A decoupling approach to integrated fault-tolerant control for linear systems with unmatched non-differentiable faults

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Abstract

This paper proposes a decoupling approach to the integrated design of fault estimation (FE) and fault-tolerant control (FTC) for linear systems in the presence of unknown bounded actuator faults and perturbations. An adaptive sliding mode augmented state unknown input observer is developed to estimate the system state, actuator faults and perturbations, based on a descriptor augmentation strategy and the *equivalent output injection* concept. Subsequently, an adaptive backstepping FTC controller is designed to compensate the effects of the faults and perturbations acting on the system to ensure robust output tracking. In the proposed observer the effects of the control system perturbations are estimated and the fault effects are compensated to ensure that the FE function is decoupled from the FTC system. This leads to satisfaction of the Separation Principle under the framework of integrated design. When compared with the existing H_∞ optimization single-step integrated FE/FTC design approach, in this paper the FE/FTC decoupling and the perturbation compensation (in the control) together contribute to a new integrated FTC strategy with more design freedom, less complexity and higher robustness. Moreover, the proposed method is shown to be applicable to a wide class of faults, which can be differentiable or non-differentiable, and matched or unmatched. Comparative simulations of the tracking control of a DC motor are provided to demonstrate the performance effectiveness of the proposed approach.

Key words: Decoupling approach, integrated fault-tolerant control, adaptive sliding mode augmented state unknown input observer, adaptive backstepping control, unmatched non-differentiable fault

1 Introduction

Fault-tolerant control (FTC) of automatic systems has attracted an increasing research interest aims to provide admissible robust system performance and improve system safety and reliability, in spite of system perturbations (including system uncertainty and/or external disturbance) and unknown faults (Blanke et al., 2006; Patton, 2015).

For the purpose of fault compensation by FTC design, fault information (magnitude, location, and time occurrence) is required. A direct and effective approach to attain fault information is fault estimation (FE), based on state observers, e.g., the sliding mode observers (SMOs) (Edwards et al., 2000; Huang et al., 2016), adaptive observers (Jiang et al., 2006), extended state observer (Gao & Ding, 2007), augmented state unknown input observer (ASUIO) (Lan & Patton, 2016), and high order SMO (de Loza et al., 2015). By a suitably designed observer the fault signals can be estimated and then be compensated within the FTC system conveniently (Patton, 2015). Previous studies show that observer-based FE methods can be very effective in FTC as long as appropriate robustness designs are considered.

A complex robustness problem arises when considering observer-based FE and FTC designs. Due to the feedforward action of system control input and output to the observer, the FE performance is affected by the perturbations. The FE feedback into the system through control action, on the other hand, introduces estimation uncertainty to the FTC system. This mutual uncertainty coupling is described as the *bi-directional robustness interactions* between the FE observer and FTC system, which break down the Separation Principle and give rise to a significant problem of integrating FE and FTC de-

⋆ Jianglin Lan acknowledges the funding support from the China Scholarship Council (No. 201406150074) and the Hull-China Scholarship for 2014 - 2017. The material in this paper was not presented at any conference.

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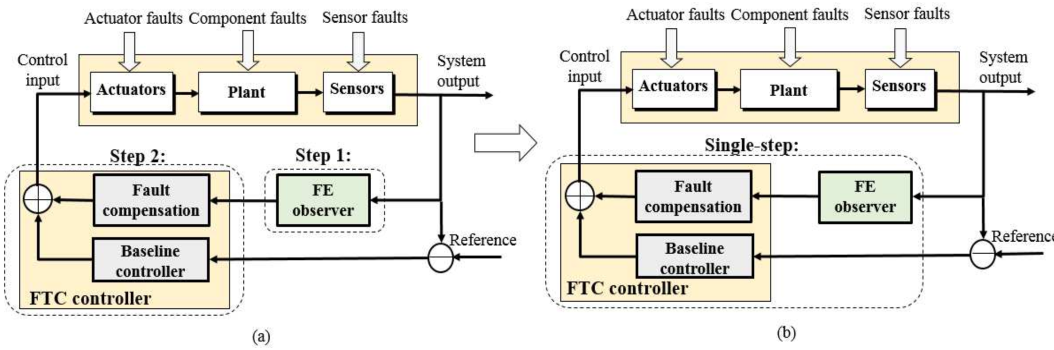

Fig. 1. Observer-based FE and FTC systems: (a) separated method and (b) integrated method (Lan & Patton, 2016)

signs to achieve required robust FTC performance (Lan & Patton, 2016).

However, most of the existing FTC systems using observer-based FE approach separate the designs of the FE observers and FTC systems (Fig. 1(a)), assuming satisfaction of the Separation Principle (e.g., Jiang et al. (2006); Gao & Ding (2007); de Loza et al. (2015)).

In Lan & Patton (2016) the concepts of *bi-directional robustness interactions* and integrated FE/FTC design (Fig. 1(b)) are defined. An effective strategy for integrated design for uncertain linear systems subject to actuator/sensor faults has been described, based on the combination of an ASUIO with sliding mode FTC and H_∞ optimization. This integrated approach effectively obtains all the observer and controller gains using a single-step linear matrix inequality (LMI) formulation. However, the approach is limited in the following aspects: 1) The faults considered are assumed to be continuously differentiable (with respect to time) and matched (with respect to the control input), which limits the applicability of the design; 2) Its solution is related to a bilinear matrix inequality (BMI) problem and the linearisation to an LMI formulation leads to an approach with low design freedom; 3) The perturbations are suppressed to minimize their effect on the FTC systems, resulting in conservative robust designs.

This paper describes an approach to overcome these limitations in order to achieve a more robust FE/FTC design which can cover a more general class of faults. The system considered here is a linear system with actuator faults and perturbations acting on both the state dynamics and system output. Contributions of this research are summarized as follows.

• *A novel FE observer is proposed to estimate more general faults.* Most FE methods in the literature assume the faults to be continuously differentiable (Jiang et al., 2006; Gao & Ding, 2007; Lan & Patton, 2016; de Loza et al., 2015). There is no such requirement in the SMOs (Edwards et al., 2000; Huang et al., 2016) by using the concept of *equivalent output injection.* However, the SMO (Edwards et al., 2000) has a canonical form in which several coordinate transformations are required. The other SMO (Huang et al., 2016) is designed based on H_∞ optimization. In this paper, a sliding mode ASUIO is proposed to estimate the system state, actuator fault and perturbation, without coordinate transformation and H_∞ optimization. An adaptive gain is introduced to cover the unknown fault bounds.

• *A decoupling FE/FTC approach is developed to offer more design freedom.* By using the descriptor approach in Lan & Patton (2015), the perturbation considered is augmented as a system state and estimated. Therefore, the proposed observer is unaffected by the control system perturbations. Moreover, with an appropriately designed switched component, the effect of the actuator fault on the estimation error dynamics is removed. By combining the above descriptor augmentation and SMO methods, the FE observer is decoupled from the FTC system, which recovers the Separation Principle and allows more freedom for the FE/FTC design. It should be noted that the proposed decoupling approach is different from the separated designs in the literature in that the *bi-directional robustness interactions* are taken in account.

• *Active perturbation cancellation contributes to a more robust FTC system.* As an alternative methodology to H_∞ robust optimization, disturbance-observer-based control has also been used to achieve robust system design (Chen et al., 2016). In the current work, instead of being suppressed, the perturbations in all the subsystems are compensated actively using adaptive backstepping control (de Loza et al., 2015). A more robust FTC system can then be achieved using this cancellation with an appropriate observer.

The paper is organized as follows. Section 2 formulates the problem. Section 3 describes the adaptive sliding mode ASUIO design and Section 4 presents the adaptive backstepping FTC design. A tutorial example of a DC motor is provided in Section 5. Finally, Section 6 concludes the study.

Notation: The symbol $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers, $\|\cdot\|$ is the Euclidean norm of a vector and the induced norm of a matrix, I_κ is a $\kappa \times \kappa$ identity matrix, $P_0^\dagger$ is the pseudo-inverse of a matrix P_0 and $\mathrm{He}(P_0) = P_0 + P_0^\top$, $\star$ is the transpose of the element on its symmetric position in a matrix, and $\mathrm{sign}(\omega)$ is the signum function of the variable ω defined by $\mathrm{sign}(\omega) = \omega/\|\omega\|$, and if $\omega = 0$, $\mathrm{sign}(\omega) = 0$.

2 Problem statement

Consider a linear system in the form of

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ff(t) + D_1 d(t),\\ y(t) &= Cx(t) + D_2 d(t),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, control input, and measure output vectors, respectively. $f \in \mathbb{R}^l$ is the actuator fault vector. $d \in \mathbb{R}^q$ is the perturbation vector including external disturbance and/or system uncertainty (Chen & Patton, 1999). The constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times l}$, $D_1 \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, and $D_2 \in \mathbb{R}^{p\times q}$ are known. To simplify the presentation the time index is omitted in the following study. The system (1) is assumed to satisfy the assumptions given below.

Assumption 2.1 *The pair (A, C) is observable, the pair (A, B) is controllable, and* $\mathrm{rank}(D_2) = q$.

Assumption 2.2 *There exists an unknown positive constant f_0 such that $\|f\| \le f_0$. The perturbation d is norm-bounded with a first-order time derivative.*

Remark 2.1 *It is rational to assume the perturbation d (including system uncertainty and/or external disturbance) to be differentiable. On the one hand, the system uncertainty is a function of the system state variables and it is continuously differentiable. On the other hand, according to the output regulation theory (Isidori, 1995), the external disturbance can be described as a differentiable exogenous system, which represents many disturbances in engineering, e.g., constant and harmonics. Although normally the distribution matrix D_1 of the perturbation cannot be obtained directly, an approximate modelling of it can be determined through several ways described in Chen & Patton (1999). The assumption of* $\mathrm{rank}(D_2) = q$ *is required for ensuring: 1) The observability of the perturbation d, i.e., the complete observability of the descriptor system (6), see Theorem 4.11 in Chapter 4.5 of Duan (2010); 2) The solvability of the matrix equation (25).*

Remark 2.2 *Compared with Lan & Patton (2016) and other works in the literature (e.g. Jiang et al. (2006); Gao & Ding (2007); de Loza et al. (2015)), this paper considers a more general class of actuator faults, which can be 1) differentiable or non-differentiable, and 2) matched or unmatched. The distribution matrix F represents the influence of faults on the system actuator and it is known if one has defined which faults are to be estimated and compensated.*

This study uses extensively the following definitions.

Definition 2.1 *With respect to the control input u, the actuator fault f can be classified as follows.*

- *Matched, if it is inside the range space spanned by u, i.e.,* $\mathrm{rank}(B, F) = \mathrm{rank}(B)$*;*

- *Unmatched, if it is outside the range space spanned by u, i.e.,* $\mathrm{rank}(B, F) \neq \mathrm{rank}(B)$.

Similar matched and unmatched definitions can also be made for the perturbation d.

This papers deals with the FE-based FTC problem for the system (1) under Assumptions 2.1 and 2.2. In order to estimate the system state x and the fault f, an ASUIO is proposed in Lan & Patton (2016), which is recalled below in brief.

Defining f as a new state and augmenting the system (1) into

$$\begin{aligned}\dot{\bar{x}}_o &= \bar{A}_o \bar{x}_o + \bar{B}_o u + \bar{D}_1 \bar{d},\\ y &= \bar{C}_o \bar{x}_o + D_2 d,\end{aligned} \tag{2}$$

where $\bar{x}_o = [x^\top \ f^\top]^\top$, $\bar{d} = [d^\top \ \dot{f}^\top]^\top$, and

$$\bar{A}_o = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}, \bar{B}_o = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} D_1 & 0 \\ 0 & I_l \end{bmatrix}, \bar{C}_o = [C \ 0].$$

The augmented state $\bar{x}_o$ is estimated by the ASUIO represented by

$$\begin{aligned}\dot{\xi}_o &= M_o \xi_o + G_o u + L_o y,\\ \hat{\bar{x}}_o &= \xi_o + H_o y,\end{aligned} \tag{3}$$

where the vectors ξ_o and $\hat{\bar{x}}_o$ are the observer system state and the augmented state estimate, respectively. M_o, G_o, L_o, and H_o are design matrices.

Define the estimation error as $e_o = \bar{x}_o - \hat{\bar{x}}_o$, then

$$\dot{e}_o = \Xi_2 e_o + \Xi_3 \xi_o + \Xi_4 u + \Xi_5 y + \chi_o, \tag{4}$$

where $\Xi_1 = I_{n+l} - H_o \bar{C}_o$, $L_o = L_{o1} + L_{o2}$, $\Xi_2 = \Xi_1 \bar{A}_o - L_{o1} \bar{C}_o$, $\Xi_3 = \Xi_1 \bar{A}_o - L_{o1} \bar{C}_o - M_o$, $\Xi_4 = \Xi_1 \bar{B}_o - G_o$, $\Xi_5 = (\Xi_1 \bar{A}_o - L_{o1} \bar{C}_o) H_o - L_{o2}$, and $\chi_o = \Xi_1 \bar{D}_1 \bar{d} - L_o D_2 d - H_o D_2 \dot{d}$.

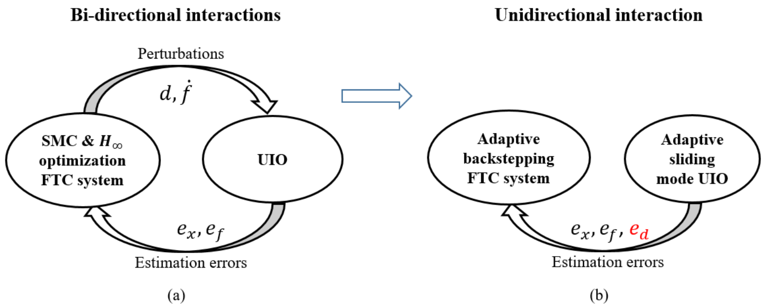

Fig. 2. Robustness interactions within (a) integrated and (b) decoupling FE/FTC systems

By designing $\Xi_i = 0$, $i = 3, 4, 5$, the error system (4) becomes

$$\dot{e}_o = \Xi_2 e_o + \chi_o. \tag{5}$$

The above ASUIO design is restrictive in the following two aspects:

1) The actuator fault f is required to be differentiable so that it can be augmented as a new system state, which limits the applicability of the ASUIO;

2) The error system (5) is affected by the uncertain term χ_0, which is a function of the system perturbations (d and $\dot{d}$) and the fault modelling errors $\dot{f}$. Since the controller uses the state and fault estimates, the estimation errors in turn affect the FTC system performance. Therefore, it is described in Lan & Patton (2016) that there exist *bi-directional robustness interactions* between the observer (3) and the FTC system. In other words, the FE and FTC functions used in Lan & Patton (2016) are coupled with each other, as shown in Fig. 2(a). As discussed in the Introduction, the strategy proposed in Lan & Patton (2016) uses H_∞ optimization to tackle with the interactions, which leads to a system design with limited design freedom.

To overcome the above limitations, this paper proposes a decoupling FE-based FTC strategy for the system (1) to handle the *bi-directional robustness interactions*. The proposed strategy includes: 1) An adaptive sliding mode ASUIO for estimating the system state, fault, and perturbation, and 2) an adaptive backstepping FTC controller for compensating the fault and perturbation to achieve satisfactory output tracking. The decoupling is realized by designing the adaptive sliding mode ASUIO to be decoupled from the control system, which enables the recovery of the Separation Principle and thus the ASUIO and the FTC controller can be designed separately.

3 Adaptive sliding mode ASUIO based FE design

The section describes the design of the adaptive sliding mode ASUIO for the system (1) to estimate the system state x, fault f, and perturbation d simultaneously.

3.1 Observer design

Inspired by Lan & Patton (2015), the perturbation d is regarded as a new system state variable and then the system (1) can be augmented into a descriptor form

$$\begin{aligned} E\dot{\bar{x}} &= \bar{A}\bar{x} + Bu + Ff, \\ y &= \bar{C}\bar{x}, \end{aligned} \tag{6}$$

where $\bar{x} = \left[x^\top \; d^\top\right]^\top$, $E = [I_n \; 0_{n\times q}]$, $\bar{A} = [A \; D_1]$, and $\bar{C} = [C \; D_2]$.

The augmented state $\bar{x}$ is estimated by the observer

$$\begin{aligned} \dot{z} &= Nz + Ju + Ly + Wv, \\ \hat{\bar{x}} &= z + Hy, \\ \hat{y} &= \bar{C}\hat{\bar{x}}, \end{aligned} \tag{7}$$

where the vector $z \in \mathbb{R}^{n+q}$ is the observer state and $\hat{\bar{x}} \in \mathbb{R}^{n+q}$ is the estimate of $\bar{x}$. $N \in \mathbb{R}^{(n+q)\times(n+q)}$, $J \in \mathbb{R}^{(n+q)\times m}$, $L \in \mathbb{R}^{(n+q)\times p}$, $W \in \mathbb{R}^{(n+q)\times p}$, and $H \in \mathbb{R}^{(n+q)\times p}$ are design matrices. The discontinuous switched component v is defined as $v = \rho_v \text{sign}(e_y)$, where $e_y = y - \hat{y}$ and ρ_v is a design scalar.

Define $\varepsilon = TE\bar{x} - z$, where $T \in \mathbb{R}^{(n+q)\times n}$ is a design matrix. It follows from (6) and (7) that

$$\begin{aligned} \dot{\varepsilon} &= N\varepsilon + (T\bar{A} - NTE - L\bar{C})\bar{x} + (TB - J)u \\ &\quad + TFf - Wv. \end{aligned} \tag{8}$$

Define the estimation error of $\bar{x}$ as $e = \bar{x} - \hat{\bar{x}}$. According to (8) the composite error system is

$$\begin{aligned} \dot{\varepsilon} &= N\varepsilon + (T\bar{A} - NTE - L\bar{C})\bar{x} + (TB - J)u \\ &\quad + TFf - Wv, \\ e &= \varepsilon + (I_{n+q} - H\bar{C} - TE)\bar{x}. \end{aligned} \tag{9}$$

Design the following matrix equations,

$$T\bar{A} - NTE - L\bar{C} = 0, \tag{10}$$
$$TB - J = 0, \tag{11}$$
$$I_{n+q} - H\bar{C} - TE = 0. \tag{12}$$

Upon the satisfaction of the above matrix equations, it follows from (9) that $e = \varepsilon$ and

$$\dot{e} = Ne + TFf - Wv. \tag{13}$$

Design $W = P^{-1}\bar{C}^\top$ and $TF = P^{-1}\bar{C}^\top Q$, where P and Q are two matrices to be designed in Section 3.2. Substituting $W = P^{-1}\bar{C}^\top$ and $TF = P^{-1}\bar{C}^\top Q$ into (13) gives

$$\dot{e} = Ne + P^{-1}\bar{C}^\top(Qf - v). \tag{14}$$

According to Definition 2.1, the fault function Qf is matched with respect to the switched component v, thus its effect on the estimation error dynamics can be totally removed by an appropriately designed v (see Section 3.2) and an idealized sliding motion can be achieved (Edwards et al., 2000). By using the *equivalent output injection* concept, the fault f can then be reconstructed through the equivalent output injection signal.

The proposed observer (7) overcomes the limitations described in Section 2, as shown below.

1) The proposed observer (7) estimates the fault f using the equivalent output injection signal, which has no requirement on the differentiation of the fault;

2) By augmenting the perturbation d as a new system state variable, the only unknown input acting on the error dynamics (14) is f. Furthermore, since Qf is matched, it can be totally compensated by the switched component v and the idealized sliding motion can be reached. In the idealized sliding motion, the error dynamics (14) are reduced to be

$$\dot{e} = Ne.$$

It can be seen that the above error dynamics are not affected by the control system and by designing N to be Hurwitz, then $e(t)$ converges to zero asymptotically. Therefore, there exists only a *unidirectional robustness interaction* between the FE observer and the FTC system (see Fig. 2(b)), rather than the bi-directional interactions in Lan & Patton (2016) (see Fig. 2(a)). This means that the observer (7) is decoupled from the FTC system and the Separation Principle is recovered for the proposed FE observer and FTC system designs, which allows more design freedom.

Remark 3.1 *This paper follows a new Separation Principle achieved by the use of a novel FE observer design (7) that is decoupled from the FTC system. It is different from the classical Separation Principle used extensively in the literature (e.g., Jiang et al. (2006); Gao & Ding (2007); de Loza et al. (2015)). Their designs cannot achieve overall robust FTC system performance since they ignore the existing bi-directional robustness interactions between the observer and control system. In this paper, however, the interactions are taken into account and eliminated in the observer and controller designs. Therefore, the Separation Principle used in this paper should be discussed under the integrated design framework.*

3.2 Estimation performance analysis

This section provides an analysis of the estimation performance of the observer (7), as given in Theorem 3.1.

Theorem 3.1 *Under Assumptions 2.1 - 2.2, the observer (7) estimates the augmented system state $\bar{x}$ and the actuator fault f accurately, if there exists a symmetric matrix $P \in \mathbb{R}^{(n+q)\times(n+q)}$, a matrix $Q \in \mathbb{R}^{p\times l}$, and a positive constant ξ, such that*

$$PN + N^\top P < -\xi I_{n+q}, \tag{15}$$
$$PTF = \bar{C}^\top Q. \tag{16}$$

The fault is estimated by: $\hat{f} = (\bar{C}P^{-1}\bar{C}^\top Q)^\dagger \bar{C}P^{-1}\bar{C}^\top v_{eq}$, where v_{eq} is the equivalent output injection signal.

Proof 3.1 *(a) Augmented system state estimation*

Consider a Lyapunov function $V_{e_0} = e^\top Pe$, where $P \in \mathbb{R}^{(n+q)\times(n+q)}$ is a symmetric positive definite matrix. By designing $W = P^{-1}\bar{C}^\top$ and the matrix equality (16), the time derivative of V_{e_0} along the error system is

$$\dot{V}_{e_0} = e^\top(PN + N^\top P)e + 2e^\top\bar{C}^\top(Qf - v). \tag{17}$$

By using $e_y = \bar{C}e$, $e_y^\top v = \rho_v\|e_y\|$, and $\|f\| \leq f_0$ (see Assumption 2.2), (17) becomes

$$\begin{aligned}\dot{V}_{e_0} &= e^\top(PN + N^\top P)e + 2(e_y^\top Qf - e_y^\top v)\\ &\leq e^\top(PN + N^\top P)e + 2\|e_y\|(\rho - \rho_v),\end{aligned} \tag{18}$$

where $\rho = \|Q\|f_0$.

In order to compensate the unknown scalar ρ, design $\rho_v = \hat{\rho} + \epsilon$, where ϵ is a positive design constant and $\hat{\rho}$ is used to estimate ρ and designed as

$$\dot{\hat{\rho}} = \sigma_0\|e_y\| \tag{19}$$

with a positive design constant σ_0.

Define the estimation error of ρ as $\tilde{\rho} = \rho - \hat{\rho}$. Consider a Lyapunov function $V_e = V_{e_0} + \frac{1}{\sigma_0}\tilde{\rho}^2$. According to (15), (18), and (19) and using the fact that $\dot{\rho} = 0$, then

$$\begin{aligned}\dot{V}_e &= \dot{V}_{e_0} + \frac{2}{\sigma_0}\tilde{\rho}\left(-\sigma_0\|e_y\|\right) \\ &\leq e^\top \left(PN + N^\top P\right) e + 2\|e_y\|\left(\rho - \hat{\rho} - \epsilon - \tilde{\rho}\right) \\ &\leq -\xi\|e\|^2 - 2\epsilon\|e_y\| \\ &\leq -\xi\|e\|^2. \end{aligned} \tag{20}$$

Taking integration on both sides of (20) from 0 to ∞ gives $V_e(t) \leq V_e(0) - \int_0^\infty \xi\|e(\tau)\|^2 d\tau$. Thus, $\int_0^\infty \xi\|e(\tau)\|^2 d\tau \leq V_e(0)$ and $\lim_{t\to\infty}\int_0^t \xi\|e(\tau)\|^2 d\tau \leq V_e(0) < \infty$. It follows from Barbalat's lemma (Slotine et al., 1991) that $\lim_{t\to\infty}\xi\|e(t)\|^2 = 0$, which implies that $\lim_{t\to\infty} e(t) = 0$. Therefore, the sliding surface $Q^\top e_y = 0$ is reachable and the observer (7) estimates the augmented system state $\bar{x}$ accurately.

(b) Actuator fault estimation

It follows from $e_y = \bar{C}e$ and (14) that

$$\dot{e_y} = \bar{C}Ne + \bar{C}P^{-1}\bar{C}^\top(Qf - v). \tag{21}$$

It is proved in (a) that the sliding motion takes place. During the sliding motion, $e_y = 0$ and $\dot{e}_y = 0$. Hence, (21) becomes

$$0 = \bar{C}Ne + \bar{C}P^{-1}\bar{C}^\top(Qf - v),$$

where v_{eq} is the so-called equivalent output injection signal that represents the average behaviour of the switched component v and the effort necessary to maintain the sliding motion (Edwards et al., 2000).

Therefore, the actuator fault f can be equivalently represented as

$$f = (\bar{C}P^{-1}\bar{C}^\top Q)^\dagger(\bar{C}P^{-1}\bar{C}^\top v_{eq} - \bar{C}Ne). \tag{22}$$

Design the actuator fault estimation as

$$\hat{f} = (\bar{C}P^{-1}\bar{C}^\top Q)^\dagger\bar{C}P^{-1}\bar{C}^\top v_{eq}. \tag{23}$$

Define the fault estimation error as $e_f = f - \hat{f}$, then it follows from (22) and (23) that

$$e_f = -(\bar{C}P^{-1}\bar{C}^\top Q)^\dagger\bar{C}Ne. \tag{24}$$

Since the estimation error e converges to zero, e_f also converges to zero. Thus, $\hat{f}$ in (23) is an accurate estimation of the actuator fault f. □

Remark 3.2 *The equivalent output injection signal v_{eq} can be obtained by passing the switched component v through an appropriately designed low-pass filter, i.e.,*

$$v_{eq} \cong \frac{1}{\tau s + 1}v,$$

where τ is a design time constant.

3.3 Observer parameters determination

In Sections 3.1 and 3.2 the FE observer (7) is described and the convergence of the estimation is analysed. A systematic way of determining the design matrices N, J, L, W, H, T, and Q is given below, by using a parametrization approach (Lan & Patton, 2015) based on the matrix equations (10) - (12) and Theorem 3.1.

The matrix equation (12) can be rearranged as

$$[T\ H]\begin{bmatrix} E \\ \bar{C} \end{bmatrix} = I_{n+q}. \tag{25}$$

Define $\Omega_1 = \begin{bmatrix} E \\ \bar{C} \end{bmatrix}$ and $\Sigma_1 = I_{n+q}$. Since $\text{rank}(\Omega_1) = \text{rank}\begin{bmatrix} \Omega_1 \\ \Sigma_1 \end{bmatrix} = n+q$, the matrix equation (25) is solvable and its general solution is

$$[T\ H] = \Sigma_1\Omega_1^\dagger - Y_1(I_{n+p} - \Omega_1\Omega_1^\dagger),$$

where Y_1 is any real matrix with the dimension of $(n+q) \times (n+p)$. Then T and H can be parametrized as

$$T = T_1 - Y_1T_2,\ H = H_1 - Y_1H_2, \tag{26}$$

with

$$\begin{aligned} T_1 &= \Sigma_1\Omega_1^\dagger\begin{bmatrix} I_n \\ 0 \end{bmatrix},\ T_2 = (I_{n+p} - \Omega_1\Omega_1^\dagger)\begin{bmatrix} I_n \\ 0 \end{bmatrix}, \\ H_1 &= \Sigma_1\Omega_1^\dagger\begin{bmatrix} 0 \\ I_p \end{bmatrix},\ H_2 = (I_{n+p} - \Omega_1\Omega_1^\dagger)\begin{bmatrix} 0 \\ I_p \end{bmatrix}. \end{aligned}$$

It follows from (12) that $TE = I_{n+q} - H\bar{C}$. Substituting it into (10) gives

$$[N\ \bar{L}]\begin{bmatrix} I_{n+q} \\ \bar{C} \end{bmatrix} = T\bar{A}, \tag{27}$$

where $\bar{L} = L - NH$.

Define $\Omega_2 = \begin{bmatrix} I_{n+q} \\ \bar{C} \end{bmatrix}$ and $\Sigma_2 = T\bar{A}$. Since $\text{rank}(\Omega_2) = \text{rank}\begin{bmatrix} \Omega_2 \\ \Sigma_2 \end{bmatrix} = n+q$, the matrix equation (27) is solvable and its general solution is

$$[N \; \bar{L}] = \Sigma_2 \Omega_2^{\dagger} - Y_2(I_{n+q+p} - \Omega_2 \Omega_2^{\dagger}),$$

where Y_2 is any real matrix with the dimension of $(n+q) \times (n+q+p)$. Hence, the matrices N and $\bar{L}$ are given by

$$N = N_1 - Y_2 N_2, \; \bar{L} = \bar{L}_1 - Y_2 \bar{L}_2, \tag{28}$$

with

$$N_1 = \Sigma_2 \Omega_2^{\dagger} \begin{bmatrix} I_{n+q} \\ 0 \end{bmatrix}, N_2 = (I_{n+q+p} - \Omega_2 \Omega_2^{\dagger}) \begin{bmatrix} I_{n+q} \\ 0 \end{bmatrix},$$
$$\bar{L}_1 = \Sigma_2 \Omega_2^{\dagger} \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \bar{L}_2 = (I_{n+q+p} - \Omega_2 \Omega_2^{\dagger}) \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

It can be seen that once the matrices Y_1 and Y_2 are determined, by using the parametrizations (26) and (28), the matrix equations (10) - (12) can be solved and all the observer design matrices can thus be obtained.

However, do such matrices Y_1 and Y_2 really exist? The answer is yes, as is shown in Lemma 3.1.

Lemma 3.1 *There exist matrices Y_1 and Y_2 such that the matrix equations (10) - (12) are solvable.*

Proof 3.2 *See Appendix A.* □

According to Lemma 3.1, by substituting the parametrizations of N and T into (15) and (16) and solving Theorem 3.2, then the matrices Y_1 and Y_2 can be obtained and so as all the observer parameters.

Theorem 3.2 *Under Assumptions 2.1 - 2.2, the observer (7) can estimate the augmented system state $\bar{x}$ and the actuator fault f accurately, if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{(n+q)\times(n+q)}$, matrices $Q \in \mathbb{R}^{p\times l}$ and $M \in \mathbb{R}^{(n+q)\times(2n+2p+q)}$, and positive constants ξ and β, such that*

$$\text{He}\left(PT_1\Phi - M \begin{bmatrix} T_2\Phi \\ N_2 \end{bmatrix} \right) < -\xi I_{n+q}, \tag{29}$$

$$\begin{bmatrix} \beta I & (PT_1 - M\hat{T}_2)F - \bar{C}^{\top} Q \\ \star & \beta I \end{bmatrix} > 0, \tag{30}$$

where $\hat{T}_2 = [T_2; 0]$. Then the matrices Y_1 and Y_2 are given by

$$Y_1 = P^{-1} M \begin{bmatrix} I_{n+p} \\ 0 \end{bmatrix}, \; Y_2 = P^{-1} M \begin{bmatrix} 0 \\ I_{n+p+q} \end{bmatrix}.$$

Proof 3.3 *Substituting (26) and (28) into (15) and (16) and defining $M = PY$, then the inequality (29) is derived from (15) directly. Moreover, by using the method described in Corless & Tu (1998), the equality constraint (16) can be converted into an inequality (30).* □

According to the above analysis, the systematic way of determining the observer design matrices is summarized as follows:
Step 1. Given ξ and β, solving the LMIs (29) and (30) yields the matrices P, Y_1, and Y_2;
Step 2. Substituting Y_1 and Y_2 into (26) and (28) gives the matrices T, H, N, and $\bar{L}$. According to the definition made in (27), $L = \bar{L} + NH$;
Step 3. Substituting T into (11) gives $J = TB$, and calculating W from $W = P^{-1}\bar{C}$.

4 Adaptive backstepping FTC design

Since in the system (1) the fault f and perturbation d considered are unmatched, their effect on the system dynamics cannot be compensated through direct control actions as described in Lan & Patton (2016). Inspired by de Loza et al. (2015) in which backstepping control is used to compensate unmatched perturbations, this section proposes an adaptive backstepping FTC controller to compensate f and d and achieve output tracking.

4.1 System reformulation

In order to use backstepping control, rearranging (1) into a strict-feedback form (de Loza et al., 2015)

$$\dot{x}_i = A_i \bar{x}_i + B_i(x_{i+1} + F_i f + S_i d), \; i = 1, 2, \ldots, r, \tag{31}$$

where $x_i \in \mathbb{R}^{n_i}$ are the new system state vectors, $x_{r+1} = u$, and x_1 is the system output. $\bar{x}_i = [x_1^{\top} \cdots x_i^{\top}]^{\top}$, $\text{rank}(B_i) = n_i$ and $\sum_{i=1}^{r} n_i = n$. The matrices A_i, F_i, and S_i are of compatible dimensions. The original system state is $x = [x_1^{\top} \cdots x_r^{\top}]^{\top}$.

Remark 4.1 *Many physical systems can be rearranged into a strict-feedback form required for backstepping control design (Krstic et al., 1995). Moreover, using the decomposition algorithm described in Polyakov (2012), a controllable system (1) can always be decomposed into the required block-controllable (strict-feedback) form. Backstepping control is also used in de Loza et al. (2015) for systems in the form of (31) for actuator fault and perturbation compensation. However, the estimation error* *effect on the control system is not taken into account and in their work a separated approach is used to obtain the FE and FTC gains.*

4.2 FTC controller design

The backstepping FTC design aims to 1) compensate the actuator fault f and perturbation d and 2) ensure that the system output x_1 can track a given reference x_d, using the system state estimate $\hat{x}_i$, fault estimate $\hat{f}$, and perturbation estimate $\hat{d}$.

Define the estimation errors as $e_{x_i} = x_i - \hat{x}_i$, $e_{\bar{x}_i} = \bar{x}_i - \hat{\bar{x}}_i$, $i = 1, 2, \ldots, r$, $e_f = f - \hat{f}$, and $e_d = d - \hat{d}$. Although it is proved in Theorem 3.1 that all these estimation errors are bounded and converge to zero, they still have side effects on the transient performance of the FTC system, which should be taken into account in the control design. Therefore, an adaptive method is incorporated with backstepping control to estimate and compensate the estimation error effect automatically.

4.2.1 Step i ($1 \leq i \leq r-1$)

Define $z_i = \hat{x}_i - \alpha_{i-1}$, where α_{i-1} is a design virtual control and $z_0 = 0$ and $\alpha_0 = x_d$. It follows from (31) that

$$\dot{z}_i = A_i\bar{x}_i + B_i\left(x_{i+1} + F_i f + S_i d\right) - \dot{e}_{x_i} - \dot{\alpha}_{i-1}. \tag{32}$$

Consider a Lyapunov function $V_{z_{i0}} = \frac{1}{2}z_i^\top z_i$, then its derivative along (32) is

$$\begin{aligned}\dot{V}_{z_{i0}} = {} & z_i^\top\left[A_i\bar{x}_i + B_i\left(\alpha_i + F_i f + S_i d\right)\right] + z_i^\top B_i z_{i+1} \\ & + z_i^\top(B_i e_{x_{i+1}} - \dot{e}_{x_i} - \dot{\alpha}_{i-1}).\end{aligned} \tag{33}$$

In order to ensure satisfactory tracking, compensate the fault and perturbation, and cancel the feed through side effects from the estimation error system (13) (i.e., the term $z_1^\top(B_i e_{x_{i+1}} - \dot{e}_{x_i})$ in (33)), α_i is designed as

$$\begin{aligned}\alpha_i = {} & -B_i^{-1}\left[c_i z_i + \rho_{z_i}\mathrm{sign}(z_i) + B_{i-1}^\top z_{i-1} + A_i\hat{\bar{x}}_i\right] \\ & - F_i\hat{f} - S_i\hat{d},\end{aligned} \tag{34}$$

where c_i is a positive design constant and ρ_{z_i} is an adaptive parameter to be determined.

Substituting (34) into (33) gives

$$\begin{aligned}\dot{V}_{z_{i0}} \leq {} & -c_i\|z_i\|^2 - z_{i-1}^\top B_{i-1} z_i + z_i^\top B_i z_{i+1} \\ & + (\rho_i - \rho_{z_i})\|z_i\|,\end{aligned} \tag{35}$$

where ρ_i is an unknown constant satisfying $\rho_i \geq \|A_i e_{\bar{x}_i} + B_i F_i e_f + B_i S_i e_d + B_i e_{x_{i+1}} - \dot{e}_{x_i} - \dot{\alpha}_{i-1}\|$, which represents the side effect of the estimation errors on the z_i subsystem.

In order to cancel ρ_i, define $\rho_{z_i} = \hat{\rho}_i + \epsilon_i$. ϵ_i is a positive design constant and $\hat{\rho}_i$ is used to estimate ρ_i with

$$\dot{\hat{\rho}}_i = \sigma_i\|z_i\|, \tag{36}$$

where σ_i is a positive design constant.

Define the estimation error of ρ_i as $\tilde{\rho}_i = \rho_i - \hat{\rho}_i$. Consider a Lyapunov function $V_{z_i} = V_{z_{i0}} + \frac{1}{2\sigma_i}\tilde{\rho}_i$. According to (35) and (36),

$$\begin{aligned}\dot{V}_{z_i} &= \dot{V}_{z_{i0}} + \frac{1}{\sigma_i}\tilde{\rho}_i(-\sigma_i\|z_i\|) \\ &\leq -c_i\|z_i\|^2 - z_{i-1}^\top B_{i-1} z_i + z_i^\top B_i z_{i+1}.\end{aligned} \tag{37}$$

For the first i steps, consider the Lyapunov function $V_i = V_{i-1} + V_{z_i}$ and define $V_0 = 0$. It follows from (37) that

$$\dot{V}_i \leq -\sum_{j=1}^{i} c_j\|z_j\|^2 + z_i^\top B_i z_{i+1}. \tag{38}$$

4.2.2 Step r

Note that

$$\dot{z}_r = A_r\bar{x}_r + B_r(u + F_r f + S_r d) - \dot{e}_{x_r} - \dot{\alpha}_{r-1}. \tag{39}$$

Consider the Lyapunov function $V_{z_{r0}} = \frac{1}{2}z_r^\top z_r$. Its derivative along (39) is

$$\begin{aligned}\dot{V}_{z_{r0}} = {} & z_r^\top\left[A_r\bar{x}_r + B_r(u + F_r f + S_r d)\right] \\ & - z_r^\top\left(\dot{e}_{x_r} + \dot{\alpha}_{r-1}\right).\end{aligned} \tag{40}$$

The FTC controller u is designed as

$$\begin{aligned}u = {} & -B_r^{-1}\left[c_r z_r + \rho_{z_r}\mathrm{sign}(z_r) + B_{r-1}^\top z_{r-1} + A_r\hat{\bar{x}}_r\right] \\ & - F_r\hat{f} - S_r\hat{d},\end{aligned} \tag{41}$$

where c_r is a design constant and ρ_{z_r} is an adaptive parameter to be designed.

Substituting (41) into (40) yields

$$\dot{V}_{z_{r0}} \leq -c_r\|z_r\|^2 - z_{r-1}^\top B_{r-1} z_r + (\rho_r - \rho_{z_r})\|z_r\|, \tag{42}$$

where ρ_r is an unknown constant such that $\rho_r \geq \|A_r e_{\bar{x}_r} + B_r F_r e_f + B_r S_r e_d - \dot{e}_{x_r} - \dot{\alpha}_{r-1}\|$.

Define $\rho_{z_r} = \hat{\rho}_r + \epsilon_r$, where ϵ_r is a positive design constant and $\hat{\rho}_r$ is the estimate of ρ_r updated by

$$\dot{\hat{\rho}}_r = \sigma_r\|z_r\|, \tag{43}$$

with a positive design constant σ_r.

Define the estimation error of ρ_r as $\tilde{\rho}_r = \rho_r - \hat{\rho}_r$. Consider a Lyapunov function $V_{z_r} = V_{z_{r0}} + \frac{1}{2\sigma_r}\tilde{\rho}_r^2$. According to (42) and (43),

$$\begin{aligned}\dot{V}_{z_r} &= \dot{V}_{z_{r0}} + \frac{1}{\sigma_r}\tilde{\rho}_r(-\sigma_r\|z_r\|) \\ &\leq -c_r\|z_r\|^2 - z_{r-1}^\top B_{r-1} z_r. \end{aligned} \tag{44}$$

Finally, for the overall control system consider the Lyapunov function $V_r = V_{r-1} + V_{z_r}$. It follows from (38) and (44) that

$$\dot{V}_r \leq -\sum_{j=1}^{r} c_j\|z_j\|^2. \tag{45}$$

Define $\Psi_z = \sum_{j=1}^{r} c_j\|z_j\|^2$. By designing $c_j > 0$, $j = 1, 2, \ldots, r$, it holds that $\Psi_z \geq 0$. Taking integration on both sides of (45) from 0 to ∞ gives $V_r(t) \leq V_r(0) - \int_0^\infty \Psi_z(\tau)d\tau$. Thus, $\int_0^\infty \Psi_z(\tau)d\tau \leq V_r(0)$ and $\lim_{t\to\infty}\int_0^t \Psi_z(\tau)d\tau \leq V_r(0) < \infty$. It follows from Barbalat's lemma (Slotine et al., 1991) that $\lim_{t\to\infty}\Psi_z(t) = 0$, which implies that $\lim_{t\to\infty} z_j(t) = 0$, $j = 1, 2, \ldots, r$. Therefore, in the presence of actuator fault and perturbation, the system output x_1 tracks the reference x_d accurately.

Remark 4.2 *Although the proposed observer (7) is decoupled from the control system, in the transient period (i.e., before the estimation errors converge to zero) the estimation errors (whose effects are defined as* ρ_i, $i = 1, 2, \ldots, r$, *in (35) and (42)) inevitably affect the closed-loop FTC performance. To improve the transient performance of the closed-loop system, the following strategies are incorporated with the proposed design.*

• *Eigenvalue assignment for the observer. The FTC system performance can be largely recovered if the observer dynamics are (much) faster than the closed-loop dynamics. To realize this, the eigenvalues of the matrix* N *are assigned to an acceptable LMI region (Chilali & Gahinet, 1996). Specifically, it is achieved by adding an eigenvalue assignment constraint (46) to the existing constraints (29) and (30) to place the eigenvalues of* N *into a strip region* (a, b)*, where* a *and* b *satisfying* $a < b < 0$.

$$\begin{bmatrix} \Pi - 2bP & 0 \\ \star & -\Pi + 2aP \end{bmatrix} < 0, \tag{46}$$

where $\Pi = \text{He}\left(PT_1\Phi - M\begin{bmatrix} T_2\Phi \\ N_2 \end{bmatrix}\right)$.

• *The estimation error effect on the FTC system is taken into account in the controller design through online estimation and compensation using the adaptive gains* $\hat{\rho}_i$, $i = 1, 2, \ldots, r$.

Remark 4.3 *In the special cases when the fault* f *and perturbation* d *are matched (i.e.,* $\text{rank}(B, F) = \text{rank}(B, D_1) = \text{rank}(B)$*), they can be compensated directly by introducing their estimates into the control action. Therefore, in such cases it is not necessary to represent the system into the triangular form (31) and FTC can be achieved through standard state-feedback control as described in (Lan & Patton, 2016).*

Remark 4.4 *The proposed FE-based FTC design achieves compensation of both the perturbation and fault. Although the work (Cao et al., 2011) also considers this kind of compensation problem, it focuses on a part of the disturbances modelled by a known linear exogenous system and requires full system state information. Moreover, the integrated FE/FTC design problem described in this paper is far beyond its concern. It is also worth noting that, in the absence of faults, the proposed approach is reduced to be a disturbance-observer-based control method which has been researched extensively and relates to significant potential industrial applications (Chen et al., 2016).*

5 A tutorial example

Consider the angular velocity tracking control of a DC motor modelled by (Lan & Patton, 2016)

$$\begin{aligned}\dot{x} &= Ax + B(u + f) + D_1 d, \\ y &= Cx + D_2 d, \end{aligned} \tag{47}$$

where $x = [w\ i_a]^\top$ is the state vector, $u = v_a$ is the control input, y is the output, f is the actuator fault, and d is the perturbation. The matrices are defined by

$$A = \begin{bmatrix} -\frac{B_0}{J_i} & \frac{K_m}{J_i} \\ -\frac{K_v}{L_a} & -\frac{R_a}{L_a} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{L_a} \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The physical parameters of the DC motor are defined as follows. w, i_a, and v_a are the the angular velocity, armature current, and armature voltage, respectively. R_a is the armature resistance. L_a is the inductance. K_v and K_m are the voltage and motor constants, respectively. J_i is the moment of inertia. B_0 is the friction coefficient.

Compared with the DC motor model in Lan & Patton (2016), in (47) the perturbation acting on the output y is also considered.

The angular velocity tracking reference is given as $x_d = 1$. The parameters of the DC motor are (Bélanger, 1995): $R_a = 1.2$, $L_a = 0.05$, $K_v = 0.6$, $K_m = 0.6$, $J_i = 0.1352$, and $B_0 = 0.3$.

Given $\gamma = 10$, $\beta = 0.1$, $a = -20$, and $b = -3$. By solving (29) and (30) in Theorems 3.2 and (46) in Remark 4.2, the obtained observer parameters are

$$P = \begin{bmatrix} 40.049 & 0 & -13.5837 \\ 0 & 26.4654 & 0 \\ -13.5837 & 0 & 40.049 \end{bmatrix},$$

$$N = \begin{bmatrix} -8.3085 & 1.7891 & -5.5415 \\ -3.4264 & -14.0715 & 3.0381 \\ -6.2116 & -1.4008 & -8.0813 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.7406 \\ 10.685 \\ 0.7425 \end{bmatrix}, \quad L = \begin{bmatrix} 0.1037 & 3.1067 \\ 0.0534 & -5.2902 \\ -0.0963 & -4.0504 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & -0.037 \\ 0 & 0.4657 \\ 1 & -0.0371 \end{bmatrix}, \quad Q = \begin{bmatrix} 19.6257 \\ 282.7828 \end{bmatrix}.$$

The other observer and controller parameters are chosen as: $c_1 = 35$, $\sigma_1 = 0.1$, $\epsilon_1 = 0.1$, $c_2 = 60$, $\sigma_2 = 0.1$, $\epsilon_2 = 0.1$, $\sigma_0 = 6000$, and $\epsilon = 1$.

Comparative simulations are performed for the DC motor (47) using the following four approaches:

• *Nominal design.* It includes a UIO (Chen & Patton, 1999) for state estimation and a state feedback controller, designed separately without FE/FTC.

• *Separated FE/FTC design.* It includes an ASUIO (Lan & Patton, 2016) for fault and state estimation and a state feedback FTC controller, designed separately by ignoring the perturbation in the observer design and the estimation errors in the control system.

• *Integrated FE/FTC design* (Lan & Patton, 2016). It includes an ASUIO for fault and state estimation and a state feedback FTC controller, designed together using a single-step LMI formulation by taking into account the effect of the perturbation and estimation errors.

• *Proposed decoupling FE/FTC design.*

In the separated and integrated designs, the perturbation d acting at the system output is treated as a sensor fault. Two cases of simulations are carried out with differentiable and non-differentiable actuator faults, respectively, using the same observer and controller parameters given above and the same zero initial conditions.

5.1 *Differentiable fault case*

Suppose the DC motor suffers from a differentiable actuator fault f and a perturbation d characterized by

$$d(t) = \begin{cases} 0.05\sin(\pi t), & 0 \text{ s} \le t \le 10 \text{ s} \\ 3\sin(4\pi t) + [0.1\ 0.5]x, & 10 \text{ s} < t \le 15 \text{ s} \end{cases},$$

$$f(t) = \begin{cases} 0, & 0 \text{ s} \le t \le 2 \text{ s} \\ 0.04(t-2)^2 + \sin(\pi(t-2)), & 2 \text{ s} < t \le 7 \text{ s} \\ 1, & 7 \text{ s} < t \le 10 \text{ s} \\ 2\sin(3\pi(t-10)) + 1, & 10 \text{ s} < t \le 15 \text{ s} \end{cases}.$$

The above f and d have different characteristics in different time periods, which are used to test the system performance under different fault and perturbation scenarios. Moreover, a Gaussian noise w with zero-mean and variance 0.001^2 is added to the measured outputs in the time interval $t \in (10, 15]$ s.

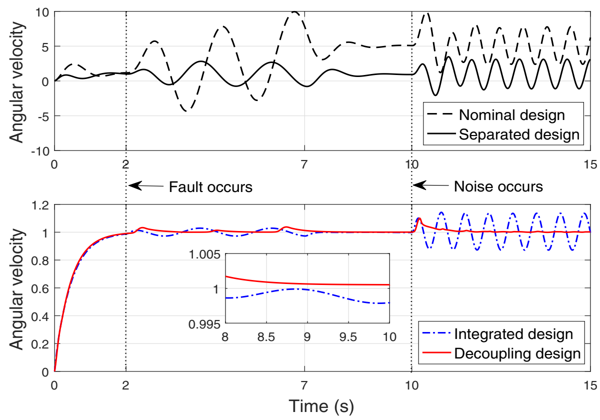

Fig. 3. Angular velocity: differentiable fault case

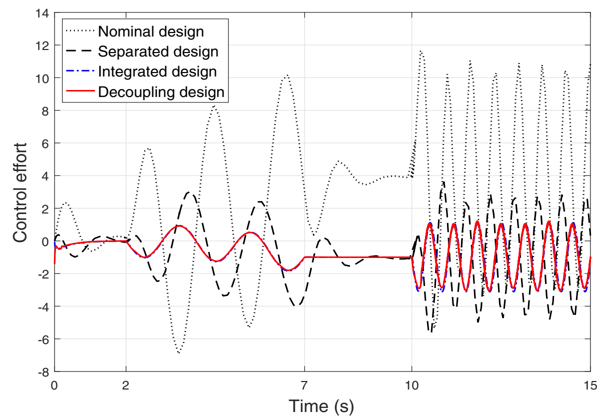

Fig. 4. Control effort: differentiable fault case

It is seen in Fig. 3 that among the four approaches simulated only the proposed decoupling approach achieves

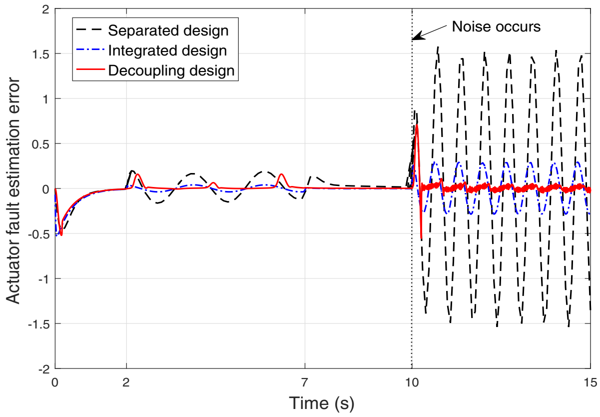

Fig. 5. Fault estimation: differentiable fault case

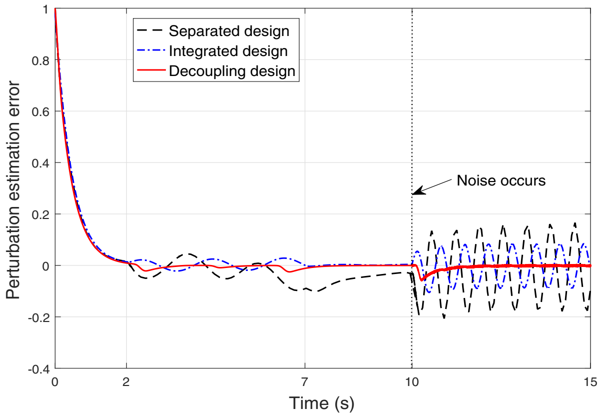

Fig. 6. Perturbation estimation: differentiable fault case

good tracking performance in the presence of actuator fault. Fig. 4 shows that the control efforts of the decoupling and integrated approaches are similar but much smaller than those of the rest two methods. As shown in Figs. 5 - 6, the decoupling method has better fault and perturbation estimation performance than the separated and integrated methods, even in the presence of measurement noise.

5.2 *Non-differentiable fault case*

Consider the case in which the DC motor is subject to a perturbation d and a non-differentiable fault f (a Weierstrass function that is smooth but nowhere differentiable (Hardy, 1916)) in the forms of

$$d(t) = 2\sin(2\pi t),\ 0 \text{ s} \leq t \leq 5 \text{ s},$$
$$f(t) = \sum_{k=0}^{50} 0.5^k \cos(3^k \pi t),\ 0 \text{ s} \leq t \leq 5 \text{ s}.$$

It is seen from Fig. 7 that neither of the nominal and sep-

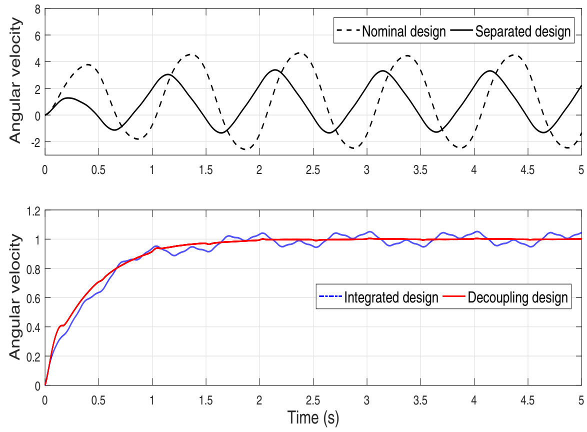

Fig. 7. Angular velocity: non-differentiable fault case

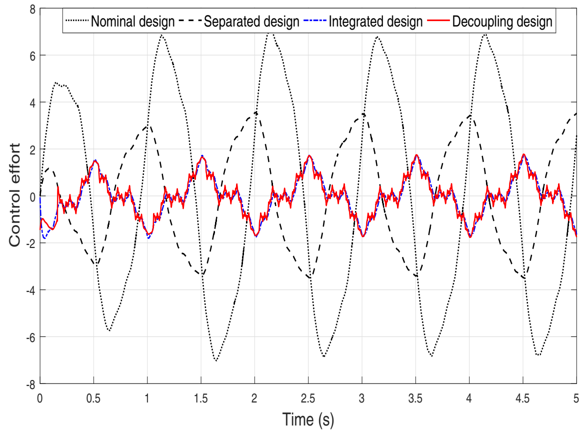

Fig. 8. Control effort: non-differentiable fault case

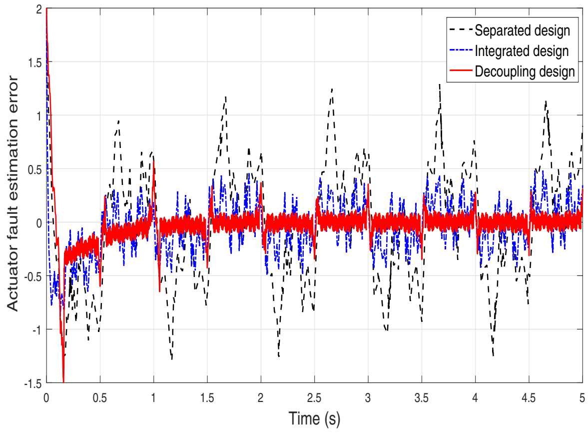

Fig. 9. Fault estimation: non-differentiable fault case

arated FE/FTC designs achieves angular velocity tracking. Although the angular velocity of the integrated design tracks the reference with small error, it has oscillatory dynamic response. Only the decoupling approach

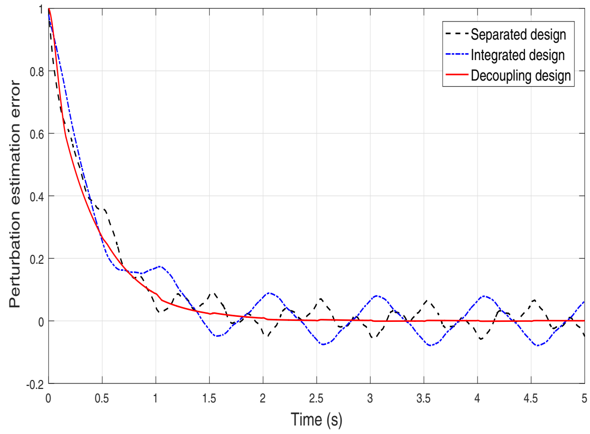

Fig. 10. Perturbation estimation: non-differentiable fault case

has good tracking performance. As shown in Fig. 8, the control efforts of the decoupling and integrated designs are similar but much smaller than those of the other two designs. Figs. 9 and 10 show that the decoupling approach has much better fault and perturbation estimation performance than the separated and integrated approaches.

Summarizing the above two simulation cases for the DC motor (47) subject to actuator faults (differentiable or non-differentiable) and perturbations, the superiority ranking of the four control designs from low to high, in terms of robust FE/FTC performance, is that 1) the nominal approach, 2) the separated approach, 3) the integrated approach, and 4) the proposed decoupling approach.

6 Conclusion

The *bi-directional robustness interactions* between the FE observer and FTC system give rise to an important integrated design problem. Although an effective single-step integrated FE/FTC strategy is proposed in Lan & Patton (2016) using H_∞ optimization, the design is conservative with low freedom. In this paper, a decoupling approach is proposed for integrated FE/FTC design for linear systems with actuator faults and perturbations. An adaptive sliding mode ASUIO is used to estimate the system state, fault, and perturbation. With the estimates an adaptive backstepping FTC controller is designed to compensate the fault and perturbation and ensure output tracking.

The proposed observer is advantageous in that it is decoupled from the FTC system, which adds great FE/FTC design freedom and the estimation uncertainty effect on the control system is handled by an adaptive method. Moreover, the actuator faults considered can be either differentiable or non-differentiable, and matched or unmatched. The comparative simulations of a DC motor demonstrate that the proposed decoupling approach has superiority over the approaches of nominal control, separated FE/FTC, and integrated FE/FTC, in the sense of acceptable robust FE and FTC performances. Future research will focus on an extension of the presented approach for nonlinear systems with perturbations and faults.

A Proof of Lemma 3.1

It follows from (26) and (28) that

$$N = (T_1 - Y_1 T_2)\bar{A}\Omega_2^\dagger \begin{bmatrix} I_{n+q} \\ 0 \end{bmatrix} - Y_2 N_2 = T_1 \Phi - Y T_{2N},$$

where $\Phi = \bar{A}\Omega_2^\dagger \begin{bmatrix} I_{n+q} \\ 0 \end{bmatrix}$, $T_{2N} = \begin{bmatrix} T_2\Phi \\ N_2 \end{bmatrix}$, $Y = [Y_1 \; Y_2]$.

Therefore, the matrix Y exists if the pair $(T_1\Phi, T_{2N})$ is observable, i.e.,

$$\text{rank} \begin{bmatrix} sI_{n+q} - T_1\Phi \\ T_2\Phi \\ N_2 \end{bmatrix} = n + q, \; \forall \, s \in \mathbb{C}. \tag{A.1}$$

A sufficient condition for (A.1) is

$$\text{rank} \begin{bmatrix} sI_{n+q} - T_1\Phi \\ T_2\Phi \end{bmatrix} = n + q. \tag{A.2}$$

Define $\bar{\Phi} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \Phi$. By using $T_1 = \Sigma_1 \Omega_1^\dagger \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, $\Sigma_1 = I_{n+q}$, and $T_2 = (I_{n+p} - \Omega_1\Omega_1^\dagger) \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, then

$$\begin{bmatrix} sI_{n+q} - T_1\Phi \\ T_2\Phi \end{bmatrix} = \begin{bmatrix} sI_{n+q} & -\Omega_1^\dagger \\ 0 & I_{n+p} - \Omega_1\Omega_1^\dagger \end{bmatrix} \begin{bmatrix} I_{n+q} \\ \bar{\Phi} \end{bmatrix}.$$

It is clear that rank $\begin{bmatrix} I_{n+q} \\ \bar{\Phi} \end{bmatrix} = n + q$. Thus, (A.2) holds if it can be proved that

$$\text{rank} \begin{bmatrix} sI_{n+q} & -\Omega_1^\dagger \\ 0 & I_{n+p} - \Omega_1\Omega_1^\dagger \end{bmatrix} = n + q. \tag{A.3}$$

Note that

$$\operatorname{rank}\begin{bmatrix} I_{n+q} & 0 \\ 0 & \Omega_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sI_{n+q} & 0 & \Omega_1^{\dagger}\Omega_1 \\ 0 & I_{n+p} & \Omega_1 \end{bmatrix}. \quad (A.4)$$

The left hand side of (A.4) is

$$\operatorname{rank}\begin{bmatrix} I_{n+q} & 0 \\ 0 & \Omega_1 \end{bmatrix} = n + q + \operatorname{rank}(\Omega_1). \quad (A.5)$$

The right hand side of (A.4) is

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} sI_{n+q} & 0 & \Omega_1^{\dagger}\Omega_1 \\ 0 & I_{n+p} & \Omega_1 \end{bmatrix} \\ &= \operatorname{rank}\left\{ \begin{bmatrix} I_{n+q} & -\Omega_1^{\dagger} \\ 0 & I_{n+p} - \Omega_1\Omega_1^{\dagger} \\ 0 & \Omega_1\Omega_1^{\dagger} \end{bmatrix} \begin{bmatrix} sI_{n+q} & 0 & \Omega_1^{\dagger}\Omega_1 \\ 0 & I_{n+p} & \Omega_1 \end{bmatrix} \right\} \\ &= \operatorname{rank}\begin{bmatrix} sI_{n+q} & -\Omega_1^{\dagger} & 0 \\ 0 & I_{n+p} - \Omega_1\Omega_1^{\dagger} & 0 \\ 0 & \Omega_1\Omega_1^{\dagger} & \Omega_1 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} sI_{n+q} & -\Omega_1^{\dagger} \\ 0 & I_{n+p} - \Omega_1\Omega_1^{\dagger} \end{bmatrix} + \operatorname{rank}(\Omega_1). \quad (A.6) \end{aligned}$$

By substituting (A.5) and (A.6) into (A.4), it can be concluded that (A.3) holds. Hence, the sufficient condition (A.2) is satisfied. This proves that the pair $(T_1\Phi, T_{2N})$ is observable and the matrices Y_1 and Y_2 exist.

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