Journal of Banking & Finance 50 (2015) 260–270

Contents lists available at ScienceDirect

Journal of Banking & Finance

journal homepage: www.elsevier.com/locate/jbf

Dynamic interaction between markets for leasing and selling automobiles

Athanasios Andrikopoulos [a,1], Raphael N. Markellos [b,*]

[a] Hull University Business School, University of Hull, Hull HU6 7RX, UK
[b] Norwich Business School, University of East Anglia, Norwich Research Park, Norwich NR4 7TJ, UK

ARTICLE INFO

Article history:
Received 6 July 2012
Accepted 21 January 2014
Available online 26 March 2014

JEL classification:
D43
L62
E32
C73

Keywords:
Interacting markets
Automobiles
Differential Games
Leasing
Valuation

ABSTRACT

We develop a model of dynamic interactions between price variations in leasing and selling markets for automobiles. Our framework assumes a differential game between multiple Bertrand-type competing firms which offer differentiated products to forward-looking agents. Empirical analysis of our model using monthly US data from 2002 to 2011 shows that variations in selling (cash) market prices lead rapidly dissipating changes of leasing market prices in the opposite direction. We discuss the practical implications of these results by augmenting a standard leasing valuation formula. The additional terms represent the leased asset value changes that can be expected on the basis of past variations in automobile selling market prices.

1. Introduction

For households in developed countries the automobile is typically the second largest asset purchased after a house and is the most commonly held non-financial asset (Aizcorbe et al., 2003). In the US, one third of all cars sold is financed via leasing (e.g., see Hendel and Lizzeri, 2002; Johnson and Waldman, 2003) while a comparable proportion of sales involves cash transactions (Mannering et al., 2002; Dasgupta et al., 2007). Despite its importance, the exact association between leasing markets and cash markets (also known as selling markets) is not yet fully understood. Although some theoretical models exist (see Bulow, 1982, 1986; Bucovetsky and Chilton, 1986; Purohit and Staelin, 1994; Purohit, 1997; Desai and Purohit, 1998, 1999; Saggi and Vettas, 2000; Huang et al., 2001), they are mostly static in nature and make the unrealistic assumption of perfect substitutability. Moreover, no study examines the empirical link between leasing and selling markets for automobiles. The objective of the present paper is to shed further light on this relationship. At a theoretical level, we make more generic assumptions which permit for dynamic interactions and imperfect substitutability. At an empirical level, we use US monthly data to model for the first time the dynamic relationship between leasing and selling market price variations. Our results motivate us to develop a new dynamic leasing asset pricing approach for automobiles whereby shocks in selling market prices are allowed to have a dissipative effect on leasing market prices and residual values.

In the next section we review the relevant literature. Section 3 lays out our model for describing the interaction between price variations for automobiles in leasing and selling markets. Section 4, estimates empirically the model using monthly US CPI data and discusses the implications of the results for leasing valuation. The final section concludes the paper.

2. Literature review

2.1. The relationship between leasing and selling markets

The earliest attempts in understanding the association between leasing and selling markets originate in the investigation of

* Corresponding author. Tel.: +44 (0)1603 597395; fax: +44 (0)1603 593343.
E-mail addresses: a.andrikopoulos@hull.ac.uk (A. Andrikopoulos), r.markellos@uea.ac.uk (R.N. Markellos).
[1] Tel.: +44 (0)1482 347547; fax: +44 (0)148246384.

http://dx.doi.org/10.1016/j.jbankfin.2014.01.032
0378-4266/

decisions made by agents in the markets for durable goods under the so-called durable goods monopoly problem (see Coase, 1972; Stokey, 1981; Bulow, 1982, 1986; Gul et al., 1986; Bucovetsky and Chilton, 1986; Purohit and Staelin, 1994; Purohit, 1997; Desai and Purohit, 1998, 1999; Saggi and Vettas, 2000). Most of these papers assume that leasing and selling are perfect substitutes with market participants that are indifferent between the two alternatives. Moreover, the focus of these studies is to investigate the conditions under which leasing is the optimal strategy in the context of different market structures. A related strand of literature examines the relationship between the markets for new and used automobiles. From a static perspective, Bresnahan (1981), Berry et al. (1995), Goldberg (1995) and Petrin (2002) gauge the market power of introducing new products in the automobile industry. However, as argued by Blanchard and Melino (1986) it is important to employ a dynamic approach for at least two reasons which are discussed below. First, dynamics may arise in durable goods models of two interacting markets where used cars constitute stock variables which are imperfect substitutes to new cars. For example, Berkovec (1985) uses the econometric estimates of a short-run model to forecast sales and other automobile industry variables. Rust (1985, 1986) concentrates on dynamic consumer demand in durable goods with new, used and scrappage markets for automobiles. Transaction costs in a dynamic setting are considered by Konishi and Sandfort (2002), Stolyarov (2002) and Schiraldi (2011). Esteban and Shum (2007) model the production decision of a firm in a discrete dynamic oligopoly setting in which automobile prices are endogenously determined. Adda and Cooper (2000a) build a dynamic stochastic discrete choice model of car ownership at the individual level in order to study the output and public finance effects of subsidies on automobile demand. Eberly (1994) and Attanasio (2000) study (S, s) models of household automobile demand with transaction costs and liquidity constraints. Second, forward-looking dynamics may arise also in the demand side of the durable goods market on the basis of consumer expectations of future prices for new cars. In this case consumers are not myopic towards the future since they consider their expected utility while making their primary decisions on if and when to buy. Chen et al. (2008, 2010) construct a calibrated equilibrium time consistent dynamic oligopoly model of a durable goods market, which incorporates both the sources of dynamics mentioned previously. In particular, Chen et al. (2008) ignore the dynamics by evaluating the bias in estimating the structural parameters of a static model. Chen et al. (2010) incorporate transaction costs in the used market which makes purchases important on the demand side.

A prominent issue in the durable goods markets is the possibility of oscillatory behavior. Sobel (1991) and Conlisk et al. (1984) consider a new group of consumers, with a heterogeneity of tastes, which enters the market sequentially and leads the monopolist to fluctuate the equilibrium price periodically (Sobel, 1984, studies the same problem in an oligopoly setting). Board (2008) considers the pricing behavior of a durable goods monopolist for a new good where agents can strategically time their purchases and where the demand fluctuates exogenously over time. Janssen and Karamychev (2002) allow for information asymmetry in a dynamic competitive model of identical generations entering the market over time. Caplin and Leahy (2006) develop an (S, s) model of oscillations in demand which reflects fluctuations in the number of consumers who purchase the durable goods as well as of variations in the demand of a single consumer. They use this model to analyze the equilibrium dynamics of prices, the number of purchases and the size of purchases of the durable goods. Empirical evidence by Bils and Klenow (1998) confirms that durable goods prices have a tendency to move procyclically relative to prices of nondurable goods. Blanchard and Melino (1986) construct a competitive equilibrium model with representative consumers and firms. Their intention is to understand the common cyclical behavior of prices and quantities in a certain market for automobiles. Finally, Adda and Cooper (2000b) concentrate on the demand side and estimate a VAR(1) model of aggregate income, relative prices of cars and consumer preference shocks. They report that the impulse response function exhibits dampened oscillations in response to an income shock. This is explained on the basis of two reasons. First, due to non-convex adjustment costs with heterogeneous consumers, the endogenous growth of the stock of cars can generate replacement cycles and subsequent oscillations in sales. Second, the oscillations can arise from the serial correlation in income and prices.

2.2. Automobile leasing and selling market structure

The literature identifies and studies different ways in which the market for durable goods can be organized. One strand argues that the optimal strategy for a durable goods firm is to try and operate in a balanced manner in both the selling and leasing market (see Bulow, 1986; Desai and Purohit, 1998, 1999; Saggi and Vettas, 2000; Huang et al., 2001; Hendel and Lizzeri, 2002; Bhaskaran and Gilbert, 2005, 2009). In this setting leasing firms function as subsidiaries of manufacturers as, for example, General Motors Lease and Ford Credit Lease. In another setup, which is relevant to the standard consumer lease agreement framework, the lessor is a financial institution which buys on behalf of the lessee a new vehicle from a licensed automobile dealer and then leases it to a lessee (Myers et al., 1976; Giaccotto et al., 2007). Since the core business of the lessor, which is usually a retail bank or a personal finance company, does not involve selling used cars, the leased car is sold through the wholesale used car market. In this way, the lessor is neither a manufacturer nor a dealer of automobiles.

Eisfeldt and Rampini (2009) argue that firms do not operate businesses in both leasing and selling spheres. This is because in line with what practitioners argue, the possibility of the lessors to take back an asset allows them to implicitly expand more credit than lenders whose claims are protected by the same asset. Dasgupta et al. (2007) describe the elements of typical dealer financing contracts and leasing contracts and how these differ. The dealer financing contract can be described by the base price of the vehicle, the annual percentage rate, the payment period (or term) and cash rebates. The alternative is leasing, which entails financing the user cost of the vehicle rather than its entire purchase price. Consequently, leasing has lower down- and monthly-payments. Lease payments could be as low as one third of those required to buy the car. This makes leasing an attractive choice for credit constrained consumers and also allows them to acquire more luxurious cars. In terms of popularity, Dasgupta et al. (2007) report that 24.2% of the transactions were leased, 35% of the sales were dealer-financed, while the remaining 40.8% were categorized as "cash" transactions (most likely these were financed elsewhere rather than being actually paid for by cash).

Another interesting case is the separate channel described by Purohit (1997). This characterizes the state of the industry in which rental agencies and dealers are licensed solely to rent and to sell cars, respectively, and compete between them. Finally, there are also the third-party independent lessors, which are neither banks nor dealers (Myers et al., 1976; Giaccotto et al., 2007; Bhaskaran and Gilbert, 2009).

3. Model formulation

In this section we build a framework for modeling in a dynamic manner the interaction between leasing and selling market prices for automobiles. Such a dynamic setting has been studied previously only at a theoretical level by Huang et al. (2001). They construct a dynamic monopoly model of leasing, selling and used goods markets, respectively, with finite duration under an infinite time horizon and nontrivial transaction costs. Although our approach does not consider transaction costs and used goods, we assume a more realistic oligopoly setting. So, for the first time we study a dynamic oligopoly model of leasing and selling in which leases and loans are imperfect substitutes. Although, our model of the leasing market is not the first dynamic model, it is the first dynamic oligopoly model of the leasing market. Our approach is also closely related to that of Esteban and Shum (2007) although they concentrate on modeling the interaction between new and used car markets. We also differ from Esteban and Shum (2007) in the focus of our models. Specifically, they assume a discrete time approach and analyze the stage in which firms determine the new car designs. Our continuous time model deals with the stage in which producers set prices conditional on product types. This distinction makes the model of Esteban and Shum (2007) backward looking, since the production choices of the firm today depend on cars produced in the past. In our setting, firms are forward looking since their price choices depend on the future.

Another important characteristic of our approach is that unlike most of the previous literature it does not treat selling and leasing of automobiles as perfect substitutes. As pointed out by Dasgupta et al. (2007), the assumption of substitutability is unrealistic for at least three reasons. First, automobile leasing contracts differ significantly from selling contracts in that the former typically comprise of several terms and conditions, such as the price, the interest rate, the installment and the maturity of the contract. Second, differences in the discount factor used by consumers can also lead to differences in the evaluation of leasing versus selling decisions. Third, in the case of leasing the decision also involves non-financial clauses related to, for example, operating and maintenance costs.

As discussed in the previous section, following Myers et al. (1976), Giaccotto et al. (2007), Eisfeldt and Rampini (2009), Purohit (1997) and Dasgupta et al. (2007), we assume the following channel structure at the retailer level: The lessor (which can be a bank, a finance company or a third-party independent lessor) purchases the automobiles from the manufacturer and then leases them out to clients. The automobile dealer or seller, we use these terms interchangeably, also buys the automobiles from the manufacturer but then sells them to clients either through loans or a cash transaction.

3.1. General case

For simplicity, we assume a market with a fixed number of n lessors and m sellers of automobiles which offer differentiated services in each market, respectively. Following Miller and Upton (1976) and Agarwal et al. (2011), we further assume that the representative firms have as control variables the rates and not the prices of their products, emphasizing in this way the financial aspect of the lease contracts. Finally, we also assume that leasing services are differentiated from selling services. So, the representative lessor (seller) in every period competes with the other lessors (sellers)-within market competition, and at the same time with the sellers (lessors) in the other market-between markets competition.

As argued by Dudine et al. (2006), the dynamic demand is driven by both the durability of the product and by the anticipation of consumers for the future prices. However, in the present setting, as Chen et al. (2008), we assume for simplicity that the consumer decisions whether to buy an automobile depend on their expectations about future market prices which create forward-looking dynamics in the demand function, and, subsequently in the decisions of the firms. In this manner, the demand depends on the current rate level as well as on its time derivative.

As Goldberg (1995), we focus on the second stage of a two stage game. Specifically, since the market is an oligopoly with differentiated products, the supply decisions and the market equilibria involve two stages. First, a long-run stage, in which firms determine the product-mix and the quality of their products, and, second, a short-run stage in which producers set prices given their product types. Since automobiles are durable goods, the representative lessor, faces the following intertemporal problem:

$$\begin{aligned}&\max_{R_{L_i}(t)} \int_0^\infty e^{-\rho t}(R_{L_i}(t) - c_{L_i})q_{L_i}(t)dt\\&\text{s.t. } q_{L_i}(R_{L_i}(t), R_{L_{-i}}(t), R_k(t), \dot{R}_{L_i}(t), \dot{R}_{L_{-i}}(t), \dot{R}_k(t))\end{aligned} \tag{1}$$

The representative seller faces the following problem:

$$\begin{aligned}&\max_{R_{k_j}(t)} \int_0^\infty e^{-\rho t}(R_{k_j}(t) - c_{k_j})q_{k_j}(t)dt\\&\text{s.t. } q_{k_j}(R_{k_j}(t), R_{k_{-j}}(t), R_L(t), \dot{R}_{k_j}(t), \dot{R}_{k_{-j}}(t), \dot{R}_L(t))\end{aligned} \tag{2}$$

where $R_i(t)$ is the rate of firm i at time t and the subscripts L and k denote leasing and selling, respectively; $R_{L_{-i}}(t)$ and $R_{k_{-j}}(t)$ are the vectors of lease and sell rates of lessor's i, sellers j, "within" rivals. In other words, the vector $R_{L_{-i}}(t)$ is the vector of all the lease rates other than the lease rate of firm i, and the vector $R_{k_{-j}}(t)$ is the vector of all the sell rates other than the sell rate of firm j. So,

$$R_{L_{-i}}(t) = \begin{bmatrix} R_{L_1}(t) \\ \vdots \\ R_{L_{i-1}}(t) \\ R_{L_{i+1}}(t) \\ \vdots \\ R_{L_n}(t) \end{bmatrix}, R_{k_{-j}}(t) = \begin{bmatrix} R_{k_1}(t) \\ \vdots \\ R_{k_{j-1}}(t) \\ R_{k_{j+1}}(t) \\ \vdots \\ R_{k_m}(t) \end{bmatrix};$$

where $R_L(t)$, $R_k(t)$ are the vectors of lease and sell rates, "between" rivals, respectively. With dots denoting time-derivatives we have:

$$\dot{R}_{L_{-i}}(t) = \begin{bmatrix} \dot{R}_{L_1}(t) \\ \vdots \\ \dot{R}_{L_{i-1}}(t) \\ \dot{R}_{L_{i+1}}(t) \\ \vdots \\ \dot{R}_{L_n}(t) \end{bmatrix}, \dot{R}_{k_{-j}}(t) = \begin{bmatrix} \dot{R}_{k_1}(t) \\ \vdots \\ \dot{R}_{k_{j-1}}(t) \\ \dot{R}_{k_{j+1}}(t) \\ \vdots \\ \dot{R}_{k_m}(t) \end{bmatrix}$$

where c_i is the opportunity cost of capital (WACC) of firm i. In order to maximize his profits the representative lessor chooses the instantaneous lease rate $R_{L_i}(t) = \frac{dP_{L_i}}{P_{L_i}} = \frac{L_i(t)}{P_{L_i}(t)}$, or, in discrete time $R_{L_i}(t) = \frac{\Delta P_{L_i}}{P_{L_{i,t-1}}} = \frac{P_{L_{i,t}} - P_{L_{i,t-1}}}{P_{L_{i,t-1}}}$. $L_i(t)$ is the default free lease payment paid at the beginning of the period; $P_{L_i}(t)$ and $P_{k_j}(t)$ represent the leased and the purchased asset prices set from i lessor and j seller, respectively, at the beginning of period. As with the lessor, the representative seller chooses the instantaneous sell rate, $R_{k_j}(t) = \frac{dP_{k_j}}{P_{k_j}}$, or, $R_{k_j}(t) = \frac{\Delta P_{k_j}}{P_{k_{j,t-1}}} = \frac{P_{k_{j,t}} - P_{k_{j,t-1}}}{P_{k_{j,t-1}}}$ in order to maximize profits.[2]

In line with Goldberg (1995), firms are assumed to be free of quantity constraints while attempting to maximize the present value of profits between consecutive market periods. They use the same discount factor $e^{-\rho t} \in (0,1)$, where ρ denotes the common

[2] Since the seller has already chosen $P_{K_j,t-1}$ in the previous period, the assumption of choosing $R_{K_j,t}$ is equivalent to the assumption of choosing $P_{K_j,t}$.

rate of discounting and corresponds to their WACC, i.e., $\rho = c_{L_i} = c_{k_j}$ for all firms. Finally, the demand functions are linear and set equal to[3]:

$$\begin{aligned} q_{L_i}(t) &= \theta_0 + \theta_{L_i}R_{L_i}(t) + \theta_{L_{-i}}R_{L_{-i}}(t) + \theta_{k_j}R_{k_j}(t) + \theta_{k_{-j}}R_{k_{-j}}(t) + \theta_{\dot{L}_i}\dot{R}_{L_i}(t) \\ &\quad + \theta_{\dot{L}_{-i}}\dot{R}_{L_{-i}}(t) + \theta_{\dot{k}_j}\dot{R}_{k_j}(t) + \theta_{\dot{k}_{-j}}\dot{R}_{k_{-j}}(t) \\ q_{k_j}(t) &= \lambda_0 + \lambda_{L_i}R_{L_i}(t) + \lambda_{L_{-i}}R_{L_{-i}}(t) + \lambda_{k_j}R_{k_j}(t) + \lambda_{k_{-j}}R_{k_{-j}}(t) \\ &\quad + \lambda_{\dot{L}_i}\dot{R}_{L_i}(t) + \lambda_{\dot{L}_{-i}}\dot{R}_{L_{-i}}(t) + \lambda_{\dot{k}_j}\dot{R}_{k_j}(t) + \lambda_{\dot{k}_{-j}}\dot{R}_{k_{-j}}(t) \end{aligned} \tag{3}$$

We now discuss the parameters of the above demand functions, by organizing them into five sets:

1. The parameters θ_0, λ_0 are the intercepts and they are always positive. θ_{L_i} and λ_{k_j} are the slopes of the demand curves and they are always negative since the demand for the specific good is downward sloping in its own rate. A demand curve with only these two parameters is a standard demand curve.
2. $\theta_{L_{-i}} = [\theta_{L_1}, \ldots, \theta_{L_{i-1}}, \theta_{L_{i+1}}, \ldots, \theta_{L_n}]$ is the vector of the coefficients of all the lease rates other than the lease rate of firm *i*. It reflects the extent to which the lease contract of lease firm *i* is a substitute of the lease contract of lease firm $s \neq i$. The vector $\lambda_{k_{-j}} = [\lambda_{k_1}, \ldots, \lambda_{k_{j-1}}, \lambda_{k_{j+1}}, \ldots, \lambda_{k_m}]$ has a similar interpretation and contains the coefficients of all the sell rates other than the sell rate of firm *j*. As for selling contracts, we expect the coefficients in both vectors to be positive since the lease contracts are substitutes.
3. The demand structure at hand allows a range of different degrees of substitutability between the contracts. The coefficient θ_{k_i} reflects the impact of the sell rates of seller *j* on the quantity of lease contracts of lessor *i*. The vector of coefficients $\theta_{k_{-j}} = [\theta_{k_1}, \ldots, \theta_{k_{j-1}}, \theta_{k_{j+1}}, \ldots, \theta_{k_m}]$ reflects the impact of the sell rates of the sellers other than *j*, on the quantity of lease contracts of lessor *i*. The interpretation is similar for the coefficient λ_{L_i} and the vector of coefficients $\lambda_{L_{-i}} = [\lambda_{L_1}, \ldots, \lambda_{L_{i-1}}, \lambda_{L_{i+1}}, \ldots, \lambda_{L_n}]$ they reflect the impact of the lease rates on the quantity of the sell contracts of seller *j*. In general, we consider that selling and leasing are substitutes when $\theta_{k_j}, \theta_{k_{-j}}, \lambda_{L_i}, \lambda_{L_{-i}} > 0$, while they are complements when $\theta_{k_j}, \theta_{k_{-j}}, \lambda_{L_i}, \lambda_{L_{-i}} < 0$. However, as recently discussed by De Jaegher (2009), the above definitions refer specifically to weak symmetric gross substitutes and weak symmetric gross complements, respectively. In general, the definition of substitutability and complementarity requires that not only the signs but also the absolute values of the coefficients $\theta_{k_j}, \theta_{k_{-j}}, \lambda_{L_i}, \lambda_{L_{-i}}$ are the same. De Jaegher considers the case when $\theta_{k_j}, \theta_{k_{-j}}, \lambda_{L_i}, \lambda_{L_{-i}}$ have opposite signs in which two goods are strong asymmetric substitutes. In our setting, selling would be a substitute for leasing and leasing would be a complement to selling, or, vice versa. If, for example, $\theta_{k_j}, \theta_{k_{-j}} < 0$ and $\lambda_{L_i}, \lambda_{L_{-i}} > 0$ then the demand for leasing is a decreasing function of the purchase rate so leasing is a gross substitute of purchasing. At the same time, the demand for purchasing is an increasing function of lease rate so that purchasing is a gross complement of leasing.
4. Specification of $\theta_{\dot{L}_i}, \theta_{\dot{L}_{-i}}, \lambda_{\dot{k}_j}$ and $\lambda_{\dot{k}_{-j}}$ depends on expectations. In our model, consumers form expectations in a perfect foresight manner according to $\frac{dR^e}{dt} = \frac{dR}{dt}$, the time-derivative of the specific rate. This constitutes the deterministic equivalent of the rational expectation hypothesis and allows us to avoid complications related to adverse selection (see Akerlof, 1970; Hendel and Lizzeri, 1999). The coefficient $\theta_{\dot{L}_i}$ reflects the impact of the expectations for lease rate *i* on the quantity of lease contracts of lessor *i*, and the vector of time derivatives $\theta_{\dot{L}_{-i}} = [\theta_{\dot{L}_1}, \ldots, \theta_{\dot{L}_{i-1}}, \theta_{\dot{L}_{i+1}}, \ldots, \theta_{\dot{L}_n}]$ gives the coefficients which reflect the impact of the expectations for the other lease rates, other than the lease rate of firm *i*, on the quantity of lease contracts of lessor *i*. The interpretation is similar for the coefficient $\lambda_{\dot{k}_j}$ and the vector of coefficients $\lambda_{\dot{k}_{-j}} = [\lambda_{\dot{k}_1}, \ldots, \lambda_{\dot{k}_{j-1}}, \lambda_{\dot{k}_{j+1}}, \ldots, \lambda_{\dot{k}_m}]$: they reflect the impact of the expectations for the lease rates on the quantity of the sell contracts of seller *j*. If consumers expect rates to continue rising, their desire to hold money is reduced so $\theta_{\dot{L}_i}, \theta_{\dot{L}_{-i}}, \lambda_{\dot{k}_j}$ and $\lambda_{\dot{k}_{-j}}$ will be positive. However, if they expect that rates will continue falling, then they restrain from buying and $\theta_{\dot{L}_i}, \theta_{\dot{L}_{-i}}, \lambda_{\dot{k}_j}$ and $\lambda_{\dot{k}_{-j}}$ will be negative.
5. Finally, the interpretation of the signs for $\theta_{\dot{k}_j}, \theta_{\dot{k}_{-j}}, \lambda_{\dot{L}_i}$, and $\lambda_{\dot{L}_{-i}}$ will depend on the substitutability of selling and leasing. More specifically, the coefficient $\theta_{\dot{k}_j}$ reflects the impact of the expectations for sell rate *j* on the quantity of lease contracts of lessor *i*. The vector of time derivatives $\theta_{\dot{k}_{-j}} = [\theta_{\dot{k}_1}, \ldots, \theta_{\dot{k}_{j-1}}, \theta_{\dot{k}_{j+1}}, \ldots, \theta_{\dot{k}_m}]$ gives the coefficients which reflect the impact of the expectations for the other sell rates, other than the sell rate of firm *j*, on the quantity of lease contracts of lessor *i*. The interpretation is similar for the coefficient $\lambda_{\dot{L}_i}$ and the vector of coefficients $\lambda_{\dot{L}_{-i}} = [\lambda_{\dot{L}_1}, \ldots, \lambda_{\dot{L}_{i-1}}, \lambda_{\dot{L}_{i+1}}, \ldots, \lambda_{\dot{L}_n}]$: they reflect the impact of the expectations for the lease rates on the quantity of the sell contracts of seller *j*. For example, when they are substitutes, if consumers expect that the sell rate will continue rising then they increase their demand now. Assuming constant supply, this leads to a rise in sell rates which in turn increases the demand for leasing services. In this case, $\theta_{\dot{k}_j}, \theta_{\dot{k}_{-j}}$ are positive. Conversely, if consumers expect sell rates to fall then $\theta_{\dot{k}_j}, \theta_{\dot{k}_{-j}}$ will be negative. A similar line of arguments can be made in interpreting the signs of $\lambda_{\dot{L}_i}, \lambda_{\dot{L}_{-i}}$.

The first order conditions are obtained by substituting the demand functions into the objective functions and then solving the maximization problems using the calculus of variations technique. More specifically, lessor's *i* and seller's *j* problems for dynamic optimization must satisfy the Euler equation according to the following proposition (the general conditions that the rate functions must satisfy for the calculus of variations technique to be valid and a detailed derivation of the first-order optimization conditions are given in Appendix A):

Proposition. (a) *Representative lessor's i Euler equation is the following:*

$$e^{-\rho t}q_{L_i}(t) + e^{-\rho t}(R_{L_i}(t) - \rho)\theta_{L_i} = -\rho e^{-\rho t}(R_{L_i}(t) - \rho)\theta_{\dot{L}_i}, \quad \forall i = 1, 2, \ldots, n$$

(b) *Representative seller's j Euler equation is the following:*

$$e^{-\rho t}q_{k_j}(t) + e^{-\rho t}(R_{k_j}(t) - \rho)\lambda_{k_j} = -\rho e^{-\rho t}(R_{k_j}(t) - \rho)\lambda_{\dot{k}_j}, \quad \forall j = 1, 2, \ldots, m$$

The first order conditions of the optimization problems underhand in matrix form, which correspond to the best response functions, are the following:

$$\begin{bmatrix} 1 & \ldots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \dot{R}_{L_1}(t) \\ \vdots \\ \dot{R}_{L_n}(t) \\ \dot{R}_{k_1}(t) \\ \vdots \\ \dot{R}_{k_m}(t) \end{bmatrix} + \begin{bmatrix} a_{L_{11}} & \ldots & \alpha_{k_{1m}} \\ \vdots & \ddots & \vdots \\ b_{L_{m1}} & \ldots & b_{L_{mm}} \end{bmatrix} \begin{bmatrix} R_{L_1}(t) \\ \vdots \\ R_{L_n}(t) \\ R_{k_1}(t) \\ \vdots \\ R_{k_m}(t) \end{bmatrix} = \begin{bmatrix} a_{L_1} \\ \vdots \\ a_{L_n} \\ b_{k_1} \\ \vdots \\ b_{k_m} \end{bmatrix}$$

[3] The linear demand structure arises from a quadratic and strictly concave utility function (see Dixit, 1979; Singh and Vives, 1984).

where

$$
\begin{aligned}
&a_{\dot{L}_{ij}} \equiv \frac{\theta_{\dot{L}_j}}{\theta_{\dot{L}_i}}, \quad a_{\dot{k}_{ij}} \equiv \frac{\theta_{\dot{k}_j}}{\theta_{\dot{L}_i}}, \quad \alpha_{L_{ii}} \equiv \frac{2\theta_{L_i} + \rho\theta_{\dot{L}_i}}{\theta_{\dot{L}_i}},\\
&\alpha_{L_{ij},i\neq j} \equiv \frac{\theta_{L_j}}{\theta_{\dot{L}_i}}, \quad \alpha_{k_{ij}} \equiv \frac{\theta_{k_j}}{\theta_{\dot{L}_i}} \quad \text{and} \quad a_{L_i} \equiv \frac{(\theta_{L_i} + \rho\theta_{\dot{L}_i})\rho - \theta_0}{\theta_{\dot{L}_i}}\\
&b_{\dot{k}_{ij}} \equiv \frac{\lambda_{\dot{k}_i}}{\lambda_{\dot{k}_j}}, \quad b_{\dot{L}_{ij}} \equiv \frac{\lambda_{\dot{L}_j}}{\lambda_{\dot{k}_j}}, \quad b_{k_{jj}} \equiv \frac{2\lambda_{k_j} + \rho\lambda_{\dot{k}_j}}{\lambda_{k_j}},\\
&b_{k_{ij},i\neq j} \equiv \frac{\lambda_{\dot{k}_i}}{\lambda_{\dot{k}_j}}, \quad b_{L_{ij}} \equiv \frac{\lambda_{L_i}}{\lambda_{\dot{k}_j}} \quad \text{and} \quad b_{k_j} \equiv \frac{(\lambda_{k_j} + \rho\lambda_{\dot{k}_j})\rho - \lambda_0}{\lambda_{\dot{k}_j}}
\end{aligned}
\tag{4}
$$

The obtained system has n + m equations and n + m unknowns and it is linear. Therefore, it has a unique solution, as long as the matrix $\begin{bmatrix} 1 & \dots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \dots & 1 \end{bmatrix}$ *is not singular. If this matrix is singular then the system has either infinite number of solutions or no solution. This general system of first order differential equations is not in normal form. We can reduce it to an equivalent first order system in normal form by assuming that the above matrix is not singular, as following:*

$$
\begin{bmatrix} \dot{R}_{L_1}(t) \\ \vdots \\ \dot{R}_{L_n}(t) \\ \dot{R}_{k_1}(t) \\ \vdots \\ \dot{R}_{k_m}(t) \end{bmatrix} = -\begin{bmatrix} 1 & \dots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{L_{11}} & \dots & a_{k_{1m}} \\ \vdots & \ddots & \vdots \\ b_{L_{m1}} & \dots & 1 \end{bmatrix} \begin{bmatrix} R_{L_1}(t) \\ \vdots \\ R_{L_n}(t) \\ R_{k_1}(t) \\ \vdots \\ R_{k_m}(t) \end{bmatrix} + \begin{bmatrix} 1 & \dots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{L_1} \\ \vdots \\ a_{L_n} \\ b_{k_1} \\ \vdots \\ b_{k_m} \end{bmatrix}
$$

or,

$$
\begin{bmatrix} \dot{R}_{L_1}(t) \\ \vdots \\ \dot{R}_{L_n}(t) \\ \dot{R}_{k_1}(t) \\ \vdots \\ \dot{R}_{k_m}(t) \end{bmatrix} = \begin{bmatrix} \mu_{1,1} & \dots & \mu_{1,n+m} \\ \vdots & \ddots & \vdots \\ \mu_{n+m,1} & \dots & \mu_{n+m,n+m} \end{bmatrix} \begin{bmatrix} R_{L_1}(t) \\ \vdots \\ R_{L_n}(t) \\ R_{k_1}(t) \\ \vdots \\ R_{k_m}(t) \end{bmatrix} + \begin{bmatrix} M_1 \\ \vdots \\ M_n \\ M_{n+1} \\ \vdots \\ M_{n+m} \end{bmatrix}
\tag{5}
$$

By setting,

$$
\begin{bmatrix} \mu_{1,1} & \dots & \mu_{1,n+m} \\ \vdots & \ddots & \vdots \\ \mu_{n+m,1} & \dots & \mu_{n+m,n+m} \end{bmatrix} = \begin{bmatrix} 1 & \dots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{L_{11}} & \dots & a_{k_{1m}} \\ \vdots & \ddots & \vdots \\ b_{L_{m1}} & \dots & 1 \end{bmatrix}
$$

and

$$
\begin{bmatrix} M_1 \\ \vdots \\ M_n \\ M_{n+1} \\ \vdots \\ M_{n+m} \end{bmatrix} = \begin{bmatrix} 1 & \dots & a_{\dot{k}_{1m}} \\ \vdots & \ddots & \vdots \\ b_{\dot{L}_{m1}} & \dots & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{L_1} \\ \vdots \\ a_{L_n} \\ b_{k_1} \\ \vdots \\ b_{k_m} \end{bmatrix}
$$

The above system gives the first time derivatives of the rates of return for each of the n lessors and m sellers in the market as a linear combination of their rates of return.

3.2. Symmetric equilibrium case

The general problem is not easily tractable. For example, if we have a leasing market with $n = 7$ lessors, and a selling market with $m = 5$ sellers, then we have to deal with a linear system of 12 linear differential equations. So, for practical reasons we focus our attention on the case of symmetric equilibrium in which all leasing firms charge the same rate and, all the selling firms as well. In the symmetric equilibrium we assume that there is no "within" market competition. In other words, we assume that all the leasing firms follow the same price strategy as do the selling firms. So, under the symmetric equilibrium we end up with the following system of differential equations in matrix form (the detailed derivation of the symmetric equilibrium is given in Appendix B):

$$
\begin{bmatrix} 1 & \frac{m}{n} a_{\dot{k}} \\ \frac{n}{m} b_{\dot{L}} & 1 \end{bmatrix} \begin{bmatrix} \dot{R}_L(t) \\ \dot{R}_k(t) \end{bmatrix} + \begin{bmatrix} a_{LL} & \frac{m}{n} a_{kk} \\ \frac{n}{m} b_{LL} & b_{kk} \end{bmatrix} \begin{bmatrix} R_L(t) \\ R_k(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{n} a_L \\ \frac{1}{m} b_k \end{bmatrix}
$$

where

$$
\begin{aligned}
&a_{LL} \equiv \frac{(n+1)\theta_L + \rho\theta_{\dot{L}}}{n\theta_{\dot{L}}}, \quad a_{kk} \equiv \frac{\theta_k}{\theta_{\dot{L}}}\\
&b_{LL} \equiv \frac{\lambda_k}{\lambda_{\dot{k}}}, \quad b_{kk} \equiv \frac{(m+1)\lambda_k + \rho\lambda_{\dot{k}}}{m\lambda_{\dot{k}}}
\end{aligned}
\tag{6}
$$

This can be reduced to an equivalent first order system in normal form as following:

$$
\begin{bmatrix} \dot{R}_L(t) \\ \dot{R}_k(t) \end{bmatrix} = -\begin{bmatrix} 1 & \frac{m}{n} a_{\dot{k}} \\ \frac{n}{m} b_{\dot{L}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_{LL} & \frac{m}{n} a_{kk} \\ \frac{n}{m} b_{LL} & b_{kk} \end{bmatrix} \begin{bmatrix} R_L(t) \\ R_k(t) \end{bmatrix} + \begin{bmatrix} 1 & \frac{m}{n} a_{\dot{k}} \\ \frac{n}{m} b_{\dot{L}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{n} a_L \\ \frac{1}{m} b_k \end{bmatrix}
$$

This assumes that the matrix $\begin{bmatrix} 1 & \frac{m}{n} a_{\dot{k}} \\ \frac{n}{m} b_{\dot{L}} & 1 \end{bmatrix}$ is not singular, i.e., that:

$$
1 - a_{\dot{k}} b_{\dot{L}} = 1 - \frac{\theta_{\dot{k}}}{\theta_{\dot{L}}} \frac{\lambda_{\dot{L}}}{\lambda_{\dot{k}}} \neq 0
$$

Finally, we obtain the following system of differential equations in normal form (the explicit solution of Eq. (7) is given in Appendix C):

$$
\begin{aligned}
\dot{R}_L(t) &= K_1 + \varphi_{11} R_L(t) + \varphi_{12} R_k(t)\\
\dot{R}_k(t) &= K_2 + \varphi_{21} R_L(t) + \varphi_{22} R_k(t)
\end{aligned}
\tag{7}
$$

where

$$
K_1 \equiv \frac{\frac{1}{n} a_L - \frac{n}{m^2} b_{\dot{L}} b_k}{1 - a_{\dot{k}} b_{\dot{L}}}, \quad K_2 \equiv \frac{\frac{1}{m} b_k - \frac{m}{n^2} a_{\dot{k}} a_L}{1 - a_{\dot{k}} b_{\dot{L}}}
$$

and

$$
\begin{aligned}
&\varphi_{11} \equiv \frac{\frac{n^2}{m^2} b_{\dot{L}} b_{LL} - a_{LL}}{1 - a_{\dot{k}} b_{\dot{L}}}, \quad \varphi_{12} \equiv \frac{\frac{n}{m} b_{\dot{L}} b_{kk} - \frac{m}{n} a_{kk}}{1 - a_{\dot{k}} b_{\dot{L}}}\\
&\varphi_{21} \equiv \frac{\frac{m}{n} a_{\dot{k}} a_{LL} - \frac{n}{m} b_{LL}}{1 - a_{\dot{k}} b_{\dot{L}}}, \quad \varphi_{22} \equiv \frac{\frac{m^2}{n^2} a_{\dot{k}} a_{kk} - b_{kk}}{1 - a_{\dot{k}} b_{\dot{L}}}
\end{aligned}
\tag{8}
$$

In the above system the two rates of return interact with each other linearly since their first time derivatives are proportional to a linear combination of their levels. The values of the coefficients φ_{ij} determine the contribution that the levels of the variables make to their growth. Specifically, φ_{12} and φ_{21} relate the growth of the return of one variable to the level of return of the other variable. So, a negative value of φ_{12} indicates the negative contribution of the level of the sell rate to the growth of the lease rate, in the sense that the

Table 1
Equilibrium points for the linear system derived in the symmetric equilibrium.

Market trend	Interaction effect	Discriminant	Roots		Determinant	Equilibrium point	Stability
$\varphi_{11} = \varphi_{22}$	φ_{12}, φ_{21} are both different than zero and have the same sign	$\Delta > 0$	Real, distinct	$\varphi_{11}\varphi_{22} > \varphi_{12}\varphi_{21}$	det(Φ)>0	Node	Stable if tr(Φ) < 0
							Unstable if tr(Φ) > 0
$\varphi_{11} \neq \varphi_{22}$, and at least one non-zero	$(\varphi_{11} - \varphi_{22})^2 > \lvert 4\varphi_{12}\varphi_{21}\rvert$ and φ_{12}, φ_{21} have the opposite sign	$\Delta > 0$	Real, distinct	$\varphi_{11}\varphi_{22} < \varphi_{12}\varphi_{21}$	det(Φ)<0	Saddle Point	Conditional stability, depends on initial conditions
$\varphi_{11} = \varphi_{22} = 0$, or $\varphi_{11} = -\varphi_{22}$,	φ_{12}, φ_{21} have opposite sign	$\Delta < 0$	Pure imaginary			Centre	
$\varphi_{11} \neq \varphi_{22}$, and at least one non-zero	$(\varphi_{11} - \varphi_{22})^2 < \lvert 4\varphi_{12}\varphi_{21}\rvert$ and φ_{12}, φ_{21} have opposite sign	$\Delta < 0$	Complex conjugate			Focus	Stable if tr(Φ) < 0
							Unstable if tr(Φ) > 0
$\varphi_{11} \neq \varphi_{22}$, and at least one non-zero	$(\varphi_{11} - \varphi_{22})^2 = -4\varphi_{12}\varphi_{21}$ and φ_{12}, φ_{21} have opposite sign	$\Delta = 0$	Real and Equal			Node	Stable if tr(Φ) < 0
$\varphi_{11} = \varphi_{22}$	φ_{12}, $\varphi_{21} = 0$ and at least one non-zero	$\Delta = 0$	Real and Equal				Unstable if tr(Φ) > 0

presence of selling reduces the growth of the lease rate. In other words, if consumers cannot buy the good then the rate of growth for the return of leasing will be higher. In this way, a negative value of φ_{12} reduces the power of the leasing firms in the market. Coefficients φ_{11} and φ_{22} indicate the effect of the level of return on its own rate of growth. The characteristic polynomial of the system's matrix $\Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ can be written as $\varphi^2 - (\varphi_{11} + \varphi_{22})\varphi + (\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}) = 0$. The signs of the coefficients, φ_{ii} and φ_{ij} allow us to classify the dynamics of the two interacting markets in four interesting cases (summarized in Table 1):

Case 1. Stable node

Arises when $\Delta = tr(\Phi)^2 - 4\det(\Phi) > 0$ where $tr(\Phi) = \varphi_1 + \varphi_2 < 0$ and $\det(\Phi) = \varphi_1\varphi_2 > 0$. There are two ways in which the trace can be negative. First, φ_{11}, φ_{22} are both negative. This means that we have a declining trend in each of the two markets. For instance, an increase in the lease rate $R_L(t)$ has an inverse impact on the growth of both $R_L(t)$ and $R_k(t)$ since the derivatives $\dot{R}_L(t), \dot{R}_k(t)$ are negative. Second, either one of φ_{11}, φ_{22} is negative while the other is positive with the negative being higher in absolute value. Although there is a declining trend within only one market, the level of decline is high enough to compensate for any growth trend in the returns of the other market. In other words, the decline is present only in one market but it is large enough to lead both markets towards their steady-state rates of returns. As an example, suppose that there is a declining trend in the leasing market, i.e., $\varphi_{11} < 0$, while in the selling market returns are increasing, i.e., $\varphi_{22} > 0$. For the determinant to be positive, the term $-\varphi_{12}$,φ_{21} must be positive since $\varphi_{11}\varphi_{22}$ is negative. So, φ_{12},φ_{21} must have opposite signs. This means that one market benefits from the growth of the other although, at the same time, it damages its growth. When real roots are positive, φ_1,$\varphi_2 > 0$ an unstable node arises. The returns in both markets arise indefinitely and system deviates from its steady state. So in order to exclude the possibility of a bubble in the markets we require that the real roots are negative.

Case 2. Saddle point

Arises only when $\Delta = tr(\Phi)^2 - 4\det(\Phi) > 0$ and $\det(\Phi) = \varphi_1\varphi_2 < 0$. This means that the interaction effect $\varphi_{12}\varphi_{21}$ is positive and greater than the product $\varphi_{11}\varphi_{22}$ and can be realized in two different ways. First, if the one market benefits the other, $\varphi_{12} > 0$, $\varphi_{21} > 0$ and this benefit dominates the system dynamics. In this manner, the interaction effect overcomes the positive combined effect, $\varphi_{11}\varphi_{22}$. Second, if the combined effect $\varphi_{11}\varphi_{22}$ is negative, i.e., there is a declining trend within one market and growth in the other making the interaction effect nonnegative.

Case 3. Focus

Arises when $\Delta = tr(\Phi)^2 - 4\det(\Phi) = (\varphi_{11} - \varphi_{22})^2 + 4\varphi_{12}\varphi_{21} < 0$ and $tr(\Phi) = \varphi_1 + \varphi_2 \neq 0$. The negative discriminant means that the term $4\varphi_{12}\varphi_{21}$ is negative and exceeds in absolute value the term $(\varphi_{11} - \varphi_{22})^2$. Consequently, φ_{12}, φ_{21} must have opposite signs. Such a situation may arise if, for example, the leasing market benefits from its interaction with the selling market, i.e., $\varphi_{12} > 0$, while the selling market is damaged by its interaction with the leasing market, i.e., $\varphi_{21} < 0$, and in the leasing market there is a declining trend ($\varphi_{11} < 0$) while the selling market is growing ($\varphi_{22} > 0$). If the discriminant is equal to zero, the interpretation is similar but the interaction of the markets follows a node rather than a focus equilibrium. In order to exclude the possibility of instability and bubbles we require that the real parts of the roots are negative and the trace is negative in the case of the focus and node, respectively.

Case 4. Centre

Arises when $\Delta = tr(\Phi)^2 - 4\det(\Phi) < 0$ and $tr(\Phi) = \varphi_1 + \varphi_2 = 0$. The negative discriminant is explained as in the previous case and the trace condition can occur in two ways. First, $\varphi_{11} = \varphi_{22} = 0$, i.e., there is no trend in the rates of returns in both markets. Second, $\varphi_{11} = -\varphi_{22}$, i.e., the intensity of the decline in the one market is equal and opposite to that of the growth in the other market.

4. Empirical application

4.1. A model of dynamic interaction between leasing and selling markets

Our sample is drawn from the Bureau of Labor Statistics (BLS) database and corresponds to the US which is the largest automobile market internationally. The period covered is January 2002–May 2011, a total of 113 monthly observations expressed in constant prices of December 2001.[4] Two city-average Consumer Price Indices (CPI) are used which correspond to seasonally adjusted price levels of New Cars and Trucks (*NEW*) and Leased Cars and Trucks (*LEAS*). The later is a component of the new and used motor vehicles expenditure class, which is part of the CPI's private transportation component in the transportation major group and it covers leases on all classes of new consumer vehicles. The CPI data collector describes each selected vehicle lease in detail including seven

[4] The discussion closely follows Labor/Bureau of Labor Statistics, www.bls.gov.

Table 2
Stationarity analysis of automobile selling prices (*NEW*) and leasing prices (*LEAS*).

Method	Test statistic	*p*-Value
H₀: Common unit root process		
Levin et al. (2002) t^*	0.4818	0.6850
H₀: Individual unit root process		
Im et al. (2003) W-stat	−0.0392	0.4844
Maddala and Wu (1999), ADF Fisher Chi-square	3.0436	0.5506
Choi (2001) PP Fisher Chi-square	3.7628	0.4391

Probabilities for Fisher tests are computed using an asymptotic Chi-square distribution. All other tests assume asymptotic normality.

Table 3
Descriptive Statistics of monthly changes in automobile selling prices (*RNEW*) and leasing prices (*RLEAS*).

	RNEW	*RLEAS*
Mean	0.0001	−0.0004
Median	0.0000	−0.0011
Maximum	0.0153	0.0367
Minimum	−0.0115	−0.0232
Std. Dev.	0.0042	0.0081
Skewness	0.3726	1.1606
Kurtosis	4.5428	7.3385
Jarque–Bera	13.6990	112.9836
p-Value	0.0011	0.0000

aspects of the lease contract: the vehicle make, nameplate, model, engine, transmission, options and lease terms. The lease terms include characteristics such as the number of months of the lease term, the down payment, the residual value, the depreciation amount and the total rent charge. The sample is updated by one model year each September through November in order to maintain the same age vehicles over time. If a production model is discontinued, it is replaced by a comparable model. A complete resampling is scheduled every 5 years. Finance charges are not included in the CPI as well as any incentives associated with low-interest financing, are excluded from the discount or rebate amount.[5] The value that the CPI uses in *LEAS* is an estimated transaction price that reflects the vehicle base price, destination charge, options, dealer preparation charges, applicable taxes, depreciation, and lease rent charge (the finance fee portion of a monthly lease payment, similar to interest on a loan). The estimated transaction price also includes the respondent's estimate for the price markup, dealer concession or discount, and consumer rebate.

A casual inspection of the CPI levels suggests some kind of inverse co-evolution between the two series under study along with a smooth variation which is consistent with nonstationarity. As shown in Table 2, panel stationarity tests of CPI levels assuming a common or separate unit root processes confirm that both *NEW* and *LEAS* are integrated. Cointegration analysis suggests that no long term equilibrium relationship exists between the levels of the two CPI series (results are available upon request by the authors). So, CPI levels are used to calculate monthly rates of returns for selling (*RNEW*) and leasing (*RLEAS*), respectively, as simple percentage changes and these are then used in the subsequent analysis. Stationarity tests show that *RNEW* and *RLEAS* are $I(0)$ indicating that the original series is $I(1)$.

Descriptive statistics of the returns appear in Table 3. The results suggest that both series are positively skewed and leptokurtic. The maximum positive (negative) change was 1.53% or 3.64 standard deviations (−1.15% or 2.74 s.d.) for *RNEW* and occurred during the recent crisis period on October 2009 (March 2008). Similarly, for RLEAS the maximum (minimum) was 3.67% or 4.5 s.d. (−2.32% or −2.86 s.d.) on February 2009 (June 2009). The Pearson correlation coefficient between the two return series is −10.84% which is statistically insignificant at the 10% level and suggests no contemporaneous relationship. However, the null hypothesis that *DNEW* does not Granger-cause *DLEAS* is rejected with a test *F*-statistic of 6.0358 for 1 lag which is significant at the 1.56% level. The hypothesis in the opposite direction cannot be rejected at conventional levels of significance.

The general system of differential Eq. (5) of our model can be written in discrete time in order to be estimable. It corresponds to a $n + m$ variable VAR of lag order one as our theory dictates in reduced form.[6] However, its estimation needs the rates of return charged by all the leasing firms and all the selling firms. So, due to data limitations, we express in discrete time the symmetric solution characterized by system (7) as following:

$$\begin{aligned} R_{L,t+1} &= K_1 + \beta_{11}R_{L,t} + \varphi_{12}R_{k,t} + u_{L,t+1} \\ R_{k,t+1} &= K_1 + \varphi_{21}R_{L,t} + \beta_{22}R_{k,t} + u_{\kappa,t+1} \end{aligned} \tag{9}$$

where $\beta_{11} \equiv (1 + \varphi_{11})$ and $\beta_{22} \equiv (1 + \varphi_{22})$. This is a again a VAR of lag order one in reduced form. Estimation of this VAR model via OLS and subsequent elimination of the insignificant coefficients led to the following results (standard errors appear in brackets below estimates):

$$R_{L,t+1} = \underset{(0.0901)}{0.2643}\ R_{L,t} - \underset{(0.1790)}{0.4408}\ R_{k,t} + u_{L,t+1}, R^2_{adj} = 0.124$$

$$R_{k,t+1} = \underset{(0.0888)}{0.3978}\ R_{k,t} + u_{k,t+1}, R^2_{adj} = 0.153$$

The estimated coefficients allow us to draw several interesting conclusions. It appears that leasing market price changes are inversely related to prices changes in the selling market from the previous month ($\varphi_{12} < 0$). From a biological perspective, this is characterized as a "predatory" relationship of selling market over the leasing market. In line with the Granger causality results obtained previously, selling market price changes do not seem to depend on past leasing market price changes ($\varphi_{21} = 0$). Both leasing and selling market price changes are moderately persistent with the autoregressive coefficients being positive. The modulus of both roots is less than unity so we have a stable equilibrium point (stable node; see Case 1 in Section 3). Since both roots are real and distinct, shocks will dissipate in a monotone rather than fluctuating manner. The explanatory power of the VAR equations is satisfactory given that we are predicting monthly changes in the variables over a relatively long time period using a parsimonious specification. Although it is beyond the scope of this paper, explanatory power could be enhanced by including additional variables related to, for example, income, inflation and interest rate levels, currency rates, oil prices, consumer sentiment, etc.

[5] The formation of the Leased cars and trucks index is based on the calculation of total monthly lease payment. The formula, which uses the U.S. Department of Labor/Bureau of Labor Statistics, for the calculation of total monthly lease payment is the following: Total Monthly Lease Payment = (Base Price of Leased Vehicle) + (Transportation to Dealer) + (Total Price of Packages & Options) + (Dealer Preparation and Miscellaneous Charges) + (Additional Dealer Markup) − (Dealer Concession or Discount), which is equal with: (Capitalized Cost) (similar to the purchase price of a vehicle) − (Down payment) − (Rebate) − (Other Capitalized Cost Reductions) + (Tax) + (Other Additions to Capitalized Cost), which in turn is equal with: (Adjusted Capitalized Cost, amount used to calculate base monthly payment) − (Residual Value, value of the vehicle at the end of the lease) and this is equal with: (Depreciation Amount, the total amount charged for the decline in value) + (Total Lease Rent Charge, the finance fee, similar to interest), which finally equals with: (Total of Base Monthly Payments/Lease Term, the number of months in the lease), or (Base Monthly Payment) + (Monthly Sales/Use Tax).

[6] The discrete form of the general model (5) is given in Appendix D. The stability conditions for the general model are given as in the symmetric case by the roots of the system. There are conditions relating the matrix coefficients with the roots of the system and its stability, the Routh-Hurwitz conditions (see Gandolfo, 2005, pp. 251–258).

4.2. Implications for leasing contract valuation

The standard framework of lease valuation (Myers et al., 1976) adopts discounted cash flow analysis to derive the equilibrium rental rate[7]:

$$P_{L,t} = P_{L,0} - \sum_{t=1}^{n} \frac{L_t}{(1+\rho)^n} - \frac{RV_n}{(1+\rho)^n} \tag{10}$$

where RV_n is the expected residual value of the asset in period n. By employing a uniform lease payment we obtain the Myers, Dill and Bautista (MDB) formula:

$$L(t) = \frac{\rho}{1-(1+\rho)^n}\left[P_{L,0} - P_{L,t} - \frac{RV_n}{(1+\rho)^n}\right]$$

or, equivalently, the lease rate:

$$\frac{L(t)}{P_L(t)} = \frac{\rho}{(1+\rho)^n - 1}\left[1 - \frac{P_{L,0}}{P_{L,t}} + \frac{RV_n}{P_{L,t}(1+\rho)^n}\right] \tag{11}$$

Given our model and empirical results, an obvious shortcoming of this valuation approach is that it treats the leasing market autonomously and ignores any interactions with the selling market. The remainder of this section will incorporate our findings concerning the interaction between the leasing and selling markets in the MBD valuation approach.

From (9) we can derive the motion for the system of lease and sell rates from the following complementary function:

$$\begin{aligned} R_{L,t} &= A_1\beta_1^t + A_2\beta_2^t \\ R_{k,t} &= B_1\beta_1^t + B_2\beta_2^t \end{aligned} \tag{12}$$

where

$$\begin{aligned} B_1 &\equiv A_1 \tfrac{\beta_1-\beta_{11}}{\varphi_{12}} \\ B_2 &\equiv A_2 \tfrac{\beta_2-\beta_{11}}{\varphi_{12}} \end{aligned} \tag{13}$$

Moreover, since $\beta_1 - \beta_{11} = 0$, we obtain:

$$\begin{aligned} R_{L,t} &= A_1\beta_1^t + A_2\beta_2^t \\ R_{k,t} &= B_2\beta_2^t \end{aligned}$$

The arbitrary constants A_i are determined by the initial conditions of the system as follows:

$$\begin{aligned} A_2 &= \tfrac{R_{k0}\varphi_{12}}{\beta_2-\beta_{11}} \\ R_{L,0} &= A_1 + A_2 \end{aligned} \tag{14}$$

where $R_{L,0}$ and $R_{k,0}$ have already been defined as $R_{L,0} = \frac{\Delta P_L}{P_{L,-1}} = \frac{P_{L,0}-P_{L,-1}}{P_{L,-1}}$ and $R_{k,0} = \frac{\Delta P_k}{P_{k,-1}} = \frac{P_{k,0}-P_{k,-1}}{P_{k,-1}}$. So, having estimated $R_{L,t}$ and $R_{k,t}$, we can obtain $P_{L,t}$ and $P_{k,t}$ as:

$$\begin{aligned} \widehat{P}_{L,t} &= P_{L,t-1}(1+\widehat{R}_{L,t}) = P_{L,t-1}(1 + A_1\beta_1^t + A_2\beta_2^t) \\ \widehat{P}_{k,t} &= P_{k,t-1}(1+\widehat{R}_{k,t}) = P_{k,t-1}(1 + B_2\beta_2^t) \end{aligned} \tag{15}$$

Now, the following quantity:

$$\begin{aligned} G_t &= P_{L,t} - \widehat{P}_{L,t} = P_{L,t} - P_{L,t-1}(1+\widehat{R}_{L,t}) \\ &= P_{L,t} - P_{L,t-1}(1 + A_1\beta_1^t + A_2\beta_2^t) \end{aligned} \tag{16}$$

represents a capital gain or loss which results from the interaction between the leasing and selling markets and could be used to augment the MBD leasing valuation formula. In other words, this term reflects an opportunity cost in the sense that the price of the leased asset changes and this is something that should be accounted for. Another reasonable adjustment that should be made concerns the residual value since this is an expectation of the stochastic value which the asset will have in the termination of the contract (e.g., Trigeorgis, 1996, assumes that the residual value follows an Ornstein–Uhlenbeck process). The residual value is corrected here on the basis of the interaction with the selling market by using the cumulative changes in the leasing market prices $\prod_{t=1}^{n}(1+\widehat{R}_{L,t})$. Finally, the overall effect of the interaction with the selling market can be captured by the following augmented lease valuation formula:

$$P_{L,t} = P_{L,0} - \sum_{t=1}^{n} \frac{L_t}{(1+\rho)^n} - \frac{RV_n \prod_{t=1}^{n}(1+\widehat{R}_{L,t})}{(1+\rho)^n} - \sum_{t=1}^{n} \frac{c_t}{(1+\rho)^n} \tag{17}$$

4.3. A numerical example

We shall use a hypothetical example in order to illustrate the application and practical importance for valuation of the interaction between leasing and selling markets. Assume that we are considering the valuation of a contract for a car with a base price $P_{L,0}$ = \$30,000 which will be leased over a 6 month period with a terminal residual value RV_n equal to \$25,000 (83.3% of the base price). Lease payments are due at the end of each month and the lease is financed at a monthly rate of ρ = 1%.

Without taking into account the interaction between the two markets, the traditional MDB formula described in (11) gives a monthly lease payment $L(t)$ equal to \$1112.74. Assume also that we are at October 2009 when the US selling market price level increased by $R_{k,t}$ = 1.535% compared to the previous month (as shown in Fig. 1). We can use this information to recursively predict lease rates and prices using Eq. (15) over the next 6 months on the basis of the VAR model parameters estimated previously using US data. The predicted lease rates are $\widehat{R}_{L,t+1} = -0.86971\%$, -0.49903%, -0.23897%, -0.10575%, -0.04489% and -0.01861%, respectively, for t = 1–6 months. This corresponds to a total compound (average) expected drop of 1.77% (0.3%). These predictions are close to the actual rates of $R_{L,t+i}$ = −0.27942%, −1.12581%, −0.19465%, 1.10043%, −0.28486% and −1.84133%, respectively, for t = 1–6 months. These changes corresponded to a total compound (average) change of −2.61% (−0.44%).

If we use these values in the augmented MDB formula described in (17) then we obtain a monthly lease payment $L(t)$ of \$1505.64

Fig. 1. Automobile selling prices (*NEW*) and leasing prices (*LEAS*).

[7] An alternative is the user cost theory approach of Miller and Upton (1976). A number of other valuation models have been proposed in order to account for credit risk in lease contracts (see, for example, Grenadier, 1996; Ambrose and Yildirim, 2008; Agarwal et al., 2011) or for various optionalities in leasing contracts (see, for example, McConnell and Schallheim, 1983; Schallheim and McConnell, 1985; Grenadier, 1995; Trigeorgis, 1996). For empirical applications see Schallheim et al. (1987), and, Giaccotto et al. (2007). A relevant literature deals with leasing of other real assets (e.g. see the theoretical framework and empirical analysis by Golbeck and Linetsky (2013)).

which is higher by \$392.9 (or 35.1%) than the previous one. If the standard MDB formula is used and the predictions of lease rates from our estimated model are realized then the lessor will underprice the lease payment. This translates into a negative monthly internal rate of return of −1.21% (instead of a positive 1%) which corresponds to an annual loss of −13.61% (instead of a 12.68% profit, i.e. the compounded return of 1% for 12 months). Using the actual rather than predicted lease rates gives an ex post fair monthly payment $L(t)$ of \$1578.70 which is close to the estimate from the augmented MDB model. These calculations suggest that our results have significant practical implication for pricing leasing contracts.

5. Conclusions

This paper describes a novel theoretical framework which leads to an interactive relationship between leasing and selling markets for automobiles. This framework extends previous approaches by allowing forward-looking firms which are set in an oligopoly while leasing and selling are not assumed to be perfect substitutes. The simplest specification justified is a VAR(1) model of lease and sell rates, which is estimated using monthly US data. Results confirm a one-way interacting relationship whereby sell rates Granger-cause lease rates. We show how this interaction can be incorporated within standard lease pricing formulas. A numerical example demonstrates that our findings have non-trivial practical implications for lease pricing.

Acknowledgements

We are grateful for useful comments received from participants of research seminars at the University of East Anglia (March 2012), the Howe School of Technology Management at Stevens Institute of Technology (April 2012) and the Royal Economic Society Annual Conference (March 2012). The usual disclaimer applies.

Appendix A. Proof of proposition

We describe the general conditions that the rate functions must satisfy for the calculus of variations technique to be valid. We also provide a detailed derivation of the first-order optimization conditions.

The representative lessor faces the intertemporal problem (1) that we reproduce here for the convenience of the reader:

$$\max_{R_{L_l}(t)} \int_0^\infty e^{-\rho t}(R_{L_l}(t) - c_{L_l})q_{L_l}(t)dt$$

By setting the integrand equal to a function F we have:

$$F \equiv e^{-\rho t}(R_{L_l}(t) - c_{L_l})q_{L_l}(t)$$

Then, the Euler equation for dynamic optimization is:

$$F_{R_{L_l}} = -F_{\dot{R}_{L_l}} + F_{\dot{R}_{L_l}\dot{R}_{L_l}}\dot{R}_{L_l} + F_{\dot{R}_{L_l}\dot{R}_{L_l}}\ddot{R}_{L_l}$$

The second and third term on the right hand side of the above Euler equation are zero and by calculating the derivative on the left hand side and the first time derivative on the right hand side of the above Euler equation we get part (a) of our proposition, which gives representative lessor's i Euler equation. The representative seller faces the intertemporal problem (2) which is symmetrical to the problem of the representative lessor. So, the steps that we have to follow in order to find the Euler equation of the representative seller are the same as described previously.

Now we can follow the following steps in order to write the first order conditions of the optimization problems in matrix form. First, we take the Euler equations for the representative lessor and the representative seller:

$$\begin{aligned} q_{L_i}(t) + (R_{L_i}(t) - c_{L_i})\theta_{L_i} &= -\rho(R_{L_i}(t) - c_{L_i})\theta_{\dot{L}_i}, \quad \forall i = 1, 2, \ldots, n \\ q_{k_j}(t) + (R_{k_j}(t) - c_{k_j})\lambda_{k_j} &= -\rho(R_{k_j}(t) - c_{k_j})\lambda_{\dot{k}_j}, \quad \forall j = 1, 2, \ldots, m \end{aligned}$$

We substitute the demand functions with their equals and $\rho = c_{L_i} = c_{k_j}$:

$$\begin{aligned} &(2\theta_{L_i} + \rho\theta_{\dot{L}_i})R_{L_i}(t) + \theta_{L_{-i}}R_{L_{-i}}(t) + \theta_{k_j}R_{k_j}(t) + \theta_{k_{-j}}R_{k_{-j}}(t) \\ &\quad + \theta_{\dot{L}_i}\dot{R}_{L_i}(t) + \theta_{\dot{L}_{-i}}\dot{R}_{L_{-i}}(t) + \theta_{\dot{k}_j}\dot{R}_{k_j}(t) + \theta_{\dot{k}_{-j}}\dot{R}_{k_{-j}}(t) \\ &\quad = (\theta_{L_i} + \rho\theta_{\dot{L}_i})\rho - \theta_0 \end{aligned} \tag{A.1}$$

$$\begin{aligned} &(2\lambda_{k_j} + \rho\lambda_{\dot{k}_j})R_{k_j}(t) + \lambda_{L_i}R_{L_i}(t) + \lambda_{L_{-i}}R_{L_{-i}}(t) + \lambda_{k_{-j}}R_{k_{-j}}(t) \\ &\quad + \lambda_{\dot{L}_i}\dot{R}_{L_i}(t) + \lambda_{\dot{L}_{-i}}\dot{R}_{L_{-i}}(t) + \lambda_{\dot{k}_j}\dot{R}_{k_j}(t) + \lambda_{\dot{k}_{-j}}\dot{R}_{k_{-j}}(t) \\ &\quad = (\lambda_{k_j} + \rho\lambda_{\dot{k}_j})\rho - \lambda_0 \end{aligned} \tag{A.2}$$

We write the above equations for every i and j, and, get the following system of equations:

$$\begin{aligned} &\theta_{\dot{L}_1}\dot{R}_{L_1}(t) + \theta_{\dot{L}_2}\dot{R}_{L_2}(t) + \ldots + \theta_{\dot{L}_n}\dot{R}_{L_n}(t) + \theta_{\dot{k}_1}\dot{R}_{k_1}(t) + \theta_{\dot{k}_2}\dot{R}_{k_2}(t) + \ldots \\ &+\theta_{\dot{k}_m}\dot{R}_{k_m}(t) + (2\theta_{\dot{L}_1} + \rho\theta_{\dot{L}_1})R_{L_1}(t) + \theta_{\dot{L}_2}R_{L_2}(t) + \ldots + \theta_{\dot{L}_n}R_{L_n}(t) \\ &+\theta_{\dot{k}_1}R_{k_1}(t) + \theta_{\dot{k}_2}R_{k_2}(t) + \ldots + \theta_{\dot{k}_m}R_{k_m}(t) = (\theta_{\dot{L}_1} + \rho\theta_{\dot{L}_1})\rho - \theta_0 \\ &\vdots \\ &\theta_{\dot{L}_1}\dot{R}_{L_1}(t) + \theta_{\dot{L}_2}\dot{R}_{L_2}(t) + \ldots + \theta_{\dot{L}_n}\dot{R}_{L_n}(t) + \theta_{\dot{k}_1}\dot{R}_{k_1}(t) + \theta_{\dot{k}_2}\dot{R}_{k_2}(t) + \ldots \\ &+\theta_{\dot{k}_m}\dot{R}_{k_m}(t) + \theta_{\dot{L}_1}R_{L_1}(t) + \theta_{\dot{L}_2}R_{L_2}(t) + \ldots + (2\theta_{\dot{L}_n} + \rho\theta_{\dot{L}_n})R_{L_n}(t) \\ &+\theta_{\dot{k}_1}R_{k_1}(t) + \theta_{\dot{k}_2}R_{k_2}(t) + \ldots + \theta_{\dot{k}_m}R_{k_m}(t) = (\theta_{\dot{L}_n} + \rho\theta_{\dot{L}_n})\rho - \theta_0 \\ &\vdots \\ &\lambda_{\dot{k}_1}\dot{R}_{k_1}(t) + \lambda_{\dot{k}_2}\dot{R}_{k_2}(t) + \ldots + \lambda_{\dot{k}_m}\dot{R}_{k_m}(t) + \lambda_{\dot{L}_1}\dot{R}_{L_1}(t) \\ &\quad +\lambda_{\dot{L}_2}\dot{R}_{L_2}(t) + \ldots + \lambda_{\dot{L}_n}\dot{R}_{L_n}(t) \\ &+(2\lambda_{k_1} + \rho\lambda_{\dot{k}_1})R_{k_1}(t) + \lambda_{k_2}R_{k_2}(t) + \ldots + \lambda_{k_m}R_{k_m}(t) + \lambda_{L_1}R_{L_1}(t) \\ &+\lambda_{L_2}R_{L_2}(t) + \ldots + \lambda_{L_n}R_{L_n}(t) = (\lambda_{k_1} + \rho\lambda_{\dot{k}_1})\rho - \lambda_0 \\ &\vdots \\ &\lambda_{\dot{k}_1}\dot{R}_{k_1}(t) + \lambda_{\dot{k}_2}\dot{R}_{k_2}(t) + \ldots + \lambda_{\dot{k}_m}\dot{R}_{k_m}(t) + \lambda_{\dot{L}_1}\dot{R}_{L_1}(t) + \lambda_{\dot{L}_2}\dot{R}_{L_2}(t) + \ldots \\ &+\lambda_{\dot{L}_n}\dot{R}_{L_n}(t) + \lambda_{k_1}R_{k_1}(t) + \lambda_{k_2}R_{k_2}(t) + \ldots + (2\lambda_{k_m} + \rho\lambda_{\dot{k}_m})R_{k_m}(t) \\ &+\lambda_{L_1}R_{L_1}(t) + \lambda_{L_2}R_{L_2}(t) + \ldots + \lambda_{L_n}R_{L_n}(t) = (\lambda_{k_m} + \rho\lambda_{\dot{k}_m})\rho - \lambda_0 \end{aligned} \tag{A.3}$$

We divide the first n equations by $\theta_{\dot{L}_i}$ and the following m equations by $\lambda_{\dot{k}_j}$ to obtain the system (5).

Appendix B. Detailed derivation of the symmetric equilibrium

In the symmetric equilibrium all leasing firms charge the same rate and all the selling firms do as well. In this case lessor's i and seller's j Euler equations become:

$$\begin{aligned} &(2\theta_L + \rho\theta_{\dot{L}})R_L(t) + (n-1)\theta_L R_L(t) + \theta_k R_k(t) + (m-1)\theta_k R_k(t) \\ &\quad + \theta_{\dot{L}}\dot{R}_L(t) + \theta_{\dot{k}}\dot{R}_k(t) + (n-1)\theta_{\dot{L}}\dot{R}_L(t) + (m-1)\theta_{\dot{k}}\dot{R}_k(t) \\ &\quad = (\theta_L + \rho\theta_{\dot{L}})\rho - \theta_0 \end{aligned} \tag{B.1}$$

$$\begin{aligned} &(2\lambda_k + \rho\lambda_{\dot{k}})R_k(t) + (n-1)\lambda_L R_L(t) + \lambda_L R_L(t) + (m-1)\lambda_k R_k(t) \\ &\quad + \lambda_{\dot{L}}\dot{R}_L(t) + \lambda_{\dot{k}}\dot{R}_k(t) + (n-1)\lambda_{\dot{L}}\dot{R}_L(t) + (m-1)\lambda_{\dot{k}}\dot{R}_k(t) \\ &\quad = (\lambda_k + \rho\lambda_{\dot{k}})\rho - \lambda_0 \end{aligned} \tag{B.2}$$

Dividing Eq. (B.1) by $n\theta_i$ and Eq. (B.2) by $m\lambda_k$, we get the first order conditions of the optimization problems, in matrix form, and the coefficients in equation (7) of the main text.

Appendix C. Explicit solution of system of differential equations in normal form

In this Appendix is given the explicit solution of Eq. (7). In matrix form, Eq. (7) can be written as:

$$\begin{bmatrix} \dot{R}_L(t) \\ \dot{R}_k(t) \end{bmatrix} = \begin{bmatrix} \varphi_{11} \varphi_{12} \\ \varphi_{21} \varphi_{22} \end{bmatrix} \begin{bmatrix} R_L(t) \\ R_k(t) \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

The solution of the above non-homogeneous system of differential equations can be obtained by adding to the solution of the corresponding homogeneous system (complementary function) the particular solution. So, the solution of the above system can be found by following three steps. Firstly, we find the complementary function. Secondly, we find the particular solution and thirdly we find the arbitrary constants for a given set of initial conditions. Finally we compose the above three steps.

The characteristic polynomial of the system's matrix $\Phi \equiv \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ can be written as $\varphi^2 - (\varphi_{11} + \varphi_{22})\varphi + (\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}) = 0$.

The above characteristic polynomial may have three different types of roots.

C.1. Distinct real roots, when $\Delta = tr(\Phi)^2 - 4\det(\Phi) > 0$

If the roots of the system are real and distinct $\left(\varphi_{1,2} = \frac{1}{2}tr(\Phi) \pm \frac{1}{2}\sqrt{tr(\Phi)^2 - 4\det(\Phi)}\right)$ then the general solution of the system is:

$$\begin{aligned} R_L(t) = {} & \left(\frac{\varphi_{12}}{\varphi_1 - \varphi_2} R_{k0} - \frac{\varphi_2 - \varphi_{11}}{\varphi_1 - \varphi_2} R_{L0}\right) e^{\varphi_1 t} \\ & + \left(\frac{\varphi_1 - 2\varphi_2 - \varphi_{11}}{\varphi_1 - \varphi_2} R_{k0} - \frac{\varphi_{12}}{\varphi_1 - \varphi_2} R_{L0}\right) e^{\varphi_2 t} \\ & + \frac{K_2\varphi_{21} - K_1\varphi_{22}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \end{aligned} \tag{C.1}$$

$$\begin{aligned} R_k(t) = {} & \left(\frac{\varphi_{12}}{\varphi_1 - \varphi_2} R_{k0} - \frac{\varphi_2 - \varphi_{11}}{\varphi_1 - \varphi_2} R_{L0}\right) \frac{\varphi_1 - \varphi_{11}}{\varphi_{12}} e^{\varphi_1 t} \\ & + \left(\frac{\varphi_1 - 2\varphi_2 - \varphi_{11}}{\varphi_1 - \varphi_2} R_{k0} - \frac{\varphi_{12}}{\varphi_1 - \varphi_2} R_{L0}\right) \frac{\varphi_2 - \varphi_{11}}{\varphi_{12}} e^{\varphi_2 t} \\ & + \frac{K_1\varphi_{21} - K_2\varphi_{11}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \end{aligned}$$

C.2. Repeated real roots, when $\Delta = tr(\Phi)^2 - 4\det(\Phi) = 0$

If the roots of the system are real and equal $(\varphi_{1,2} = \varphi^* = \frac{1}{2}tr(\Phi))$ then the general solution of the system is:

$$\begin{aligned} R_L(t) &= \left[R_{L0} + \varphi_{12}R_{k0}t - \tfrac{\varphi_{22}-\varphi_{11}}{2} R_{L0}t\right] e^{\varphi * t} + \tfrac{K_2\varphi_{21} - K_1\varphi_{22}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \\ R_k(t) &= \left[R_{k0} + \tfrac{\varphi_{22}-\varphi_{11}}{2} R_{k0}t - \tfrac{(\varphi_{22}-\varphi_{11})^2}{4\varphi_{12}} R_{L0}t\right] e^{\varphi * t} + \tfrac{K_1\varphi_{21} - K_2\varphi_{11}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \end{aligned} \tag{C.2}$$

C.3. Complex conjugate roots, when $\Delta = tr(\Phi)^2 - 4\det(\Phi) < 0$

Finally, if the roots of the system are complex conjugates $(\varphi_{1,2} = \frac{1}{2}tr(\Phi) \pm \frac{1}{2}i\sqrt{4\det(\Phi) - tr(\Phi)^2} = a \pm i\theta$, where $a = \frac{1}{2}tr(\Phi)$ and $\theta = \frac{1}{2}\sqrt{4\det(\Phi) - tr(\Phi)^2}$), then the general solution of the system is:

$$\begin{aligned} R_k(t) &= e^{at}\left[R_{k_0}\cos(\theta t) + \left(\tfrac{\varphi_{12}}{\theta} R_{k_0} - \tfrac{a-\varphi_{11}}{\theta} R_{L_0}\right)\sin(\theta t)\right] + \tfrac{K_2\varphi_{21} - K_1\varphi_{22}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \\ R_k(t) &= e^{at}\Bigg[R_{k_0}(t)\cos(\theta t) \\ & \quad + \left[\tfrac{a-\varphi_{11}}{\theta} R_{k_0} - \tfrac{(a-\varphi_{11})^2}{\theta\varphi_{12}} R_{L_0} + \tfrac{\theta}{\varphi_{12}} R_{L_0}\right]\sin(\theta t)\Bigg] \\ & \quad + \tfrac{K_1\varphi_{21} - K_2\varphi_{11}}{\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}} \end{aligned} \tag{C.3}$$

For a more systematic treatment of Simultaneous Systems of Differential Equations see Gandolfo (2005, pp. 237–245).

Appendix D. Discrete form of the general model

We present the discrete form of the general model (5) which is a $n + m$ variable VAR model of lag order one:

$$\begin{bmatrix} R_{L_{1,t+1}} \\ \vdots \\ R_{L_{n,t+1}} \\ R_{k_{1,t+1}} \\ \vdots \\ R_{k_{m,t+1}} \end{bmatrix} = \begin{bmatrix} 1 + \mu_{1,1} & \cdots & \mu_{1,n+m} \\ \vdots & \ddots & \vdots \\ \mu_{n+m,1} & \cdots & 1 + \mu_{n+m,n+m} \end{bmatrix} \begin{bmatrix} R_{L_{1,t}} \\ \vdots \\ R_{L_{n,t}} \\ R_{k_{1,t}} \\ \vdots \\ R_{k_{m,t}} \end{bmatrix} + \begin{bmatrix} M_1 \\ \vdots \\ M_n \\ M_{n+1} \\ \vdots \\ M_{n+m} \end{bmatrix}$$

References

Adda, J., Cooper, R., 2000a. Balladurette and Jupette: a discrete analysis of scrapping subsidies. Journal of Political Economy 108, 778–806.

Adda, J., Cooper, R., 2000b. The Dynamics of Car Sales: A Discrete-Choice Approach. NBER Working Paper No. 7785.

Agarwal, S., Ambrose, B., Huang, H., Yildirim, Y., 2011. The term structure of lease rates with endogenous default triggers and tenant capital structure: theory and evidence. Journal of Financial and Quantitative Analysis 46, 553–584.

Aizcorbe, A., Kennickell, A., Moore, K., 2003. Recent changes in U.S. family finances: evidence from the 1998 and 2001 survey consumer finances. Federal Reserve Bulletin 80, 1–31.

Akerlof, G., 1970. The market for 'lemons': quality uncertainty and the market mechanism. Quarterly Journal of Economics 84, 488–500.

Ambrose, B.W., Yildirim, Y., 2008. Credit risk and the term structure of lease rates: a reduced form approach. Journal of Real Estate Finance and Economics 37, 281–298.

Attanasio, O., 2000. Consumer durables and inertial behavior: estimation and aggregation of (S, s) rules. Review of Economic Studies 67, 667–696.

Berkovec, J., 1985. New car sales and used car stocks: a model of the automobile market. RAND Journal of Economics 16, 195–214.

Berry, S., Levinsohn, J., Pakes, A., 1995. Automobile prices in market equilibrium. Econometrica 63, 841–890.

Bhaskaran, S.R., Gilbert, S.M., 2005. Selling and leasing strategies for durable goods with complementary products. Management Science 51, 1278–1290.

Bhaskaran, S.R., Gilbert, S.M., 2009. Implications of channel structure for leasing or selling durable goods. Marketing Science 28, 918–934.

Bils, M., Klenow, P., 1998. Using consumer theory to test competing business cycle models. Journal of Political Economy 106, 233–261.

Blanchard, O., Melino, A., 1986. The cyclical behavior of prices and quantities: the case of the automobile market. Journal of Monetary Economics 17, 379–407.

Board, S., 2008. Durable-goods monopoly with varying demand. Review of Economic Studies 75, 391–413.

Bresnahan, T., 1981. Departures from marginal cost pricing in the American automobile industry. Journal of Econometrics 17, 201–227.

Bucovetsky, S., Chilton, J., 1986. Concurrent renting and selling in a durable goods monopoly under threat of entry. RAND Journal of Economics 17, 261–278.

Bulow, J., 1982. Durable goods monopolists. Journal of Political Economy 90, 314–332.

Bulow, J., 1986. An economic theory of planned obsolescence. Quarterly Journal of Economics 101, 729–749.

Caplin, A., Leahy, J., 2006. Equilibrium in a durable goods market with lumpy adjustment. Journal of Economic Theory 128, 187–213.

Chen, J., Esteban, S., Shum, M., 2008. Demand and supply estimation biases due to omission of durability. Journal of Econometrics 147, 247–257.

Chen, J., Esteban, S., Shum, M., 2010. How Much Competition is a Secondary Market? IMDEA Working Paper 2010-06.

Choi, I., 2001. Unit root tests for panel data. Journal of International Money and Finance 20, 249–272.
Coase, R., 1972. Durability and monopoly. Journal of Law and Economics 15, 143–149.
Conlisk, J., Gerstner, E., Sobel, J., 1984. Cyclic pricing by a durable goods monopolist. Quarterly Journal of Economics 99, 489–505.
Dasgupta, S., Siddarth, S., Risso, J., 2007. To Lease or to buy? A structural model of a consumer's vehicle and contract choice decisions. Journal of Marketing Research 44, 490–502.
De Jaegher, K., 2009. Asymmetric substitutability: theory and some applications. Economic Inquiry 47, 838–855.
Desai, P., Purohit, D., 1998. Leasing and selling: optimal marketing strategies for a durable goods firm. Management Science 44, 19–34.
Desai, P., Purohit, D., 1999. Competition in durable goods markets: the strategic consequences of leasing and selling. Marketing Science 18, 42–58.
Dixit, A., 1979. A model of duopoly suggesting a theory of entry barriers. Bell Journal of Economics 10, 20–32.
Dudine, P., Hendel, I., Lizzeri, A., 2006. Storable good monopoly: the role of commitment. American Economic Review 96, 1706–1719.
Eberly, J., 1994. Adjustment of consumers' durables stocks: evidence from automobile purchases. Journal of Political Economy 102, 403–436.
Eisfeldt, L.A., Rampini, A.A., 2009. Leasing, ability to repossess, and debt capacity. Review of Financial Studies 22, 1621–1657.
Esteban, S., Shum, M., 2007. Durable goods oligopoly with secondary markets: the case of automobiles. RAND Journal of Economics 38, 332–354.
Gandolfo, G., 2005. Economic Dynamics-Study Edition. Springer, Berlin, Heidelberg, New York.
Giaccotto, C., Goldberg, G.M., Hegde, S.P., 2007. The value of embedded real options: evidence from consumer automobile lease contracts. Journal of Finance 62, 411–445.
Golbeck, S., Linetsky, V., 2013. Asset financing with credit risk. Journal of Banking and Finance 37, 43–59.
Goldberg, P., 1995. Product differentiation and oligopoly in international markets: the case of the U.S. Automobile Industry. Econometrica 63, 891–951.
Grenadier, S.R., 1995. Valuing lease contracts: a real-options approach. Journal of Financial Economics 38, 297–331.
Grenadier, S., 1996. Leasing and credit risk. Journal of Financial Economics 42, 333–364.
Gul, F., Sonneschein, H., Wilson, R., 1986. Foundations of dynamic monopoly and the coase conjecture. Journal of Economic Theory 39, 155–190.
Hendel, I., Lizzeri, A., 1999. Adverse selection in durable goods market. American Economic Review 89, 1097–1115.
Hendel, I., Lizzeri, A., 2002. The role of leasing under adverse selection. Journal of Political Economy 110, 113–143.
Huang, S., Yang, Y., Anderson, K., 2001. A theory of finitely durable goods monopoly with used-goods market and transaction costs. Management Science 47, 1515–1532.
Im, K., Pesaran, M., Shin, Y., 2003. Testing for unit roots in heterogeneous panels. Journal of Econometrics 115, 53–74.
Janssen, M., Karamychev, V., 2002. Cycles and multiple equilibria in the market for durable lemons. Economic Theory 20, 579–601.
Johnson, J., Waldman, M., 2003. Leasing, lemons and buybacks. RAND Journal of Economics 34, 247–265.
Konishi, H., Sandfort, M., 2002. Existence of stationary equilibrium in the markets for new and used durable goods. Journal of Economic Dynamics & Control 26, 1029–1052.
Levin, A., Lin, C., Chu, C., 2002. Unit root tests in panel data: asymptotic and finite-sample properties. Journal of Econometrics 108, 1–24.
Maddala, G., Wu, S., 1999. A comparative study of unit root tests with panel data and a new simple test. Oxford Bulletin of Economics and Statistics 61, 631–652.
Mannering, F., Winston, C., Starkey, W., 2002. An exploratory analysis of automobile leasing by US households. Journal of Urban Economics 52, 154–176.
McConnell, J.J., Schallheim, J.S., 1983. Valuation of asset leasing contracts. Journal of Financial Economics 12, 237–261.
Miller, M., Upton, C., 1976. Leasing, buying and the cost of capital services. Journal of Finance 31, 761–786.
Myers, S., Dill, D., Bautista, A., 1976. Valuation of financial lease contracts. Journal of Finance 31, 799–819.
Petrin, A., 2002. Quantifying the benefits of new products: the case of the minivan. Journal of Political Economy 110, 705–729.
Purohit, D., 1997. Dual distribution channels: the competition between rental agencies and dealers. Marketing Science 16, 228–245.
Purohit, D., Staelin, R., 1994. Rentals, sales, and buybacks: managing secondary distribution channels. Journal of Marketing Research 31, 325–338.
Rust, J., 1985. Stationary equilibrium in a market for durable assets. Econometrica 53, 783–805.
Rust, J., 1986. When is it optimal to kill off the market for used durable goods? Econometrica 54, 65–86.
Saggi, K., Vettas, N., 2000. Leasing versus selling and firm efficiency in oligopoly. Economics Letters 66, 361–368.
Schallheim, J.S., McConnell, J.J., 1985. A model for the determination of "Fair" premiums on lease cancellation insurance policies. Journal of Finance 40, 1439–1457.
Schallheim, J.S., Johnson, R.E., Lease, R.C., McConnell, J.J., 1987. The determinants of yields on financial leasing contracts. Journal of Financial Economics 19, 45–67.
Schiraldi, P., 2011. Automobile replacement: a dynamic structural approach. RAND Journal of Economics 42, 266–291.
Singh, N., Vives, X., 1984. Price and quantity competition in a differentiated duopoly. Rand Journal of Economics 15, 546–554.
Sobel, J., 1984. The timing of sales. Review of Economic Studies 51, 353–368.
Sobel, J., 1991. Durable goods monopoly with entry of new consumers. Econometrica 59, 1455–1485.
Stokey, N., 1981. Rational expectations and durable goods pricing. Bell Journal of Economics 12, 112–128.
Stolyarov, D., 2002. Turnover of used durables in a stationary equilibrium: are older goods traded more? Journal of Political Economy 110, 1390–1413.
Trigeorgis, L., 1996. Evaluating leases with complex operating options. European Journal of Operational Research 91, 315–329.