frac{1}{2} g(u(\sigma))(\nabla_\sigma u, \nabla_\sigma u), \ \ \sigma \in S^1, \tag{6.2.2}$$

where g is the metric on $\mathbb{R}^d$ as constructed in Chapter 3.

The following inequality gives us the estimate we require for our term F and the result is particular to our metric.

Corollary 6.2.3 *For $u \in W^{s,p}$, $s > 1 + \frac{1}{p}$,*

$$| F(u) |_{L^p_d} \leq C(p,d,R) | e(u) |_{L^p} \tag{6.2.3}$$

where $C(p,d,R)$ is a constant independent of u. $R > 0$ depends on the manifold M.

Proof: Recall that our metric g is a smooth function which coincides with the Euclidean metric outside some closed ball, $B(0,R) \subset \mathbb{R}^d$, of radius R, which contains both the manifold M and its tubular neighbourhood U. It follows that

$$g(x)(\xi,\xi) = <\xi,\xi>, \ \ \xi \in \mathbb{R}^d, \ \ x \in B(0,R)^c. \tag{6.2.4}$$

Now as g is smooth it attains its minimum on the ball $B(0,R)$, i.e. there exists $y \in B(0,R)$ such that

$$g(y)(\xi,\xi) \leq g(x)(\xi,\xi), \ \ x \in B(0,R), \ \ \xi \in \mathbb{R}^d. \tag{6.2.5}$$

As g is a metric on $\mathbb{R}^d$ and y depends on R, there exists a constant $C(d,R) > 0$ such that

$$|\ \xi\ |^2 \leq C(d,R)g(x)(\xi,\xi), \ \ x \in B(0,R), \ \ \xi \in \mathbb{R}^d. \tag{6.2.6}$$

Putting (6.2.4), (6.2.5) and (6.2.6) together, we deduce that

$$|\ \xi\ |^2 \leq C(d,R)g(x)(\xi,\xi), \ \ x,\xi \in \mathbb{R}^d. \tag{6.2.7}$$

Using (6.2.7) we have

$$|\ \nabla_\sigma u\ |^2 \leq C(d,R)g(u(\sigma))(\nabla_\sigma u, \nabla_\sigma u). \tag{6.2.8}$$

The result now follows from (6.2.8), (6.2.2) and (6.2.1).

♠

The following result will play an important rôle.

Corollary 6.2.4 *For* $u \in W^{s,p}(S^1,M)$, $s > 1 + \frac{1}{p}$,

$$|\ u\ |_{1,2p}^{2p} \ \leq C(p,d,R)\,(1+ |\ e(u)\ |_{L^p}^p) \tag{6.2.9}$$

where $C(p,d,R)$ *is a constant independent of* u. $R > 0$ *depends on the manifold* M *and* $|\cdot|_{1,2p}$ *is the norm on the space* $W^{1,2p}$.

Proof : Note first that $|\ u\ |_{1,2p}^{2p} \ = \ |\ u\ |_{L_d^{2p}}^{2p} + |\ \nabla u\ |_{L_d^{2p}}^{2p}$. As $W^{s,p}(S^1,M)$, then the range of u is contained in M. It follows that

$$|\ u\ |_{L^{2p}}^{2p} \ \leq C(p,R),$$

where $C(p,R)$ is a constant independent of u and depends on the manifold M. The result now follows as in Corollary 6.2.3.

♠

The estimates (6.2.3) and (6.2.9) will be essential in proving the following result:

- For the energy process $e(u(t))$, $t < \tau$, where $u(t)$, $t < \tau$, is our maximal solution, we have, for each $t \in [0,T)$ and $k \in \mathbb{N}$, the following estimate

$$\mathbb{E}\ |\ e(u(t\wedge\sigma_k))\ |_{L^p}^p \leq C(p,d,T)\left\{\mathbb{E}\ |\ e(u(0))\ |_{L^p(S^1,\mathbb{R})}^p + 1\right\} e^{C(p,d,T)} \tag{6.2.10}$$

  where the constant $C(p,d,T)$ is independent of k and u and $\{\sigma_k\}$ is an increasing sequence of bounded stopping times with $\sigma_k < \tau$ and $\sigma_k \to \tau$.

This is the fundamental inequality needed to prove global existence of our solution. The proof of the estimate is quite lengthy and technical. So as not to lose sight of our goal, i.e. that of global existence, we first assume the above estimate and show how to use it to obtain our result. The estimate is then proved in section 4 of this chapter.

♡

6.3 Existence Of A Global Solution

Theorem 6.3.1 *Given any $T > 0$, fixed but arbitrary, then our maximal solution is defined on [0, T], i.e. our solution is a global solution.*

Proof : To prove the Theorem we just need to show that $\tau = T$ a.s.. We follow the method used in [Br/El,98] where in place of their linear growth condition we use the energy estimate above. To ease notation we will ignore constants unless they depend on t or k. C will denote a generic constant. We begin with quoting the following result from [Br/El,98]:

Proposition 6.3.2 *Suppose that a.s on a measurable set $\Omega_1 \subset \mathcal{F}_0$, $\tau < T$ and $\limsup_{t\nearrow\tau} | u(t) |_{s,p} < \infty$ then $\mathbb{P}(\Omega_1) = 0$.*

♠

For each $k \in \mathbb{N}$ define $\tau_k := \inf\{t < \tau :| u(t) |_{s,p} > k\}$, where $| \cdot |_{s,p}$ is the norm on $W^{s,p}$. Proposition 6.3.2 implies τ_k is a well-defined stopping time. Moreover, in view of Remark 5.2.2, our maximal solution satisfies the following Itô integral equation:

$$
\begin{aligned}
u(t \wedge \tau_k) &= e^{-(t\wedge\tau_k)A}u(0) + \int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A}\left(F(u(s)) + V(u(s))\right) ds \\
&\quad + \int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A} H(u(s))dw(s) \\
&\quad + \frac{1}{2}\int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A} tr\{H'(u(s))H(u(s))\}ds
\end{aligned}
$$

for each $t \in [0, T)$ and $k \in \mathbb{N}$, a.s.. Henceforth we omit the terms V and $\frac{1}{2}tr(H'H)$ to ease notation. Using the triangle inequality, we may write

$$
\begin{aligned}
\mathbb{E} | u(t \wedge \tau_k) |_{s,p}^p &\leq C\mathbb{E} | e^{-(t\wedge\tau_k)A}u(0) |_{s,p}^p \\
&\quad + C\mathbb{E} \left| \int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A} F(u(s))ds \right|_{s,p}^p \\
&\quad + C\mathbb{E} \left| \int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A} H(u(s))dw(s) \right|_{s,p}^p . \qquad (6.3.11)
\end{aligned}
$$

Using the estimate of Theorem 4.3.2 and (6.2.3) we have

$$
\begin{aligned}
\left| \int_0^{t\wedge\tau_k} e^{-(t\wedge\tau_k-s)A} F(u(s))ds \right|_{s,p}^p &\leq C_T \int_0^{t\wedge\tau_k} | F(u(s)) |_{L_d^p} ds \\
&\leq C_T \int_0^{t\wedge\tau_k} | e(u(s)) |_{L^p}^p ds. \qquad (6.3.12)
\end{aligned}
$$

Recall that the semigroup $\left\{e^{-tA}\right\}_{t\geq 0}$ may be viewed as a semigroup on $W^{s,p}$, see Chapter 3. In particular for any $t \in [0, T)$ and $k \in \mathbb{N}$

$$
| e^{-(t\wedge\tau_k)A}u |_{s,p}^p \leq C(T) | u |_{s,p}^p, \; u \in W^{s,p}. \qquad (6.3.13)
$$

Applying (6.3.13) and (6.3.12) to (6.3.11), followed by the Burkholder inequality and the linear growth property of H, we obtain the following sequence of inequalities, which hold for $t \in [0,T)$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E} \mid u(t\wedge\tau_k) \mid_{s,p}^p &\leq C(T)\mathbb{E} \mid u(0) \mid_{s,p}^p + C(T)\mathbb{E}\int_0^{t\wedge\tau_k} \mid e(u(s)) \mid_{L^p}^p ds \\
&\quad + C(T)T^{1-\frac{p}{2}}\mathbb{E}\int_0^{t\wedge\tau_k} \mid H(u(s)) \mid_{L^p_d}^p ds \\
&\leq C(T)\mathbb{E} \mid u(0) \mid_{s,p}^p + C(T)\mathbb{E}\int_0^{t\wedge\tau_k} \mid e(u(s\wedge\tau_k)) \mid_{L^p}^p ds \\
&\quad + C(T)T^{1-\frac{p}{2}}\mathbb{E}\int_0^{t\wedge\tau_k} 1+ \mid u(s) \mid_{s,p}^p ds \\
&\leq C(T)\mathbb{E} \mid u(0) \mid_{s,p}^p + C(T)\mathbb{E}\int_0^{t} \mid e(u(s\wedge\tau_k)) \mid_{L^p}^p ds \\
&\quad + C(T) + C(T)T^{1-\frac{p}{2}}\mathbb{E}\int_0^{t} \mid u(s\wedge\tau_k) \mid_{s,p}^p ds.
\end{aligned}$$

Using the estimate (6.2.10) it then follows that

$$\mathbb{E} \mid u(t\wedge\tau_k) \mid_{s,p}^p \leq \tilde{C}(T) + C(T)\int_0^t \mathbb{E} \mid u(s\wedge\tau_k) \mid_{s,p}^p ds$$

where $\tilde{C}(T) = C(T)\{\mathbb{E} \mid e(u(0)) \mid_{L^p}^p +1\}e^{C(T)}$. So by Gronwall we have, for each $t \in [0,T)$ and $k \in \mathbb{N}$,

$$\mathbb{E} \mid u(t\wedge\tau_k) \mid_{s,p}^p \leq \tilde{C}(T)e^{C(T)}. \tag{6.3.14}$$

Note that for fixed $t \in [0,T)$

$$\mathbb{E} \mid u(t\wedge\tau_k) \mid_{s,p}^p = \int_{\{t>\tau_k\}} \mid u(\tau_k) \mid_{s,p}^p d\mathbb{P} + \int_{\{t\leq\tau_k\}} \mid u(t) \mid_{s,p}^p d\mathbb{P}.$$

This implies $k^p\mathbb{P}(t>\tau_k) \leq \mathbb{E} \mid u(t\wedge\tau_k) \mid_{s,p}^p$, which in conjunction with (6.3.14) gives

$$\mathbb{P}(t>\tau_k) \leq \frac{\tilde{C}(T)e^{C(T)}}{k^p}.$$

Note though that as $\tau_k < \tau$ a.s. for each k then $\{t>\tau\} \subset \{t>\tau_k\}$. So we have for each $k \in \mathbb{N}$

$$\mathbb{P}(t>\tau) \leq \frac{\tilde{C}(T)e^{C(T)}}{k^p}.$$

This implies, by letting $k \to \infty$, $\mathbb{P}\{t>\tau\} = 0$ and so $t \leq \tau$ a.s.. This holds for any $t \in [0,T)$ and so we deduce that $\tau = T$ a.s.. This completes the proof.

♠

It thus follows from Theorem 6.3.1 that on any time interval $[0,T]$, $T < \infty$ there exists a unique mild solution to the Stochastic Nonlinear Heat Equation, (SNHE), with our solution belonging to the loop manifold $\mathcal{M} = W^{s,p}(S^1, M)$, $s > 1 + \frac{1}{p}$.

♡

6.4 Calculation Of The Energy Estimates

We are left with proving the following theorem.

Theorem 6.4.1 *For the energy process $e(u(t))$, $t < \tau$, where $u(t)$, $t < \tau$, is our maximal solution, we have, for each $t \in [0,T)$ and $k \in \mathbb{N}$, the following estimate*

$$\mathbb{E} \mid e(u(t \wedge \sigma_k)) \mid_{L^p}^p \leq C(p,d,T) \{\mathbb{E} \mid e(u(0)) \mid_{L^p}^p + 1\} e^{C(p,d,T)} \tag{6.4.15}$$

where the constant $C(p,d,T)$ is independent of k and u.

Proof : We briefly explain the idea behind the proof. We need to somehow calculate, using Itô formula, an expression for the process $e(u(t))$, $t < \tau$ and then prove that the estimate (6.4.15) holds. The problem is that our maximal solution only satisfies a mild integral equation which involves the semigroup operators e^{-tA}, $t \geq 0$ and this is very restrictive for calculating the estimate (6.4.15). To overcome this difficulty, we approximate our maximal solution with a sequence of processes, $\{u_n(t)\}_{n \geq 1}$, $t < \tau$, which are of a higher regularity in the space variable. These approximation processes satisfy a strict integral equation. We calculate the energy of these processes using the Itô formula. As they are strict solutions the integral equations they satisfy do not involve the semigroup operators. We then calculate estimates for the energy of the approximation processes, using results of Eells and Sampson. These estimates are similar to the estimate (6.4.15) above. (6.4.15) follows by taking limits as $n \to \infty$. The proof will consist of a number of steps.

Step 1: We use the same approximation processes as in Chapter 5, i.e. for each $n \in \mathbb{N}$, $u^n(t) = n(n+A)^{-1}u(t)$, $t < \tau$. For each n, $u^n(t)$, $t < \tau$, is an admissible $W^{s+2,p}$-valued process. In particular, as $u^n(t) \in D(A)$ a.s., then for $t \in [0,T)$ and $k \in \mathbb{N}$, u^n satisfies

$$\begin{aligned} u^n(t \wedge \sigma_k) &= u^n(0) - \int_0^{t\wedge\sigma_k} Au^n(s)ds + \int_0^{t\wedge\sigma_k} F_n(u(s))ds \\ &\quad + \int_0^{t\wedge\sigma_k} H_n(u(s))dw(s) \text{ a.s.,} \end{aligned}$$

where $F_n = n(n+A)^{-1}F$ and $H_n = n(n+A)^{-1}H$.

Step 2: We now wish to obtain integral equations satisfied by the sequence of energy processes $e(u^n(t))$, $t < \tau$. In the subsequent, we will let u, v and w will denote elements of $W^{s,p}$.

The metric g on $\mathbb{R}^d$ is a smooth map $g : \mathbb{R}^d \to \mathcal{L}_2^s(\mathbb{R}^d; \mathbb{R})$. The Nemytski map G of g, defined by $G(u) := g \circ u$, is a smooth map satisfying

$$G : W^{s,p} \to W^{s,p}(S^1, \mathcal{L}_2^s(\mathbb{R}^d; \mathbb{R})),$$

where $\mathcal{L}_2^s$ is the space of bilinear symmetric maps. The first and second derivatives, G' and G'', act through the following formulas:

$$G'(u)(v)(\sigma) = g'(u(\sigma))(v(\sigma))(\cdot,\cdot), \tag{6.4.16}$$

$$\begin{aligned} G''(u)(v,w)(\sigma) &= g''(u(\sigma))(v(\sigma),w(\sigma))(\cdot,\cdot) + 2g'(u(\sigma))(v(\sigma))(\cdot,\cdot) \\ &\quad + 2g'(u(\sigma))(w(\sigma)(\cdot,\cdot) + 2g(u(\sigma))(\cdot,\cdot), \end{aligned} \tag{6.4.17}$$

where u, v and $w \in W^{s,p}$ and $\sigma \in S^1$. Note that for each $\sigma \in S^1$, $G'(u)(v)(\sigma)$ and $G''(u)(v,w)(\sigma)$ belong to $\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})$. We omit the proofs of (6.4.16) and (6.4.17) which are quite lengthy, referring the reader to [Br/El,98] where calculations of a similar nature are carried out. Define the map ψ on $W^{s,p}$ through the formula

$$\psi(u)(\sigma) := e(u)(\sigma) = \frac{1}{2}g(u(\sigma))(\nabla_\sigma u, \nabla_\sigma u), \quad \sigma \in S^1. \tag{6.4.18}$$

The following Lemma will be needed. It is important as it gives us explicit formulae for the first and second derivates of ψ.

Lemma 6.4.2 *For $s > 1 + \frac{1}{p}$, ψ is a smooth map*

$$\psi : W^{s,p} \to W^{s-1,p}(S^1, \mathbb{R}).$$

Moreover its derivatives, ψ' and ψ'', act through the following formulas

$$\psi'(u)(v)(\sigma) = g'(u_\sigma)(v_\sigma))(\nabla_\sigma u, \nabla_\sigma u) + 2g(u_\sigma)(\nabla_\sigma u, \nabla_\sigma v), \tag{6.4.19}$$

$$\begin{aligned}\psi''(u)(v,w)(\sigma) &= g''(u_\sigma)(v_\sigma, w_\sigma)(\nabla_\sigma u, \nabla_\sigma u) + 2g'(u_\sigma)(v_\sigma)(\nabla_\sigma w, \nabla_\sigma u)\\ &+ 2g'(u_\sigma)(w_\sigma)(\nabla_\sigma v, \nabla_\sigma u) + 2g(u_\sigma)(\nabla_\sigma v, \nabla_\sigma w),\end{aligned} \tag{6.4.20}$$

where u, v and $w \in W^{s,p}$ and $\sigma \in S^1$.

Proof : $\nabla : W^{s,p} \to W^{s-1,p}$ is a bounded linear map, see Chapter 3. As G is smooth it follows that the map

$$\mathcal{A} : W^{s,p} \to W^{s,p}(S^1, \mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})) \times W^{s-1,p} \times W^{s-1,p},$$

given by $\mathcal{A}(u) = (\frac{1}{2}G(u), \nabla u, \nabla u)$, is smooth. Define the trilinear map

$$\Gamma : W^{s,p}(S^1, \mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})) \times W^{s-1,p} \times W^{s-1,p} \to W^{s-1,p}(S^1, \mathbb{R})$$

by

$$\Gamma(L, x, y)(\sigma) = L(\sigma)\,(x(\sigma), y(\sigma)), \text{ for } \sigma \in S^1. \tag{6.4.21}$$

As $s - 1 > \frac{1}{p}$ then, see [Am,91], Γ is bounded and hence smooth. Finally note that $\psi = \Gamma \circ \mathcal{A}$ and so ψ is smooth. The first and second derivatives of ψ satisfy

$$\begin{aligned}\psi' &: W^{s,p} \to L(W^{s,p}; W^{s-1,p}(S^1, \mathbb{R})),\\ \psi'' &: W^{s,p} \to \mathcal{L}_2(W^{s,p}; W^{s-1,p}(S^1, \mathbb{R})).\end{aligned}$$

We now prove (6.4.19). Using the chain rule we have

$$\psi'(u)v = \Gamma'(\mathcal{A}(u))\mathcal{A}'(u)(v).$$

Recalling that ∇ is linear and bounded then $\mathcal{A}'(u)(v) = (G'(u)(v), \nabla v, \nabla v)$. Moreover as Γ is a bounded trilinear map we have, see [Ca,71],

$$\psi'(u)(v) = \Gamma(G(u), \nabla v, \nabla u) + \Gamma(G(u), \nabla u, \nabla v) + \Gamma(G'(u)(v), \nabla u, \nabla u). \tag{6.4.22}$$

Now (6.4.22), (6.4.21) and (6.4.16) together give us (6.4.19). The proof of (6.4.20) follows in an identical manner.

♠

For each $n \in \mathbb{N}$, define the process $e_n(t)$, $t < \tau$, by

$$e_n(t) := \psi(u^n(t)), \;\; t < \tau. \tag{6.4.23}$$

From Lemma 6.4.2, with s replaced by $s+2$, we deduce that $e_n(t)$, $t < \tau$, is an admissible $W^{s+1,p}$-valued process. Moreover, by applying the Itô formula we have for each $k \in \mathbb{N}$ and $t \in [0,T)$,

$$\begin{aligned} e_n(t \wedge \sigma_k) &= e_n(0) + \int_0^{t\wedge\sigma_k} \psi'(u^n(s))[F_n(u(s)) - Au^n(s)]ds \\ &+ \int_0^{t\wedge\sigma_k} \psi'(u^n(s))H_n(u(s))dw(s) \\ &+ \frac{1}{2}\int_0^{t\wedge\sigma_k} tr\{\psi''(u^n(s))\,(H_n(u(s)), H_n(u(s)))\}ds. \end{aligned} \tag{6.4.24}$$

We aim to obtain an estimate for $\mathbb{E} \mid e_n(t \wedge \sigma_k) \mid_{L^p}^p$. The difficulty lies with the integrand $\psi'(u^n(s))[F_n(u(s)) - Au^n(s)]$, where for $u \in W^{s,p}$, with $u^n = n(n+A)^{-1}u$,

$$\psi'(u^n)[F_n(u) - Au^n] = \psi'(u^n)[n(n+A)^{-1}F(u) - An(n+A)^{-1}u].$$

This may be rewritten as

$$\begin{aligned} \psi'(u^n)[F_n(u) - Au^n] &= \psi'(u^n)[F_n(u) - F(u^n)] \\ &+ \psi'(u^n)[F(u^n) - Au^n]. \end{aligned} \tag{6.4.25}$$

In view of the fact that we will be taking limits as $n \to \infty$ we leave the first term on the RHS for the moment. To deal with the second term we have the following crucial Lemma, which relies on a result from [Ee/Sa,64], see also [Ot,84]. We first state this result.

Theorem 6.4.3 *For each $u \in C^\infty(S^1, \mathbb{R}^d)$ there exists a unique solution $f : [0,\infty) \times S^1 \to \mathbb{R}^d$ to the deterministic nonlinear heat equation*

$$\frac{\partial f_s}{\partial s}(\sigma) = \underline{\Delta} f_s(\sigma) \tag{6.4.26}$$

with $f_0 = u$, where we have denoted $f_s(\cdot) := f(s,\cdot)$, $s \geq 0$. Moreover the following equality holds

$$\frac{\partial e(f_s)}{\partial s}(\sigma) = \Delta e(f_s)(\sigma) - \mid \underline{\Delta} f_s(\sigma) \mid^2 \tag{6.4.27}$$

for $s > 0$, $\sigma \in S^1$.

Remark 6.4.4 In Theorem 6.4.3, Δ denotes the Laplacian acting on real-valued loops, i.e. $y : S^1 \to \mathbb{R}$, see Chapter 3.

Lemma 6.4.5 *For $u \in W^{\eta,p}$, $\eta > 3 + \frac{1}{p}$,*

$$\psi'(u)(\sigma)[(-A + F)(u)](\sigma) = \Delta e(u)(\sigma) - \mid \underline{\Delta} u(\sigma) \mid^2, \;\; \sigma \in S^1. \tag{6.4.28}$$

Remark 6.4.6 Note that the terms on both sides of the equality (6.4.28) belong at least to $W^{\eta-3,p}$ and hence, as $\eta > 3 + \frac{1}{p}$, are continuous. It thus makes sense to talk about equality holding for all $\sigma \in S^1$. $\diamondsuit$

Proof : Let $v \in C^\infty(S^1, \mathbb{R}^d)$ be given and let f_s be the solution to (6.4.26) with initial value $f_0 = v$. For $s > 0$

$$\frac{\partial e(f_s)}{\partial s} = \frac{\partial \psi(f_s)}{\partial s} = \psi'(f_s)\frac{\partial f_s}{\partial s} = \psi'(f_s)\underline{\Delta} f_s = \psi'(f_s)(-A+F)(f_s). \tag{6.4.29}$$

Using (6.4.27) and (6.4.29) we have for $\sigma \in S^1$ and $s > 0$

$$\psi'(f_s)(\sigma)[(-A+F)(f_s)](\sigma) = \Delta e(f_s)(\sigma) - \mid \underline{\Delta} f_s(\sigma) \mid^2 . \tag{6.4.30}$$

Now as f is a continuous function in time then all the terms in (6.4.30) are. So by continuity (6.4.30) will hold for $s = 0$. So, recalling that $f_0 = v$, we have

$$\psi'(u)(\sigma)[(-A+F)(u)](\sigma) = \Delta e(u)(\sigma) - \mid \underline{\Delta} u(\sigma) \mid^2 .$$

As $C^\infty(S^1, \mathbb{R}^d)$ is dense in $W^{\eta,p}$, then (6.4.28) holds for all $u \in W^{\eta,p}$.

♠

Before proceeding we note that $e_n(t)$ is a $W^{s+1,p}(S^1, \mathbb{R})$-valued process and so in particular $e_n(t)$ is a $D(\Delta)$-valued process where Δ is the Laplacian on $L^p(S^1, \mathbb{R})$ with $D(\Delta) = W^{2,p}(S^1, \mathbb{R})$. Thus $\Delta e_n(t)$ is well-defined and moreover, for $\eta \geq 2$, $\Delta : W^{\eta,p}(S^1, \mathbb{R}) \to W^{\eta-2,p}(S^1, \mathbb{R})$ is linear and bounded. As a result the process $\Delta e_n(t)$, $t < \tau$, is pathwise continuous and thus pathwise integrable. Using (6.4.25) and (6.4.28), (6.4.24) may be written

$$\begin{aligned} e_n(t \wedge \sigma_k) = e_n(0) \; &+ \int_0^{t\wedge\sigma_k} \Delta e_n(s) ds \\ &+ \int_0^{t\wedge\sigma_k} \psi'(u^n(s))[F_n(u(s)) - F(u^n(s))] ds \\ &+ \int_0^{t\wedge\sigma_k} \{\psi'(u^n(s))[-Au^n(s) + F(u^n(s))] - \Delta e_n(s)\} \, ds \\ &+ \int_0^{t\wedge\sigma_k} \psi'(u^n(s)) H_n(u(s)) dw(s) \\ &+ \int_0^{t\wedge\sigma_k} \frac{1}{2} tr\{\psi''(u^n(s)) \left(H_n(u(s)), H_n(u(s))\right)\} ds. \end{aligned}$$

As $u^n(t) \in W^{s+2,p}(S^1, \mathbb{R}^d)$ we may apply Lemma 6.4.5 to obtain

$$\begin{aligned} e_n(t \wedge \sigma_k) = e_n(0) \; &+ \int_0^{t\wedge\sigma_k} \Delta e_n(s) ds \\ &+ \int_0^{t\wedge\sigma_k} \psi'(u^n(s))[F_n(u(s)) - F(u^n(s))] ds \\ &- \int_0^{t\wedge\sigma_k} \mid \underline{\Delta} u^n(s) \mid^2 ds + \int_0^{t\wedge\sigma_k} \psi'(u^n(s)) H_n(u(s)) dw(s) \\ &+ \int_0^{t\wedge\sigma_k} \frac{1}{2} tr\{\psi''(u^n(s)) \left(H_n(u(s)), H_n(u(s))\right)\} ds. \end{aligned} \tag{6.4.31}$$

To simplify notation we define for $u \in W^{s,p}$

$$\begin{aligned} K_1^n(u) &:= \psi'(u^n)[F_n(u) - F(u^n)], \\ K_2^n(u) &:= \psi'(u^n)H_n(u), \\ K_3^n(u) &:= \frac{1}{2} tr\psi''(u^n)\,(H_n(u), H_n(u)). \end{aligned}$$

$\triangle$ is the generator of an analytic semigroup $\{T_t\}_{t\geq 0}$ on $L^p(S^1, \mathbb{R})$, which in fact is a contraction semigroup, with domain $D(\triangle) = W^{2,p}(S^1, \mathbb{R})$, see Chapter 3. Moreover, (6.4.31) can be considered as a linear equation in $e_n(s)$ and so it may be written as an equation in mild form, i.e.

$$\begin{aligned} e_n(t \wedge \sigma_k) &= T_{t\wedge\sigma_k} e_n(0) + \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} K_1^n(u(s))ds \\ &+ \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} K_2^n(u(s))dw(s) + \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} K_3^n(u(s))ds \\ &- \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} \mid \underline{\triangle} u^n(s) \mid^2 ds. \end{aligned} \tag{6.4.32}$$

Step 3: We are now in a position to calculate estimates for $\mathbb{E} \mid e_n(t\wedge\sigma_k) \mid_{L^p}^p$. We have the following Proposition:

Proposition 6.4.7 *The following estimate holds for each k, $n \in \mathbb{N}$ and for each $t \in [0,T)$,*

$$\begin{aligned} \mathbb{E} \mid e_n(t\wedge\sigma_k) \mid_{L^p}^p &\leq C(p,d,T)\left\{\mathbb{E} \mid e_n(0) \mid_{L^p}^p + 1 + \mathbb{E}\int_0^{t\wedge\sigma_k} \mid e(u(s)) \mid_{L^p}^p ds\right\} \\ &+ C(p,d,t)\mathbb{E}\int_0^{t\wedge\sigma_k} \Lambda_n(u(s))ds \end{aligned} \tag{6.4.33}$$

where $C(p,d,T)$ is a constant independent of n and k and

$$\Lambda_n(u) = \left\{\mid e(u) \mid_{L^p}^p + \mid u \mid_{s,p}^p\right\} \mid F_n(u) - F(u^n) \mid_{1,p}^p, \quad u \in W^{s,p}. \tag{6.4.34}$$

Proof : The proof of this proposition will be carried out in a number of lemmas. Recall that our base space is $L^p(S^1, \mathbb{R}^d)$. Now fix $t \in [0,T)$ and $k \in \mathbb{N}$ and denote

$$\begin{aligned} \mathcal{V}^n(t\wedge\sigma_k) &:= T_{t\wedge\sigma_k} e_n(0) + \int_0^{t\wedge\sigma_k} \{T_{t\wedge\sigma_k - s} K_1^n(u(s)) + T_{t\wedge\sigma_k - s} K_3^n(s)\}ds \\ &- \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} \mid \underline{\triangle} u^n(s) \mid^2 ds. \end{aligned}$$

In the following Lemma we find estimates for the terms $\mathbb{E} \mid \mathcal{V}^n(t\wedge\sigma_k) \mid_{L^p}^p$. As earlier we ignore constants unless they depend on n, k and t.

Lemma 6.4.8 *For k, $n \in \mathbb{N}$ and $t \in [0,T)$ the following estimate holds*

$$\begin{aligned} \mathbb{E} \mid \mathcal{V}^n(t\wedge\sigma_k) \mid_{L^p}^p &\leq \mathbb{E} \mid e_n(0) \mid_{L^p}^p + T^{1-\frac{1}{p}}\mathbb{E}\int_0^{t\wedge\sigma_k} \mid K_1^n(u(s)) \mid_{L^p}^p ds \\ &+ T^{1-\frac{1}{p}}\mathbb{E}\int_0^{t\wedge\sigma_k} \mid K_3^n(u(s)) \mid_{L^p}^p ds. \end{aligned} \tag{6.4.35}$$

Proof : Note that if $u(\sigma) \geq 0$, $\sigma \in S^1$ then $(T_t u)(\sigma) \geq 0$ for all $t \geq 0$. Thus we have for each $\sigma \in S^1$, $\left(T_t \mid \underline{\Delta} u \mid^2\right)(\sigma) \geq 0$, $t \geq 0$. Moreover, as our maximal solution is $W^{s,p}$-valued and has, almost surely, continuous paths, then it follows that, almost surely, the processes $K_1^n(u(s))$ and $K_3^n(u(s))$ defined above, are $W^{s-1,p}(S^1, \mathbb{R})$-valued, at least, with continuous paths. Working pathwise we may then deduce that for each, $\sigma \in S^1$,

$$\left(\int_0^{t\wedge\sigma_k} K_1^n(u(s))ds\right)(\sigma) = \int_0^{t\wedge\sigma_k} K_1^n(u(s))(\sigma)ds \text{ a.s..}$$

Similarly for the integral involving K_3^n. It follows that, for $\sigma \in S^1$,

$$\begin{aligned}\mathcal{V}^n(t \wedge \sigma_k)(\sigma) \;\leq\; & (T_{t\wedge\sigma_k} e_n(0))(\sigma) + \int_0^{t\wedge\sigma_k} (T_{t\wedge\sigma_k - s} K_1^n(u(s)))(\sigma)ds \\ & + \int_0^{t\wedge\sigma_k} (T_{t\wedge\sigma_k - s} K_3^n(u(s)))(\sigma)ds \text{ a.s..}\end{aligned}$$

Using the Hölder inequality followed by the Fubini Theorem gives

$$\begin{aligned}\int_{S^1} \mid v_3(\sigma) \mid^p d\sigma \;\leq\; & \int_{S^1} \mid (T_{t\wedge\sigma_k} e_n(0))(\sigma) \mid^p d\sigma \\ & + T^{1-\frac{1}{p}} \int_0^{t\wedge\sigma_k} \int_{S^1} \mid (T_{t\wedge\sigma_k - s} K_1^n(u(s)))(\sigma) \mid^p d\sigma ds \\ & + T^{1-\frac{1}{p}} \int_0^{t\wedge\sigma_k} \int_{S^1} \mid (T_{t\wedge\sigma_k - s} K_3^n(u(s)))(\sigma) \mid^p d\sigma ds,\end{aligned}$$

which holds a.s.. (6.4.35) now follows by taking expectations and noting that $\{T_t\}_{t\geq 0}$ is a contraction semigroup.

♠

Using the triangle inequality and (6.4.35), it follows from (6.4.32) that for $t \in [0, T)$ and $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E} \mid e_n(t \wedge \sigma_k) \mid_{L^p}^p \;\leq\; & \mathbb{E} \mid e_n(0) \mid_{L^p}^p + \mathbb{E} \left| \int_0^{t\wedge\sigma_k} T_{t\wedge\sigma_k - s} K_2^n(u(s))dw_s \right|_{L^p}^p \\ & + T^{1-\frac{1}{p}} \mathbb{E} \int_0^{t\wedge\sigma_k} (\mid K_1^n(u(s)) \mid_{L^p}^p + \mid K_3^n(u(s)) \mid_{L^p}^p) ds.\end{aligned}$$

An application of the Burkholder inequality and using the contraction property of $\{T_t\}_{t\geq 0}$ then gives

$$\begin{aligned}\mathbb{E} \mid e_n(t \wedge \sigma_k) \mid_{L^p}^p \;\leq\; & \mathbb{E} \mid e_n(0) \mid_{L^p}^p + T^{1-\frac{2}{p}} \mathbb{E} \int_0^{t\wedge\sigma_k} \| K_2^n(u(s)) \|^2 ds \\ & + T^{1-\frac{1}{p}} \mathbb{E} \int_0^{t\wedge\sigma_k} (\mid K_1^n(u(s)) \mid_{L^p}^p + \mid K_3^n(u(s)) \mid_{L^p}^p) ds\end{aligned}$$

where $\| \cdot \|$ is the norm on $L(W^{\theta,p}, L^p(S^1, \mathbb{R}))$.
We need now to calculate estimates for the terms involving $\mid K_1^n(u(s)) \mid_{L^p}$, $\mid K_3^n(u(s)) \mid_{L^p}$ and $\| K_1^n(u(s)) \|$. For this we need the formulae (6.4.19) and (6.4.20) for the first and second derivatives of ψ, see Lemma 6.4.2, and the following Lemma, which again is particular to the extension of our metric.

Lemma 6.4.9 *The metric g satisfies the following estimates: for any a, b, x, y and $z \in \mathbb{R}^d$*

$$g(x)(a,b) \leq C \mid b \mid\mid c \mid, \tag{6.4.36}$$

$$g'(x)(y)(a,b) \leq C \mid y \mid\mid a \mid\mid b \mid, \tag{6.4.37}$$

$$g''(x)(y,z)(a,b) \leq C \mid y \mid\mid z \mid\mid a \mid\mid b \mid, \tag{6.4.38}$$

where $C(d,R)$ is independent a, b, x, y and $z \in \mathbb{R}^d$, and R depends on the manifold M.

Proof : Recall that

$$\begin{aligned} g : \mathbb{R}^d &\rightarrow \mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R}) \\ g' : \mathbb{R}^d &\rightarrow L(\mathbb{R}^d;\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})) \\ g'' : \mathbb{R}^d &\rightarrow L_2(\mathbb{R}^d;\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})) \end{aligned}$$

are all smooth maps and so on the closed ball $B(0,R)$ (see the proof of Corollary 6.2.3) we have

$$\sup_{x\in B(0,R)} \left\{\mid g(x) \mid_{\mathcal{L}_2^s} + \mid g'(x) \mid_L + \mid g''(x) \mid_{L_2}\right\} \leq C \tag{6.4.39}$$

where $\mid \cdot \mid_{\mathcal{L}_2^s}$, $\mid \cdot \mid_L$ and $\mid \cdot \mid_{L_2}$ are the norms on $\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R})$, $L(\mathbb{R}^d;\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R}))$ and $L_2(\mathbb{R}^d;\mathcal{L}_2^s(\mathbb{R}^d;\mathbb{R}))$ respectively. Furthermore g coincides with the Euclidean metric outside $B(0,R)$, i.e. for $x \in B(0,R)^c$

$$g(x)(v,w) = < v,w >, \quad v,w \in \mathbb{R}^d, \tag{6.4.40}$$

which implies that g' and g'' vanish outside the ball $B(0,R)$. This observation along with (6.4.39) and (6.4.40) imply the estimates (6.4.36), (6.4.37) and (6.4.38). This completes the proof.

♠

We begin with estimating $\mid K_1^n(u) \mid_{L^p}$, for $u \in W^{s,p}(S^1,M)$, where we recall

$$K_1^n(u) = \psi'(u^n)[F_n(u) - F(u^n)]$$

with $u^n = n(n+A)^{-1}u$ and $F_n = n(n+A)^{-1}F$.

Lemma 6.4.10 *For $u \in W^{s,p}(S^1,M)$, $s > 1 + \frac{1}{p}$, the following estimate holds*

$$\mid K_1^n(u) \mid_{L^p}^p \leq C(p,d,R)\,(1 + \mid e(u) \mid_{L^p}^p + \Lambda_n(u)) \tag{6.4.41}$$

where $C(p,d,R)$ is independent of u and n and Λ_n is given by (6.4.34).

Proof : Using (6.4.19), (6.4.36) and (6.4.37) we have

$$\begin{aligned} \mid K_1^n(u) \mid_{L^p}^p \;\leq\; & \int_{S^1} \{g'(u_\sigma^n)(F_n(u_\sigma) - F(u_\sigma^n)(\nabla u_\sigma^n, \nabla u_\sigma^n)\}^p \, d\sigma \\ & + \int_{S^1} \{g(u_\sigma^n)(\nabla u_\sigma^n, \nabla F_n(u_\sigma) - \nabla F(u_\sigma^n)\}^p \, d\sigma \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{S^1} | F_n(u_\sigma) - F(u^n_\sigma) |^p | \nabla u^n_\sigma |^{2p} \, d\sigma \\
&\quad + \int_{S^1} | \nabla u^n_\sigma |^p | \nabla F_n(u_\sigma) - \nabla F(u^n_\sigma) |^p \, d\sigma \\
&\leq \sup_{\sigma \in S^1} | F_n(u_\sigma) - F(u^n_\sigma) || \nabla u^n |^{2p}_{L^{2p}_d} \\
&\quad + \sup_{\sigma \in S^1} | \nabla u^n_\sigma |^p | \nabla (F_n(u) - F(u^n)) |^p_{L^p_d} \\
&\leq \left\{ | \nabla u^n |^{2p}_{L^{2p}_d} + | u^n |^p_{s,p} \right\} | F_n(u) - F(u^n) |^p_{1,p} .
\end{aligned}
$$

The result follows by noting that

$$
| \nabla u^n |^{2p}_{L^{2p}_d} \leq | u^n |^{2p}_{1,2p} \leq | u |^{2p}_{1,2p}, \tag{6.4.42}
$$

and then applying (6.2.9).

♠

We now turn to estimating $| K^n_3(u) |^p_{L^p}$ for $u \in W^{s,p}(S^1, M)$, where we recall

$$
K^n_3(u) = \frac{1}{2} tr\{\psi''(u^n)(H_n(u), H_n(u))\}.
$$

Note also that $\psi''(u^n)(H_n(u), H_n(u)) \in L_2(W^{\theta,p}; L^p(S^1, \mathbb{R}))$ and for $(x, y) \in W^{\theta,p} \times W^{\theta,p}$ we have

$$
\psi''(u^n)(H_n(u), H_n(u))(x, y) = \psi''(u^n)(H_n(u)x, H_n(u)y) .
$$

Lemma 6.4.11 *For $u \in W^{s,p}(S^1, M)$, $s > 1 + \frac{1}{p}$, we have the following estimate*

$$
| K^n_3(u) |_{L^p} \leq C(p, d, R)(1 + | e(u) |_{L^p}), \tag{6.4.43}
$$

where $C(p, d, R)$ is a constant independent of n and u.

Proof : Recalling that $tr : L_2(W^{\theta,p}; L^p(S^1, \mathbb{R})) \to L^p(S^1, \mathbb{R}))$ is linear and bounded, then we have, again ignoring constants depending on p only,

$$
\begin{aligned}
| K^n_3(u) |_{L^p} &\leq \| \frac{1}{2} \psi''(u^n)(H_n(u), H_n(u)) \|^p_{L_2(W^{\theta,p}; L^p(S^1, \mathbb{R}))} \\
&= \sup_{|x|_{\theta,p} = |y|_{\theta,p} = 1} | \frac{1}{2} \psi''(u^n)(H_n(u)x, H_n(u)y) |_{L^p} \\
&= \sup_{|x|_{\theta,p} = |y|_{\theta,p} = 1} \int_{S^1} | \frac{1}{2} \psi''(u^n)(H_n(u)x, H_n(u)y)(\sigma) |^p \, d\sigma.
\end{aligned}
$$

Denoting $X := H_n(u)x$ and $Y := H_n(u)y$, then using (6.4.20), (6.4.36), (6.4.37) and (6.4.38), we have

$$
\begin{aligned}
| K^n_3(u) |^p_{L^p} &\leq \sup_{|x|=|y|=1} \int_{S^1} (| 2g(u^n_\sigma)(\nabla Y_\sigma, \nabla X_\sigma) \\
&\qquad + 2g'(u^n_\sigma)(X_\sigma)(\nabla Y_\sigma, \nabla u^n_\sigma) \\
&\qquad + 2g'(u^n_\sigma)(Y_\sigma)(\nabla X_\sigma, \nabla u^n_\sigma) \\
&\qquad + g''(u^n_\sigma)(X_\sigma, Y_\sigma)(\nabla u^n_\sigma, \nabla u^n_\sigma) |^p) \, d\sigma \\
&\leq \sup_{|x|=|y|=1} \int_{S^1} \{ 2 | \nabla X_\sigma |^p | \nabla Y_\sigma |^p \\
&\qquad + 2 | X(\sigma) |^p | \nabla Y_\sigma |^p | \nabla u^n_\sigma |^p \\
&\qquad + 2 | Y(\sigma) |^p | \nabla X_\sigma |^p | \nabla u^n_\sigma |^p \\
&\qquad + | X_\sigma |^p | Y_\sigma |^p | \nabla u^n_\sigma |^{2p} \} \, d\sigma.
\end{aligned} \tag{6.4.44}
$$

To complete the proof we need the following:

Lemma 6.4.12 *For $u \in W^{s,p}$, $s > 1 + \frac{1}{p}$, $x, y \in W^{\theta,p}$ with $| x |_{\theta,p} = | y |_{\theta,p} = 1$, there exist constants $C_1(p)$ and $C_2(p)$ independent of x, y, u and n such that for all $\sigma \in S^1$*

$$| X(\sigma) | + | \nabla X(\sigma) | \leq C_1(p) \tag{6.4.45}$$

and

$$| Y(\sigma) | + | \nabla Y(\sigma) | \leq C_2(p). \tag{6.4.46}$$

Proof : It is enough to prove the estimate (6.4.45). Thus note that in particular $X \in W^{2,p}$ and so ∇X is continuous with the following sequence of inequalities holding

$$| X_\sigma | + | \nabla X_\sigma | \leq \sup_{\sigma \in S^1} \{| X_\sigma | + | \nabla X_\sigma |\} \leq C_p | X |_{2,p} . \tag{6.4.47}$$

Note that

$$| X |_{2,p} \leq C_p | H(u)x |_{L^p} .$$

Recall that H is the Nemytski map corresponding to $\tilde{h}$ which is smooth with compact support. As $| x |_{\theta,p} = 1$, we have

$$| H(u)x |_{L^p}^p \leq \int_{S^1} | \tilde{h}(u(\sigma))x(\sigma) |^p \, d\sigma \leq C_p \int_{S^1} | x(\sigma) |^p \, d\sigma \leq C_p.$$

This along with (6.4.47) proves (6.4.45).

♠

Using (6.4.45) and (6.4.46) our estimate (6.4.44) for $| K_3^n(u) |_{L^p}^p$ now reads, for some constant $C(p)$ depending on p only:

$$| K_3^n(u) |_{L^p}^p \leq C(p) \int_{S^1} \left\{| \nabla u_\sigma^n |^{2p} + | \nabla u_\sigma^n |^p + 1\right\} d\sigma.$$

Note that, for all $x \in \mathbb{R}$, $x^{\frac{1}{2}} \leq 1 + x$, and so

$$| \nabla u_\sigma^n |^p \leq 1 + | \nabla u_\sigma^n |^{2p} . \tag{6.4.48}$$

Using the estimate (6.4.42) then gives us the estimate (6.4.43).

$$| \nabla u^n |_{L^{2p}}^{2p} \leq C(p, d, R) \{1 + | e(u) |_{L^p}^p\} . \tag{6.4.49}$$

♠

In a similar manner we obtain the estimate for

$$\| K_2^n(u) \| = | \psi'(u^n) H_n(u) |_{L(W^{\theta,p}, L^p)} .$$

Lemma 6.4.13 *For $u \in W^{s,p}(S^1, M)$, $s > 1 + \frac{1}{p}$, the following estimate holds*

$$\| K_2^n(u) \| \leq C(p, d, R) \{1 + | e(u) |_{L^p}^p\} , \tag{6.4.50}$$

where $C(p, d, R)$ is a constant independent of u and n.

Proof: It follows from (6.4.19), (6.4.36), (6.4.37), (6.4.45) and (6.4.48) that

$$\begin{aligned}\| K_2^n(u(s)) \|^p &= \sup_{|x|_{\theta,p}=1} \int_{S^1} \{| g'(u_\sigma^n)(X_\sigma)(\nabla u_\sigma^n, \nabla u_\sigma^n) \\ &\quad + 2g(u_\sigma^n)(\nabla u_\sigma^n, \nabla X_\sigma) |^p \, d\sigma\} \\ &\leq C(p) \int_{S^1} | \nabla u_\sigma^n |^{2p} + | \nabla u_\sigma^n |^p \, d\sigma \\ &\leq C(p) \int_{S^1} \left\{1 + | u_\sigma^n |^{2p}\right\} d\sigma.\end{aligned}$$

Now applying (6.4.42) and (6.2.9) gives us the result.

♠

The estimates (6.4.41), (6.4.43) and (6.4.50) imply

$$\begin{aligned}\mathbb{E} | e_n(t \wedge \sigma_k) |_{L^p}^p &\leq C(p,d,T)\mathbb{E}\{1 + | e_n(0) |_{L^p}^p\} \\ &\quad + C(p,d,T)\mathbb{E}\int_0^{t\wedge\sigma_k} | e(u(s)) |_{L^p}^p \, ds \\ &\quad + C(p,d,T)\mathbb{E}\int_0^{t\wedge\sigma_k} \Lambda_n(u(s)) ds.\end{aligned}$$

This completes the proof of Proposition 1.

♠

Step 4: We now prove the estimate (6.4.15) for the energy process $e(u(t))$, $t < \tau$. We first prove the following:

Lemma 6.4.14 *For each $t \in [0,T)$ and $k \in \mathbb{N}$*

$$\mathbb{E} | e_n(t \wedge \sigma_k) - e(u(t \wedge \sigma_k)) |_{L^p}^p \to 0 \qquad as\ n \to \infty. \tag{6.4.51}$$

Proof : Recall that $e_n(\cdot) = \psi(u^n(\cdot))$ and $e(u(\cdot)) = \psi(u(\cdot))$. As ψ is continuous and $u^n = n(n+A)^{-1}u \to u$, as $n \to \infty$, then we have, for each $t \in [0,T)$ and $k \in \mathbb{N}$, a.s.

$$| e_n(t \wedge \sigma_k) - e(u(t \wedge \sigma_k)) |_{L^p}^p \to 0 \qquad \text{as } n \to \infty.$$

Furthermore, noting that

$$\begin{aligned}| e_n(t \wedge \sigma_k) - e(u(t \wedge \sigma_k)) |_{L^p}^p &\leq C(p) | e_n(t \wedge \sigma_k) |_{L^p}^p + | e(u(t \wedge \sigma_k)) |_{L^p}^p \\ &\leq C(p) | e(u(t \wedge \sigma_k)) |_{L^p}^p,\end{aligned}$$

then as $\mathbb{E} \ | \ e(u(t \wedge \sigma_k)) \ |_{L^p}^p < \infty$ we may apply the LDC Theorem to obtain our result.

♠

The proof of the following Lemma follows in exactly the same manner as in Lemma 5.3.7 and so is omitted.

Lemma 6.4.15 *For each $t \in [0, T)$ and $k \in \mathbb{N}$,*

$$\mathbb{E}\int_0^{t\wedge\sigma_k} \Lambda_n(u(s))ds \to \quad \text{as } n \to \infty \tag{6.4.52}$$

where $\Lambda_n(u)$ is given by (6.4.34), i.e.

$$\Lambda_n(u) = \left\{|\, e(u)\,|_{L^p}^p + |\, u\,|_{s,p}^p\right\} |\, F_n(u) - F(u^n)\,|_{1,p}^p, \quad u \in W^{s,p}.$$

♠

Recall the estimate (6.4.33) for $\mathbb{E}\,|\, e_n(t\wedge\sigma_k)\,|_{L^p}^p$:

$$\begin{aligned}\mathbb{E}\,|\, e_n(t\wedge\sigma_k)\,|_{L^p}^p &\leq C(p,d,T)\left\{\mathbb{E}\,|\, e_n(0)\,|_{L^p}^p + 1 + \mathbb{E}\int_0^{t\wedge\sigma_k} |\, e(u(s))\,|_{L^p}^p\, ds\right\} \\ &\quad + C(p,d,T)\mathbb{E}\int_0^{t\wedge\sigma_k} \Lambda_n(u(s))ds.\end{aligned}$$

Then, in view of (6.4.51) and (6.4.52), we let $n \to \infty$ and deduce that

$$\mathbb{E}\,|\, e(u(t\wedge\sigma_k))\,|_{L^p}^p \leq C(p,d,T)\left\{\mathbb{E}\,|\, e(u(0))\,|_{L^p}^p + 1 + \mathbb{E}\int_0^{t\wedge\sigma_k} |\, e(u(s))\,|_{L^p}^p\, ds\right\}.$$

Note the following

$$\begin{aligned}\mathbb{E}\int_0^{t\wedge\sigma_k} |\, e(u(s))\,|_{L^p}^p\, ds &= \mathbb{E}\int_0^{t\wedge\sigma_k} |\, e(u(s\wedge\sigma_k))\,|_{L^p}^p\, ds \\ &\leq \mathbb{E}\int_0^t |\, e(u(s\wedge\sigma_k))\,|_{L^p}^p\, ds \\ &= \int_0^t \mathbb{E}\,|\, e(u(s\wedge\sigma_k))\,|_{L^p}^p\, ds.\end{aligned}$$

It then follows that for each $t \in [0, T)$ and $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\,|\, e(u(t\wedge\sigma_k))\,|_{L^p}^p &\leq C(p,d,T)\{\mathbb{E}\,|\, e(u(0))\,|_{L^p}^p + 1\} \\ &\quad + C(p,d,T)\int_0^t \mathbb{E}\,|\, e(u(s\wedge\sigma_k))\,|_{L^p}^p\, ds.\end{aligned}$$

By applying the Gronwall Lemma to the function

$$\varphi(t) = \mathbb{E}\,|\, e(u(t\wedge\sigma_k))\,|_{L^p}^p,$$

we have for $t \in [0, T)$ and $k \in \mathbb{N}$,

$$\mathbb{E}\,|\, e(u(t\wedge\sigma_k))\,|_{L^p}^p \leq C(p,d,T)\{\mathbb{E}\,|\, e(u(0))\,|_{L^p}^p + 1\}\, e^{C(p,d,T)}.$$

This completes the proof of Theorem 2.

♠

This concludes our work on this particular problem.

6.5 Further Work

There are still avenues left open for further research. The line of work the author wishes to continue in would involve investigating the qualitative behaviour and ergodic properties of the solution to the SNHE. This should proceed along the following lines of thought; solution flows, Lyapunov exponents, attractors and/or invariant measures. In particular, suppose $u \in \mathcal{M}$ and $u(t)$, $t \geq 0$ is our solution starting at u. Define the transition operator P_t by

$$(P_t f)(u) := \mathbb{E}[f(u(t))], \quad f \in C_b(X), \tag{6.5.53}$$

where $C_b(X)$ is the space of real-valued bounded continuous functions defined on $X := W^{s,p}(S^1, \mathbb{R}^d)$. The family $\{P_t\}_{t\geq 0}$ is the transition semigroup corresponding to $u(t)$, $t \geq 0$. We would be interested in studying Feller and strong Feller properties, as well as irreducibility of this transition semigroup, see [DP/Z,96] and references therein. Existence and uniqueness of invariant measures for $\{P_t\}_{t\geq 0}$ would be a challenging problem. Indeed existing theory does not meet our criteria since our process takes values in the loop manifold $\mathcal{M}$. For example, uniqueness of invariant measures needs to be stated properly as $\mathcal{M}$ is not a connected manifold.

It would also be of interest consider the SNHE and similar problems in the case when the starting manifold S^1 is replaced with a higher dimensional compact manifold, possibly with boundary. One of the motivations for this comes from Quantum Field Theory, see [Br/Le,99] and references therein. In this paper, they are concerned with diffusion processes over spaces of maps $u : N \to M$, where N is two dimensional manifold with boundary.

♡

Chapter 7

An Approximation Result on Stratonovich Integrals

7.1 Introduction

In this chapter we prove an approximation result of the Wong-Zakai type for S-tratonovich integrals in M-type 2 Banach spaces. In particular, consider a process $x : [0,T] \times \Omega \to X$, $T < \infty$, given by

$$x(t) = x(0) + \int_0^t h(x(s)) \circ dw(s) \tag{7.1.1}$$

where $w(t)$, $t \geq 0$ is a Banach space valued Wiener process, h is a suitable function of C^1 class and the integral in (7.1.1) is the Stratonovich integral, as defined in Chapter 2. By considering piecewise linear approximations of $w(t)$, we prove that x is the almost sure limit of solutions to certain ordinary differential equations in $C(0,T;X)$, the space of X-valued continuous functions on $[0,T]$. The first result of this type was proved in [Wo/Za,] in finite dimensions. Since then there has been considerable work done in this area relating to stochastic ordinary and partial d-ifferential equations, both in finite and infinite dimensions, see [Ma,84], [Mo,88], [Tw,92], [Br/Fl,95], for example, and references therein. Our result is a generalisation of a result in [Dow,80], where they treat the Hilbert space case. Before ending this subsection we recall certain facts about abstract Wiener spaces, AWSs, and M-type 2 Banach spaces.

Let $i : H \hookrightarrow E$ be an AWS and $\{w(t)\}_{t\geq 0}$ the canonical E-valued Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the standard filtration induced by $\{w(t)\}_{t\geq 0}$. Note that the law of the random function $(t-s)^{-\frac{1}{2}}(w(t)-w(s)) : \Omega \to E$ equals μ where μ is the canonical Gaussian probability measure on E. It is straightforward to show that for $p \geq 0$,

$$m_p := \mathbb{E} \left| \frac{w(t) - w(s)}{(t-s)^{\frac{1}{2}}} \right|_E^p = \int_E | z |_E^p \, d\mu(z). \tag{7.1.2}$$

Furthermore by the Fernique-Landau-Shepp Theorem, see [Kuo,75], $m_p < \infty$ for each $p \geq 0$. Let X be an M-type 2 Banach space. Recall that, under this assumption, we have the following Burkholder inequalities, see [De,91]:

- For any X-valued martingale $\{M_n\}_{n\in\mathbb{N}}$ and $p \in [2,\infty)$ we have

$$\mathbb{E}\sup_{n\in\mathbb{N}} \mid M_n \mid_X^p \leq C(p,X)\mathbb{E}\{\sum_n \mid M_n - M_{n-1} \mid_X^2\}^{\frac{p}{2}}, \tag{7.1.3}$$

where $C(p,X)$ is independent of n.

- For any progressively measurable process ξ taking values in $L(E,X)$ with $\mathbb{E}\int_0^T \mid \xi(t) \mid_{L(E,X)}^2 dt < \infty$ we have

$$\mathbb{E}\sup_{0\leq t\leq T} \mid \int_0^t \xi(s)dw(s) \mid_X^p \leq \tilde{C}(p,X)\mathbb{E}[\int_0^T \mid \xi(s) \mid_{L(E,X)}^2 ds]^{\frac{p}{2}}, \tag{7.1.4}$$

where $\tilde{C}(p,X)$ is independent of T and ξ.

♡

7.2 The Approximation Result

Suppose $f : X \to X$ a Lipschitz map which is bounded. Let $h : X \to L(E,X)$ be a C^1 map , i.e. h is differentiable with continuous derivative $h' : X \to L(X, L(E,X))$. Assume that h and h' are Lipschitz in X and that they are bounded. As a consequence the map $tr(h'h) : X \to X$ is Lipschitz and bounded where $tr(h'h)(x) := tr\{h'(x)h(x)\}$, $x \in X$. Here $tr : L_2(E;X) \to X$ is a bounded linear map relating to the AWS $i : H \to E$, see Chapter 2. For each $T < \infty$, fixed but arbitrary, there exists a unique continuous progressively measurable process $x : [0,T]\times\Omega \to X$ satisfying, for $t \in [0,T]$,

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(x(s))ds + \int_0^t h(x(s))dw(s) \\ &\quad + \frac{1}{2}\int_0^t tr\{h'(x(s))h(x(s))\}ds, \text{ a.s.}, \end{aligned} \tag{7.2.5}$$

where $x(0) = x_0 \in L^p(\Omega, X)$ and $p \geq 2$, see [Ne,78]. In particular x is a solution to the Stratonovich equation

$$dx(t) = f(x(t)) + h(x(t)) \circ dw(t) \tag{7.2.6}$$

and x may be written as

$$x(t) = x(0) + \int_0^t f(x(s))ds + \int_0^t h(x(s)) \circ dw(s), \tag{7.2.7}$$

where the last integral on the RHS is a Stratonovich integral.

Using the Burkholder inequality (7.1.4) and the boundedness of the maps f, h and $tr(h'h)$ one may show directly that for $s,t \in [0,T]$

$$\mathbb{E} \mid x(t) - x(s) \mid_X^p \leq C(p,T,X) \mid t - s \mid^{\frac{p}{2}}.$$

Thus x is $\frac{p}{2}$-Hölder continuous considered as a map $x : [0,T] \to L^p(\Omega, X)$.

For $n \in \mathbb{N}$, let π_n be a partition of $[0,T]$, i.e.

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N(n)} = T.$$

We assume the partition satisfies

$$\text{mesh}\pi_n := \max_{0 \le k \le N(n)-1} | t_{k+1} - t_k | \le \frac{C_1}{n}, \tag{7.2.8}$$

$$N(n) \le C_2 n, \tag{7.2.9}$$

where C_1 and C_2 are constants independent of n. For a fixed partition $\pi = \pi_n$, we consider the following piecewise linear approximation of $w(t)$:

$$w_\pi(t) = w(t_i) + \frac{t - t_i}{t_{i+1} - t_i}(w(t_{i+1}) - w(t_i)), \ \ t \in [t_i, t_{i+1}].$$

For each partition, $\pi = \pi_n$, of $[0,T]$, let $x_\pi : [0,T] \times \Omega \to X$ be the solutions to the family of ODEs, indexed by $\omega \in \Omega$,

$$\frac{dx_\pi(t)}{dt} = h(x_\pi(t))\frac{dw_\pi(t)}{dt} + f(x_\pi(t)), \tag{7.2.10}$$

where $x_\pi(0) = x_0$ and $0 \le t \le T$. In particular, for $t \in (t_i, t_{i+1})$, x_π takes the form

$$x_\pi(t) = x_\pi(t_i) + \int_{t_i}^t h(x_\pi(s)) \left(\frac{w(t_{i+1}) - w(t_i)}{t_{i+1} - t_i} \right) ds + \int_{t_i}^t f(x_\pi(s))ds. \tag{7.2.11}$$

Theorem 7.2.1 *For $p \ge 2$ and $n \in \mathbb{N}$*

$$\mathbb{E} \sup_{0 \le t \le T} | x(t) - x_{\pi_n}(t) |_X^p \le C n^{-\frac{p}{2}}. \tag{7.2.12}$$

Furthermore

$$x_{\pi_n}(\cdot) \to x(\cdot) \ \ in \ C(0,T;X), \ \ a.s. \tag{7.2.13}$$

as $n \to \infty$. Here $C(0,T;X)$ is the space of X valued continuous functions on the interval $[0,T]$.

Remark 7.2.2 The constant C appearing in (7.2.12) depends on the space X, p, T, m_p and the bounds and Lipschitz constants of f, h, h' and tr. ◇

Remark 7.2.3 This theorem is an extension of a result proved in [Dow,80]. There, the case $p = 2$ with X being a Hilbert space was treated. Most of the proof presented in [Dow,80], which itself is a generalisation of a similar result in [McS,74], carries over with no difficulty to the case where $p \ge 2$ and X is an M-type 2 Banach space. The Burkholder inequality (7.1.4) is the main tool we use here. Even so, there are still problems that arise which require more work. Although Dowell was familiar with stochastic integration in 2-uniformly smooth Banach spaces and the inequality (7.1.4) (for $p = 2$) he was not able to deal with the Banach space case because of the term involving the tr map. There is a considerable level of difficulty in dealing with the tr map in Banach spaces as opposed to Hilbert spaces. To deal with this we make use of the M-type 2 property of our space, in particular, the inequality (7.1.3).

◇

Proof: Fix a partition $\pi = \pi_n = \{0 \le t_0 \le t_1 ... \le t_{N(n)} = T\}$ and denote x_π by y. Set $x_j = x(t_j)$, $y_j = y(t_j) = x_\pi(t_j)$, $\Delta_j t = t_{j+1} - t_j$ and $\Delta_j w = w(t_{j+1}) - w(t_j)$. To simplify notation we put f identically zero. This will not affect the result owing to the conditions put on f. Moreover, C will denote a generic constant depending only on the space X, p, T, m_p the bounds and Lipschitz constants of h, h' and tr. For each $t \in [0,T]$, let k be the largest integer such that $t \in [t_k, t_{k+1})$. Moreover, for $r > t$, $r \in [0,T]$, set $R(n) = \max\{m : t_m \le r\}$. Then, using the triangle inequality, we have

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le r} | x(t) - y(t) |_X^p \le \; & C\mathbb{E} \sup_{0\le t\le r} \{| x(t) - x(t_k) |_X^p + | y(t_k) - y(t) |_X^p\} \\ & + C\mathbb{E} \sup_{0\le k\le R(n)} | x(t_k) - y(t_k) |_X^p . \end{aligned} \tag{7.2.14}$$

Suppose, for the time being, we have the following estimates

$$\mathbb{E} \sup_{0\le t\le r} \{| x(t) - x(t_k) |_X^p + | y(t_k) - y(t) |_X^p\} \le C\eta(\pi), \tag{7.2.15}$$

$$\mathbb{E} \sup_{0\le k\le R(n)} | x(t_k) - y(t_k) |_X^p \le \eta(\pi) + C\int_0^r \mathbb{E}(\gamma(s))ds, \tag{7.2.16}$$

where

$$\gamma(s) = \sup_{0\le l\le s} | x(l) - y(l) |_X^p \tag{7.2.17}$$

and $\eta(\pi)$ is independent of k and satisfies

$$\eta(\pi) \le Cn^{-\frac{p}{2}}.$$

(Note, for example, that $(\text{mesh}\pi)^{\frac{p}{2}}$ is a term of the form $\eta(\pi)$.) From (7.2.14), (7.2.15), (7.2.16) and (7.2.17) we may deduce that for all $r \in [0,T]$:

$$\mathbb{E}\gamma(r) = \mathbb{E} \sup_{0\le t\le r} | x(t) - y(t) |_X^p \le C\eta(\pi) + C\int_0^r \mathbb{E}(\gamma(s))ds.$$

An application of Gronwall's Lemma implies that

$$\mathbb{E}(\gamma(T)) \le C\eta(\pi) \exp^{CT},$$

i.e.

$$\mathbb{E} \sup_{0\le t\le T} | x(t) - y(t) |_X^p \le Cn^{-\frac{p}{2}}.$$

To complete the proof of Theorem 7.2.1 we need to prove the estimates (7.2.15) and (7.2.16). We begin with (7.2.15).

Lemma 7.2.4 *With the above notation*

$$\mathbb{E} \sup_{0\le t\le r} \{| x(t) - x(t_k) |_X^p + | y(t_k) - y(t) |_X^p\} \le C(mesh\pi)^{\frac{p}{2}}. \tag{7.2.18}$$

Proof: Note first that from (7.2.5) and the boundedness of the maps h and $tr(h'h)$ we have

$$\mathbb{E} \sup_{0\le t\le r} | x(t) - x(t_k) |_X^p \le \; C(\text{mesh}\pi)^p + C\mathbb{E} \sup_{0\le t\le r} \left| \int_{t_k}^t h(x(s))dw(s) \right|_X^p .$$

It then follows using the Burkholder inequality and the boundedness of h that

$$\mathbb{E}\sup_{0\leq t\leq r} \mid x(t)-x(t_k)\mid_X^p\leq CT^{\frac{p}{2}}(\text{mesh}\pi)^{\frac{p}{2}}.$$

Recall Taylor's formula in integral form, see [Ca,71]:

$$y(a)-y(b)=\int_0^1 (a-b)y'(b+r(a-b))dr. \tag{7.2.19}$$

For some $0\leq s\leq 1$ we have, using (7.2.19), (7.2.10) and the boundedness of h,

$$\begin{aligned}\mid y(t)-y(t_k)\mid_X^p &= \mid y(t_k+s\Delta_k t)-y(t_k)\mid_X^p\\ &= \mid \int_0^1 y'(t_k+r(s\Delta_k t))(s\Delta_k t)dr\mid_X^p\\ &= \mid \int_0^s y'(t_k+r\Delta_k t)(\Delta_k t)dr\mid_X^p\\ &= \mid \int_0^s h(y(t_k+r\Delta_k t))(\Delta_k w)dr\mid_X^p\\ &\leq C\mid \Delta_k w\mid_E^p . \end{aligned} \tag{7.2.20}$$

Using (7.1.2) we infer that

$$\mathbb{E}\sup_{0\leq t\leq r} \mid y(t)-y(t_k)\mid_X^p\leq C(\text{mesh}\pi)^{\frac{p}{2}}.$$

This completes the proof of Lemma 7.2.4.

♠

Fix an interval $[t_i,t_{i+1}]$ in the partition π. We quote another form of Taylor's formula, see [Ca,71]:

$$y(a)-y(b)=(a-b)y'(b)+\int_0^1(1-s)y''(b+s(a-b))(a-b,a-b)ds. \tag{7.2.21}$$

Using (7.2.21), the chain rule and (7.2.10) we obtain

$$\begin{aligned}y(t_{i+1})-y(t_i) &= \Delta_j t y'(t_j)+\int_0^1(1-s)y''(t_j+s\Delta_j t)(\Delta_j t,\Delta_j t)ds\\ &= h(y_j)\Delta_j w\\ &+ \int_0^1(1-s)[h'(y(t_j+s\Delta_j t))h(y(t_j+s\Delta_j t))(\Delta_j w,\Delta_j w)]ds.\end{aligned}$$

It then follows, denoting $s_j:=t_j+s\Delta_j t$, that

$$\begin{aligned}y(t_k)-y(t_0) &= \sum_{j=0}^{k-1}(y_{j+1}-y_j)\\ &= \sum_{j=0}^{k-1}\{h(y_j)\Delta_j w+\frac{1}{2}h'(y_j)h(y_j)(\Delta_j w,\Delta_j w)\}\\ &\quad +\sum_{j=0}^{k-1}\int_0^1(1-s)h'(y(s_j))h(y(s_j))(\Delta_j w,\Delta_j w)ds\\ &\quad -\sum_{j=0}^{k-1}\int_0^1(1-s)h'(y_j)h(y_j)(\Delta_j w,\Delta_j w)ds.\end{aligned}$$

Recalling that

$$x(t_k) = x(0) + \int_0^{t_k} h(x(s))dw(s) + \frac{1}{2}\int_0^{t_k} tr\{h'(x(s))h(x(s))\}ds,$$

we may write

$$y(t_k) - x(t_k) = A_k + B_k + \frac{1}{2}\overline{C}_k + D_k + \frac{1}{2}E_k + \frac{1}{2}F_k,$$

where

$$\begin{aligned}
A_k &= \sum_{j=0}^{k-1}\int_0^1 (1-s)\{h'(y(s_j))h(y(s_j)) - h'(y_j)h(y_j)\}(\Delta_j w, \Delta_j w)ds \\
B_k &= \sum_{j=0}^{k-1}(h(y_j) - h(x_j))\Delta_j w \\
\overline{C}_k &= \sum_{j=0}^{k-1}(h'(y_j)h(y_j) - h'(x_j)h(x_j))(\Delta_j w, \Delta_j w) \\
D_k &= \sum_{j=0}^{k-1} h(x_j)\Delta_j w - \int_0^{t_k} h(x(s))dw(s) \\
E_k &= \sum_{j=0}^{k-1}\{h'(x_j)h(x_j)(\Delta_j w, \Delta_j w) - tr\{h'(x_j)h(x_j)\}\Delta_j t\} \\
F_k &= \sum_{j=0}^{k-1} tr h'(x_j)h(x_j)\Delta_j t - \int_0^{t_k} tr\{h'(x(t))h(x(t))\}dt.
\end{aligned}$$

We begin with proving:

Lemma 7.2.5 *Using the above notation we have*

$$\mathbb{E}\sup_{1\le k\le R(n)} | A_k + D_k + E_k + F_k |_X^p \le C(mesh\pi)^{\frac{p}{2}}.$$

Proof: Consider the term $A_k = \sum_{j=0}^{k-1}\Gamma_j$, where

$$\Gamma_j := \int_0^1 (1-s)\{h'(y(s_j))h(y(s_j))(\Delta_j w, \Delta_j w) - h'(y_j)h(y_j)(\Delta_j w, \Delta_j w)\}ds.$$

The boundedness and Lipschitz properties of h' and h, along with (7.2.20) imply that

$$\begin{aligned}
| \Gamma_j |_X &\le \int_0^1 | \{h'(y(s_j)) - h'(y_j)\}h(y(s_j))(\Delta_j w, \Delta_j w) | ds \\
&\quad + \int_0^1 | h'(y(s_j))(h(y(s_j)) - h(y_j))(\Delta_j w, \Delta_j w) |_X ds \\
&\le C | \Delta_j w |_E^2 | y(s_j) - y_j | \\
&\le C | \Delta_j w |_E^3 . \qquad (7.2.22)
\end{aligned}$$

Using (7.2.22) and Hölder's inequality for sums we have

$$\mathbb{E}\sup_{1\le k\le N(n)} | A_k |_X^p \le CN(n)^{p-1}\mathbb{E}\sum_{j=0}^{N(n)-1} | \Delta_j w |_E^{3p} .$$

Applying (7.1.2) (with p replaced by $3p$) gives us

$$\mathbb{E} \sup_{1\leq k\leq N(n)} | A_k |_X^p \leq CN(n)^{p-1} \sum_{j=0}^{N(n)-1} | \Delta_j t |^{\frac{3p}{2}} .$$

It then follows, using (7.2.8) and (7.2.9), that

$$\mathbb{E} \sup_{1\leq k\leq N(n)} | A_k |_X^p \leq Cn^{p-1}(\text{mesh}\pi)^{\frac{3p}{2}} n \leq C(\text{mesh}\pi)^{\frac{p}{2}}. \tag{7.2.23}$$

Consider the term $D_k = \sum_{j=0}^{k-1} h(x_j)\Delta_j w - \int_0^{t_k} h(x(s))dw(s)$. Define

$$\tilde{h}(s) = \begin{cases} h(x_j) & \text{for } t_j \leq s < t_{j+1} \\ 0 & \text{if } s > t_k \end{cases} .$$

$\tilde{h}(s)$ is well-defined, adapted to the filtration $\{\mathcal{F}_s\}_{s\geq 0}$ and moreover, $\int_0^t \tilde{h}(s)dw(s)$ makes sense for all $t \in [0,T]$. We may write

$$D_k = \int_0^{t_k} [\tilde{h}(s) - h(x(s))]dw(s).$$

Using the Burkholder inequality, the Lipschitz property of h and the properties (7.2.8) and (7.2.9), it follows that

$$\begin{aligned}
\mathbb{E} \sup_{1\leq k\leq R(n)} | D_k |_X^p &\leq \mathbb{E} \sup_{0\leq t\leq r} | \int_0^t [\tilde{h}(s) - h(x(s))]dw(s) |_X^p \\
&\leq C\mathbb{E}\{\int_0^r | \tilde{h}(s) - h(x(s)) |_{L(E,X)}^2 ds\}^{\frac{p}{2}} \\
&= C\mathbb{E}\{ \sum_{j=0}^{R(n)-1} \int_{t_j}^{t_{j+1}} | h(x_j) - h(x(s)) |_{L(E,X)}^2 ds\}^{\frac{p}{2}} \\
&\leq C\mathbb{E}\{ \sum_{j=0}^{R(n)-1} \int_{t_j}^{t_{j+1}} | x_j - x(s) |_X^2 ds\}^{\frac{p}{2}} \\
&\leq C\mathbb{E} \sup_{0\leq t\leq r} | x(t_l) - x(t) |_X^p
\end{aligned}$$

where l is such that $t \in [t_l, t_{l+1})$. Using Lemma 7.2.4 we deduce that

$$\mathbb{E} \sup_{1\leq k\leq R(n)} | D_k |_X^p \leq C\text{mesh}\pi^{\frac{p}{2}}. \tag{7.2.24}$$

Consider the term F_k. Then

$$\begin{aligned}
| F_k |_X &= | \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} [tr\{h'(x_j)h(x_j)\} - tr\{h'(x(t))h(x(t))\}]dt |_X \\
&\leq \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} | tr\{h'(x_j)h(x_j) - h'(x(t))h(x(t))\} |_X dt \\
&\leq C\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} | h'(x_j)h(x_j) - h'(x(t))h(x(t)) |_{L_2(E,X)} dt \\
&\leq C\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Big\{| h'(x_j)h(x_j) - h'(x_j)h(x(t)) |_{L_2(E,X)} \\
&\qquad + | h'(x_j)h(x(t)) - h'(x(t))h(x(t)) |_{L_2(E,X)}\Big\} dt.
\end{aligned}$$

Using the boundedness and Lipschitz properties of g and h', we deduce that

$$\begin{aligned} | F_k |_X &\leq C\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} | x_j - x(t) |_X \, dt \\ &\leq CT \sup_{0\leq t\leq r} | x(t_l) - x(t) |_X \, dt, \end{aligned}$$

where l is such that $t \in [t_l, t_{l+1})$. Again, using Lemma 7.2.4, we conclude that

$$\mathbb{E} \sup_{1\leq k\leq R(n)} | F_k |_X^p \leq C\text{mesh}\pi^{\frac{p}{2}}. \tag{7.2.25}$$

Finally, we deal with the term E_k and we will prove

$$\mathbb{E} \sup_{1\leq k\leq R(n)} | E_k |_X^p \leq C\text{mesh}\pi^{\frac{p}{2}}. \tag{7.2.26}$$

This part of the proof differs considerably from [Dow,80]. Dowell proves (7.2.26) using the properties of the inner product on a Hilbert space and the proof is quite straightforward. We do not have an inner product to work with and instead we make use of the M-type 2 property of our space X. Let $E_k = \sum_{j=0}^{k-1} \Lambda_j$ where

$$\Lambda_j = h'(x_j)h(x_j)(\Delta_j w, \Delta_j w) - tr(h'(x_j)h(x_j))\Delta_j t \tag{7.2.27}$$

We first show that E_k is an X-valued martingale with respect to the discrete filtration $\{\mathcal{F}_{t_k}\}_{1\leq k\leq R(n)}$. For $0 \leq j \leq k-1$, $x_j : \Omega \to X$ and $w(t_{j+1}) - w(t_j) : \Omega \to E$ are $\mathcal{F}_{t_{j+1}}$ measurable. Using the continuity of the maps h, h' and $tr(h'h)$ it follows that each Λ_j is $\mathcal{F}_{t_{j+1}}$ measurable. We deduce that E_k is $\mathcal{F}_{t_k}$ measurable. To prove E_k is a martingale we are left with showing that $\mathbb{E}(E_k \mid \mathcal{F}_{t_{k-1}}) = E_{k-1}$. For this it suffices to prove that $\mathbb{E}(\Lambda_{k-1} \mid \mathcal{F}_{t_{k-1}}) = 0$.
Denote

$$\Psi_{k-1} := h'(x_{k-1})h(x_{k-1})(\Delta_{k-1}w, \Delta_{k-1}w).$$

Then

$$\begin{aligned} \mathbb{E}(\Psi_{k-1} \mid \mathcal{F}_{k-1}) &= \mathbb{E}[h'(x_{k-1})h(x_{k-1})(\Delta_{k-1}w, \Delta_{k-1}w) \mid \mathcal{F}_{t_{k-1}}] \\ &= \mathbb{E}[h'(x_{k-1})h(x_{k-1})(\Delta_{k-1}w, \Delta_{k-1}w)] \\ &= (t_k - t_{k-1})\int_E h'(x_{k-1})h(x_{k-1})(e,e)d\mu(e) \\ &= (\Delta_{k-1}t)tr\{h'(x_{k-1})h(x_{k-1})\} \\ &= \mathbb{E}[(\Delta_{k-1}t)tr\{h'(x_{k-1})h(x_{k-1})\} \mid \mathcal{F}_{t_{k-1}}]. \end{aligned} \tag{7.2.28}$$

As x_{k-1} is $\mathcal{F}_{t_{k-1}}$ measurable, then so is $tr\{h'(x_{k-1})h(x_{k-1})\}$, which explains the final step. Thus (7.2.27) and (7.2.28) imply that $\mathbb{E}(\Lambda_{k-1} \mid \mathcal{F}_{t_{k-1}}) = 0$. We conclude that $\{E_k\}_{k=1}^{R(n)}$ is an X-valued martingale with respect to the dicrete filtration $\{\mathcal{F}_{t_k}\}_{1\leq k\leq R(n)}$. Since X is an M-type 2 Banach space it follows that

$$\mathbb{E} \sup_{1\leq k\leq R(n)} | E_k |_X^p \leq C\mathbb{E}\{ \sum_{j=1}^{R(n)-1} | E_j - E_{j-1} |_X^2\}^{\frac{p}{2}}.$$

Thus

$$\mathbb{E}\sup_{1\le k\le R(n)} \mid \sum_{j=0}^{k-1}\Lambda_j \mid_X^p \le C\mathbb{E}\{\sum_{j=1}^{R(n)} \mid \Lambda_{j-1}\mid_X^2\}^{\frac{p}{2}}.$$

Applying the Hölder inequality for sums gives

$$\begin{aligned}\mathbb{E}\sup_{1\le k\le R(n)} \mid \sum_{j=0}^{k-1}\Lambda_j \mid_X^p &\le CR(n)^{\frac{p}{2}-1}\mathbb{E}\sum_{j=1}^{R(n)} \mid \Lambda_{j-1}\mid_X^p \\ &\le CN(n)^{\frac{p}{2}-1}\sum_{j=1}^{N(n)}\mathbb{E}\mid \Lambda_{j-1}\mid_X^p . \qquad (7.2.29)\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}\mid \Lambda_j\mid_X^p &\le \mathbb{E}\{(\mid h'(x_j)h(x_j)(\Delta_j w,\Delta_j w)\mid_X + \mid tr\{h'(x_j)h(x_j)\}(\Delta_j t)\mid_X\}^p \\ &\le C\mathbb{E}\{\mid \Delta_j w\mid_E^{2p} + \mid \Delta_j t\mid^p\} \\ &\le C(\Delta_j t)^p. \qquad (7.2.30)\end{aligned}$$

It follows from (7.2.29) and (7.2.30) that

$$\begin{aligned}\mathbb{E}\sup_{1\le k\le R(n)} \mid E_k\mid_X^p &\le CN(n)^{\frac{p}{2}-1}\sum_{j=1}^{N(n)}(\Delta_j t)^p \\ &\le CN(n)^{\frac{p}{2}-1}\sum_{j=1}^{N(n)}(\text{mesh}\pi)^p \\ &\le CN(n)^{\frac{p}{2}}(\text{mesh}\pi)^p \\ &\le C\text{mesh}\pi^{\frac{p}{2}}. \qquad (7.2.31)\end{aligned}$$

Lemma 7.2.5 now follows from (7.2.23), (7.2.24), (7.2.25) and (7.2.31).

♠

Lemma 7.2.6 *For a constant C independent of k and r*

$$\mathbb{E}\sup_{1\le k\le R(n)} \mid B_k + \overline{C}_k\mid_X^p \le C\int_0^r \mathbb{E}\gamma(s)ds \qquad (7.2.32)$$

Proof: As in the proof of Lemma 7.2.5, define

$$Y(s) = \begin{cases} h(y_j) - h(x_j) & \text{if } t_j \le s < t_{j+1}, \text{ where } 0\le j\le k-1 \\ 0 & \text{if } s > t_k. \end{cases}$$

$Y(s)$ is well-defined, adapted to the filtration $\{\mathcal{F}_s\}_{s\ge 0}$ and $\int_0^t Y(s)dw(s)$ makes sense for all $t\in[0,T]$. Moreover,

$$\mid B_k\mid_X^p = \mid \int_0^{t_k} Y(s)dw(s)\mid_X^p .$$

Using the Burkholder inequality and the Lipschitz properties of h, it follows that

$$\mathbb{E}\sup_{1\le k\le R(n)} \mid B_k\mid_X^p = \mathbb{E}\sup_{0\le t\le r} \mid \int_0^t Y(s)dw(s)\mid_X^p$$

$$
\begin{aligned}
&\leq C\mathbb{E}\Big(\int_0^r \mid Y(s)\mid^2_{L(E,X)} ds\Big)^{\frac{p}{2}} \\
&= C\mathbb{E}\Big(\sum_{j=0}^{R(n)-1}\int_{t_j}^{t_{j+1}} \mid h(y_j)-h(x_j)\mid^2_{L(E,X)} ds\Big)^{\frac{p}{2}} \\
&\leq C\mathbb{E}\Big(\sum_{j=0}^{R(n)-1} \mid y_j - x_j\mid^2_X \Delta_j t\Big)^{\frac{p}{2}} \\
&\leq C\mathbb{E}\Big(\sum_{j=0}^{R(n)-1} \gamma(t_j)^{\frac{2}{p}}\Delta_j t\Big)^{\frac{p}{2}}.
\end{aligned}
$$

Applying the Hölder inequality for sums gives

$$
\begin{aligned}
\mathbb{E}\sup_{1\leq k\leq R(n)} \mid B_k\mid^p_X &\leq CR(n)^{\frac{p}{2}-1}\mathbb{E}\sum_{j=0}^{R(n)-1}\gamma(t_j)(\Delta_j t)^{\frac{p}{2}} \\
&\leq CN(n)^{\frac{p}{2}-1}(\text{mesh}\pi)^{\frac{p}{2}-1}\sum_{j=0}^{R(n)-1}\mathbb{E}(\gamma(t_j)\Delta_j t) \\
&\leq C\int_0^r \mathbb{E}(\gamma(s))ds,
\end{aligned}
$$

which constitutes the first in proving Lemma 7.2.6. Consider the final term $\overline{C}_k$. Then

$$
\begin{aligned}
\mid \overline{C}_k\mid_X &= \Big|\sum_{j=0}^{k-1}(h'(y_j)h(y_j)-h'(x_j)h(x_j))(\Delta_j w,\Delta_j w)\Big|_X \\
&\leq \sum_{j=0}^{k-1}\{\mid (h'(y_j)-h'(x_j))h(x_j)(\Delta_j w,\Delta_j w)\mid_X \\
&\qquad + \mid h'(y_j)(h(y_j)-h(x_j))(\Delta_j w,\Delta_j w)\mid_X\} \\
&\leq C\sum_{j=0}^{k-1}\mid x_j-y_j\mid_X\mid \Delta_j w\mid^2_E .
\end{aligned}
$$

Applying the Hölder inequality gives

$$
\mid \overline{C}_k\mid^p \leq CN(n)^{p-1}\sum_{j=0}^{k-1}\mid x_j-y_j\mid^p_X\mid \Delta_j w\mid^{2p}_E .
$$

On taking supremum over k and then expectations we get

$$
\mathbb{E}\sup_{1\leq k\leq R(n)}\mid \overline{C}_k\mid^p_X \leq CN(n)^{p-1}\sum_{j=0}^{R(n)-1}\mathbb{E}\mid x_j-y_j\mid^p_X\mid \Delta_j w\mid^{2p}_E .
$$

Since both x_j and y_j are $\mathcal{F}_{t_j}$-measurable and $\Delta_j w$ is independent of $\mathcal{F}_{t_j}$ then using the properties of conditional expectation and (7.1.2) we have

$$
\begin{aligned}
\mathbb{E}(\mid x_j-y_j\mid^p_X\mid \Delta_j w\mid^{2p}_{E_0}) &= \mathbb{E}[\mathbb{E}(\mid x_j-y_j\mid^p_X\mid \Delta_j w\mid^{2p}_E\mid \mathcal{F}_{t_j})] \\
&= \mathbb{E}[\mid x_j-y_j\mid^p_X \mathbb{E}(\mid \Delta_j w\mid^{2p}_E\mid \mathcal{F}_{t_j})] \\
&= \mathbb{E}[\mid x_j-y_j\mid^p_X \mathbb{E}(\mid \Delta_j w\mid^{2p}_E)] \\
&\leq C\mid \Delta_j t\mid^p \mathbb{E}(\mid x_j-y_j\mid^p_X). \qquad (7.2.33)
\end{aligned}
$$

It then follows using (7.2.33), (7.2.8) and (7.2.9) that

$$\begin{aligned}
\mathbb{E} \sup_{1\leq k\leq R(n)} |\, \overline{C}_k \,|_X^p &\leq CN(n)^{p-1} \sum_{j=0}^{R(n)-1} \mathbb{E} \,|\, x_j - y_j \,|_X^p |\, \Delta_j t \,|^p \\
&\leq CN(n)^{p-1}(\text{mesh}\pi)^{p-1} \sum_{j=0}^{R(n)-1} (\Delta_j t)\mathbb{E} \sup_{0\leq r\leq t_j} |\, x(r) - y(r) \,|_X^p \\
&\leq C \sum_{j=0}^{R(n)-1} \mathbb{E}(\gamma(t_j)\Delta_j t).
\end{aligned}$$

Since $\gamma(s)$ is nondecreasing we can conclude that

$$\mathbb{E} \sup_{1\leq k\leq R(n)} |\, \overline{C}_k \,|_X^p \leq C \int_0^r \mathbb{E}(\gamma(s))ds,$$

which concludes the proof of Lemma 7.2.6. The proof of Theorem 7.2.1 is now complete.

♠

Chapter 8

Appendix: A Result On Fractional Powers

Introduction

The aim of this Appendix is to prove that the generalised factorisation operator, introduced in Chapter 4, is the fractional power of a certain abstract parabolic operator.
Let Y be a Banach space and Λ a positive operator on Y, i.e. $(\lambda+\Lambda)^{-1}$ is bounded for $\lambda \geq 0$ and for some $C \geq 1$ we have

$$| (\lambda+\Lambda)^{-1} |_{L(Y)} \leq \frac{C}{1+\lambda}, \quad \lambda \geq . \tag{8.0.1}$$

One may then define the fractional powers Λ^z, $z \in \mathbb{C}$, of Λ, through the formula

$$\Lambda^z x = \frac{\Gamma(m)}{\Gamma(z+n)\Gamma(m-n-z)} \int_0^\infty \lambda^{z+n-1} \Lambda^{m-n} (\lambda+\Lambda)^{-m} x d\lambda, \tag{8.0.2}$$

where $x \in Y$, m, $n \in \mathbb{N}_0$ with $-n < Re z < m-n$ and Γ is the Euler Gamma function, see Chapter 4. As Λ is positive, then A^z is a bounded linear operator. In particular, noting that $\frac{\sin(\pi\alpha)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$, the fractional power $\Lambda^{-\alpha}$, $\alpha \in (0,1)$, is given by the formula

$$\Lambda^{-\alpha} x = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda+\Lambda)^{-1} x d\lambda, \quad x \in Y. \tag{8.0.3}$$

Let X be a Banach space. Fix $0 < T < \infty$ and $p \in [1,\infty)$. Consider the operator Λ_T defined on a subspace of $L^p(0,T;X)$ (whose norm we denote $|\cdot|_{L^p}$) through the formula

$$(\Lambda_T u)(t) = (\mathcal{A}_T u)(t) + (\mathcal{B}_T u)(t) = Au(t) + u'(t), \quad \text{a.e. } t \in [0,T],$$

where $-A$ is the generator of an analytic semigroup on X and u' denotes the weak derivative of $u \in L^p(0,T;X)$. We aim to prove that for $\lambda \geq 0$, $\alpha \in (0,1)$ and $u \in L^p(0,T;X)$

$$(\lambda+\Lambda_T)^{-\alpha} u = \frac{1}{\Gamma(\alpha)} R_\alpha^\lambda u, \tag{8.0.4}$$

where R^λ_α is the factorisation operator, given by

$$(R^\lambda_\alpha u)(t) = \int_0^t (t-s)^{\alpha-1} e^{-(t-s)\lambda} e^{-(t-s)A} u(s) ds. \qquad (8.0.5)$$

To prove (8.0.4), we first show that Λ_T generates a semigroup $\{M_t\}_{t\geq 0}$ on $L^p(0,T;X)$ and then we will use the representation

$$(\lambda + \Lambda)^{-1} = \int_0^\infty e^{-\lambda s} M_s ds.$$

Given this, we show that Λ_T is positive and then we apply the formula (8.0.3) above.

Remark 8.0.7 This result was essentially proven in [Br,97]. We collect together here the relevant background material and results needed to prove (8.0.4), which were not explicitly stated in [Br,97]. Although most of the results given here are considered well known, we often provide proofs which could not be found in the literature.

Throughout this Appendix we assume that $-A$ generates a C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ on X. It will be stated when we use the following additional assumptions:

(a1) A is a positive operator, i.e. $(\lambda + A)^{-1}$ is bounded for all $\lambda \geq 0$. In particular there exists $M \geq 1$ such that

$$| (\lambda + A)^{-1} | \leq \frac{M}{1+\lambda}, \quad \lambda \geq 0.$$

(a2) For all $s \in \mathbb{R}$, A^{is} is bounded and there exists $K \geq 1$ and $0 \leq \nu < \frac{\pi}{2}$ such that

$$| A^{is} | \leq K e^{\nu |s|}.$$

Remark 8.0.8 $(a2)$ will only ever be used in conjunction with $(a1)$, so that the imaginary powers appearing in $(a2)$ will make sense. Furthermore, $(a1)$ and $(a2)$ together imply that $-A$ is the generator of a uniformly bounded analytic semigroup on X, see [Pr/So,90]. ◇

The Operator $\mathcal{A}_T$

For $p \in [1,\infty)$, $T \in (0,\infty)$ define a linear operator $\mathcal{A}_T$ on $L^p := L^p(0,T;X)$ through the formula

$$(\mathcal{A}_T u)(t) = Au(t), \qquad \text{a.e. } t \in [0,T],\ u \in D(\mathcal{A}_T), \qquad (8.0.6)$$

where

$$D(\mathcal{A}_T) := \{u \in L^p(0,T;X) : \int_0^T | Au(t) |^p dt < \infty\}. \qquad (8.0.7)$$

We endow $D(\mathcal{A}_T)$ with the graph norm. Note that as A is closed then $D(A)$ is a Banach space endowed with the graph norm and hence $L^p(0,T;D(A))$ is also a Banach space. Noting that $D(\mathcal{A}_T) = L^p(0,T;D(A))$ with equivalent norms then $D(\mathcal{A}_T)$ is also a Banach space and therefore $\mathcal{A}_T$ is closed.
We now define a C_0-semigroup $\{P_t\}_{t\geq 0}$ on $L^p(0,T;X)$ which acts through the formula

$$[P_t u](r) = e^{-tA}(u(r)), \quad \text{a.e. } r \in [0,T],\ u \in L^p(0,T;X)\,. \qquad (8.0.8)$$

$\{P_t\}_{t\geq 0}$ is a C_0-semigroup on $L^p(0,T;X)$, which is a simple consequence of the fact that $\{e^{-tA}\}_{t\geq 0}$ is a C_0-semigroup on X. Moreover we have the following

Proposition 8.0.9 $-\mathcal{A}_T$ *is the generator of the* C_0*-semigroup* $\{P_t\}_{t\geq 0}$.

Proof: Let C be the generator of $\{P_t\}_{t\geq 0}$. We need to show that $D(-\mathcal{A}_T) = D(C)$ and $-\mathcal{A}_T u = Cu$ for $u \in D(-\mathcal{A}_T) = D(C)$. Let $f \in D(-\mathcal{A}_T)$ and consider

$$\mid \frac{P_t f - f}{t} + \mathcal{A}_T f \mid_{L^p}^p = \int_0^T \mid \frac{e^{-tA}(f(s)) - f(s)}{t} + Af(s) \mid_X^p ds.$$

Since $f \in D(-\mathcal{A}_T)$, then $f(s) \in D(A)$ a.e. $s \in [0,T]$. Thus as $-A$ generates $\{e^{-tA}\}_{t\geq 0}$ the above integrand tends to zero a.e. as $t \to 0$. Using the following inequality

$$\mid e^{-tA}(x) - x \mid_X^p \leq K(p,T) t^p \mid Ax \mid_X^p, \quad x \in D(A),$$

where $K(p,T)$ is a constant, see Chapter 2, we have, a.e.,

$$\mid \frac{e^{-tA}(f(s)) - f(s)}{t} + Af(s) \mid_X^p \leq K(p,T) \mid Af(s) \mid_X^p .$$

The RHS is independent of t and is integrable since $f \in D(\mathcal{A}_T)$. The Lebesgue Dominating Convergence Theorem then implies

$$\mid \frac{P_t f - f}{t} + \mathcal{A}_T f \mid_{L^p} \to 0 \text{ as } t \to 0,$$

i.e. $D(-\mathcal{A}_T) \subset D(C)$ and for $f \in D(-\mathcal{A}_T), -\mathcal{A}_T f = Cf$.

For the converse let $u \in D(C)$. Then for any $\lambda > 0$, $u \in R((\lambda - C)^{-1})$ and so there exists a unique $f \in L^p(0,T;X)$ such that

$$u = (\lambda - C)^{-1} f = \int_0^\infty e^{-\lambda t} P_t f dt.$$

Assume that f is continuous and so, in particular, $(\lambda - C)^{-1} f$ is continuous. Thus u has a continuous representative in $L^p(0,T;X)$. It follows that for each $s \in [0,T]$

$$u(s) = \int_0^\infty e^{-tA} P_t f(s) dt = \int_0^\infty e^{-\lambda t} e^{-tA} f(s) dt = (\lambda + A)^{-1}(f(s)). \tag{8.0.9}$$

Moreover for each $s \in [0,T]$, $u(s) \in D(A)$ with the following equality holding

$$Au(s) = A(\lambda + A)^{-1}(f(s)) = f(s) - \lambda(\lambda + A)^{-1} f(s). \tag{8.0.10}$$

The RHS of this equality defines a function belonging to $L^p(0,T;X)$ and so $\int_0^T \mid Au(s) \mid_X^p ds < \infty$, i.e. $u \in D(-\mathcal{A}_T)$.
Now consider the case where $u = (\lambda - C)^{-1} f$ with $f \in L^p(0,T;X)$. Let $\{f_n\}_{n\geq 1} \subset C(0,T;X)$ with $f_n \to f$ in $L^p(0,T;X)$. By what we have just proved

$$u_n = (\lambda - C)^{-1} f_n \in D(\mathcal{A}_T).$$

As $(\lambda - C)^{-1}$ is bounded then

$$\mid u - u_n \mid_{L^p} \to 0 \text{ as } n \to \infty.$$

Using (8.0.10) we deduce that

$$\mid \mathcal{A}_T u_n - \mathcal{A}_T u_m \mid_{L^p} \to 0 \text{ as } n, m \to \infty,$$

i.e. $\{\mathcal{A}_T u_n\}_{n\geq 1}$ is Cauchy in $L^p(0,T;X)$ and so has a limit, y, say. Thus $(u_n, \mathcal{A}_T u_n) \to (u,y)$ and so by the closedness of $\mathcal{A}_T$ we deduce that $u \in D(\mathcal{A}_T)$. It follows that $D(C) \subset D(\mathcal{A}_T)$, which completes the proof.

♠

(8.0.9) and (8.0.10) imply the following equalities for $f \in L^p(0,T;X)$

$$(\lambda + \mathcal{A}_T)^{-1} f(s) = (\lambda + A)^{-1}(f(s)) \text{ a.e.} \tag{8.0.11}$$

$$\mathcal{A}_T(\lambda + \mathcal{A}_T)^{-1} f(s) = A(\lambda + A)^{-1}(f(s)) \text{ a.e.} \tag{8.0.12}$$

Remark 8.0.10 $(a1) \Rightarrow (\mathcal{A}_T 1)$ and $(a2) \Rightarrow (\mathcal{A}_T 2)$ where $(\mathcal{A}_T 1)$ and $(\mathcal{A}_T 2)$ are the same assumptions just with A replaced by $\mathcal{A}_T$. Moreover constants M, K and ν appearing in $(a1)$ and $(a2)$ are the same for $(\mathcal{A}_T 1)$ and $(\mathcal{A}_T 2)$. To see this just note that $(a1) \Rightarrow (\mathcal{A}_T 1)$ follows from Proposition 8.0.9 and (8.0.11). $(a2) \Rightarrow (\mathcal{A}_T 2)$ follows from (8.0.12) and the general formula for fractional powers, see (8.0.2). ◇

Remark 8.0.11 The two conditions $(\mathcal{A}_T 1)$ and $(\mathcal{A}_T 2)$ together imply that the semigroup $\{P_t\}_{t\geq 0}$ generated by $-\mathcal{A}_T$ is a uniformly bounded analytic C_0-semigroup, see Remark 8.0.8. ◇

The Operator B_T

For $p \in [1,\infty)$ and $T \in (0,\infty)$ fixed, let B_T be the linear unbounded operator in $L^p(0,T;X)$ defined by

$$B_T = u' \quad u \in D(B_T), \tag{8.0.13}$$

$$\text{where } D(B_T) := \{u \in W^{1,p}(0,T;X) : u(0) = 0\}. \tag{8.0.14}$$

Recall that u' is the weak derivative of u.
It is well known, see [G/G/K,90] for example, that the operator $-B_T$ generates a C_0-semigroup of contractions on the Banach space $L^p(0,T;X)$ and this semigroup, denoted by $\{S_t\}_{t\geq 0}$, acts through the formula

$$[S_t u](r) = \begin{cases} u(r-t) & \text{if } 0 \leq t \leq r \\ 0 & \text{otherwise} \end{cases} \tag{8.0.15}$$

for a.e. $r \in [0,T]$, $u \in L^p(0,T;X)$
It follows that $-B_T$ is a densely defined, closed, linear operator. In fact even more is true from the following result, see [Do/Ve,87]:

$(\mathbf{B_T 1})$ B_T is positive i.e. for $\lambda \geq 0$, $(\lambda + B_T)^{-1}$ is bounded and there exists M_1 such that

$$|(\lambda + B_T)^{-1}| \leq \frac{M_1}{1+\lambda}, \quad \lambda \geq 0. \tag{8.0.16}$$

$(\mathbf{B_T 2})$ If X is an UMD Banach space, then for $s \in \mathbb{R}$, B_T^{is} is bounded and for some $K_2 > 0$

$$|B_T^{is}| \leq K_2(1+s^2)e^{\frac{\pi}{2}|s|}. \tag{8.0.17}$$

♡

The Semigroup $\{M_t\}_{t\geq 0}$

The two semigroups $\{S_t\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$ enjoy the property that they are commuting semigroups, i.e. for s, $t \geq 0$,

$$S_t P_s = P_s S_t. \tag{8.0.18}$$

To see this, let f be continuous and so for all $r \in [0, T]$ we have

$$\begin{aligned}
[S_t(P_s f)](r) &= \begin{cases} (P_s f)(r-t) & \text{if } 0 \leq t \leq r \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} e^{-sA} f(r-t) & \text{if } 0 \leq t \leq r \\ 0 & \text{otherwise} \end{cases} \\
&= e^{-sA}[(S_t f)(r)] \\
&= [P_s(S_t f)](r).
\end{aligned}$$

The case of a general $f \in L^p(0,T;X)$ follows by the standard limit argument. There are two important consequences of this fact. The first follows directly from the Fubini Theorem and property (8.0.18). The second is a consequence of (8.0.18) and the semigroup properties of $\{P_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$.

Lemma 8.0.12 *$-\mathcal{A}_T$ and $-\mathcal{B}_T$ are resolvent commuting, i.e. for all λ, $\mu > 0$*

$$(\lambda + \mathcal{A}_T)^{-1}(\mu + \mathcal{B}_T)^{-1} = (\mu + \mathcal{B}_T)^{-1}(\lambda + \mathcal{A}_T)^{-1}.$$

♠

Proposition 8.0.13 *Set $M_t = P_t S_t = S_t P_t$, $t \geq 0$. Then $\{M_t\}_{t\geq 0}$ is a C_0-semigroup on $L^p(0,T;X)$.*

♠

Remark 8.0.14 Under the assumptions $(a1)$ and $(a2)$ then, see Remark 8.0.11, $\{P_t\}_{t\geq 0}$ is uniformly bounded, i.e.

$$\exists M_2 \geq 1 \text{ such that } \mid P_t \mid\leq M_2, \quad \forall t \geq 0$$

As $\{S_t\}_{t\geq 0}$ is a contraction semigroup on $L^p(0,T;X)$ we then have

$$\mid M_t \mid\leq\mid S_t \mid\mid P_t \mid\leq M_2$$

i.e. $\{M_t\}_{t\geq 0}$ is a uniformly bounded C_0-semigroup with the same uniform bound as $\{P_t\}_{t\geq 0}$. ◊

The Parabolic Operator Λ_T

Define the linear operator Λ_T on $L^p(0,T;X)$ by

$$\Lambda_T := B_T + \mathcal{A}_T, \tag{8.0.19}$$

$$D(\Lambda_T) := D(B_T) \cap D(\mathcal{A}_T). \tag{8.0.20}$$

Assuming A is injective and hence also $\mathcal{A}_T$, we endow $D(\Lambda_T)$ with the graph norm

$$| u |^p_{D(\Lambda_T)} = | \mathcal{A}_T u |^p_{L^p} + | B_T u |^p_{L^p} .$$

Since B_T^{-1} is bounded then $D(\Lambda_T)$ is complete with respect to this norm.

We first show that the operator $-\Lambda_T$ is the generator of the semigroup $\{M_t\}_{t\geq 0}$. The proof of the following Theorem relies on the closedness of the operator Λ_T. Although Λ_T is the sum of two closed operators $\mathcal{A}_T$ and B_T, in the case of general Banach spaces, it does not necessarily follow that their sum is closed. Dore and Venni first considered the problem of the closedness of the operator Λ_T, see [Do/Ve,87]. They showed that one needs to impose conditions on both the Banach space X and the operators $\mathcal{A}_T$ and B_T to guarantee closedness of Λ_T. The conditions imposed on the operators are: positivity, see $(\mathcal{A}_T 1)$ and $(B_T 1)$; resolvent commutativity, see Lemma 8.0.12; boundedness of the imaginary powers, see $(\mathcal{A}_T 2)$ and $(B_T 2)$. For the Banach space we need to impose the so-called UMD condition, or equivalently the ζ-convexity condition, see [Bu,86] and references therein. This property relates, in some sense, to the geometry of the Banach space. A necessary and sufficient condition for a Banach space X to be UMD is that the Hilbert transform is a continuous operator from $L^p(\mathbb{R}, X)$ to itself. Any Hilbert space is UMD. Moreover the interpolation spaces of UMD spaces are again UMD. Finally, if X is UMD, then so is $L^p(0,T;X)$, $p \geq 1$.
For a discussion on the motivations for the conditions sufficient for the closedness of Λ_T, see [Do/Ve,87] and references therein.

Theorem 8.0.15 (Dore-Venni) *Let X be an UMD Banach space and assume $(a1)$ and $(a2)$ hold. Then Λ_T is a densely defined, closed, nonnegative operator with Λ_T^{-1} bounded. In particular for each $f \in L^p(0,T;X)$ there exists a unique $u \in D(\Lambda_T)$ with $\Lambda_T u = f$ and*

$$| \mathcal{A}_T u |_{L^p_T} + | B_T u |_{L^p_T} \leq C \, | \Lambda_T u |_{L^p_T} \tag{8.0.21}$$

where $C = C(p, \nu, \tilde{K}, T, X)$ is a constant independent of f.

The following Theorem is fundamental in what we aim to eventually prove.

Theorem 8.0.16 *Assume that X is an UMD Banach space, $(a1)$ and $(a2)$. Then $-\Lambda_T$ is the generator of the C_0-semigroup $\{M_t\}_{t\geq 0}$.*

<u>Proof :</u> Let Q be the generator of $\{M_t\}_{t\geq 0}$. We first show that $D(\Lambda_T) \subset D(Q)$ and $-\Lambda_T f = Qf$ for $f \in D(\Lambda_T)$. Let $f \in D(\Lambda_T)$ then we have

$$\begin{aligned} \left| \frac{M_t f - f}{t} + \Lambda_T f \right| &= \left| \frac{S_t P_t f - f}{t} + \Lambda_T f \right| \\ &\leq \left| \frac{S_t P_t f - S_t f}{t} + S_t \mathcal{A}_T f \right| + | \mathcal{A}_T f - S_t \mathcal{A}_T f | \\ &\quad + \left| \frac{S_t f - f}{t} + B_T f \right| \\ &\to 0 \text{ as } t \to 0 \end{aligned}$$

since $\{S_t\}_{t\geq 0}$ is strongly continuous and bounded and $-\mathcal{A}_T$, $-B_T$ generate $\{P_t\}_{t\geq 0}$ and $\{S_t\}_{t\geq 0}$ respectively.
For the converse, note that Q (being the generator) is closed and Λ_T is closed by Theorem 8.0.15. Thus there exists $\lambda > 0$ such that $(\lambda - Q)^{-1}$ and $(\lambda + \Lambda_T)^{-1}$ are both everywhere defined bounded linear operators, see [Yo,69]. Let $f \in D(Q)$ and so $(\lambda - Q)f \in X$. There exists $g \in D(\mathcal{A}_T)$ such that

$$(\lambda + \Lambda_T)g = (\lambda - Q)f.$$

In particular, as $D(\Lambda_T) \subset D(Q)$ then $(\lambda + \Lambda_T)g = (\lambda - Q)g$. We deduce that

$$f = (\lambda - Q)^{-1}(\lambda - Q)f = (\lambda - Q)^{-1}(\lambda - Q)g = g \in D(\Lambda_T),$$

which implies $D(Q) \subset D(\Lambda_T)$. This completes the proof of Theorem 8.0.10.

♠

We can say even more about our operator Λ_T:

Corollary 8.0.17 *Assume that X is an UMD Banach space, (a1) and (a2). Then Λ_T is positive and for $\mu > 0$, $f \in L^p(0,T;X)$, we have*

$$[(\mu + \Lambda_T)^{-1} f](t) = \int_0^t e^{-\mu(t-u)} e^{-(t-u)A} f(u)du \ \ a.e.. \tag{8.0.22}$$

Proof: Since $-\Lambda_T$ generates $\{M_t\}_{t\geq 0}$ then for $\mu > 0$ we have

$$(\mu + \Lambda_T)^{-1} f = \int_0^\infty e^{-\lambda t} M_t f dt. \tag{8.0.23}$$

Let $f \in C(0,T;X)$, then $(\mu+\Lambda_T)^{-1}f$ is continuous. Using (8.0.23) and the defintion of M_t, one may show by calculation that for each $s \in [0,T]$

$$[(\mu + \Lambda_T)^{-1} f](s) = \int_0^s e^{-\mu(s-u)} e^{-(s-u)A} f(u)du.$$

Using the boundedness of $(\mu + \Lambda_T)^{-1}$, (8.0.22) follows for general $f \in L^p(0,T;X)$ by the standard limit argument. To prove positivity, recall that, by Theorem 8.0.15, $(\mu + \Lambda_T)^{-1}$ is bounded for $\mu \geq 0$. We are thus left with proving the inequality of the form (8.0.1). Using (8.0.22) and the Young inequality we have for $\mu > 0$

$$\begin{aligned} | (\mu + \Lambda_T)^{-1} f |_{L^p}^p &\leq e^{pT}\tilde{M}^p \int_0^T \left(\int_0^s e^{-(\mu+1)(s-u)} \mid f(u) \mid_X du\right)^p ds \\ &\leq C(p,T,\tilde{M}) \frac{| f |_{L^p}^p}{1+\mu}. \end{aligned}$$

As Λ_T^{-1} is bounded then we can deduce that $\exists\ C_1 \geq 1$ such that

$$| (\mu + \Lambda_T)^{-1} | \leq \frac{C_1}{1+\mu}, \quad \mu \geq 0.$$

♠

The Fractional Powers $\Lambda_T^{-\alpha}$

We will now prove that R_α^λ is actually the fractional power $\Lambda_T^{-\alpha}$ of the operator Λ_T, modulo the constant $\Gamma(\alpha)$.

Theorem 8.0.18 *Assume that X is an UMD Banach space and that (a1) and (a2) hold. Then for $\alpha \in (0,1)$, $\lambda \geq 0$ and $f \in L^p(0,T;X)$*

$$(\lambda + \Lambda_T)^{-\alpha} f = \frac{1}{\Gamma(\alpha)} R_\alpha^\lambda f,$$

where $R_\alpha^\lambda f$ is given by (8.0.5).

Proof : Since $-\Lambda_T$ generates $\{M_t\}_{t\geq 0}$ then for any $\lambda > 0$, $-\Lambda_T^\lambda := -(\lambda + \Lambda_T)$ generates $\{e^{-\lambda t} M_t\}_{t\geq 0}$. Thus for $\mu > 0$ we have for $f \in C(0,T;X)$

$$[(\mu + \Lambda_T^\lambda)^{-1} f](t) = \left(\int_0^\infty e^{-(\lambda+\mu)s} M_s f ds\right)(t) = \int_0^t e^{-(\lambda+\mu)s} e^{-tA} f(t-s) ds.$$

Using (8.0.3), the Fubini Theorem and noting that $\frac{\sin(\pi\alpha)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$, we have, for $\alpha \in (0,1)$,

$$\begin{aligned} ((\lambda + \Lambda_T)^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \lambda^{-\alpha} \int_0^t e^{-(\lambda+\mu)s} e^{-sA} f(t-s) ds d\mu \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t e^{-\lambda s} e^{-sA} f(t-s) (\int_0^\infty \lambda^{-\alpha} e^{-\mu s} d\mu) dt. \end{aligned}$$

Making the substitution $u = \mu t$, then

$$\int_0^\infty \mu^{-\alpha} e^{-\mu s} d\mu = s^{\alpha-1} \int_0^\infty u^{-\alpha} e^{-u} du = s^{\alpha-1}\Gamma(1-\alpha).$$

It then follows

$$\begin{aligned} ((\lambda + \Lambda_T)^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} e^{-\lambda s} e^{-sA} f(t-s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)\lambda} e^{-(t-s)A} f(s) ds \\ &= R_\alpha^\lambda f(t). \end{aligned}$$

The case of general $f \in L^p(0,T;X)$ follows by the standard limit argument.

♠

Remark 8.0.19 The results of this chapter rely essentially on the Dore-Venni Theorem. The assumptions necessary to apply this result are quite strong. In particular, one needs to assume that both $\mathcal{A}_T$ and $\mathcal{B}_T$ are invertible with bounded inverse. In the paper [Pr/So,90], Prüss-Sohr were able to prove the same result (i.e. Theorem 8.0.15) under the weaker assumption that $\mathcal{A}_T$ is only nonnegative. This reduces to

assuming that the operator A is nonegative, i.e. we may replace (a1) with
(a1*) $(\lambda + A)^{-1}$ is bounded $\forall\lambda > 0$ and there exists $M \geq 1$ such that

$$|(\lambda + A)^{-1}| \leq \frac{M}{\lambda}, \quad \lambda > 0.$$

The assumption (a2) is still needed. For a nonnegative operator, one may still define the fractional powers, see [Kom,66], but the formulas for them are more involved than the formula (8.0.2). On inspection of our proof though, we do not explicity use the representation (8.0.2) for the imaginary powers A^{is}, we only need to know they are bounded. Thus our result holds also in the case where our operator A satisfies (a1*) and (a2). In particular, see [Se,71], the operator $-A := \frac{d^2}{d\sigma^2}$ on $L^p(S^1, \mathbb{R}^d)$ used in the problem of the stochastic nonlinear heat equation satisfies the required assumptions.

♡

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